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Geometrical aspects of D-branes on noncompact varieties

Doctoral Dissertation of

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a tutta la mia famiglia

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Introduction

The first part of this thesis concerns the study of homological local mirror symmetry applied to the specific example of the noncompact toric orbifold $\mathbb{C}^3/\mathbb{Z}_6$ (this work is the subject of the paper [CaCo]).
From the physical point of view, mirror symmetry looks like a generalization of T-duality equivalence between different perturbative limits of the, supposed to exist, unique M-theory. This defines a map between different Calabi-Yau (CY) varieties. The case of noncompact CY was studied quite recently in [HoIqVa]. The mirror manifold of a toric variety X_t, with t specifying the Kähler moduli, results to be defined as the zero locus

$$Y_a = \{(\vec{u},\vec{v}) \in \mathbb{C}^2 \times \mathbb{C}^2_* | F_a(\vec{u},\vec{v}) = 0\},$$

where a determines a point in the complex structures moduli space of the mirror, and $F_a(\vec{u},\vec{v}) = u_1^2 + u_2^2 + f_a(\vec{v})$ is a certain polynomial fully determined by the toric data describing the starting orbifold.
Some non perturbative enhancements are provided by adding D-brane configurations. At a semiclassical level, D-branes are described by closed cycles (with bundles) which the branes are supposed to wrap on. For IIB strings on a CY, stable configurations are represented by holomorphic (then evendimensional) cycles, whereas for type IIA branes one finds Lagrangian submanifolds (with respect to the Kähler form) as brane representatives, that are halfdimensional subvarieties. In the case in our interest the lasts are three dimensional surfaces. An astounding advance in the extension of mirror symmetry in order to take account of nonperturbative brane configurations has been proposed by M. Kontsevich [Kon] who introduced the concept of homological mirror symmetry. In this contest type B branes are described in terms of bounded derived categories of coherent sheaves whereas A branes are substituted by derived Fukaya categories. [Fuk]. Brane charges (central charges or masses) are thus described in terms of K-theory groups (even if there are many other indications for this beyond and independently from mirror symmetry, see [GrHaMo, Wi]). At homological level, mirror symmetry is conjectured to define a map

$$\mathrm{Mir} : D^\flat Coh(X) \longrightarrow DFuk^o(Y,\omega)$$

where the symplectic form ω is a fixed Kähler form on Y corresponding to a fixed complex structure on X. The Kähler moduli on X dual to complex moduli on Y are responsible for monodromies, that at the categorical level are described as autoequivalences of the triangulated categories.
From the physical point of view, some new insight in the direction of Homological mirror symmetry for the case of noncompact CY manifolds was done by X. de la Ossa, B. Florea and H. Skarke [dOFS] who were able to select a distinguished K-theory basis for B-branes configurations adapted to support monodromy correspondence, generalizing (at least at a conjectural level) the corresponding results quite well established in the compact case. Further progress is due to S. Hosono [Hos1] who found an elegant way to describe local mirror symmetry in terms of cohomology valued hypergeometric series. Mirror symmetry identifies the Kähler moduli of a CY variety with the periods of its mirror, which as functions of the complex structure moduli must satisfy a set of Picard-Fuchs equations, the Gel'fand-Kapranov-Zelevinski system [GKZ]. It results that a particular cohomology valued hypergeometric series w arises naturally providing a basis of solutions for the GKZ system [HKTY1], [HKTY2], [HLY1], [HLY2]. Hosono was able to recognize such series as a formula identifying the BPS states of the associated physical theory, and proposed an intriguing conjecture, which we dub "*the Hosono conjecture*", see conjecture 6.3 in [Hos1], which, beyond identifying the central charge of a brane configuration $F \in K^c(X)$ in terms of w, interprets the monodromy of the periods via a naturally associated symplectic form on $K^c(X)$. Hosono checked very carefully his conjecture for the toric quotients $\mathbb{C}^2/G$, and for the examples $\mathbb{C}^3/\mathbb{Z}_3$, $\mathbb{C}^3/\mathbb{Z}_5$ in three complex dimensions. Among others, a consequence of the Hosono conjecture is to provide a closed formulation of a prepotential for noncompact quotients also. Indeed, at cohomological level, mirror symmetry provides a map

$$\mathrm{mir} : K^c(X) \xrightarrow{\sim} H_3(Y,\mathbb{Z}),$$

transferring the symplectic form on $K^c(X)$ to a symplectic structure on $H_3(Y,\mathbb{Z})$. This is the noncompact analog of the symplectic structure that, combined with Griffiths transversality, ensures the existence of a prepotential in the compact cases. However, due to non compactness, the symplectic structure is generically degenerate. On the X side, it defines a correspondence between $H^2(X,\mathbb{Q})$ and $H^4(X,\mathbb{Q})$ which permits a complete determination of the prepotential (and correspondingly of all GW invariants) only when it arises as a vector space isomorphism.

In the thesis we apply the Hosono conjecture to compute the prepotential and the GW invariants for the family of noncompact toric CY varieties obtained as crepant resolutions X of the orbifold quotient $\mathbb{C}^3/\mathbb{Z}_6$. We have chosen this model because it has quite general properties which make it very interesting to test the conjecture. In fact we have that the second and fourth Betti numbers are $b_2 = 4$ and $b_4 = 1$, so that the symplectic structure result to be highly degenerate. Thus it defines a quite poor correspondence between $H^2(X,\mathbb{Q})$ and $H^4(X,\mathbb{Q})$. Nevertheless, we will see that the Hosono procedure permits to define a partial prepotential containing a lot of information about local geometry. Indeed, from it we are able to read out almost all GW invariants, leaving out only a three dimensional subcone of the fourdimensional Mori cone. We will discuss in the conclusions the possible origin for the ambiguity in defining the lacking GW invariants from an enumerative point of view. The first natural extension of our work should be the resolution of such indeterminateness. In this sense we could explore the proposal of B. Forbes and M. Jinzenji [FoJi1], [FoJi2], that gives a possible way to extend the GKZ system obtaining a complete determination of all GW invariants.

A second interesting peculiarity of our model is that it admits five distinct crepant resolutions, which differ by flops. Going from a resolution to another passing through the singular orbifold realizes a geometrical transition. Geometrically, the transition is obtained by moving the Kähler moduli t through an orbifold point, where the manifold becomes singular with a curve which shrinks down and reemerges as a flopped curve. Mirror symmetry says that varying the moduli t corresponds to varying the complex moduli a of the mirror manifold. However, whereas X undergoes a flop transition, Y simply changes smoothly its complex moduli. Therefore in string theory such transition correspond to smooth physical processes. At the homological and categorical level one expects monodromy to relate different resolutions by means of different Fourier Mukay transforms which are expected to realize a (quiver) representation of the quotient group by the McKay correspondence. The study of the details of the mirror isomorphism *mir* in this situation represents the most promising prosecution of the thesis. In fact we remember that the Hosono conjecture works at the level of derived categories too, therefore it gives some hints to get information about the largely unknown mirror map *Mir*.

The organization of this part of the thesis is as follows.

In chapter 1 we include a short overview of the main steps which lead to the introduction of local mirror symmetry and GW–invariants to arrive to the Hosono conjecture.

In chapter 2 we present a detailed analysis of the first resolution, that is the G-Hilbert resolution. We will use the Hosono's conjecture to construct the cohomological hypergeometric series generating the periods of the mirror manifold. We define a partial prepotential which generates all GW–invariants associated to the curves in the Mori cone, excluding a codimension one subcone.

In chapter 3 we repeat the previous analysis for all the other resolutions, deriving a partial determination of the GW–invariants for all of them.

The results will be commented in chapter 4.

The second part of the thesis is devoted to the study of the geometrical properties of the orbifold $\mathbb{C}^3/\Delta_{27}$ recently object of attentions in the contest of the bottom-up approach to string phenomenology.

Over the past 10 years, fueled by the deepened understanding of duality and D-brane physics, open string theory has evolved into an increasingly successful tool for building 4-d supersymmetric field theories. In particular, it is now understood that by taking a judicious low energy limit of the world-volume theory on a stack of D3-branes, one recovers a purely 3+1-dimensional gauge theory, decoupled from gravity and all extra dimensional

dynamics. In this decoupling limit, the closed string background freezes into a set of non-dynamical, tunable gauge invariant couplings. By placing one or more D-branes near various types of geometric singularities, realizations of large classes of gauge theories have been uncovered, and a detailed dictionary between geometric and gauge theory data is emerging. Open string theory has become a preferred duality frame for representing weakly coupled 3+1-d Quantum Field Theories in string theory.
The study of D-branes at Calabi-Yau singularities provides an alternative route towards string phenomenology, known as "bottom-up" approach. In this program, after establish a sufficiently general dictionary between gauge theory quantities and local and global properties of singular CY threefold, one tries to find explicit realizations of world-volume gauge theories on D-branes at singularities that reproduce the Standard Model of particle physics. Though clearly a non-trivial challenge, this question is still much less ambitious, and thus easier to answer, than finding a fully realistic closed string background via the conventional top-down approach.
Verlinde and Wijnholt in their celebrated article [VeWi] proposed an explicit realization of the supersymmetric extension of the Standard Model as the world-volume theory on a D3-brane living in a specific geometry. Actually they study the quiver gauge theory associated to a D3-brane located near the singularity of a noncompact Calabi-Yau variety given by a complex cone over a collapsing del Pezzo surface dP_8. Such quiver gauge theory is just rich enough to contain the SM gauge group and matter content. Then they look for a well-chosen symmetry breaking process that reduces the gauge group and matter content realistically close to that of the Standard Model.
During their work, they observed that the starting quiver gauge theory on the dP_8 singularity is identical to that associated to a single D3-brane on the $\mathbb{C}^3/\Delta_{27}$ orbifold singularity, the model considered earlier in [BJL] as a possible starting point for a string realization of a Standard Model-like gauge theory. They partially explain this correspondence giving an explicit isomorphism from the $\mathbb{C}^3/\Delta_{27}$ orbifold to the cone over a singular degree six hypersurface in the weighted projective space $\mathbb{P}_{1123}$. Since any such nonsingular hypersurface is isomorphic to a dP_8 surface they argued that the $\mathbb{C}^3/\Delta_{27}$ orbifold should have been obtained as some sort of limit in the moduli space of cones over del Pezzo surfaces.

