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BIVARIATE QUASI-INTERPOLATING SPLINES WITH APPLICATIONS IN NUMERICAL INTEGRATION

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ABSTRACT. Numerical methods for the evaluation of 2D integrals, based on bivariate quasi-interpolating splines, with a four directional mesh, are presented and convergence results are derived. Moreover an application to 2D singular integrals, defined in the Hadamard finite part sense, is proposed and studied.

SUBJECT CLASSIFICATION AMS (MOS): 65D32, 41A15.
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1 Introduction

Univariate and multivariate spline theory can have an essential influence on numerical integration [1, 5, 6, 7, 8, 13].

The purpose of this paper is to present some recent results that we obtained on 2D spline-based numerical integration.

We propose cubatures based on bivariate local quadratic quasi-interpolating (q-i)C^1splines with a four directional mesh [2], introduced in [3], for which we can show some interesting computational features and convergence properties.

Moreover we consider the problem of the evaluation of 2D integrals, defined in the Hadamard part sense, by integration rules based on the bivariate splines above introduced. For such rules convergence results are presented.

2 On the bivariate quasi-interpolating spline approximation

Let $S = \{(x, y) : 0 \le x, y \le 1\}$ and let $\Delta_{mn}^{(2)}$ be a uniform grid partition of S with type-2 triangulation (Fig. 1) defined by the restriction on S of the $\mathbf{R}^2$ partition consisting of lines:

$$\begin{cases} mx - i = 0 \\ ny - i = 0 \\ ny - mx - i = 0 \\ ny + mx - i = 0 \qquad i = \dots, -1, 0, 1, \dots \end{cases}$$

where m, n are given integers.

We consider the set $S_2^1(\Delta_{mn}^{(2)})$ of local bivariate C^1 spline functions that in every triangular cell of $\Delta_{mn}^{(2)}$ are polynomials of total degree two.

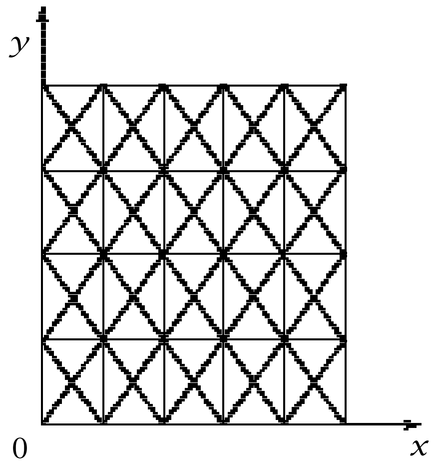

Fig. 1

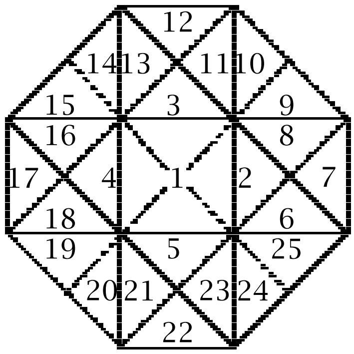

Fig. 2

Let $B(x,y)$ be the B-spline [3, 15] supported on the octagon Q with center at $(0,0)$ and vertices at

$$(3/2,1/2),\ (1/2,3/2),\ (-1/2,3/2),(-3/2,1/2),$$
$$(-3/2,-1/2),\ (-1/2,-3/2),\ (1/2,-3/2),\ (3/2,-1/2).$$

It is strictly positive inside its support, that has been partitioned into twentyfive cells (Fig. 2).

On every cell Δ_i, labeled by i, the function $B(x,y)$ is a polynomial $p_i(x,y) \in \mathbf{P}_2, i = 1,\ldots,25$ where $\mathbf{P}_2$ is the class of polynomials of total degree two. In particular it results [3] that:

$$\begin{aligned}
p_1(x,y) &= \frac{1}{2} - \frac{1}{2}x^2 - \frac{1}{2}y^2, \\
p_2(x,y) &= \frac{5}{8} - \frac{1}{2}x - \frac{1}{2}y^2 \\
p_6(x,y) &= \left(\frac{7}{8} - x + \frac{1}{4}x^2\right) + \left(\frac{1}{2} - \frac{1}{2}x\right)y - \frac{1}{4}y^2 \\
p_7(x,y) &= \frac{9}{8} - \frac{3}{2}x + \frac{1}{2}x^2 \\
p_9(x,y) &= \left(1 - x + \frac{1}{4}x^2\right) + \left(-1 + \frac{1}{2}x\right)y + \frac{1}{4}y^2
\end{aligned}$$

and the other p_i's are obtained from the above ones by simmetry.

