

LUCIO DEMEIO[†] and GIOVANNI FROSALI[‡]

[†] Dipartimento di Matematica "V. Volterra"
Università di Ancona
60131 Ancona, Italy

[‡] Dipartimento di Matematica Applicata "G. Sansone"
Università di Firenze
50139 Firenze, Italy

**DIFFUSION LIMITS OF THE LINEAR BOLTZMANN
EQUATION IN EXTENDED KINETIC THEORY:
WEAK AND STRONG INELASTIC COLLISIONS.**

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ABSTRACT. We consider the asymptotic analysis for the linear Boltzmann equation with elastic and inelastic scattering. The physical model describes the motion of test particles propagating by elastic and inelastic collisions through a host medium in the Lorentz gas limit. The background is in thermodynamical equilibrium with only two internal energy levels. We apply the compressed Chapman-Enskog procedure to derive the diffusive-type approximations in the cases of dominant elastic and dominant inelastic collisions. Then we present numerical examples showing the time evolution of the distribution function in some physically relevant cases. In the appendix the successive overrelaxation method is briefly outlined.

1 Introduction

Asymptotic analysis is commonly used in kinetic theory to solve the Boltzmann equation by perturbation expansion and to approximate

the solutions of kinetic equations by the solutions of suitable fluidodynamical equations. When the kinetic equation is written in dimensionless form, the coefficients of each term can be compared with each other and one or more coefficients can be significantly larger than the others. Numerous systems are governed by two (or more) independent competitive processes. In order to illustrate the role of asymptotic analysis in linear kinetic theory, we consider a linear transport model in which the collisional mechanism with the background is dominant (this justifies the linearity) and which models elastic and inelastic scattering. Just to give an idea, let us consider an evolution initial value problem for a kinetic equation in abstract form

$$\frac{\partial f}{\partial t} = Sf + \frac{1}{\epsilon_1} C_e f + \frac{1}{\epsilon_2} C_i f$$

where C_e and C_i , as we will explain in detail in the next section, are the operators modeling elastic and inelastic collisions respectively. The parameters ϵ_1 and ϵ_2 characterize the importance of the two phenomena; they may be of the same order (both of them small) or one of them may be small and the other of order 1. For example, if we take $\epsilon_1 = 1$ and $\epsilon_2 = \epsilon$, then inelastic collisions are the mechanism which prevails when the system tends to equilibrium.

In a mathematical framework, we can suppose to have on the right-hand side a family of evolution operators depending on the parameter ϵ acting in a suitable Banach space X with a given initial datum.

The classical asymptotic analysis suggests to look for a solution in the form of a truncated power series

$$f_\epsilon^{(n)}(t) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + \cdots + \epsilon^n f_n(t)$$

and builds an algorithm to determine the coefficients $f_0, f_1, f_2, \dots, f_n$. $f_\epsilon^{(n)}(t)$ is an approximation of order n to the solution $f_\epsilon(t)$ of the original equation in the sense that

$$\|f_\epsilon(t) - f_\epsilon^{(n)}(t)\|_X = o(\epsilon^n),$$

for $0 \leq t \leq T$, where $T > 0$.

Sometimes this approximation does not hold in a neighborhood of $t = 0$, because of the existence of an initial layer where the estimate

is not uniform with respect to t . For this reason it is necessary to introduce an initial layer correction.

Our point of view is to find, in a systematic way, a new (simpler) family of operators still depending on ϵ , say B_ϵ , and a new evolution problem

$$\frac{\partial \varphi_\epsilon}{\partial t} = B_\epsilon \varphi_\epsilon.$$

such that the solutions $\varphi_\epsilon(t)$ of the new evolution problem satisfy

$$\|f_\epsilon(t) - \varphi_\epsilon(t)\|_X = o(\epsilon^n),$$

for $0 \leq t \leq T$, where $T > 0$.

In this work we will apply the Chapman-Enskog procedure in the version modified by J. Mika in [13], to a transport problem with elastic and inelastic scattering which will be introduced in Section 2. The main feature of the modified Chapman-Enskog procedure is that of decomposing the initial value problem into two problems for the kinetic and hydrodynamic parts of the solution respectively and of expanding only the kinetic part in series of ϵ , while leaving the hydrodynamic part unexpanded. This decomposition is performed by a projection of the unknown solution on the null-space of the dominant collision operator and on its complement. Then, a two time scaling is introduced to obtain the initial layer corrections. The asymptotic algorithm permits to derive in a natural way the solution of the hydrodynamic equation, the initial value for the hydrodynamic equation and the initial layer corrections. Hence, it is possible to prove, under suitable assumptions, that the error of the approximating solution is of order ϵ^2 , uniformly in $t \geq 0$.

The reader interested in understanding the essential advantages of the compressed Chapman-Enskog procedure is referred to the book by J. Mika and J. Banasiak [14]. Using the compressed procedure a rigorous asymptotic analysis of a linear Boltzmann equation with inelastic scattering is provided in [3, 4].

In this paper we consider the linear Boltzmann equation with elastic and inelastic collisions, describing the time evolution of an ensemble of test particles propagating through a background medium assumed at thermodynamical equilibrium. We consider the two limit cases of dominant elastic and dominant inelastic collisions, perform

the asymptotic analysis of the Boltzmann equation and derive the two hydrodynamic approximations by the compressed Chapman-Enskog method.

A similar analysis for a transport equation with down-scattering as the only effective inelastic mechanism was studied in [8]. This paper generalizes those results.

Finally, we look at some numerical examples which illustrate the relaxation of the isotropic component of the distribution function towards the kernel of the collision operator, the isotropization of the distribution function with time and the time behavior of the error of the diffusion approximation in the case of dominant inelastic collisions. We also report the numerical methods used in the solution of the Boltzmann equation.

2 The physical model

In this section we introduce a model in the framework of extended kinetic theory to describe the motion of an ensemble of test particles of mass m diffusing by elastic and inelastic collisions through a host medium of particles having mass M . In the literature of kinetic theory inelastic phenomena have attracted much attention, because inelastic collisions are very important in various traditional fields, and more recently in electron transport at low energies and in semiconductor theory [5, 6]. In [11], Garibotti and Spiga developed a formalism to include inelastic collisions in the Boltzmann equation.

We consider a gas of test particles endowed only with translational degrees of freedom, moving in a medium with internal degrees of freedom. We assume that the background medium consists of particles having only two energy levels, a ground level and an excited level, spaced by an energy jump ΔE . This assumption of only two significant energy levels is reasonable in the case of a particle gas at low temperature. Moreover we assume that the background medium is in thermodynamical equilibrium with temperature T . If n_1 and n_2 are the number densities of the background particles in the ground state and in the excited state respectively, the thermodynamical equi-

librium implies the following relation

$$b = \frac{n_2}{n_1} = \exp\left(-\frac{\Delta E}{kT}\right) < 1,$$

where k is the Boltzmann constant.

We are interested in the so-called Lorentz gas, obtained with the limit $m/M \rightarrow 0$ which describes a physical situation in which light test particles collide with a heavy background assumed at rest. The test particles can be deflected elastically or can attain a loss or a gain of a fixed amount of energy through inelastic collisions. Mass, momentum and energy of the interacting particles are conserved during the collision, but a quantity of energy is transferred from one type of particle to another by up-scattering or by down-scattering. Moreover we neglect particle-particle interactions, and the Boltzmann transport equation becomes linear.

In this paper we only consider a one dimensional model, but many of the papers quoted below on this subject are in the more realistic three dimensional setting. We denote by x the one-dimensional space variable, by $\xi = v^2$ the energy variable, by v the velocity modulus, and by μ the cosine of the polar angle. The space variable is rescaled by L and the time variable by the related characteristic time L/δ , where L is a typical macroscopic length and $\delta^2 = 2\Delta E/m$. Hence the dimensionless variable $\xi = \frac{v^2}{\delta^2}$ is used instead of the adimensionalized speed obtaining an energy jump equal to unity.