In this thesis we review the construction of [VeWi] and in the spirit of [AD] we explain the exact correspondence between the orbifold and the del Pezzo geometry, trying to pursue the scope of furnishing a readable work for a theoretical physicists audience. As a first step we analyze the discrete group $\Gamma = \Delta_{27}$ (the Heisenberg group of order 27) and the natural action of its abelianization $\tilde{\Gamma}$ on the projective plane $\mathbb{P}^2$. The singular quotient $\mathbb{P}^2/\tilde{\Gamma}$ admits a finite group of automorphisms of order 256 which is strictly related to the automorphisms group of the well known Hesse pencil of cubics. The study of the properties of the pencil leads to the discovery that $\mathbb{P}^2/\tilde{\Gamma}$ admits as minimal resolution the blow up of $\mathbb{P}^2$ in eight of the nine base points of the Hesse pencil. Since these points are in particular position (any three of them are collinear), the resolution is actually a particular limit variety in the moduli space of Del Pezzo surfaces.
There are many possible developments arising from this work. In connection with the first part of the thesis where we study the homological mirror symmetry on the orbifold $\mathbb{C}^3/\mathbb{Z}_6$, we could try to investigate the mirror symmetry on $\mathbb{C}^3/\Gamma$. Indeed in this case we have not a toric variety but many facts are well known for del Pezzo surfaces [AKO]. The exact relation that we found between the del Pezzo and orbifold constructions therefore could help us to find improvements in defining mirror symmetry in the non abelian orbifold case. From a more physical point of view one could investigate the role of Γ as the automorphisms group of the quiver and the consequences in the contest of AdS/CFT correspondence [BLZ]. Directly related to the original paper [VeWi] it should be interesting to purse the symmetry breaking program towards the Standard Model. This require to identify a particular partial resolution of the singularity and then a Calabi-Yau compactification of this local geometry, in order to explicitly realize the program delineated in general terms in [VeWi, BMMVW].

The organization of this part of the thesis is as follows.
In chapter 6 we include an overview of the Verlinde and Wijnholt model, following their original papers. In

particular we recall the relation between the del Pezzo and the orbifold geometry that the same authors suggest.
In chapter 7 we analyze the group Δ_{27} and its action on $\mathbb{C}^3$. We show how the essential properties of the orbifold are contained in the geometry of the surface obtained as the quotient of the projective plane $\mathbb{P}^2$ by the abelianization of Δ_{27}. Then we prove the isomorphism between the orbifold $\mathbb{C}^3/\Delta_{27}$ and the hypersurface in $\mathbb{C}^4$ proposed by Verlinde and Wijnholt.
In chapter 8 we study a particular subgroup of the automorphism group of the surface $\mathbb{P}^2/\tilde{\Gamma}$, showing how this is related to the Hessian group, the subgroup of $PGL(3,\mathbb{C})$ of order 216 preserving the Hesse pencil of plane cubic curves. Then we explain how the group $\tilde{\Gamma}$ acts on the pencil and how this suggest the right relation between the orbifold and the del Pezzo geometry.
In chapter 9 we recall the fundamental properties of the del Pezzo surfaces, in particular the relation between dP_8 surfaces and hypersurfaces of degree two in the weighted projected space $\mathbb{P}_{1,1,2,3}$. We explain the difference between such general case and ours, then we conclude giving the explicit resolution map from the particular limit of del Pezzo surface defined in the second section to the quotient surface $\mathbb{P}^2/\tilde{\Gamma}$.
The final chapter 10 contains discussion and conclusions.

Part I

D-Branes on $\mathbb{C}^3_6$

Chapter 1

Local mirror symmetry and the Hosono conjecture

Here we will recall some main step leading to the conjectures we are testing. The literature on the subject is quite huge, so that we will mainly refer to [Horietal] and references therein.

1.1 Dualities and mirror symmetry

Let us consider a type A string theory having a toric Calabi-Yau variety X as target space. Thus, there is a nice interpretation of mirror symmetry as a T-duality transformation. Indeed, string theory on X can be described in terms of a two dimensional $U(1)^m$ supersymmetric gauge theory, the so called "gauged linear sigma model" (see [Horietal], sections 7.3, 7.4). It contains a certain number $n > m$ of complex scalar fields $Z = \{Z_\alpha\}_{\alpha=1}^n$ having charges $Q_{\alpha,r}$ $r = 1,2,\dots,m$ with respect to the gauge group $U(1)^m$, and with potential energy

$$U(Z) = \frac{1}{2}\sum_{r=1}^m g_r^2 \left(\sum_{\alpha=1}^n Q_{\alpha,r} Z_\alpha \overline{Z}_\alpha - \mathrm{r}_r\right)^2 .$$

Here g_r and r_r are the gauge couplings and the Fayet-Iliopoulos terms respectively. Supersymmetric ground states require the vanishing of the potential energy:

$$\sum_{\alpha=1}^n Q_{\alpha,r} Z_\alpha \overline{Z}_\alpha = \mathrm{r}_r.$$

For a fixed choice of the F–I parameters, these equations define a toric variety associated to a fan, in an $(n-m)$–dimensional lattice N, generated by an opportune set $\Sigma(1) = v_1,\dots,v_n$ of vectors in N. Actually we have that the supersymmetric vacua are identified with the points of a toric variety that coincides with the starting X. Each vector v_α determines an invariant divisor[1], D_{v_α}. It is not hard to show (see [Horietal], sec. 7.4) that one can choose a basis $\{C_r\}_{r=1}^m$ of irreducible curves of $H_2(X,\mathbb{Z})$ (which indeed result to be m–dimensional) such that the charges are given by the intersection numbers $Q_{\alpha,r} = D_{v_\alpha} \cdot C_r$. Also note that the F–I parameters rescale as $|Z|^2$ so that, if chosen to be positive, they indeed parameterize the points of the Kähler cone of X. This means that the supersymmetric configurations are completely characterized in geometrical terms.
At this point mirror symmetry can be realized as a T–duality transformation ([Horietal], sec. 20). Indeed, recall

[1] t.i. invariant under the toric action

that roughly speaking T–duality on a circle transforms a type A string theory on a circle of radius R in a type B string theory on a circle of radius α'/R. If Z_α are taking value on a complex variety (indeed the toric variety in the vacuum configuration) then we can T–dualize their phases which define circles in the target manifold. The result ([Horietal], sec. 13) is a Landau-Ginzburg theory with superpotential

$$W(Y,t) = \sum_{\alpha=1}^{n} e^{-Y_\alpha},$$

for a set of chiral superfields related by the set of constraints

$$\sum_{\alpha=1}^{n} Q_{\alpha,r} Y_\alpha = t_r$$

where t_r are the complexified Kähler parameters of the original variety X ($\mathrm{Re}(t_r) = \mathrm{r}_r$). In this way, the mirror transformation applied to the twodimensional sigma model gives rise to a Landau–Ginzburg model with superpotential $W(Y,t)$. To take contact with the Batyrev's geometric construction of mirror manifolds for toric varieties, let us proceed as follows (see [HoIqVa]) for the cases when the starting linear sigma model describes strings on a crepant resolution of some abelian quotient $\mathbb{C}_3/G$. Being crepant, it will be described by a set of vectors $v_1,\dots,v_n$ in a three-dimensional lattice such that for some isomorphism $\phi : N \longrightarrow \mathbb{Z}^3$ one has $\phi(v_\alpha) = (n_{\alpha,1}, n_{\alpha,2}, 1)$. The solutions of the constraints can thus be written in terms of three independent fields y_0, y_1, y_2 as $Y_\alpha = y_0 + n_{\alpha,1}y_1 + n_{\alpha,2}y_2 + c_\alpha$ where c_α are some constant satisfying $\sum_{\alpha=1}^{n} Q_{\alpha,r}Y_\alpha = t_r$. These linear redefinitions do not affect the functional measure, and setting $w_a = \exp(-y_a)$, $a = 0,1,2$ and $a_\alpha = \exp(-c_\alpha)$ we get for the superpotential

$$W(w,a) = w_0 \sum_{\alpha=1}^{n} a_\alpha w_1^{n_{\alpha,1}} w_2^{n_{\alpha,2}}, \qquad w_a \in \mathbb{C}^*.$$

As discussed in [HoIqVa], we can note that, for what concerns the BPS configurations, this LG model is equivalent to another one, where $w_0 \in \mathbb{C}$ and with two extra chiral fields $U,V \in \mathbb{C}$, whose superpotential is

$$\tilde{W}(U,V;w;a) = W(w,a) - w_0 UV.$$

Integrating the field w_0 thus gives a delta function $\delta(\sum_{\alpha=1}^{n} a_\alpha w_1^{n_{\alpha,1}} w_2^{n_{\alpha,2}} - UV)$. This mirror LG model is equivalent to a type B string theory on the Calabi-Yau manifold

$$Y = \{(\vec{u},\vec{w}) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 | F_a(\vec{u},\vec{w}) = 0\},$$

where

$$F_a(\vec{u},\vec{w}) = u_1^2 + u_2^2 + f_a(\vec{u},\vec{w}) = u_1^2 + u_2^2 + \sum_{\alpha=1}^{n} a_\alpha w_1^{n_{\alpha,1}} w_2^{n_{\alpha,2}}.$$

The Kähler parameters t now parameterize the complex moduli of Y. This is indeed local mirror symmetry as discovered for the first time at physical level in [KKLMV], [KLMVW]. It defines a map from the Kähler moduli space of the starting variety X of A type theory to the complex moduli space of the mirror variety Y of B type theory. Symmetrically we could consider a B theory on X and its dual A theory on Y, defining a map from the complex moduli space of X to the Kähler moduli space of Y.