We recall that B is in $C^1(\mathbf{R}^2)$. Moreover $B(x,y) = 0$ for all (x,y) outside Q and it is strictly positive inside Q. Following [3] we consider

$$B_{ij}(x,y) = B\left(mx - i + \frac{1}{2}, ny - j + \frac{1}{2}\right) \qquad \begin{array}{l} i = 0,\dots,m+1 \\ j = 0,\dots,n+1 \end{array}$$

whose support $Q_{ij} = \bigcup_{k=1}^{25} \Delta_k^{(ij)}$ has center at

$$(x_i, y_j) = \left(\frac{2i-1}{2m}, \frac{2j-1}{2n}\right)$$

and 'radius'

$$\delta'_{mn} = \frac{1}{2mn} \max\left[\sqrt{9m^2+n^2}, \sqrt{m^2+9n^2}\right],$$

where $\Delta_k^{(ij)}$ is the k-th cell of Q_{ij}. Then the set of the above bivariate B-splines $\{B_{ij},\ i = 0,\dots,m+1,\ j = 0,\dots,n+1\}$ spans all splines of $S_2^1(\Delta_{mn}^{(2)})$.

Now we consider the following bivariate 'variation diminishing' spline operator

$$V_{mn} : C(\Omega) \to S_2^1(\Delta_{mn}^{(2)})$$

defined by

$$V_{mn}(f;x,y) = \sum_{i=0}^{m+1} \sum_{j=0}^{n+1} f(x_i,y_j) B_{ij}(x,y),$$

where Ω is an open set containing S and

$$V_{mn}(f) = f,$$

for $f(x,y) = 1, x, y, xy$.

Let:

- K be a compact set which is the closure of Ω;
- $\omega_K(f;\delta) = \sup\{|f(x,y) - f(u,v)| : (x,y), (u,v) \in K, |(x,y) - (u,v)| < \delta\}$ be the modulus of continuity of f on K, where $|(x,y)| = (x^2+y^2)^{1/2}$;

- $\| \cdot \|_S$ be as usual the supremum norm over S;
- $\delta_{mn} = \max\left[\frac{1}{m}, \frac{1}{n}\right]$;
- f_1, f_2 be the partial derivatives of f with respect to its first and second variable, respectively;
- $\| D^2 f \|$ be the maximum over S of the norm of the linear transformation

$$D^2 f(x,y) : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$$

defined by:

$$D^2 f(x,y)((u_1,u_2),(v_1,v_2)) = f_{11}(x,y)u_1v_1 + \\ + f_{12}(x,y)u_1v_2 + f_{21}(x,y)u_2v_1 + f_{22}(x,y)u_2v_2\,,$$

where f_{11} is the partial derivative of f_1 with respect to its first variable, etc.

Then, from [3], we recall the following theorem.

THEOREM 2.1 *Let $f \in C(K)$. Then for all sufficiently large m, n, say $m, n \ge N_0$,*

$$\| f - V_{mn}(f) \|_S \le \omega_K(f, \delta'_{mn})\,.$$

Furthermore, if in addition $f \in C^1(S)$, then

$$\| f - V_{mn}(f) \|_S \le \delta_{mn} \max\left[\omega_S(f_1, \delta_{mn}/2), \omega_S(f_2, \delta_{mn}/2)\right].$$

Finally, if in addition $f \in C^2(S)$, then

$$\| f - V_{mn}(f) \|_S \le \frac{1}{4}\delta_{mn}^2 \| D^2 f \|\,,$$

for all $m, n \ge N_0$.