Let $f = f(x, \xi, \mu, t)$ be the distribution function of the test particles and let $\Phi = \Phi(x, \xi, \mu, t) = v f(x, \xi, \mu, t)$ be the flux function. By using the new dimensionless variables introduced above, the Boltzmann equation for the flux function is:

$$\begin{aligned} \frac{\partial \Phi}{\partial t}(x, \xi, \mu, t) + \sqrt{\xi} \mu \frac{\partial \Phi}{\partial x}(x, \xi, \mu, t) = \\ = \frac{n_1 L}{\delta} v_1^i(\xi + 1) \frac{1}{2} \int_{-1}^{+1} \Phi(x, \xi + 1, \mu', t) d\mu' \\ + b \frac{n_1 L}{\delta} H(\xi - 1) v_1^i(\xi) \sqrt{\frac{\xi}{\xi - 1}} \frac{1}{2} \int_{-1}^{+1} \Phi(x, \xi - 1, \mu', t) d\mu' \end{aligned}$$

$$\begin{aligned}
& - \frac{n_1 L}{\delta} H(\xi' - 1) v_1^i(\xi) \Phi(x, \xi, \mu, t) \\
& - b \frac{n_1 L}{\delta} v_1^i(\xi + 1) \sqrt{\frac{\xi + 1}{\xi}} \Phi(x, \xi, \mu, t) \\
& + \frac{n_1 L}{\delta} [v_1^e(\xi) + b v_2^e(\xi)] \frac{1}{2} \int_{-1}^{+1} \Phi(x, \xi, \mu', t) d\mu' \\
& - \frac{n_1 L}{\delta} [v_1^e(\xi) + b v_2^e(\xi)] \Phi(x, \xi, \mu, t),
\end{aligned}$$

where v_1^e , v_2^e and v_1^i , v_2^i are the elastic and the inelastic collision frequencies for the scattering of the test particles with the background particles in the fundamental and excited state respectively and we have used that the collision frequency v_2^i can be determined by the microreversibility conditions [11, 9]

$$v v_1^i(v) = H(v - \delta) \sqrt{v^2 - \delta^2} v_2^i(\sqrt{v^2 - \delta^2}) \quad (2.1)$$

$$v v_2^i(v) = \sqrt{v^2 + \delta^2} v_1^i(\sqrt{v^2 + \delta^2}). \quad (2.2)$$

In the following we assume that the collision frequencies v_1^e , v_2^e and v_1^i are constant. We now introduce the parameters

$$\epsilon_1 = \frac{\delta}{n_1 L (v_1^e + b v_2^e)} \quad (2.3)$$

and

$$\epsilon_2 = \frac{\delta}{n_1 L v_1^i} \quad (2.4)$$

which characterize the importance of the two collisional mechanisms. When elastic collisions are dominant, we have $\epsilon_1 \ll 1$ and $\epsilon_2 = 1$, and when inelastic collisions are dominant we have $\epsilon_2 \ll 1$ and $\epsilon_1 = 1$.

Then the transport equation for the flux takes the following form:

$$\begin{aligned}
 \frac{\partial \Phi}{\partial t}(x, \xi, \mu, t) + \sqrt{\xi} \mu \frac{\partial \Phi}{\partial x}(x, \xi, \mu, t) = & \\
 = \frac{1}{\epsilon_1} \left[\frac{1}{2} \int_{-1}^{+1} \Phi(x, \xi + 1, \mu', t) d\mu' \right. & \\
 + bH(\xi - 1) \sqrt{\frac{\xi}{\xi - 1}} \frac{1}{2} \int_{-1}^{+1} \Phi(x, \xi - 1, \mu', t) d\mu' & \\
 - H(\xi - 1) \Phi(x, \xi, \mu, t) - b \sqrt{\frac{\xi + 1}{\xi}} \Phi(x, \xi, \mu, t) & \\
 \left. + \frac{1}{\epsilon_2} \left[\frac{1}{2} \int_{-1}^{+1} \Phi(x, \xi, \mu', t) d\mu' - \Phi(x, \xi, \mu, t) \right] \right], &
 \end{aligned}$$

where ϵ_1 and ϵ_2 have been introduced in (2.3) and (2.4).

By discretizing the velocity space according to the energy jump intervals and by defining

$$\Phi_n(x, \xi, \mu, t) = \Phi(x, \xi + n, \mu, t), \text{ with } 0 \leq \xi \leq 1, \text{ and } n = 0, 1, 2, \dots,$$

the Boltzmann equation for the flux becomes:

$$\begin{aligned}
 \frac{\partial \Phi_n}{\partial t}(x, \xi, \mu, t) + \sqrt{\xi + n} \mu \frac{\partial \Phi_n}{\partial x}(x, \xi, \mu, t) = & \\
 = \frac{1}{\epsilon_1} [C^e \Phi]_n(x, \xi, \mu, t) + \frac{1}{\epsilon_2} [C^i \Phi]_n(x, \xi, \mu, t) & \quad (2.5)
 \end{aligned}$$

where the n -th components of the elastic and inelastic collision op-

erators C^e and C^i are given by

$$\begin{aligned}
 [C^e\Phi]_n(x, \xi, \mu, t) &= \frac{1}{2} \int_{-1}^{+1} \Phi_n(x, \xi, \mu', t) d\mu' \\
 &\quad - \Phi_n(x, \xi, \mu, t), \quad \text{for } n \geq 0, \\
 [C^i\Phi]_0(x, \xi, \mu, t) &= \frac{1}{2} \int_{-1}^{+1} \Phi_1(x, \xi, \mu', t) d\mu' \\
 &\quad - b \sqrt{\frac{\xi+1}{\xi}} \Phi_0(x, \xi, \mu, t), \\
 [C^i\Phi]_n(x, \xi, \mu, t) &= \frac{1}{2} \int_{-1}^{+1} \Phi_{n+1}(x, \xi, \mu', t) d\mu' - \Phi_n(x, \xi, \mu, t) \\
 &\quad + b \sqrt{\frac{\xi+n}{\xi+n-1}} \frac{1}{2} \int_{-1}^{+1} \Phi_{n-1}(x, \xi, \mu', t) d\mu' \\
 &\quad - b \sqrt{\frac{\xi+n+1}{\xi+n}} \Phi_n(x, \xi, \mu, t), \quad \text{for } n \geq 1,
 \end{aligned}$$

and $0 \leq \xi \leq 1$.

3 Abstract formulation of the problem

Before introducing some spaces and operators, we remark that in the original three dimensional problem the particle distribution function $f = f(\mathbf{x}, \mathbf{v}, t)$ represents the expected number of test particles at time t in $d\mathbf{x}$ at \mathbf{x} and in $d\mathbf{v}$ at \mathbf{v} . Here $d\mathbf{x}d\mathbf{v} = d\mathbf{x}d\Omega v^2 dv$, where Ω is the solid angle variable and v the particle speed, is the volume element in phase space. Hence, the corresponding particle density $\rho = \rho(\mathbf{x}, t)$ is obtained with the weighted integral

$$\rho(\mathbf{x}, t) = \frac{\delta^3}{2} \int_0^{+\infty} \xi^{1/2} \int_{S^2} f(\mathbf{x}, \xi, \Omega, t) d\Omega d\xi$$

where $\xi = \frac{v^2}{\delta^2}$ is the dimensionless energy variable, δ the characteristic speed, and Ω the velocity direction. Obviously the remaining

variables x and t can be regarded as hydrodynamic macroscopic variables.

In the one-dimensional case, the previous physical consideration suggests introducing the following weighted norm for the particle distribution function f :

$$\|f\| = \int_{-\infty}^{+\infty} dx \int_0^{+\infty} d\xi \int_{-1}^{+1} \sqrt{\xi} |f(x, \xi, \mu)| d\mu.$$

Consequently, it is natural for the flux function $\Phi = \nu f$ to introduce the Banach space \mathbb{Z} defined as $L_1(\mathbb{R}) \otimes_{\pi} L_1(0, +\infty) \otimes_{\pi} L_1(-1, 1)$, where \otimes_{π} denotes the projective tensor product, with the norm

$$\|\Phi\|_{\mathbb{Z}} = \int_{-\infty}^{+\infty} dx \int_0^{+\infty} d\xi \int_{-1}^{+1} |\Phi(x, \xi, \mu)| d\mu.$$

Let \mathbb{X} be the Banach space defined as $L_1(\mathbb{R}) \otimes_{\pi} L_1(0, 1) \otimes_{\pi} L_1(-1, 1)$ with the usual norm

$$\|\Phi_i\|_{\mathbb{X}} = \int_{-\infty}^{+\infty} dx \int_0^1 d\xi \int_{-1}^{+1} |\Phi_i(x, \xi, \mu)| d\mu,$$