1.2 Branes and homological mirror symmetry

The intuitive picture described above does not takes into account the presence of brane configurations. Because we are looking for supersymmetric vacua, we need to know what kind of brane configurations are admitted on a Calabi-Yau manifold. In other words, one must search for boundary condition compatible with supersymmetry. This is described for example in [HoIqVa], sec. 3. The answer depends on the type of string theory. Let us consider a type B theory on the variety X with a chosen complex structure. One finds that supersymmetric branes must wrap holomorphic cycles of X supporting holomorphic vector bundles. In our models it means that type B brane configurations will be described classically by compact divisors, curves of the Mori cone and points. For the dual type A theory on the mirror variety Y with a chosen Kähler form ω, supersymmetric branes are represented (at classical level) by halfdimensional subvarieties S, $\iota : S \hookrightarrow Y$, where ω vanishes, $\iota^*\omega = 0$, and supporting flat vector bundles. Thus A–branes are Lagrangian submanifolds with respect to the symplectic structure ω.

Mirror symmetry maps BPS states of a model into the BPS states of the mirror model, converting B–branes to A–branes and viceversa. However, there is an odd asymmetry between B and A configurations: indeed all A–branes have the same dimensions, whereas this does not happen for B–branes. Now, the point is that B–branes on X and A–branes on Y can change respectively when Kähler moduli on X and complex moduli on Y vary. In particular if one completes certain loops in the (isomorphic) moduli spaces, monodromy transformations can give rise to new brane configurations ([HoIqVa]). The boundary states corresponding to the A–branes are described by the periods of the holomorphic three-form Ω of Y (in the geometric picture). The monodromy thus acts on a basis of cycles recasting them in some linear recombination or equivalently on the periods in the same linear recombination. On the mirror X it should correspond to a recombination of the holomorphic cycles, hard to understand in the naïve geometrical picture where they have different dimensions.

To solve this point a first aid comes from a K-theoretical description, where lower dimensional branes can be described in terms of the top dimensional branes and a tachyon field [Wi]. K-theory mainly captures topological aspects of the problem, carrying important information on the admissible brane configurations, but it is quite poor from the geometrical point of view. In [Dou] it was argued that a deeper geometrical understanding of (stable) brane configuration in (topological) type B superstring can be understood in terms of triangulated categories, in particular the derived category of coherent sheaves on the manifold (see also [Asp], or [Br] for a more mathematical point of view). This provided a deep contact between physics and the “homological mirror symmetry” conjectured by Kontsevich [Kon] who proposed that the usual geometrical mirror symmetry should enhance to homological level as an equivalence between triangulated categories: the derived category of coherent sheaves on a CY manifold X with a fixed complex structure on one side[2] and the derived $\mathcal{A}^\infty$ Fukaya's category over the mirror manifold Y on the other side, essentially generated by the Lagrangian submanifolds of $\{Y, \omega\}$, where the symplectic structure ω is given by the fixed Kähler form on Y, dual to the complex form on X:

$$\mathrm{Mir} : D^\flat Coh(X) \xrightarrow{\simeq} DFuk^o(Y, \omega).$$

1.3 The Hosono conjecture

As we said, in the type A string model on Y, BPS states are described by periods that are integrals of the holomorphic three form Ω on Y over the Lagrangian cycles. For a crepant resolution X of the noncompact

[2] for clarity we confine ourselves to the case of Calabi-Yau varieties

quotient we are describing, the holomorphic three form on the mirror Y is

$$\Omega = \frac{1}{4\pi^3}\mathrm{Res}_{F=0}\left[\frac{du^1\wedge du^2\wedge dw^1\wedge dw^2}{w^1w^2F(\vec{u};\vec{w};a)}\right].$$

Here we have fixed the Kähler form, however Ω depends explicitly on the complex moduli of Y (as shown by the explicit dependence on a of the polynomial F) so that the periods

$$\Pi_{C_i}(a) = \int_{C_i}\Omega,$$

of any set of Lagrangian cycles C_i, will be locally holomorphic functions of the moduli. Indeed, they are forced to satisfy a set of hypergeometric differential equations known as the GKZ hypergeometric system, largely studied in [GKZ].

For compact varieties the knowledge of a complete set of solutions for the GKZ system correspond to an exhaustive description of the set of BPS brane configurations on the A side. Furthermore, the special Kähler geometry of the complex structure moduli space of a C-Y manifold can be described in terms of periods [St]. If a parameterizes the structure complex moduli of Y then the Kähler potential of the moduli space can be written as

$$K(a,\bar{a}) = -\log\left[i\sum_{I=0}^{h^{2,1}(Y)}\left(X^I\frac{\partial\overline{G}}{\partial X^I} - \overline{X}^I\frac{\partial G}{\partial X^I}\right)\right],$$

where

$$X^I(a) = \int_{A^I}\Omega$$

are the periods with respect to a canonical symplectic basis $\{A^I, B_I\}$ of $H_3(Y,\mathbb{Z})$. Finally $G(a)$ is the prepotential

$$G(a) = \frac{1}{2}\sum_{I=0}^{h^{1,2}(Y)}\int_{A^I}\Omega\int_{B_I}\Omega.$$

Mirror symmetry gives a correspondence between Kähler moduli t_i of X and complex moduli of Y so that

$$t_i = \frac{X^i}{X^0}, \qquad i = 1,\dots,h^{1,2}(Y) = h^{1,1}(X).$$

On the other side, also the Kähler moduli space of X is a special Kähler manifold which can thus be described in terms of a prepotential function $F(t)$. At classical level such geometry is described by the prepotential

$$F^c(t) = \frac{1}{6}d_{ijk}t^it^jt^k$$

where $d_{ijk} = J_i\cdot J_j\cdot J_k$ are the intersection numbers of the Kähler cone generators. Physically they determine the Yukawa couplings of the chiral fields [CdOGP]. However these couplings receive quantum corrections which come from D–brane instantons. At lowest order they corresponds to wrapping of the D–branes on rational curves in X. The energy of such a wrapping is given by the volume of the wrapped cycle as measured by the Kähler metric. Any given (class of) rational curve of degree $\vec{d}$ results to contribute to the prepotential with a term

$$n_{\vec{d}}\,Li_3(e^{2\pi i\vec{d}\cdot t}),$$

$n_{\vec{d}}$ being the number of classes of curves with the given degree, so that it can be shown that the quantum corrected prepotential takes the form [CKYZ]

$$F(t) = \frac{1}{6}d_{ijk}t^it^jt^k - \frac{1}{24}c_2(X)\cdot J_i t^i - i\frac{\zeta(3)}{16\pi^3}c_3(X) + \sum_{d\in\mathbb{Z}_>^{h^{1,1}}} n_{\vec{d}}\, Li_3(e^{2\pi i\vec{d}\cdot t}).$$

The $n_{\vec{d}}$ are the Gopakumar-Vafa (GV) integral invariants related to the Gromov-Witten (GW) fractional invariants of the curves of degree $\vec{d}$ in X [GoVaI, GoVaII]). See [CKYZ] for a mathematical enumerative interpretation. By means of the identification (making use of the Griffith transversality, [CKYZ], [St])

$$\left\{\int_{A^I}\Omega; \int_{B_I}\Omega\right\}_{I=0}^{h^{2,1}(Y)} = \left\{1,\ t^i;\ \partial_{t^i}F,\ 2F - \sum_{j=1}^{h^{1,1}(X)} t^j\partial_{t^i}F\right\}_{i=1}^{h^{1,1}(X)}$$

mirror symmetry thus gives a simple way to compute the GW–invariants of X.

In a series of papers (see for example [HKTY1], [HKTY2], [HLY1], [HLY2]) it was provided an efficient strategy to characterize a complete set of the GKZ system for a C-Y hypersurface, which is summarized in [Ho]. In particular there was introduced a cohomological valued power series whose expansion in the Chow ring

$$A^*(X)\otimes\mathbb{C}[[x]][\log x]$$

gives a basis for the period integrals of the mirror manifold Y in the large complex structure limit (LCSL), (see [Ho], Claim 5.11). Thus the cohomological series encodes many geometrical information on both the manifolds X and Y so summarizing several fundamental aspects of mirror symmetry.
In [Hos2], [Hos1] Hosono extended this picture to local mirror symmetry for noncompact C-Y manifolds, in particular for resolutions of abelian quotients $\mathbb{C}^k/G$, with $k = 2,3$. For convenience we will state the conjecture in section 2.6. In [Hos1], Hosono verified his conjecture carefully for the case $k = 2$ and reported the analysis for the cases $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_5$, where it was shown the existence of a prepotential for the noncompact cases also.

Here we use Hosono conjecture to analyze the geometry of a quotient $X = \mathbb{C}^3/\mathbb{Z}_6$, which we call for simplicity $\mathbb{C}^3_6$. As stated in the introduction, this singular orbifold admits five distinct crepant resolutions. All these resolutions are related by flop transformations. To noncompactness of the manifold X it corresponds the ambiguity in defining the GW–invariants. In our model this reflects in the fact that the symplectic structure on the half dimensional homology of the mirror Y is degenerate. On the mirror, such structure should determine a pairing between two dimensional and four dimensional cohomology, permitting the reconstruction of the prepotential, but which now becomes degenerate. We determine the LCSL cohomological series for all the resolutions. From each of them, using Hosono's prescriptions we will able to partially determine a prepotential which codifies all the GW–invariants of the (four dimensional) Mori cone excluding a three dimensional subcone.

Chapter 2

The orbifold $\mathbb{C}^3_6$ and the G-Hilb resolution

2.1 Definition of $\mathbb{C}^3_6$

We briefly review the homogeneous coordinates construction of toric varieties [Cox]. The data of a d-dimensional toric variety $X(\Delta)$ can always be specified in terms of a fan Δ in a lattice N isomorphic to $\mathbb{Z}^d$. Let $\rho_1,\ldots,\rho_r$ be the 1-dimensional cones of Δ and let $v_i \in \mathbb{Z}^n$ denote the primitive element of ρ_i, i.e. the generator of $\rho_i \cap \mathbb{Z}^n$. Then introduce variables x_i for $i = 1,\ldots,r$ in the affine complex space $\mathbb{C}^r$. The homogeneous coordinates construction represents $X(\Delta)$ as the quotient

$$X(\Delta) = (\mathbb{C}^r \backslash Z)/G$$

for a certain variety Z and some abelian group $G \subset (C^*)^r$.