3 Cubature rules based on the spline operator V_{mn}

For any function $g \in C(\Omega)$ we consider the numerical evaluation of the integral

$$I(g) = \int_S g(x,y)dxdy\,. \tag{3.1}$$

by cubature rules defined by

$$I(g) \simeq I(V_{mn}(g)) = \sum_{i=0}^{m+1} \sum_{j=0}^{n+1} w_{ij} g(x_i, y_j)\,, \tag{3.2}$$

where

$$w_{ij} = \int_{Q_{ij}\cap S} B_{ij}(x,y)dxdy\,.$$

We remark that, for all i and j, it results $w_{ij} > 0$.

Moreover we can show the following theorems [4].

THEOREM 3.1 *The weights of the cubature (3.2) satisfy the following symmetric properties:*

- *for $m, n \geq 5$ it results that*

  i) $w_{ij} = w_{m-i+1,j} = w_{i,n-j+1} = w_{m-i+1,n-j+1} =$
  $w_{ji} = w_{m-j+1,i} = w_{j,n-i+1} = w_{m-j+1,n-i+1}$
  $i = 0, \dots, 2 \quad j = 0, \dots, i;$

  ii) $w_{ij} = w_{i,n-j+1} = w_{2j}, \quad j = 0, 1$
  $w_{i2} = w_{i,n-1} = w_{22} \quad i = 3, \dots, m-2;$

  iii) $w_{ij} = w_{m-i+1,j} = w_{i2}, \quad i = 0, 1 \quad j = 3, \dots, n-2;$

  iv) $w_{ij} = w_{22}, \quad i = 2, \dots, m-1 \quad j = 3, \dots, n-2;$

- *for $3 \leq m < 5$, $n \geq 5$, the above i),iii),iiii) hold;*

- *for $m \geq 5$, $3 \leq n < 5$, the above i),ii) hold;*

- *for $3 \leq m, n < 5$, the above i) holds.*

THEOREM 3.2 *For any $g \in C(\Omega)$ and for $m, n \geq 3$ the rule (3.2) can be written in the following simplified form:*

$$I(V_{mn}(g)) = \sum_{i=0}^{2} \sum_{j=0}^{i} w_{ij} z_{ij}(g) \tag{3.3}$$

defined only by six weights, given by Table 1, with

$z_{kk}(g) = g_{kk} + g_{m-k+1,k} + g_{k,n-k+1} + g_{m-k+1,n-k+1};$
$z_{2k}(g) = \sum_{i=2}^{m-1} (g_{ik} + g_{i,n-k+1}) + \sum_{j=2}^{n-1} (g_{kj} + g_{m-k+1,j}), \quad k = 0, 1$
$z_{10}(g) = g_{10} + g_{m,0} + g_{1,n+1} + g_{m,n+1} + g_{01} + g_{m+1,1} + g_{0n} + g_{m+1,n};$
$z_{22}(g) = \sum_{i=2}^{m-1} \sum_{j=2}^{n-1} g_{ij}$
and $g_{ij} = g(x_i, y_j)$.

w_{00}	w_{10}	w_{11}	w_{20}	w_{21}	w_{22}
$\frac{1}{48mn}$	$\frac{7}{48mn}$	$\frac{33}{48mn}$	$\frac{1}{6mn}$	$\frac{5}{6mn}$	$\frac{1}{mn}$

Table 1

In order to study the convergence of $\{I(V_{mn}(g))\}$ to $I(g)$, from Theorem 2.1 we can deduce the following

THEOREM 3.3 *Let $g \in C(K)$. Then*

$$I(V_{mn}(g)) \to I(g) \quad as \quad m, n \to \infty .$$

In particular

$$|E_{mn}(g)| = |I(g) - I(V_{mn}(g))| = O(\omega_k(g, \delta'_{mn})) .$$

If in addition $g \in C^k(S)$, $k = 1, 2$, then

$$E_{mn}(g) = O(\delta_{mn}^k) .$$

4 Numerical evaluation of 2-D singular integrals defined in the Hadamard finite part sense

In some physical and engineering problems [9, 10, 11, 14] we have to deal with 2-D singular integrals of the form

$$J(f) = \int_{\theta_1}^{\theta_2} \fint_0^{R(\theta)} \frac{f(r, \theta)}{r} \, dr d\theta \, , \tag{4.1}$$

where the domain of integration is the triangle

$$T = \{(r,\theta) : 0 \le r \le R(\theta), \theta_1 \le \theta \le \theta_2\}$$

as in Fig. 3:

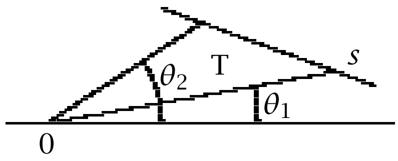

Fig. 3

and $⨍$ indicates the Hadamard finite part [11, 12].