\mathbb{X}^N the Banach space defined as $\mathbb{X}^N = L_1(\mathbb{R}) \otimes_{\pi} L_1^{N+1}(0, 1) \otimes_{\pi} L_1(-1, 1)$, whose elements Φ are vectors with components $(\Phi_0, \Phi_1, \Phi_2, \dots, \Phi_N)$, with $\Phi_i \in \mathbb{X}$, and the norm given by

$$\|\Phi\|_{\mathbb{X}^N} = \sum_{i=0}^N \|\Phi_i\|_{\mathbb{X}},$$

and let \mathbb{Y} be the Banach space of all sequences of functions Φ_i of \mathbb{X} , whose elements Φ are vectors with components $(\Phi_0, \Phi_1, \Phi_2, \dots, \Phi_n, \dots)$ with $\Phi_i \in \mathbb{X}$, such that

$$\|\Phi\|_{\mathbb{Y}} = \sum_{i=0}^{\infty} \|\Phi_i\|_{\mathbb{X}} < +\infty.$$

In the \mathbb{Y} Banach setting our problem can be written in the abstract form as

$$\begin{cases} \frac{\partial \Phi}{\partial t} = S\Phi + \frac{1}{\epsilon_1} C^e \Phi + \frac{1}{\epsilon_2} C^i \Phi \\ \Phi(x, \xi, \mu, 0) = \Phi^0 \end{cases} \quad (3.1)$$

where the streaming operator S is given by

$$\begin{pmatrix} -\sqrt{\xi}\mu \frac{\partial}{\partial x} & 0 & 0 & \dots & 0 & \dots \\ 0 & -\sqrt{\xi+1}\mu \frac{\partial}{\partial x} & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -\sqrt{\xi+n-1}\mu \frac{\partial}{\partial x} & 0 & \dots \\ 0 & \dots & 0 & 0 & -\sqrt{\xi+n}\mu \frac{\partial}{\partial x} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

The elastic operator C^e can be written in the form

$$\begin{pmatrix} -I+P & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & -I+P & 0 & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -I+P & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & -I+P & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

and the inelastic operator C^i in the form

$$\begin{pmatrix} -b\sqrt{\frac{\xi+1}{\xi}} & P & 0 & 0 & \dots & 0 & \dots \\ b\sqrt{\frac{\xi+1}{\xi}}P & -I - b\sqrt{\frac{\xi+2}{\xi+1}} & P & 0 & \dots & 0 & \dots \\ 0 & b\sqrt{\frac{\xi+2}{\xi+1}}P & -I - b\sqrt{\frac{\xi+3}{\xi+2}} & P & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b\sqrt{\frac{\xi+n}{\xi+n-1}}P & -I - b\sqrt{\frac{\xi+n+1}{\xi+n}} & P & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

where $P \cdot = \frac{1}{2} \int_{-1}^{+1} \cdot d\mu$.

4 The compressed Chapman-Enskog procedure

In this section we recall the principal steps of the compressed Chapman-Enskog procedure. One of the main ingredient of the compressed procedure is the projection of the unknown function onto the hydrodynamic subspace.

In our problem we consider the system (3.1) in two cases: in the first case, when elastic collisions are dominant, with $\epsilon_1 = \epsilon$ and $\epsilon_2 = 1$,

$$\frac{\partial \Phi}{\partial t} = S\Phi + \frac{1}{\epsilon} C^e \Phi + C^i \Phi, \quad (4.1)$$

and in the second case, when inelastic collisions are dominant, with $\epsilon_1 = 1$ and $\epsilon_2 = \epsilon$

$$\frac{\partial \Phi}{\partial t} = S\Phi + C^e \Phi + \frac{1}{\epsilon} C^i \Phi. \quad (4.2)$$

Let us begin with the first case. If we put formally $\epsilon = 0$ in equation (4.1) we have out into

$$C^e f = 0.$$

Similarly to the standard case in kinetic theory, the eigenspace of C^e corresponding to the 0-eigenvalue is one-dimensional and is spanned by the equilibrium function depending only on the energy variable, i.e.

$$\begin{aligned} \ker(C^e) &= \{ \Phi : \Phi_n(x, \xi, \mu) = \phi_n(x, \xi), \text{ arbitrary,} \\ &\quad \text{but independent of } \mu, \text{ for } n \geq 0 \} \end{aligned}$$

and the range of C^e is given by the functions f such that

$$\int_{-1}^{+1} f(x, \xi, \mu) d\mu = 0.$$

The hydrodynamic quantity, i.e. the quantity that is conserved in the absence of streaming, in this case is given by

$$\rho_\Phi^e(x, \xi, t) = \frac{1}{2} \int_{-1}^{+1} \Phi(x, \xi, \mu, t) d\mu, \quad \xi \in [0, +\infty) \quad (4.3)$$

and the spectral projection \mathcal{P}^e onto the kernel of the elastic collision operator, in the Banach setting \mathbb{Y} , is defined by

$$[\mathcal{P}^e\Phi]_k(x, \xi) = \frac{1}{2} \int_{-1}^{+1} \Phi_k(x, \xi, \mu) d\mu, \quad k \geq 0. \quad (4.4)$$

Let us consider now the case of dominant inelastic collisions, which corresponds to the second case. Putting formally $\epsilon = 0$ in equation (4.2), we have

$$C^i f = 0.$$

This case differs from the classical case, because the eigenspace of C^i corresponding to the 0-eigenvalue is infinite dimensional. We can prove that the null space of C^i consists of functions which are periodic for $\xi > 1$ a part from the factor $b^n \sqrt{\frac{\xi+n}{\xi}}$, i.e.

$$\ker(C^i) = \left\{ \Phi : \Phi_0(x, \xi, \mu) = \phi_0(x, \xi), \text{ arbitrary, but independent of } \mu \text{ and } \Phi_n(x, \xi, \mu) = b^n \sqrt{\frac{\xi+n}{\xi}} \phi_0(x, \xi), \text{ for } n \geq 1 \right\}.$$

In other words the kernel of the inelastic collision operator is made of functions which are arbitrary in the first interval of energy $(0, 1)$, and whose other components are multiplied by the factor $b^n \sqrt{\frac{\xi+n}{\xi}}$. We remark that if we reformulate the problem in terms of the particle distribution function, instead of the flux, the slightly different form of the operator C^i leads to a kernel of periodic functions, a part from the only constant factor b^n .

In this case, the hydrodynamic quantity is given by

$$\rho_\Phi^i(x, \xi, t) = \sum_{n=0}^{\infty} \frac{1}{2} \int_{-1}^{+1} \Phi_n(x, \xi, \mu, t) d\mu, \quad \xi \in [0, 1] \quad (4.5)$$

and the projection \mathcal{P}^i on the kernel of C^i is given by

$$[\mathcal{P}^i\Phi]_k(x, \xi) = \frac{b^k \sqrt{\xi+k}}{\sum_{j=0}^{\infty} b^j \sqrt{\xi+j}} \sum_{n=0}^{\infty} \frac{1}{2} \int_{-1}^{+1} \Phi_n(x, \xi, \mu) d\mu, \quad k \geq 0. \quad (4.6)$$

For the spectral properties of such collision operators see [1].

In the following, we shall denote by $\mathcal{Q}^e = \mathcal{I} - \mathcal{P}^e$ and $\mathcal{Q}^i = \mathcal{I} - \mathcal{P}^i$ the complementary projections in the Banach setting \mathbb{Y} , where \mathcal{I} is the identity operator. The projections (4.4) and (4.6) can be written in matrix form. For example, we have for the projection \mathcal{P}^i on $\ker(C^i)$, as an operator acting in \mathbb{Y} :

$$\mathcal{P}^i \Phi = \frac{\sum_{j=0}^{\infty} P\Phi_j}{\sum_{j=0}^{\infty} b^j \sqrt{\xi + j}} \begin{pmatrix} \sqrt{\xi} \\ b\sqrt{\xi + 1} \\ \vdots \\ b^n \sqrt{\xi + n} \\ \vdots \end{pmatrix},$$

where $P\Phi_j \equiv \frac{1}{2} \int_{-1}^{+1} \Phi_j(x, \xi, \mu) d\mu$, and it is easy to see that

$$\mathcal{Q}^i \Phi = \begin{pmatrix} \Phi_0(x, \xi, \mu) - \frac{\sqrt{\xi}}{\sum_{j=0}^{\infty} b^j \sqrt{\xi + j}} \sum_{j=0}^{\infty} P\Phi_j \\ \Phi_1(x, \xi, \mu) - \frac{b\sqrt{\xi + 1}}{\sum_{j=0}^{\infty} b^j \sqrt{\xi + j}} \sum_{j=0}^{\infty} P\Phi_j \\ \dots\dots\dots \\ \Phi_n(x, \xi, \mu) - \frac{b^n \sqrt{\xi + n}}{\sum_{j=0}^{\infty} b^j \sqrt{\xi + j}} \sum_{j=0}^{\infty} P\Phi_j \\ \dots\dots\dots \end{pmatrix}.$$

The projections \mathcal{P}^e and \mathcal{Q}^e for their simplicity have a diagonal matrix form.