Z is determined as follows. We say that a set of edge generators $I = \{v_{i_1},\ldots,v_{i_s}\}$ is primitive if they don't lie in any cone of Δ but every proper subset does. Then

$$Z = \bigcup_{I \text{ primitive}} \{x_{i_1} = 0,\ldots,x_{i_s} = 0\} \ .$$

If $\{e_1,\ldots,e_d\}$ is the standard basis of the dual lattice M and $<,>: M \times N \to \mathbb{Z}$ is the natural pairing, the group G is defined as the kernel of the following homomorphism

$$\Phi : (\mathbb{C}^*)^r \to (\mathbb{C}^*)^d, \quad (\lambda_1,\ldots,\lambda_r) \mapsto \left(\prod_{i=1}^r \lambda_i^{\langle e_1, v_i\rangle},\ldots,\prod_{i=1}^r \lambda_i^{\langle e_n, v_i\rangle}\right)$$

and its actions on $\mathbb{C}^r \backslash Z$ is by multiplication

$$(\lambda_1,\ldots,\lambda_r)\cdot(x_1,\ldots,x_r) := (\lambda_1 x_1,\ldots,\lambda_r x_r) \ .$$

In this paper we study the threedimensional orbifold $\mathbb{C}^3_6$ defined as the toric variety associated to the fan generated by the vectors

$$v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \tag{2.1.1}$$

in $N \simeq \mathbb{Z}^3$. In this case $Z = \emptyset$ and the associated homomorphism is

$$\Phi : (\mathbb{C}^*)^3 \to (\mathbb{C}^*)^3, \quad (\lambda_1,\lambda_2,\lambda_3) \mapsto (\ \lambda_1^{-1}\lambda_2^2\lambda_3^{-1}, \lambda_1^{-1}\lambda_2^{-1}\lambda_3, \lambda_1\lambda_2\lambda_3) \ , \tag{2.1.2}$$

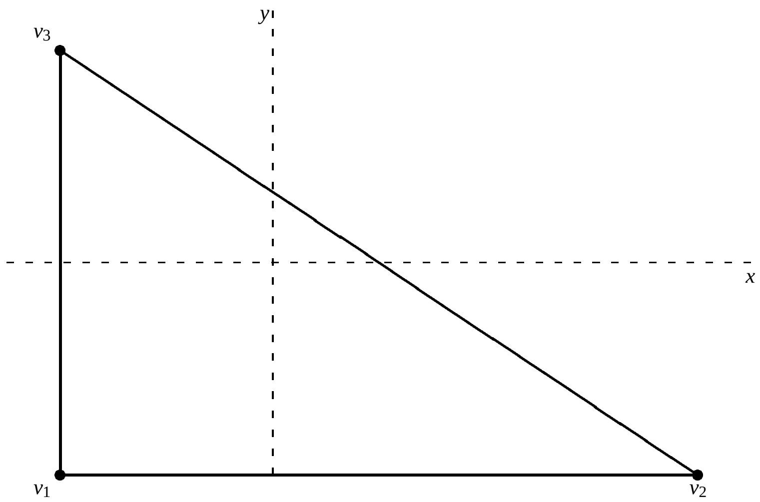

Figure 2.1: $\mathbb{C}^3_6$ fan

which has kernel

$$G := \ker\Phi =< (\varepsilon,\varepsilon^2,\varepsilon^3) >\subset (\mathbb{C}^*)^3 \ , \text{ with } \varepsilon = e^{\frac{2\pi i}{6}} \ . \tag{2.1.3}$$

Thus $G \simeq \mathbb{Z}_6$ and $\mathbb{C}^3_6 = \mathbb{C}^3/\mathbb{Z}_6$ where the action on the coordinates is

$$\varepsilon \cdot (x_1,x_2,x_3) = (\varepsilon x_1, \varepsilon^2 x_2, \varepsilon^3 x_3) \ . \tag{2.1.4}$$

$\mathbb{C}^3_6$ is a non compact Calabi-Yau ($K_{\mathbb{C}^3_6}$ is trivial) threefold with an isolated quotient singularity at the origin, because all vectors v_i lie in the plane $z = 1$ (if (x,y,z) are the coordinates on the lattice).[1] In this way, all relevant information is included in the two dimensional intersection of the fan Δ with the plane $z = 1$. In the figure 2.1 we have drawn this section for the fan of $\mathbb{C}^3_6$.

2.2 Crepant resolutions of $\mathbb{C}^3_6$

A crepant resolution of a variety X is a smooth variety Y together with a a proper birational morphism $\tau : Y \to X$ such that $K_Y = \tau^* K_X$. If X is a Calabi-Yau variety this means that K_Y has to be trivial. Any crepant resolution of a toric Calabi-Yau orbifold $X(\Delta) = \mathbb{C}^3/G$ can be obtained in two simple steps (see [Ful2, Oda]). First, add to Δ all possible edges ρ_i that are generated by the integer vectors $v_i \in N$ intersecting the fan and lying on the plane determined by v_1, v_2, v_3. Next, let one completely triangulate Δ, to obtain the regular fan Δ' of the toric resolution $X(\Delta')$. If there exist several complete triangulations this means that the orbifold admits multiple crepant resolutions, all related by flops of curves.

Therefore, to obtain the resolutions of the $\mathbb{C}^3_6$ singular variety we add to Δ the four vectors

$$v_4 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} , \qquad v_5 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} , \qquad v_6 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} , \qquad v_7 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{2.2.1}$$

It is easy to show that we have five admissible complete triangulations.

[1] We refer to section 2.3.1 for an explanation about this CY condition.

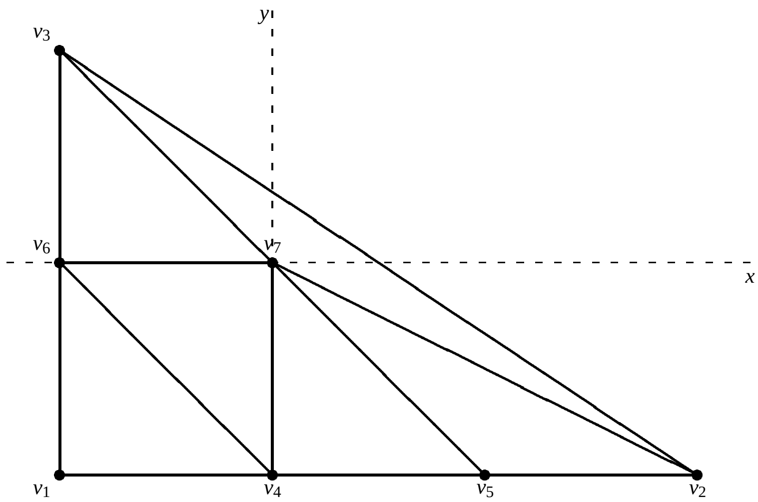

Figure 2.2: Fan of the G-Hilbert resolution of $\mathbb{C}^3_6$

2.2.1 Toric *G*-Hilbert resolution

We start considering the *G*-Hilbert resolution, which we call $\mathrm{G}-\mathbb{C}^3_6$. Its general toric construction is given in [CR] and we refer to it for a detailed explanation. We can think to *G*-Hilb fan as the "more symmetric" triangulation. We try to illustrate this concept in our case. First we add to Δ the two dimensional cones generated by (v_2, v_7) and (v_3, v_7), that are necessary to obtain any complete triangulation. Then we extend the line (v_3, v_7) to v_5 so obtaining a subdivision of the fan into regular triangles, three of them with edges of length one and the bigger one with edges of length two. Finally we complete the triangulation subdividing this last triangle with a regular tessellation, obtained by drawing all possible internal lines parallel to its edges.

2.2.2 *G*-Hilbert resolution as the moduli space of *G*-clusters of $\mathbb{C}^3$

Given an algebraic variety M and a finite group G with an action on M, the G-Hilb(M) is defined as the moduli space of G-clusters $Z \subset M$. A G-cluster is a G-invariant zero dimensional subscheme Z, with defining ideal $I_Z \subset O_M$ and structure sheaf $O_Z = O_M/I_Z$ isomorphic to the regular representation of G, i.e. $H^0(Z, O_Z) \simeq R(G)$ with $\dim H^0(Z, O_Z) = |G|$. The simplest example of G-cluster is a general orbit of G consisting of N distinct point.

We will study the simple example of $\mathbb{Z}_2$-Hilb$(\mathbb{C}^2)$. Let us consider $\mathbb{C}^2 = \mathrm{Spec}\,\mathbb{C}[X,Y]$ and the action of $\mathbb{Z}_2$, with generator $\varepsilon = -1$, defined on the coordinates as

$$\varepsilon \cdot (X,Y) = (\varepsilon X, \varepsilon Y)\ . \tag{2.2.2}$$

The orbits of $\mathbb{Z}_2$ are the sets of couple of points

$$\{(p_1, p_2) \in \mathbb{C}^2 \times \mathbb{C}^2 \,|\, X(p_1) = -X(p_2), Y(p_1) = -Y(p_2)\}\ . \tag{2.2.3}$$

If p_1 has coordinates (a,b), on the open set $X \neq 0$ the $\mathbb{Z}_2$-cluster Z with support over (p_1, p_2) is defined by the equations

$$X^2 = a, \quad Y = \tfrac{b}{a}X \qquad \Longrightarrow \qquad O_Z = \frac{\mathbb{C}[X,Y]}{(X^2-a, Y-\frac{b}{a}X)} \simeq \mathbb{C} \oplus \mathbb{C}\cdot X\ , \tag{2.2.4}$$

and on the open set $Y \neq 0$ by

$$Y^2 = b, \quad X = \tfrac{a}{b} Y \qquad \Longrightarrow \qquad O_Z = \tfrac{\mathbb{C}[X,Y]}{(Y^2 - b, X - \frac{a}{b} Y)} \simeq \mathbb{C} \oplus \mathbb{C} \cdot Y \ . \tag{2.2.5}$$

It is easy to verify all the properties of the $\mathbb{Z}_2$-clusters. Thus we have a bijective relation between generic orbits and $\mathbb{Z}_2$-cluster having support on them. On the set $\{X = 0, Y = 0\}$ we have $\mathbb{Z}_2$-clusters Z of type

$$X^2 = 0, \quad X = \tfrac{\beta}{\alpha} Y \qquad \Longrightarrow \qquad O_Z = \tfrac{\mathbb{C}[X,Y]}{(X^2, X - \frac{\beta}{\alpha} Y)} \simeq \mathbb{C} \oplus \mathbb{C} \cdot X \ , \tag{2.2.6}$$

for any (α, β) with $\alpha \neq 0$, or, in alternative, of type

$$Y^2 = 0, \quad Y = \tfrac{\alpha}{\beta} X \qquad \Longrightarrow \qquad O_Z = \tfrac{\mathbb{C}[X,Y]}{(Y^2, Y - \frac{\alpha}{\beta} X)} \simeq \mathbb{C} \oplus \mathbb{C} \cdot Y \ , \tag{2.2.7}$$

for any (α, β) with $\beta \neq 0$. It is evident that the $\mathbb{Z}_2$-Hilb($\mathbb{C}^2$) has the structure of the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ at the origin and, with the map

$$\tau : \mathbb{Z}_2\text{-Hilb}(\mathbb{C}^2) \longrightarrow \mathbb{C}^2/G \qquad Z\,(p_1, p_2) \longmapsto (p_1, p_2) \ , \tag{2.2.8}$$

it becomes the (crepant) resolution of the orbifold. Let us prove this fact explicitly using toric geometry.