We remark that in practical applications $R(\theta)$ is always analytic in $[\theta_1,\theta_2]$ and we suppose $f \in C^1(\overline{\Omega})$, with $\overline{\Omega}$ an open set containing $[0,R] \times [\theta_1,\theta_2]$, $R = \max_{\theta_1 \le \theta \le \theta_2} |R(\theta)|$.

We can write (4.1) as follows:

$$J(f) = J^{(0)}(f) + J^{(1)}(f)\,, \tag{4.1'}$$

where

$$J^{(0)}(f) = \int_{\theta_1}^{\theta_2} \int_0^{R(\theta)} \frac{f(r,\theta) - f(0,\theta)}{r}\, dr d\theta$$

and

$$J^{(1)}(f) = \int_{\theta_1}^{\theta_2} f(0,\theta) \ln R(\theta)\, d\theta\,.$$

Since $J^{(1)}$ is a regular one-dimensional integral, it can be accurately evaluated.

For the regular double integral $J^{(0)}$ we can write

$$J^{(0)}(f) = (\theta_2 - \theta_1) I(\Psi)\,, \tag{4.2}$$

where

$$\Psi(\rho,\overline{\theta}) = \frac{f(R(\xi(\overline{\theta}))\rho, \xi(\overline{\theta})) - f(0,\xi(\overline{\theta}))}{\rho}$$

and

$$\xi(\overline{\theta}) = \theta_1 + (\theta_2 - \theta_1)\overline{\theta}\,.$$

Since $f \in C^1(\overline{\Omega})$, we can deduce that $\Psi \in C(\Omega)$, with Ω open set containing S.

Therefore we can estimate $I(\Psi)$ in (4.2) by (3.2) and we obtain

$$J^{(0)}(f) = J^{(0)}_{mn}(f) + E^{(0)}_{mn}(f)\,,$$

where

- $J^{(0)}_{mn}(f) = (\theta_2 - \theta_1)I(V_{mn}(\Psi)) = (\theta_2 - \theta_1)\sum_{i=0}^{2}\sum_{j=0}^{i} w_{ij}z_{ij}(\Psi)$
- $E^{(0)}_{mn}(f) = E_{mn}(\Psi)$

with $\{w_{ij}\}$, $\{(x_i, y_j)\}$ respectively weights and nodes of (3.2) and $\{z_{ij}\}$ defined in section 3.

Now we can show the following convergence theorem [4].

THEOREM 4.1 *Let $\overline{J}_1$ be the closure of an open set containing $[0, R]$ and $\overline{J}_2$ the closure of an open set containing $[\theta_1, \theta_2]$. Let $f \in C^1(\overline{J}_1 \times \overline{J}_2)$. Then*

$$E^{(0)}_{mn} \to 0 \;\; as \;\; m, n \to \infty\,. \tag{9}$$

If in addition $f \in C^2([0, R] \times [\theta_1, \theta_2])$, then

$$|E^{(0)}_{mn}(f)| = o(\delta_{mn})\,. \tag{4.3}$$

If also $f \in C^3([0, R] \times [\theta_1, \theta_2])$, then

$$|E^{(0)}_{mn}(f)| = O(\delta^2_{mn})\,. \tag{4.4}$$

5 Conclusion

In this note we have considered cubatures based on a class of bivariate C^1 local polynomial splines, for which we have proved some computational features and convergence properties.

We remark that formulas of this kind, based on 'non' tensor product splines, can be 'better' than the product integration ones for functions f with oscillations in the directions $nx \pm my = 0$, with n and m integers.

Moreover we have presented an application of our local method to the evaluation of 2-D integrals, defined in the Hadamard finite part sense, arising in engineering problems.

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