The next step in the compressed Chapman-Enskog procedure is to decompose the Banach space \mathbb{Y} by the projections \mathcal{P}^e and \mathcal{Q}^e , when the elastic collisions are dominant and by the projections \mathcal{P}^i and \mathcal{Q}^i , when the inelastic collisions are dominant.

According to this decomposition, we can write the unknown function as the sum of a function in the kernel of the dominant collision operator (hydrodynamic solution) and the complementary part (kinetic solution). For example in the case of system (4.1), we can write the flux function $\Phi \in \mathbb{Y}$ as

$$\Phi = \mathcal{P}^e \Phi + \mathcal{Q}^e \Phi = \varphi + \psi, \tag{4.7}$$

where φ is the hydrodynamic part of the solution and ψ is the kinetic part.

The subsequent step in the procedure is to derive new equations for the new unknowns φ and ψ , and this can be performed by operating formally on both sides of (4.1) with the projections \mathcal{P}^e and \mathcal{Q}^e .

In the following section we illustrate formally this modified Chapman-Enskog procedure, in the two different scalings of the kinetic equation corresponding physically to having the ratio of the elastic to inelastic collision frequencies very large or very small.

To make the treatment easier theoretically and numerically, we introduce a cut-off in the range of the energy variable, assuming that $\xi \in [0, N + 1]$ or equivalently that, if $\xi_n \equiv \xi + n$, we have $n = 0, 1, \dots, N$. Physically, the cut-off corresponds to the assumption that the number of particles having energies larger than $N\Delta E$ is negligible.

5 The diffusion approximations

In this section we derive formally the diffusion approximations of systems (4.1) and (4.2).

Let us begin with the first case, when elastic collisions are dominant. Making use of the decomposition of the unknown function as in (4.7), we operate formally on both sides of (4.1) with the projection operators \mathcal{P}^e and \mathcal{Q}^e and we obtain the following system of equations:

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \mathcal{P}^e S \mathcal{P}^e \varphi + \mathcal{P}^e S \mathcal{Q}^e \psi + \mathcal{P}^e C^i \mathcal{P}^e \varphi \\ \frac{\partial \psi}{\partial t} = \mathcal{Q}^e S \mathcal{P}^e \varphi + \mathcal{Q}^e S \mathcal{Q}^e \psi + \mathcal{Q}^e C^i \mathcal{Q}^e \psi + \frac{1}{\epsilon} \mathcal{Q}^e C^e \mathcal{Q}^e \psi, \end{cases} \quad (5.1)$$

with initial conditions

$$\begin{cases} \varphi(0) = \varphi_0 = \mathcal{P}^e \Phi^0 \\ \psi(0) = \psi_0 = \mathcal{Q}^e \Phi^0 \end{cases} .$$

Now we split the unknown functions φ and ψ into the sums of the "bulk" parts $\bar{\varphi}$ and $\bar{\psi}$ and of the initial layer parts $\tilde{\varphi}$ and $\tilde{\psi}$ which

take account of the rapid changes of Φ for small times:

$$\begin{aligned}\varphi(t) &= \bar{\varphi}(t) + \tilde{\varphi}\left(\frac{t}{\epsilon}\right) \\ \psi(t) &= \bar{\psi}(t) + \tilde{\psi}\left(\frac{t}{\epsilon}\right).\end{aligned}\quad (5.2)$$

According to the Chapman-Enskog procedure, we expand the unknown functions $\bar{\varphi}$, $\bar{\psi}$, and $\tilde{\psi}$ in series of ϵ , and we leave the bulk part $\bar{\varphi}$ unexpanded because we are interested in the complete behaviour of the hydrodynamic part:

$$\begin{aligned}\tilde{\psi}(t) &= \tilde{\psi}_0(t) + \epsilon\tilde{\psi}_1(t) \dots \\ \bar{\varphi}(\tau) &= \bar{\varphi}_0(\tau) + \epsilon\bar{\varphi}_1(\tau) \dots \\ \tilde{\psi}(\tau) &= \tilde{\psi}_0(\tau) + \epsilon\tilde{\psi}_1(\tau) \dots,\end{aligned}\quad (5.3)$$

where $\tau = t/\epsilon$. Now we have to obtain equations for the terms of the expansions introduced above and we substitute the bulk and initial layer parts separately into the second equation of (5.1).

For the bulk part we have:

$$\tilde{\psi}_0(t) \equiv 0 \quad (5.4)$$

$$\tilde{\psi}_1(t) = -(\mathcal{Q}^e C^e \mathcal{Q}^e)^{-1} \mathcal{Q}^e S \mathcal{P}^e \bar{\varphi}(t) \quad (5.5)$$

and the diffusion equation, correct up to first order in the bulk kinetic part, is

$$\frac{\partial \bar{\varphi}}{\partial t} = \mathcal{P}^e S \mathcal{P}^e \bar{\varphi} + \mathcal{P}^e C^i \mathcal{P}^e \bar{\varphi} - \epsilon \mathcal{P}^e S \mathcal{Q}^e (\mathcal{Q}^e C^e \mathcal{Q}^e)^{-1} \mathcal{Q}^e S \mathcal{P}^e \bar{\varphi}. \quad (5.6)$$

In this case, it is easy to see that $\mathcal{P}^e S \mathcal{P}^e \bar{\varphi} = 0$, while the evolution of $\bar{\varphi}$ is affected by the inelastic collision operator by $\mathcal{P}^e C^i \mathcal{P}^e$ and the contributions of elastic collisions appear with the correction of first order in ϵ . It could be possible to improve such approximation and obtain equations for the hydrodynamic part $\bar{\varphi}$ which takes account also of the second order corrections, but this is out of our aim.

Hence, the diffusion equation reduces to

$$\frac{\partial \bar{\varphi}}{\partial t} = \mathcal{P}^e C^i \mathcal{P}^e \bar{\varphi} - \epsilon \mathcal{P}^e S \mathcal{Q}^e (\mathcal{Q}^e C^e \mathcal{Q}^e)^{-1} \mathcal{Q}^e S \mathcal{P}^e \bar{\varphi}. \quad (5.7)$$

If we want to give a more explicit form to equation (5.7), some algebra is necessary to invert the operator $Q^e C^e Q^e$ and to get the forms of the operators $P^e C^i P^e$, $P^e S Q^e$, and $Q^e S P^e$.

Finally the diffusion equation (5.7) takes the form

$$\frac{\partial}{\partial t} \begin{pmatrix} P\Phi_0 \\ P\Phi_1 \\ P\Phi_2 \\ \vdots \\ P\Phi_N \end{pmatrix} = \epsilon \frac{2}{3} \begin{pmatrix} \xi \\ \xi+1 \\ \xi+2 \\ \dots \\ \xi+N \end{pmatrix} \frac{\partial^2}{\partial x^2} \begin{pmatrix} P\Phi_0 \\ P\Phi_1 \\ P\Phi_2 \\ \vdots \\ P\Phi_N \end{pmatrix} \quad (5.8)$$

$$+ \begin{pmatrix} -b\sqrt{\frac{\xi+1}{\xi}}P\Phi_0 + P\Phi_1 \\ b\sqrt{\frac{\xi+1}{\xi}}P\Phi_0 - P\Phi_1 - b\sqrt{\frac{\xi+2}{\xi+1}}P\Phi_1 + P\Phi_2 \\ b\sqrt{\frac{\xi+2}{\xi+1}}P\Phi_1 - P\Phi_2 - b\sqrt{\frac{\xi+3}{\xi+2}}P\Phi_2 + P\Phi_3 \\ \dots \dots \\ b\sqrt{\frac{\xi+N}{\xi+N-1}}P\Phi_{N-1} - P\Phi_N \end{pmatrix}.$$

In order to equip the previous equation with the appropriate initial condition, we need to consider the initial layer part. Referring the reader to [14], here we limit ourselves to say that expanding in series of ϵ the initial layer equations corresponding to (5.1) and equating the corresponding powers of ϵ , we get a set of equations for $\tilde{\varphi}_0, \tilde{\varphi}_1, \dots$ and $\tilde{\psi}_0, \tilde{\psi}_1, \dots$. Then we have to find the appropriate initial conditions for such equations, keeping in mind that at the end we want to obtain an estimate of the error of order ϵ^2 .