We will follow the construction of toric orbifold given in [Ful2]. Let $L = \mathbb{Z}^2 + \frac{1}{2}(1,1)$ be the lattice over $\mathbb{Z}^2$; in L the fan of $\mathbb{C}^2/\mathbb{Z}_2$ is the junior simplex Δ_{junior} generated by the standard base (e_1, e_2) of $\mathbb{Z}_2$.

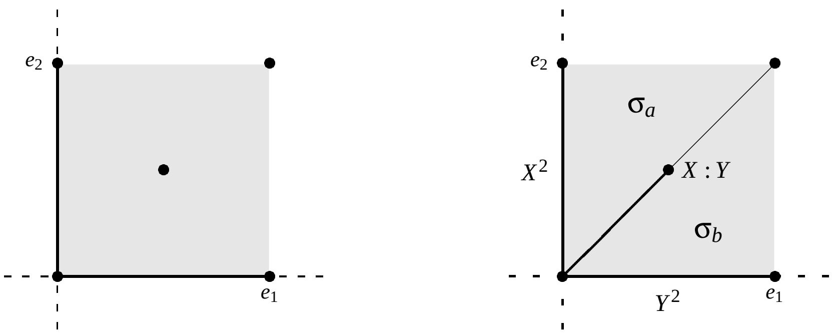

Figure 2.3: Fan for $C^2/\mathbb{Z}_2$ and $\mathbb{Z}_2$-Hilb(C^2) in $L = \mathbb{Z}^2 + \frac{1}{2}(1,1)$

Using the "old construction" of toric variety [Ful2], we have

$$X_{\Delta_{\text{junior}}} = \text{Spec}\ \mathbb{C}[X^2, XY, Y^2] = \text{Spec}\ \frac{\mathbb{C}[U,V,W]}{(UW - V^2)} \ . \tag{2.2.9}$$

The toric resolution of $X_{\Delta_{\text{junior}}}$ is obtained adding to Δ_{junior} the edge generated by $\frac{1}{2}(e_1 + e_2)$. In the right side of the figure 2.3 we have drawn the toric fan of the resolution marked with the coordinates related to the toric curves, expressed as $\mathbb{Z}_2$-invariant ratios of monomials in the orbifold coordinates. Geometrically this is the blow up of $X_{\Delta_{\text{junior}}} = \mathbb{C}^2/\mathbb{Z}_2$ in the origin. The two affine open sets are

$$U_{\sigma_a} = \text{Spec}\ \mathbb{C}[X^2, Y/X] \ , \qquad U_{\sigma_b} = \text{Spec}\ \mathbb{C}[Y^2, X/Y] \ . \tag{2.2.10}$$

Thus U_{σ_a}, for example, parameterizes equations of the form

$$X^2 = \xi_a \qquad Y = \eta_a X \ , \tag{2.2.11}$$

which define the $\mathbb{Z}_2$-clusters (2.2.4). Similar U_{σ_b} parameterizes clusters (2.2.5) and their intersection $U_{\sigma_a\cap\sigma_b}$ the clusters (2.2.6,2.2.7). Therefore the crepant toric resolution of $\mathbb{C}^2/\mathbb{Z}_2$ is exactly $\mathbb{Z}_2$-Hilb($\mathbb{C}^2$).

In a similar way it has been proved in [Ito-Nak, CR] that the toric resolutions of $\mathbb{C}^3/G$ defined in the previous section (for $G \subset SL(3,\mathbb{C})$ abelian) are exactly the G-Hilb($\mathbb{C}^3$). In figure 2.4 we report the $\mathbb{Z}_6$-Hilb($\mathbb{C}^3$) fan marked with the $\mathbb{Z}_6$-invariant ratios associated to the curves of the resolution.

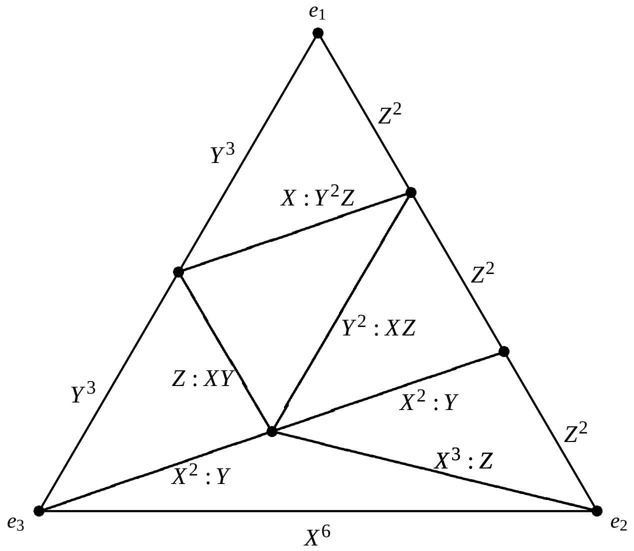

Figure 2.4: Fan for $\mathbb{Z}_6$-Hilb(C^3) in lattice $L = \mathbb{Z}^3 + \frac{1}{6}(1,2,3)$

2.3 Intersection theory for $\mathrm{G}-\mathbb{C}^3_6$

We are interested in finding the Chow ring $A^*(\mathrm{G}-\mathbb{C}^3_6)$, the module $A^c_*(\mathrm{G}-\mathbb{C}^3_6)$ and the intersection pairing $A^*(\mathrm{G}-\mathbb{C}^3_6)\otimes A^c_*(\mathrm{G}-\mathbb{C}^3_6)\to A^c_*(\mathrm{G}-\mathbb{C}^3_6)$ (for this section we refer to [Ful1, Ful2, Oda]).

2.3.1 Chow ring $A^*(\mathrm{G}-\mathbb{C}^3_6)$

On any variety X the Chow group $A_k(X)$ is defined to be the free abelian group on the k-dimensional irreducible closed subvarieties of X, modulo the subgroup generated by the cycles of the form (f), where f is a nonzero rational function on a $k+1$-dimensional subvariety of X. [2] For a toric variety $X = X(\Delta)$, the Chow group $A_k(X)$ is generated by the classes of the closures $V(\sigma) = \overline{O(\sigma)}$ of orbits of the $n-k$ dimensional cones $\sigma \in \Delta$ under the action of the torus $\mathbb{C}^n_*$. If τ is a cone of Δ, we define $N_\tau := \mathbb{Z}\cdot\tau$ and $N(\tau) := N/N_\tau$. The relations in $A_k(X)$ are generated by the cycles of the form $(\chi^u) := \sum_i \langle u, v_i\rangle V(\rho_i)$, where u is an element in the dual lattice $M(\tau) = N(\tau)^*$, ρ_i are the one dimensional subcones of the projection of τ in $N(\tau)$ with primitive vectors v_i, $\langle\ ,\ \rangle$ is the natural pairing between $M(\tau)$ and $N(\tau)$, for any cone $\tau \in \Delta$ of dimension $n-k-1$.

We will study explicitly this construction for $X = \mathrm{G}-\mathbb{C}^3_6$.
Let us first decorate the fan in figure 2.5 with labels for the toric invariant subvarieties related to the cones of Δ:
$A_3(X)$ has only one generator, corresponding to the unique zero dimensional cone of Δ, and obviously without

[2]Recall that (f) is the cycle obtained as the sum of the zeros of f minus the poles of f, each counted with its multiplicity.

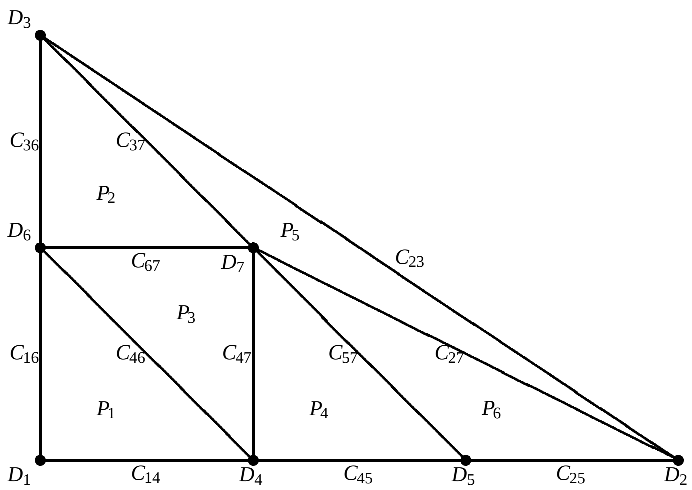

Figure 2.5:

relations.

$$A_3(X) = \mathbb{Z} \cdot X \qquad X = V(0) \tag{2.3.1}$$

$A_2(X)$ has seven generators, related to the seven one dimensional cones of the fan. The relations are generated by the cycles (χ^u) for u in $M(0) = M$:

$$A_2(X) = \frac{\bigoplus_{i=1}^{7} \mathbb{Z} \cdot D_i}{< (\chi^u) >}, \qquad D_i = V(\rho_i), \qquad \rho_i = \mathbb{R}_{\geq 0} \cdot v_i \tag{2.3.2}$$

We choose as u the standard basis of the lattice M, e_1^*, e_2^*, e_3^*, so we obtain these three independent relations

$$-D_1 + 2D_2 - D_3 + D_5 - D_6 = 0 \,, \tag{2.3.3}$$
$$-D_1 - D_2 + D_3 - D_4 - D_5 = 0 \,, \tag{2.3.4}$$
$$D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 = 0 \,. \tag{2.3.5}$$

In $A_2(X)$ the divisors $D_5,\ D_6,\ D_7$ can be expressed in terms of the others

$$D_5 = -D_1 - D_2 + D_3 - D_4 \,, \tag{2.3.6}$$
$$D_6 = -2D_1 + D_2 - D_4 \,, \tag{2.3.7}$$
$$D_7 = 2D_1 - D_2 - 2D_3 + D_4 \,. \tag{2.3.8}$$

It follows that

$$A_2(X) = \bigoplus_{i=1}^{4} \mathbb{Z} \cdot D_i \simeq \mathbb{Z}^4. \tag{2.3.9}$$

Since the variety $\mathrm{G}-\mathbb{C}^3_6$ is non singular, we have

$$\mathrm{Pic}(\mathrm{G}-\mathbb{C}^3_6) \simeq A_2(\mathrm{G}-\mathbb{C}^3_6) \simeq \mathbb{Z}^4 \,. \tag{2.3.10}$$

Let us make a remark about the canonical divisor. It is a standard fact in toric geometry that the canonical divisor of a variety X is given by $K_X = -\sum_i D_i$ where the sum is over all toric invariant divisors. By relation (2.3.5) it then follows that $K_{G-\mathbb{C}^3_6} = 0$ in $\mathrm{Pic}(G-\mathbb{C}^3_6)$ so that $G-\mathbb{C}^3_6$ is a Calabi-Yau variety. This is true for any toric variety with all integer vectors generating the fan lying on the same (hyper)plane.
$A_1(X)$ has twelve generators, the toric invariant curves. The relations are generated by the cycles (χ^u) for u in $M(\rho_i)$:

$$A_1(X) = \frac{\bigoplus \mathbb{Z}\cdot C_{ij}}{<(\chi^u)>}. \tag{2.3.11}$$

Therefore, to find the relations we have to study the geometry of any toric invariant divisor. Recall that $D_i = V(\rho_i)$ is a toric variety for any i; its fan is called $Star(\rho_i)$ and is obtained by the projection of the cones containing ρ_i into the quotient lattice $N(\rho_i)$. As an example we plot $Star(\rho_7)$ in figure 2.6.

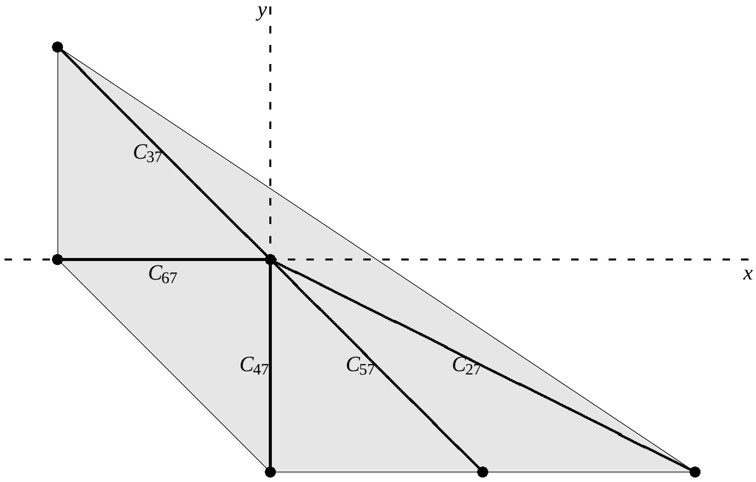

Figure 2.6: Fan for D_7 $(Star(\rho_7))$

D_7 has five toric invariant divisors and two relations between them

$$-C_{27} + C_{37} - C_{47} - C_{57} = 0\ , \tag{2.3.12}$$
$$2C_{27} - C_{37} + C_{57} - C_{67} = 0\ . \tag{2.3.13}$$

Doing the same for any divisor D_i we obtain all relations between curves. At the end we find that any two given curves are equivalent:

$$A_1(X) = \mathbb{Z}\cdot C, \qquad C = [C_{46}]. \tag{2.3.14}$$

Any other invariant curve is related to C by the relations expressed in the decorated fan of figure 2.7.
$A_0(X)$ is generated by the six toric invariant points of X. Every toric variety contains only two kind of toric curves: compact curves isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ and noncompact curves isomorphic to $\mathbb{A}^1_{\mathbb{C}}$. Any compact curve gives a rational relation between two invariant points, and in such way we find that any two given points are rationally equivalent. Finally linear equivalence on affine curves says us that points are rational equivalent to zero. Therefore $A_0(X)$ is the trivial group (this is true for any noncompact toric variety).

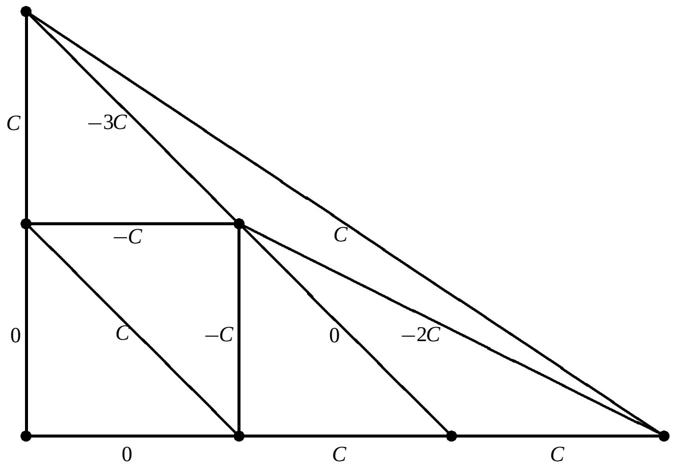

Figure 2.7:

On a nonsingular n-dimensional variety X, one sets $A^p(X) := A_{n-p}(X)$. There is an intersection product $A^p(X) \times A^q(X) \to A^{p+q}(X)$, making $A^*(X) := \bigoplus A^p(X)$ into a commutative graded ring. For a general toric variety $X(\Delta)$, if σ and τ are cones in Δ, then

$$V(\sigma) \cap V(\tau) = \begin{cases} V(\gamma) & \text{if } \sigma \text{ and } \tau \text{ span the cone } \gamma, \\ \varnothing & \text{if } \sigma \text{ and } \tau \text{ do not span a cone in } \Delta. \end{cases}$$

If $X(\Delta)$ is nonsingular and the intersection is proper, i.e. each component of the intersection has codimension equal to the sum of the codimension of the two subvarieties, or empty, then $V(\sigma)$ and $V(\tau)$ meet transversally in $V(\gamma)$ (or $\varnothing$). In this case we define $[V(\sigma)] \cdot [V(\tau)] = [V(\gamma)]$ (or 0). Otherwise if $V(\sigma)$ and $V(\tau)$ do not meet properly, we can always use rational equivalence to replace in $A^*(X)$ a subvariety (i.e. $V(\sigma)$) with another one in the same class and such that it meets $V(\tau)$ in a proper way.
Again, let us apply these considerations to our example $X = \mathrm{G}{-}\mathbb{C}^3_6$.
First, note that the intersection between X and any subvarieties $V(\sigma)$ is obviously equal to $V(\sigma)$. Therefore X is the multiplicative identity in $A^*(X)$.
Any divisor D_i meets each other properly (or not at all), and so their products give the curves $D_i \cdot D_j = [C_{ij}]$ (or 0). We have to use the linear equivalences (2.3.3) only to find the autointersections $D_i \cdot D_i$.
Finally, when we intersect divisors and curves we obtain a point (or $\varnothing$), but, as we have seen, they are rational equivalent to zero. Any other intersection is always equivalent to the empty set.
The intersection products in $A^*(X)$ are summarized in table 2.1.
We can see that the product is symmetric and respects the grading. If we call R the set of relations given by the intersection product we find

$$A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R \,. \tag{2.3.15}$$

2.3.2 Group $A^c_*(\mathrm{G}{-}\mathbb{C}^3_6)$ of compactly supported subvarieties

On a noncompact variety X the group $A^c_*(X)$ is defined to be the direct limit of the groups $A^*(Z)$, where Z are the closed and compact subvarieties of X ordered by inclusion. This means $A^c_*(X) = \bigoplus A^*(Z)/R$, where

	X	D_1	D_2	D_3	D_4	C
X	X	D_1	D_2	D_3	D_4	C
D_1	D_1	0	0	0	0	0
D_2	D_2	0	0	C	0	0
D_3	D_3	0	C	C	0	0
D_4	D_4	0	0	0	$-C$	0
C	C	0	0	0	0	0

Table 2.1: Intersection product in $A^*(X)$

the direct sum is over all compact subvarieties of X and the relations R say that two elements $[Z_1]$ and $[Z_2]$ of $\bigoplus A^*(Z)$ must be identified if exists a compact subvariety Z_3 that contains them and such that in $A^*(Z_3)$ they represent the same cycle class. As usual in toric geometry we can restrict our analysis to compact toric invariant subvarieties $V(\sigma)$; recall that $X(\Delta)$ is compact in the classical topology if and only if its support $|\Delta|$ is the whole space $N_{\mathbb{R}}$.