We now study the case of eq. (4.2), when inelastic collisions are dominant. Here, the mathematical difficulties require a more delicate treatment.

Making use of the projections \mathcal{P}^i and \mathcal{Q}^i , we obtain from (4.2) the following system of equations:

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \mathcal{P}^i S \mathcal{P}^i \varphi + \mathcal{P}^i S \mathcal{Q}^i \psi \\ \frac{\partial \psi}{\partial t} = \mathcal{Q}^i S \mathcal{P}^i \varphi + \mathcal{Q}^i S \mathcal{Q}^i \psi + \mathcal{Q}^i C^e \mathcal{Q}^i \psi + \frac{1}{\epsilon} \mathcal{Q}^i C^i \mathcal{Q}^i \psi, \end{cases} \quad (5.9)$$

with initial conditions

$$\begin{cases} \varphi(0) = \varphi_0 = \mathcal{P}^i \Phi^0 \\ \psi(0) = \psi_0 = \mathcal{Q}^i \Phi^0. \end{cases}$$

Splitting the unknown functions into the sums of the “bulk” and “initial layer” parts and expanding the unknown functions in series of ϵ , with the bulk part $\bar{\varphi}$ unexpanded, we obtain for the bulk parts, corrected up to first order in the kinetic part

$$\bar{\psi}_0 \equiv 0, \tag{5.10}$$

$$\bar{\psi}_1 = -(\mathcal{Q}^i C^i \mathcal{Q}^i)^{-1} \mathcal{Q}^i S \mathcal{P}^i \bar{\varphi}, \tag{5.11}$$

$$\frac{\partial \bar{\varphi}}{\partial t} = \mathcal{P}^i S \mathcal{P}^i \bar{\varphi} - \epsilon \mathcal{P}^i S \mathcal{Q}^i (\mathcal{Q}^i C^i \mathcal{Q}^i)^{-1} \mathcal{Q}^i S \mathcal{P}^i \bar{\varphi}. \tag{5.12}$$

We see that, up to first order in ϵ , the evolution of $\bar{\varphi}$ is not affected by the elastic collision term. The effect of the elastic term appears only after correcting the evolution of $\bar{\varphi}$ with terms of the second order in ϵ . Since $\mathcal{P}^i S \mathcal{P}^i \bar{\varphi} = 0$, because $S \mathcal{P}^i \Phi$ is an odd function of μ , the diffusion equation reduces to

$$\frac{\partial \bar{\varphi}}{\partial t} = -\epsilon \mathcal{P}^i S \mathcal{Q}^i (\mathcal{Q}^i C^i \mathcal{Q}^i)^{-1} \mathcal{Q}^i S \mathcal{P}^i \bar{\varphi}. \tag{5.13}$$

It is easy to give an explicit form to the operators $\mathcal{Q}^i S \mathcal{P}^i$ and $\mathcal{P}^i S \mathcal{Q}^i$.

Then, it is necessary to solve $(\mathcal{Q}^i C^i \mathcal{Q}^i) \Phi = \mathbf{g}$. Because $(\mathcal{Q}^i C^i \mathcal{Q}^i) \Phi = C^i(\Phi - \mathcal{P}^i \Phi)$, we have

$$C^i(\Phi - \mathcal{P}^i \Phi) = \begin{pmatrix} -b\sqrt{\frac{\xi+1}{\xi}} \Phi_0 + P\Phi_1 \\ b\sqrt{\frac{\xi+1}{\xi}} P\Phi_0 - \Phi_1 - b\sqrt{\frac{\xi+2}{\xi+1}} \Phi_1 + P\Phi_2 \\ b\sqrt{\frac{\xi+2}{\xi+1}} P\Phi_1 - \Phi_2 - b\sqrt{\frac{\xi+3}{\xi+2}} \Phi_2 + P\Phi_3 \\ \dots \dots \dots \dots \\ b\sqrt{\frac{\xi+N}{\xi+N-1}} P\Phi_{N-1} - \Phi_N \end{pmatrix}.$$

In order to solve the system $(Q^i C^i Q^i)\Phi = \mathbf{g}$, we apply P to all equations, we use that $Pg_j = 0$ and get

$$\left\{ \begin{array}{l} P\Phi_1(x, \xi) = b\sqrt{\frac{\xi+1}{\xi}}P\Phi_0(x, \xi) \\ P\Phi_2(x, \xi) = b^2\sqrt{\frac{\xi+2}{\xi}}P\Phi_0(x, \xi) \\ \dots \quad \dots \quad \dots \\ P\Phi_{N-1}(x, \xi) = b^{N-1}\sqrt{\frac{\xi+N-1}{\xi}}P\Phi_0(x, \xi) \\ P\Phi_N(x, \xi) = b^N\sqrt{\frac{\xi+N}{\xi}}P\Phi_0(x, \xi) \end{array} \right.$$

But Φ has to belong to the range of C^i , hence

$$\left(1 + b\sqrt{\frac{\xi+1}{\xi}} + b^2\sqrt{\frac{\xi+2}{\xi}} + \dots + b^N\sqrt{\frac{\xi+N}{\xi}}\right)P\Phi_0 = 0$$

implies $P\Phi_0 = 0$, and consequently $P\Phi_i = 0$, for all $i = 1, \dots, N$. Then, the solution of $C^i Q^i \Phi = \mathbf{g}$ in the range of C^i is simply given by

$$\left\{ \begin{array}{l} \Phi_0(x, \xi, \mu) = -\left(b\sqrt{\frac{\xi+1}{\xi}}\right)^{-1} g_0(x, \xi, \mu) \\ \Phi_1(x, \xi, \mu) = -\left(1 + b\sqrt{\frac{\xi+2}{\xi+1}}\right)^{-1} g_1(x, \xi, \mu) \\ \dots \quad \dots \quad \dots \\ \Phi_{N-1}(x, \xi, \mu) = -\left(1 + b\sqrt{\frac{\xi+N}{\xi+N-1}}\right)^{-1} g_{N-1}(x, \xi, \mu) \\ \Phi_N(x, \xi, \mu) = -g_N(x, \xi, \mu) \end{array} \right.$$

In conclusion, the diffusion equation takes the form

$$\frac{\partial}{\partial t} \left(\sum_{i=0}^N P\Phi_i \right) = \epsilon \frac{2 \sum_{j=0}^N b^j \sqrt{\xi+j} (\xi+j) \left[\bar{\delta}_{j0} + \bar{\delta}_{jN} b \sqrt{\frac{\xi+j+1}{\xi+j}} \right]^{-1}}{\sum_{j=0}^N b^j \sqrt{\xi+j}} \cdot \frac{\partial^2}{\partial x^2} \left(\sum_{i=0}^N P\Phi_i \right), \quad (5.14)$$

where $\bar{\delta}_{ij} = 1 - \delta_{ij}$, δ_{ij} is the Kronecker delta and $\sum_{i=0}^N P\Phi_i$ is the hydrodynamic quantity suggested by our scaling. Analogously to the previous case we have to supply equation (5.14) with the initial condition obtained correctly after having solved the initial layer problem.

We remark that the diffusion equations (5.8) and (5.14) are different, both because of the different diffusion coefficients and because of the presence of the linear term $\mathcal{P}^e C^i \mathcal{P}^e$ in eq. (5.8). Such difference has to be ascribed mainly to the set theoretical inclusion relation of the kernels of the two operators, due to the more general form of the solutions of $C^e = 0$.

6 Numerical results

In this section we present a numerical example illustrating the time evolution of the distribution function in a particular case, when the initial datum is chosen with compact support in one of the energy intervals and inelastic collisions are taken to be dominant. We shall see that the asymptotic evolution is characterized by a homogenization in the space variable x and in the energy variable ξ within each interval. More extensive numerical results have already been presented in [2] and [10]. The numerical methods used for the numerical solution of the Boltzmann equation are standard, but we describe them here for completeness, making explicit reference to the present problem. The rigorous results contained in the previous sections have been presented for the flux function. The computer programs used for the numerical solution, instead, are written for the distribution function, mainly for historical reasons.

In analogy with (2.5), the Boltzmann equation for the distribution function is:

$$\frac{\partial f_n}{\partial t}(x, \xi, \mu, t) + \sqrt{\xi + n} \mu \frac{\partial f_n}{\partial x}(x, \xi, \mu, t) = \frac{1}{\epsilon_1} [C^e f]_n(x, \xi, \mu, t) + \frac{1}{\epsilon_2} [C^i f]_n(x, \xi, \mu, t), \quad (6.1)$$

$n = 0, 1, \dots$, where the components of the elastic and inelastic colli-

sion operators C^e and C^i are given by

$$\begin{aligned}
[C^e f]_n(x, \xi, \mu, t) &= \frac{1}{2} \int_{-1}^{+1} f_n(x, \xi, \mu', t) d\mu' \\
&\quad - f_n(x, \xi, \mu, t), \quad \text{for } n \geq 0, \\
[C^i f]_0(x, \xi, \mu, t) &= \sqrt{\frac{\xi+1}{\xi}} \frac{1}{2} \int_{-1}^{+1} f_1(x, \xi, \mu', t) d\mu' \\
&\quad - b \sqrt{\frac{\xi+1}{\xi}} f_0(x, \xi, \mu, t), \\
[C^i f]_n(x, \xi, \mu, t) &= \sqrt{\frac{\xi+n+1}{\xi+n}} \frac{1}{2} \int_{-1}^{+1} f_{n+1}(x, \xi, \mu', t) d\mu' \\
&\quad + b \frac{1}{2} \int_{-1}^{+1} f_{n-1}(x, \xi, \mu', t) d\mu' - f_n(x, \xi, \mu, t) \\
&\quad - b \sqrt{\frac{\xi+n+1}{\xi+n}} f_n(x, \xi, \mu, t), \quad \text{for } n \geq 1,
\end{aligned}$$

and $0 \leq \xi \leq 1$.

6.1 Numerical solution of the Boltzmann equation

The Boltzmann equation (6.1) is solved numerically by expanding the unknown distribution function in Legendre polynomials in the angular variable and, assuming periodic boundary conditions, by Fourier collocation in space. For the time discretization we have adopted a simple backward Euler scheme.

The expansion in Legendre polynomials is given by:

$$f_n(x, \xi, \mu, t) = \sum_{k=0}^K f_{nk}(x, \xi, t) L_k(\mu) \quad (6.2)$$

where $L_k(\mu)$ is the Legendre polynomial of order k . If we multiply

the Boltzmann equation by $L_k(\mu)$ and integrate over μ we obtain:

$$\begin{aligned} \frac{\partial f_{nk}}{\partial t} + \sqrt{\xi_n} \left(\alpha_k \frac{\partial f_{n,k+1}}{\partial x} + \beta_k \frac{\partial f_{n,k-1}}{\partial x} \right) = \\ = \left\{ \frac{1}{\epsilon_1} \left[\sqrt{\frac{\xi_{n+1}}{\xi_n}} f_{n+1,0} + b f_{n-1,0} \right] + \frac{1}{\epsilon_2} f_{n0} \right\} \delta_{k0} + \\ - \left\{ \frac{1}{\epsilon_1} \left[1 - \delta_{n0} + b \sqrt{\frac{\xi_{n+1}}{\xi_n}} \right] + \frac{1}{\epsilon_2} \right\} f_{nk}, \end{aligned} \quad (6.3)$$

$n = 0, 1, \dots, N, k = 0, 1, \dots, K,$

where N is the highest energy interval considered and we have introduced

$$\begin{aligned} \alpha_k &= \frac{k+1}{2k+1} \frac{2}{2k+3}, \\ \beta_k &= \frac{k}{2k+1} \frac{2}{2k-1} \end{aligned}$$

and

$$\xi_n = \xi + n.$$

In deriving (6.3) we have used the recurrence relation of the Legendre polynomials and of their derivatives and their orthogonality properties. The spatial discretization is performed by using Fourier collocation points, namely $x = x_j = 2\pi j/M, j = 0, \dots, M-1$ [7]. In this representation the derivative of a function $f(x)$ is expressed in the form

$$\left(\frac{\partial f}{\partial x} \right)_{x=x_i} \sim \sum_{j=0}^{M-1} D_{ij}^M f_j$$

where $f_j = f(x_j)$ and the matrix D^M is defined by [7]

$$D_{ij}^M = \begin{cases} \frac{1}{2}(-1)^{i+j} \cot \frac{(i-j)\pi}{M} & i \neq j \\ 0 & i = j \end{cases}.$$

Finally, we introduce a backward Euler scheme in time and replace the time derivative on the left hand side of (6.3) with

$$\frac{\partial f}{\partial t} \sim \frac{f(t + \Delta t) - f(t)}{\Delta t},$$

where Δt is the time step, and evaluate the function f or its spatial derivatives at the time $t + \Delta t$ in all other places where it appears. After all the discretizations have been introduced, the unknown function $f_n(x, \xi, \mu, t + \Delta t)$ at the new time step is represented by a set of discrete values f_{nkj} , $n = 0, \dots, N$, $k = 0, \dots, K$, $j = 0, \dots, M$. After some algebra, equation (6.3) for the unknowns f_{nkj} becomes:

$$\begin{aligned} & \left\{ 1 + \frac{\Delta t}{\epsilon_1} \left(1 - \delta_{n0} + b \sqrt{\frac{\xi_{n+1}}{\xi_n}} \right) + \frac{\Delta t}{\epsilon_2} (1 - \delta_{k0}) \right\} f_{nkj} + \\ & + \Delta t \sqrt{\xi_n} \left(\alpha_k \sum_l D_{jl}^M f_{n,k+1,l} + \beta_k \sum_l D_{jl}^M f_{n,k-1,l} \right) + \\ & - \frac{\Delta t}{\epsilon_2} \left(\sqrt{\frac{\xi_{n+1}}{\xi_n}} f_{n+1,0,j} + b f_{n-1,0,j} \right) \delta_{k0} = f_{nkj}^* \end{aligned} \quad (6.4)$$

where f^* denotes f at the old time step. Equation (6.4) gives an $N \times M \times K$ sparse linear system which we have solved by the successive overrelaxation method (SOR). For this purpose, we write (6.4) in the following block-matrix form:

$$\mathcal{U}_{nk} \mathbf{f}_{nk} + \mathcal{A}_{nk} \mathbf{f}_{n,k+1} + \mathcal{B}_{nk} \mathbf{f}_{n,k-1} + \mathcal{E}_{nk} \mathbf{f}_{n+1,0} + \mathcal{G}_{nk} \mathbf{f}_{n-1,0} = \mathbf{f}_{nk}^*, \quad (6.5)$$

where \mathcal{U}_{nk} , \mathcal{A}_{nk} , \mathcal{B}_{nk} , \mathcal{E}_{nk} and \mathcal{G}_{nk} are $M \times M$ matrices whose elements are given by

$$\begin{aligned} \mathcal{U}_{nk;ij} &= 1 + \frac{\Delta t}{\epsilon_1} \left(1 - \delta_{n0} + b \sqrt{\frac{\xi_{n+1}}{\xi_n}} \right) + \frac{\Delta t}{\epsilon_2} (1 - \delta_{k0}) \delta_{ij} \\ \mathcal{A}_{nk;ij} &= \Delta t \sqrt{\xi_n} \alpha_k D_{ij}^M \\ \mathcal{B}_{nk;ij} &= \Delta t \sqrt{\xi_n} \beta_k D_{ij}^M \\ \mathcal{E}_{nk;ij} &= -\frac{\Delta t}{\epsilon_2} \sqrt{\frac{\xi_{n+1}}{\xi_n}} \delta_{k0} \\ \mathcal{G}_{nk;ij} &= -\frac{\Delta t}{\epsilon_2} b \delta_{k0} \end{aligned}$$

and the \mathbf{f}_{nk} and \mathbf{f}_{nk}^* are the M -dimensional vectors given by $\mathbf{f}_{nk;j} = f_{nkj}$ and $\mathbf{f}_{nk;j}^* = f_{nkj}^*$, respectively.