Our example has one compact invariant divisor D_7, six compact curves (C_{27}, C_{37},C_{47}, C_{57},C_{67}, C_{46}) and six points P_i. It's easy to see that (as a group)

$$A^*(P_i) = \mathbb{Z}\cdot P_i^c \ , \qquad A^*(C_{ij}) = \mathbb{Z}\cdot C_{ij}^c \oplus \mathbb{Z}\cdot P_{ij}^c \ , \tag{2.3.16}$$

where P_{ij}^c represents the point class in the curve C_{ij}^c.
The divisor D_7 is the toric variety associated to the fan of figure 2.6, therefore

$$A^*(D_7) = \mathbb{Z}\cdot D_7^c \oplus \mathbb{Z}\cdot C_{47}^c \oplus \mathbb{Z}\cdot C_{57}^c \oplus \mathbb{Z}\cdot C_{67}^c \oplus \mathbb{Z}\cdot P_7^c \tag{2.3.17}$$

and the relations with other curves are

$$C_{27}^c = C_{47}^c + C_{67}^c \ , \tag{2.3.18}$$

$$C_{37}^c = 2C_{47}^c + C_{57}^c + C_{67}^c \ . \tag{2.3.19}$$

Now we have to sum all these groups and find relations between different generators. It results that all point classes have to be identified, exactly as the classes of the same curve. The group of compact subvarieties of G$-\mathbb{C}^3_6$ is then isomorphic to $\mathbb{Z}^6$:

$$A_*^c(X) = \mathbb{Z}\cdot D_7^c \oplus \mathbb{Z}\cdot C_{46}^c \oplus \mathbb{Z}\cdot C_{57}^c \oplus \mathbb{Z}\cdot C_{67}^c \oplus \mathbb{Z}\cdot C_{47}^c \oplus \mathbb{Z}\cdot P^c \ . \tag{2.3.20}$$

2.3.3 Intersection pairing

There is a well defined intersection pairing $A^*(X) \otimes A_*^c(X) \to A_*^c(X)$. For any two generators $[Z_1] \in A^*(X)$ and $[Z_2] \in A_*^c(X)$ it is possible to find two representatives which meet properly. Their intersection is a compact subvariety and defines the above pairing $[Z_1].[Z_2] := [Z_1 \cap Z_2] \in A_*^c(X)$, which is extendable by linearity to all elements in $A^*(X) \otimes A_*^c(X)$. This product gives the group $A_*^c(X)$ the structure of an $A^*(X)-$module.

For $X = \mathrm{G}{-}\mathbb{C}^3_6$ we obtain the intersection pairing of table 2.2: [3]

	D_7^c	C_{46}^c	C_{57}^c	C_{67}^c	C_{47}^c	P^c
X	D_7^c	C_{46}^c	C_{57}^c	C_{67}^c	C_{47}^c	P^c
D_1	0	P^c	0	0	0	0
D_2	C_{27}^c	0	P^c	0	0	0
D_3	C_{37}^c	0	0	P^c	0	0
D_4	C_{47}^c	$-P^c$	P^c	P^c	$-P^c$	0
C	P^c	0	0	0	0	0

Table 2.2: Intersection pairing $A^*(X) \otimes A^c_*(X) \to A^c_*(X)$

2.4 Homology, cohomology, Mori and Kähler cones

For any compact smooth variety X we have two natural homomorphisms

$$cl_X : A_*(X) \to H_*(X,\mathbb{Z}) \ , \qquad cl^X : A^*(X) \to H^*(X,\mathbb{Z}) \ .$$

The map cl_X sends the representative V of an algebraic cycle to the homological cycle $[V]$; it's well defined because algebraic equivalence implies homological equivalence. The map cl^X is defined by composition of cl_X with Poincaré duality, which associate to an homological k-cycle V the $(n-k)$-form η_V such that

$$\int_V \theta = \int_X \theta \wedge \eta_V \ .$$

In the case of crepant resolutions of toric orbifolds it is possible to prove [Cr] that it exists the following module isomorphism

$$A^c_*(X) \simeq H^c_*(X,\mathbb{Z}) \ , \qquad A^*(X) \simeq H^*(X,\mathbb{Z})$$

which respects the intersection product [4]

$$H^*(X,\mathbb{Z}) \otimes H^c_*(X,\mathbb{Z}) \to H^c_*(X,\mathbb{Z}) \ .$$

We are interested in determining the Kähler cone of X, which is the set of all forms J in $H^2(X,\mathbb{Q})$ such that

$$\int_C J \geq 0$$

for all effective cycles in $H^c_2(X,\mathbb{Q})$. We describe the Kähler cone using the module isomorphism of the previous paragraph. We begin defining the Mori cone, i.e. the polyhedral cone in $A^c_2(X) \otimes \mathbb{Q}$ generated by effective toric

[3] We can quickly obtain the product between divisors and curves which do not intersect properly in this way: suppose v_1, v_2 are the minimal lattice points on the edges of σ_C and let v', v'' be the minimal lattice points of the threedimensional cones containing σ_C, then $v' + v'' = a_1 v_1 + a_2 v_2$ and $D_k \cdot C = -a_k P^c$.

[4] The restriction to compact homology is necessary because of the problem in defining integration over non compact cycles.

invariant compact curves of X, which are the compact algebraic cycles $\sum_{i=1}^{l} a_{ij} C^c_{ij}$ where all the a_{ij} are non-negative. Now we can think at the Kähler cone of X as the dual polyhedral cone in $A^2(X) \otimes \mathbb{Q}$ of the Mori cone with respect to the intersection pairing.

For $X = \mathrm{G}{-}\mathbb{C}^3_6$, in view of the relations (2.3.18,2.3.19), the Mori cone is generated by:

$$C_1 := C^c_{46}\,, \qquad C_2 := C^c_{57}\,, \qquad C_3 := C^c_{67}\,, \qquad C_4 := C^c_{47}\,. \tag{2.4.1}$$

Then the Kähler cone has the following dual generators, that satisfied $T_a \cdot C_b = \delta_{ab} P^c$:

$$T_1 := D_1\,, \quad T_2 := D_2\,, \quad T_3 := D_3\,, \quad T_4 := -D_1 + D_2 + D_3 - D_4\,. \tag{2.4.2}$$

For completeness we report in table 2.3 the products between the T_i in the Chow ring $A^*(X)$:

	T_1	T_2	T_3	T_4
T_1	0	0	0	0
T_2	0	0	C	C
T_3	0	C	C	$2C$
T_4	0	C	$2C$	$2C$

Table 2.3: Products between the Kähler generators in $A^*(X)$.

If we call J_i the Kähler generators in $H^2(X,\mathbb{Q})$ corresponding to the T_i, then we find the cohomology ring

$$H^*(X,\mathbb{Q}) = \frac{\mathbb{Q}[J_1,J_2,J_3,J_4]}{(J_1^2, J_1J_2, J_1J_3, J_1J_4, J_2^2, J_2J_4 - J_2J_3, J_3^2 - J_2J_3, J_3J_4 - 2J_2J_3, J_4^2 - 2J_2J_3)}. \tag{2.4.3}$$

2.5 K-theory

2.5.1 Preliminaries

The K-theory ring of a variety X is related to the cohomology via the Chern character map. More precisely this is an injective homomorphism of rings from $K(X)$ to the Chow ring with rational coefficients $A^*(X)_{\mathbb{Q}}$. Then composition with cl^X gives the homomorphism with $H^*(X,\mathbb{Q})$:

$$\mathrm{ch}\ :\ K(X) \to A^*(X) \otimes \mathbb{Q} \simeq H^*(X,\mathbb{Q})\,.$$

Here we summarize some general properties of Chern map that will be useful in the next sections. The Chern class c_i is a map from $K(X)$ to $A^i(X) \otimes \mathbb{Q}$; the total Chern class is defined as the sum of all Chern class

$$\mathrm{c}(\mathcal{F}) := \mathrm{c}_0(\mathcal{F}) + \mathrm{c}_1(\mathcal{F}) + \ldots + \mathrm{c}_n(\mathcal{F})\,,$$

where n is the dimension of X. A divisorial $\mathcal{O}_X(D)$ sheaf has a very simple total Chern class

$$\mathrm{c}(\mathcal{O}_X(D)) = X + D$$

and, using multiplicative properties of the Chern classes, this implies

$$\mathcal{F} = \bigoplus_{i=1}^{r} \mathcal{O}_X(D_i) \qquad \Longrightarrow \qquad \mathrm{c}(\mathcal{F}) = \prod_{i=1}^{r} (X + D_i)\,.$$

The Chern character is defined for such sheaf as

$$\mathrm{ch}(\mathcal{F}) := \sum_{i=1}^{r} e^{D_i} = rX + \mathrm{c}_1(\mathcal{F}) + \frac{1}{2}(\mathrm{c}_1(\mathcal{F})^2 - 2\mathrm{c}_2(\mathcal{F})) \ ,$$

where the expansion is stopped to second order in view of the cohomology ring structure of our non compact threefold varieties. In particular for a divisorial sheaf we have

$$\mathrm{ch}(O_X(D)) = X + D + \frac{1}{2}D^2 \ .$$

We recall also the definition of the Todd class:

$$\mathrm{td}(\mathcal{F}) := \prod_{i=1}^{r} \frac{D_i}{1-e^{-D_i}} = X + \frac{1}{2}\mathrm{c}_1(\mathcal{F}) + \frac{1}{12}(\mathrm{c}_1(\mathcal{F})^2 + \mathrm{c}_2(\mathcal{F})) \ .$$

In particular, we need the Todd class of the tangent bundle T_X; for a toric variety its total Chern class is

$$\mathrm{c}(T_X) = \sum_{\sigma\in\Delta} [V(\sigma)]$$

and so if X is a Calabi-Yau non compact toric threefold we have

$$\mathrm{c}(T_X) = X + \sum [C_{ij}] \quad \Rightarrow \quad \mathrm{td}(X) = \mathrm{td}(T_X) = X + \frac{1}{12}\sum [C_{ij}] \ , \tag{2.5.1}$$

where the sum is over all (compact and non compact) toric curves in X.