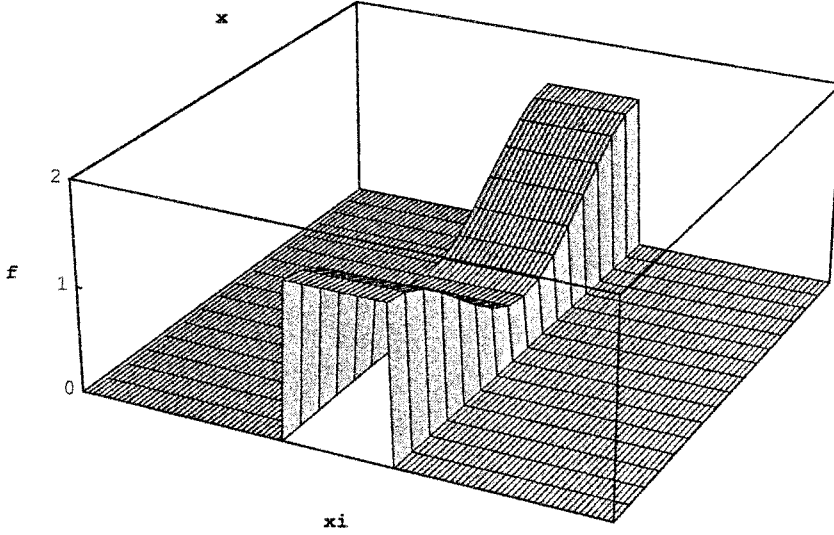


Figure 6.1: Isotropic component of the initial distribution of f , $0 \leq x \leq 2\pi$, $0 \leq \xi \leq 5b$, and $\epsilon_1 = 1$, $\epsilon_2 = 0.05$ (dominant inelastic case).

Equation (6.5) is then solved by the SOR method [15], which we recall briefly in the Appendix. In order to apply the SOR method to equation (6.5), we rewrite the equation itself in the form

$$\mathbf{f}_{nk;j} = \frac{1}{\mathcal{U}_{nk;ii}} \left\{ \mathbf{f}_{nk;j}^* - \sum_l (\mathcal{A}_{nk;jl} \mathbf{f}_{n,k+1;l} + \mathcal{B}_{nk;jl} \mathbf{f}_{n,k-1;l}) + \right. \\ \left. - \mathcal{E}_{nk;jj} \mathbf{f}_{n+1,0;j} - \mathcal{G}_{nk;jj} \mathbf{f}_{n-1,0;j} \right\},$$

which gives the s -th iterate for $\mathbf{f}_{nk;j}$, with the understanding that updated values for the components of the \mathbf{f}_{nk} vectors are used wherever available (see equation (7.1) of the Appendix).

The parameter ω of the SOR method was adjusted ad hoc in each case.

6.2 Numerical examples

In this section we illustrate the time evolution of the isotropic component (the $k = 0$ term in the Legendre polynomial expansion) of the

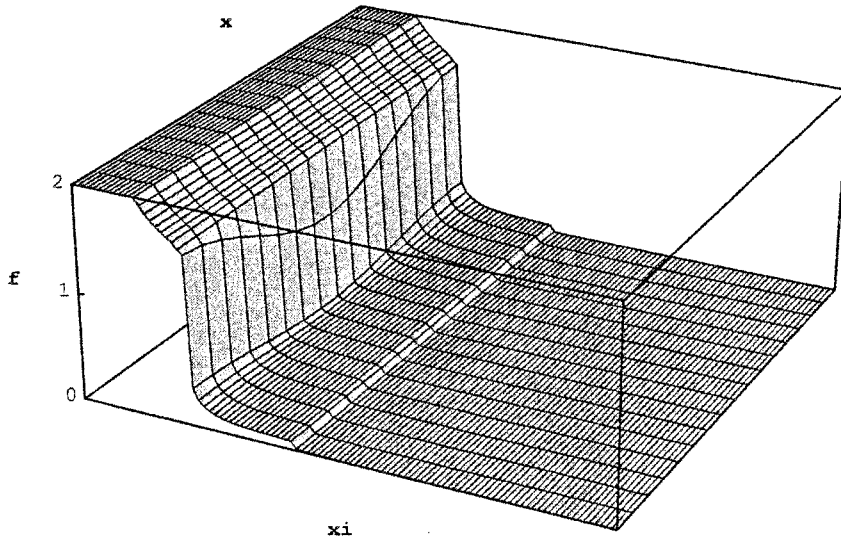


Figure 6.2: Isotropic component of f at $t = 10$, $0 \leq x \leq 2\pi$, $0 \leq \xi \leq 5b$ and $\epsilon_1 = 1$, $\epsilon_2 = 0.05$ (dominant inelastic case).

distribution function in a case when the initial datum is chosen with compact support in a specific energy interval and inelastic collisions are considered as dominant. In particular, we choose an initial distribution which is nonzero only in the $n = 2$ energy interval, where it is given by

$$f_n(x, \xi, \mu) = \begin{cases} 1 + A_0(1 + \mu) \cos x & \text{for } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Also, we set $\epsilon_1 = 1.0$ and $\epsilon_2 = 0.05$ and the Boltzmann factor is $b = 3$. This initial distribution is far, in any sense, from any element of the kernel of the inelastic or of the total collision operator. It was shown numerically in [2], where only inelastic collision processes were included, that the solution of the kinetic equations tends, as time proceeds, to some element of the kernel of the inelastic collision operator. The initial distribution is shown in Figure 6.1.

Figures 6.2 and 6.3 show the same distribution at $t = 10$. Figure 6.2 shows the energy range from $\xi = 0$ to $\xi = 5b$, while Fig-

Figure 6.3 shows the first two energy intervals, with a change of scale in the ordinate to better show the behavior at energies close to zero.

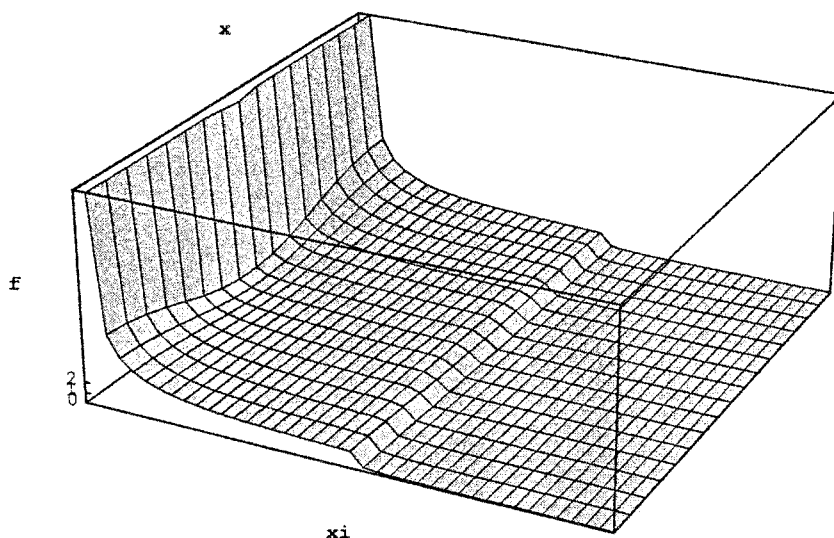


Figure 6.3: Isotropic component of f at $t = 10$, $0 \leq x \leq 2\pi$, $0 \leq \xi \leq 2b$ and $\epsilon_1 = 1$, $\epsilon_2 = 0.05$ (dominant inelastic case).

Figure 6.4 shows the distribution at $t = 20$ and for $2b < \xi < 5b$, also with a change of scale in the ordinate to better follow the behavior at high energies. The distribution at $t = 20$ doesn't differ from the distribution at $t = 10$ in any appreciable way, so we can assume that by $t = 10$ it has settled to its asymptotic shape.

In a space homogeneous case, since both elastic and inelastic collisions don't mix the populations at different energies within each energy interval, we would expect to see the distribution reaching a piecewise constant shape in energy as time goes to infinity, with the values of the distribution itself within each interval following a Boltzmann-like profile. The distribution would also remain independent of space. The space inhomogeneity of the initial datum changes this picture. The streaming term now provides some mixing of the populations within each energy interval, while the diffusion mechanism homogenizes the distribution in space. The resulting time asymptotic shape

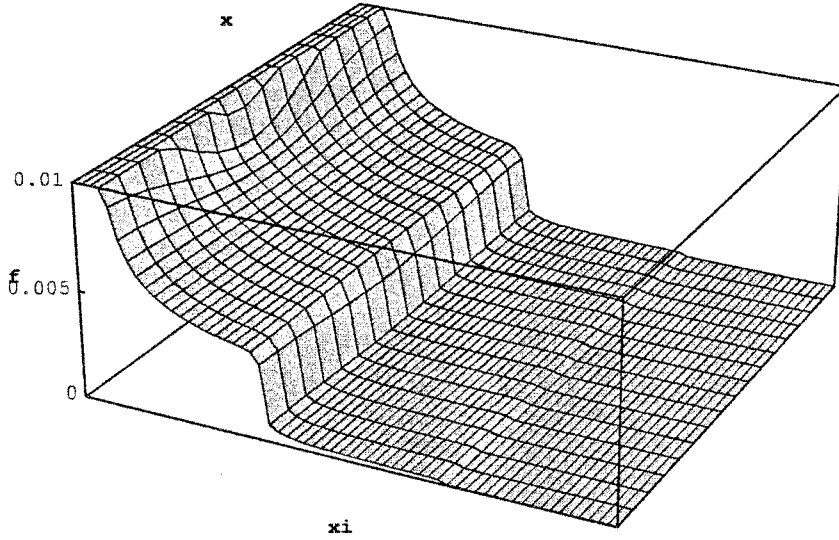


Figure 6.4: Isotropic component of f at $t = 20$, $0 \leq x \leq 2\pi$, $2b < \xi < 5b$ and $\epsilon_1 = 1$, $\epsilon_2 = 0.05$ (dominant inelastic case).

of the distribution function is homogeneous in space but it is not flat in energy within each interval, it rather shows a gentle fall. We believe that this is to be ascribed to the mixing effect of the streaming term.

In the second example we illustrate the isotropization of the distribution function in energy space. The initial condition is

$$f_n(x, \xi, \mu) = M(\xi_n)[1 + (A_0 + A_1 L_2(\mu)) \cos x], \quad (6.6)$$

where $M(\xi)$ is the Maxwellian distribution expressed in terms of the energy, $L_2(\mu)$ is the Legendre polynomial of order 2 and $A_0 = 0.5$ and $A_1 = 0.2$ are constants. In this case, we first report the error of the diffusion approximation to the Boltzmann equation as function of time for two different values of ϵ . Figure 6.5 shows the error $e(t)$, given by

$$e(t) = \int_0^{2\pi} dx \int_0^1 d\xi |\rho_{BE}(x, \xi, t) - \rho_{DIFF}(x, \xi, t)|, \quad (6.7)$$

for $\epsilon_2 = 0.05$ (solid line) and $\epsilon_2 = 0.1$ (dashed line) and $\epsilon_1 = 1$ (dominant inelastic case). We note that the error is smaller for the smaller

value of ϵ , as expected. Also, the error decays with time, that is the diffusion approximation becomes better and better as time proceeds.

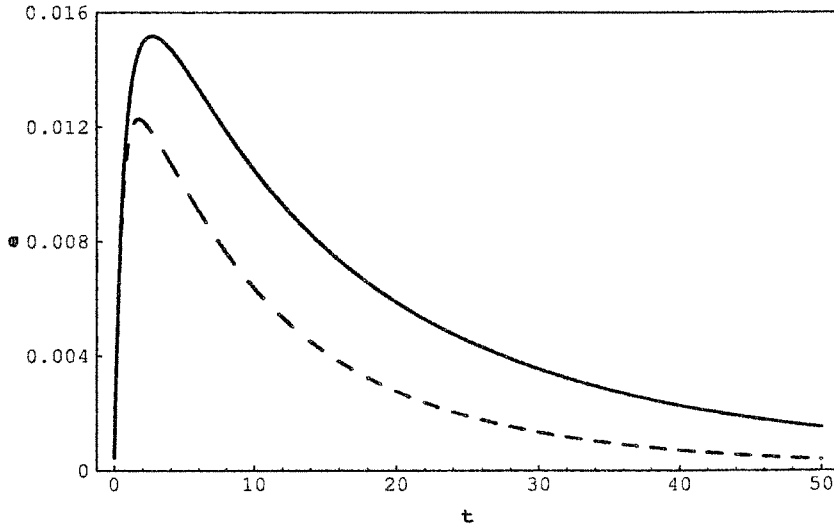


Figure 6.5: $e(t)$, $0 < t < 50$ for $\epsilon_1 = 1$, $\epsilon_2 = 0.05$ (solid line) and $\epsilon_2 = 0.1$ (dashed line).

Finally, in Figure 6.6 we show the time evolution of the isotropic component of the distribution function as function of time for two values of ϵ . We see that such function relaxes to a constant value as time proceeds. The relaxation appears somewhat faster for the smaller value of ϵ .

7 Appendix: The Successive Overrelaxation Method

The successive overrelaxation method (SOR) is an iterative method for solving linear and nonlinear systems. We briefly outline here the main steps of the method in the case of an n by n linear system of equations. Let

$$\sum_{j=1}^n a_{ij}x_j = b_i, \quad i = 1, \dots, n$$

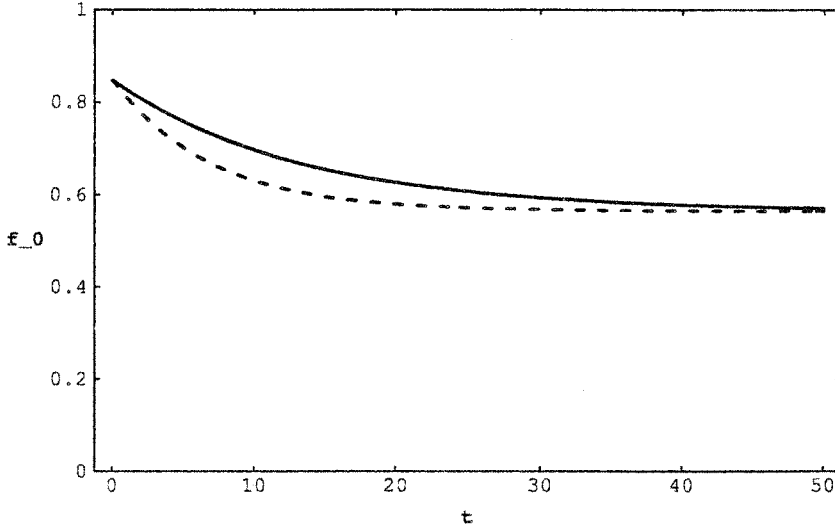


Figure 6.6: Isotropic component of f , $0 < t < 50$ for $x = \pi/4$, $\xi \approx 0$ and $\epsilon_1 = 1$, $\epsilon_2 = 0.05$ (solid line) and $\epsilon_2 = 0.1$ (dashed line).

be a linear system in the n unknowns x_1, x_2, \dots, x_n , with coefficients a_{ij} and constant terms b_i . We first rewrite the system in the form:

$$a_{ii}x_i = b_i - \sum_{j=1, j \neq i}^n a_{ij}x_j. \quad (7.1)$$

Suppose that the values of all variables x_i , $i = 1, \dots, n$, at the s -th iterate are known, say $x_i = x_i^s$. Equation (7.1) can then be used to update the value of the variable x_i , by using the x_j^s values in the right hand side (this gives the Jacobi iteration) or by using, for each variable x_j on the right hand side, its updated value when available, and the old value x_j^s otherwise (this gives the Gauss-Seidel iteration). Let x_i^* , $i = 1, \dots, n$, be the solution of (7.1) obtained with the Gauss-Seidel iteration, namely

$$x_i^* = \frac{b_i}{a_{ii}} - \sum_{j=1, j \neq i}^n \frac{a_{ij}}{a_{ii}} x_j^t,$$

where $t = *$ if x_j has already been updated, and $t = s$ otherwise. The SOR method, instead of taking the values x_i^* , $i = 1, \dots, n$ as the

values for the $s + 1$ -th iterate, considers a weighted average of the x_i^* with the values x_i^s at the previous iterate, namely

$$x_i^{s+1} = \omega x_i^* + (1 - \omega)x_i^s,$$

where ω is a relaxation parameter satisfying $0 \leq \omega \leq 2$ and which accelerates the convergence. In principle, the optimal value for ω can be determined on theoretical grounds [15], but it can also be adjusted ad hoc by trial and error if the speed of convergence is not of primary importance.

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