In the context of non compact varieties we also have to work with the compactly supported K-theory group $K^c(X)$. This group is related to the compactly supported Chow group with rational coefficients $A^c_*(X)_{\mathbb{Q}}$, and therefore to $H^c_*(X,\mathbb{Q})$, via the local Chern character map [Iv]:

$$\mathrm{ch}^c \ : \ K^c(X) \to A^c_*(X) \otimes \mathbb{Q} \ \simeq H^c_*(X,\mathbb{Q}) \ .$$

Let us briefly review its definition and properties. Any element S of $K^c(X)$ can be represented by coherent sheaves S_V on a compact subvariety V of X. If $i : V \hookrightarrow X$ is the embedding of V in X, we can define the local Chern character of S by

$$\mathrm{ch}^c(S) = \mathrm{ch}(i_* S_V) \ .$$

Actually we can compute the local Chern characters with the help of the Grothendieck-Riemann-Roch theorem:

$$i_*(\mathrm{ch}(S_V)\mathrm{td}(V)) = \mathrm{ch}(i_* S_V)\mathrm{td}(X)$$

for any compact subvarieties V of X, which implies

$$\mathrm{ch}^c(S) = \mathrm{td}(X)^{-1}\, i_*(\mathrm{ch}(S_V)\mathrm{td}(V)) \ .$$

The local Chern classes of the divisorial sheaves over the compact subvarieties in a Calabi-Yau non compact toric threefold X are:

$$\begin{aligned} &\mathrm{ch}^c(O_{p^c}) = p^c \qquad \mathrm{ch}^c(O_{C^c}(n)) = C^c + (n+1)p^c \\ &\mathrm{ch}^c(O_{D^c}(C)) = i_*\Bigg(D^c + \Big(C + \frac{1}{2}\mathrm{c}_1(D^c)\Big) + \\ &\qquad\qquad + \frac{1}{2}\Big(C^2 + \mathrm{c}_1(D^c)C + \frac{1}{6}\big(\mathrm{c}_1(D^c)^2 + \mathrm{c}_2(D^c)\big)\Big)\Bigg) - \frac{1}{12}\mathrm{c}_2(X)\, D^c. \end{aligned} \tag{2.5.2}$$

In the last character, C is a divisor in D^c and the $\mathrm{c}_i(D^c)$ are the Chern classes $\mathrm{c}_i(T_{D^c})$, which naturally live in $A^*(D^c)$ and that can be calculated using formula (2.5.1). Moreover all the products excepted the last are in $A^*(D^c)$.

2.5.2 K-theory generators

Let G be an abelian subgroup of $SL(3,\mathbb{C})$ which acts on the affine space $\mathbb{C}^3$. We write $\pi : \mathbb{C}^3 \to Y = \mathbb{C}^3/G$ for the quotient, $X = G$-Hilb($\mathbb{C}^3$) for the Hilbert scheme with crepant resolution $\tau : X \to Y$ and the universal scheme $\mathcal{Z} = \{(Z(x),x) \in X \times \mathbb{C}^3\}$ where $Z(x)$ is the G-cluster over x (see section 2.2.2). Thus we have the commutative diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & \mathbb{C}^3 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\tau} & Y \end{array}$$

Let us consider the sheaf $\mathcal{R} := p_* \mathcal{O}_{\mathcal{Z}}$ on the resolution X. Over any point $x \in X$ the fiber of $\mathcal{R}$ is $H^0(Z(x), \mathcal{O}_{Z(x)})$ which supports the regular representation of G. In particular the rank of $\mathcal{R}$ is equal to the order of the group G. The decomposition of the regular representation into irreducible submodules induces the decomposition

$$\mathcal{R} = \bigoplus_k \mathcal{R}_k \otimes \rho_k \qquad \text{for } \mathcal{R}_k = \mathrm{Hom}_G(\rho_k, \mathcal{R})$$

into locally free sheaves of rank $\mathcal{R}_i = \dim \rho_i = 1$. We called $\mathcal{R}_k$ the tautological line bundle on X associated to the irreducible representation ρ_k of G. At the level of K-theory the McKay correspondence states the equivalence of the G-equivariant K-theory of $\mathbb{C}^n$ and the K-theory of the crepant resolutions. In [Ito-Nak] it has been determined the ring isomorphism

$$\varphi : K^G(\mathbb{C}^3) \xrightarrow{\sim} K(X)$$

showing that $\varphi(\rho_i \otimes \mathcal{O}_{\mathbb{C}^3}) = \mathcal{R}_i$ and therefore, that the tautological line bundles form a $\mathbb{Z}$-basis of $K(X)$.

In section 2.2.2 we studied the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ and its crepant resolution $\mathbb{Z}^2$-Hilb($\mathbb{C}^2$). We have given a description of $\mathcal{R}$ and its decomposition into line bundles on the two open sets U_{σ_a} and U_{σ_b}. In figure 2.8 we report the monomial generators of $\mathcal{R}_i$ on the affine pieces.

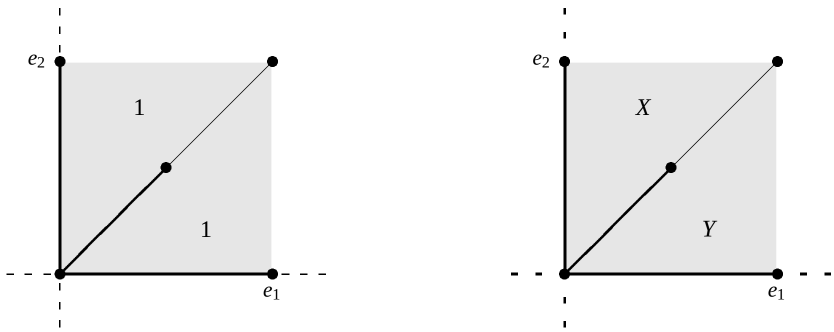

Figure 2.8: Monomial generators of $\mathcal{R}_0$ and $\mathcal{R}_1$ for $\mathbb{Z}_2$-Hilb($\mathbb{C}^2$)

With the same procedure we can give the generators of $\mathcal{R}_i$ on G-Hilb($\mathbb{C}^3$) for any abelian $G \subset SL(3,\mathbb{C})$. We report in the figure 2.9 the monomial generators for $\mathbb{Z}_6$-Hilb($\mathbb{C}^3$).
The action of $\mathbb{Z}_6$ on the coordinate ring of $\mathbb{C}^3$ is

$$\varepsilon \cdot (X,Y,Z) = (\varepsilon X, \varepsilon^2 Y, \varepsilon^3 Z), \qquad \varepsilon = e^{\frac{2\pi i}{6}}. \tag{2.5.3}$$

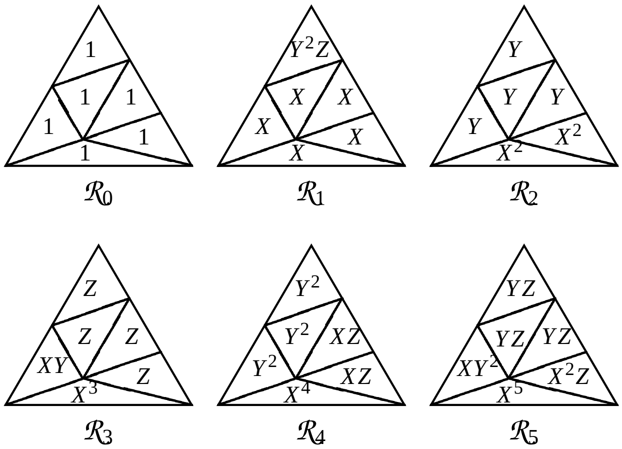

Figure 2.9: Monomial generators of $\mathcal{R}_i$ for $\mathbb{Z}_6$-Hilb($\mathbb{C}^3$)

Therefore, it is simple to verify that each $\mathcal{R}_i$ supports the irreducible representation ρ_i of $\mathbb{Z}_6$.

Any line bundle on a smooth algebraic variety is a divisorial bundle. We briefly sketch the standard procedure to find the divisor related to the $\mathcal{R}_i$ defined by Reid and proved by Craw, and refer to [Cr] for a detailed explanation.

The first step consists in decorating the G-Hilb fan with the characters of the group. Any curve has to be marked with the character of the monomials in its associated ratio. For any internal vertex v there exists a recipe to associate one or two characters of G, depending primarily on the valency of v (i.e the number of lines meeting at v). For a G-Hilb fan this is always $3,4,5$ or 6. There are the following cases:

- A vertex v of valency 3 defines an exceptional $\mathbb{P}^2$. A single character χ_k marks all three lines meeting at v. Mark the vertex v with the character $\chi_m := \chi_k \otimes \chi_k$.
- A vertex v of valency 4 defines an exceptional Hirzebruch surface $\mathbb{F}_r$. There are distinct characters χ_k and χ_l each one marking a pair of lines meeting at v. Mark the vertex v with the character $\chi_m := \chi_k \otimes \chi_l$.
- A vertex v of valency 5 or 6 (excluding three straight lines meeting at a point) defines an Hirzebruch surface $\mathbb{F}_r$ blown-up in one or two points. There are uniquely determined characters χ_k and χ_l each one marking a pair of lines meeting at v. Mark the vertex v with $\chi_m := \chi_k \otimes \chi_l$.
- A vertex v at the intersection of three straight lines defines an exceptional Del Pezzo surface of degree six, denoted dP_6. The monomials defining the pair of morphisms $\mathrm{dP}_6 \to \mathbb{P}^2$ lie in uniquely determined character spaces χ_l and χ_m satisfying

$$\chi_l \otimes \chi_m = \chi_i \otimes \chi_j \otimes \chi_k \ ,$$

  where χ_i, χ_j and χ_k mark the straight lines through the vertex v. Mark the vertex v with both χ_l and χ_m.

Each character of G appears once on the fan Δ.

By analyzing of the monomial generators of the tautological line bundles $\mathcal{R}_i$, in [Cr] the author proved that:

- If χ_k marks the line defining the compact curve $C_k \in H_2^c(X,\mathbb{Z})$ on the resolution X, the first Chern class $c_1(\mathcal{R}_k)$ is the dual to C_k in $H^2(X,\mathbb{Z})$:

$$\int_{C_l} c_1(\mathcal{R}_k) = \delta_{lk} \ .$$

This means that $R_k = \mathcal{O}_X(T_k)$, where T_k is the generator of the Kähler cone dual to C_k.

- In Pic(X) all relations between tautological line bundles are of the following forms:
  - $\mathcal{R}_m = \mathcal{R}_k \otimes \mathcal{R}_k$ when $\chi_m = \chi_k \otimes \chi_k$ marks a vertex v of valency 3;
  - $\mathcal{R}_m = \mathcal{R}_k \otimes \mathcal{R}_l$ when $\chi_m = \chi_k \otimes \chi_l$ marks a vertex v of valency 4;
  - $\mathcal{R}_m = \mathcal{R}_k \otimes \mathcal{R}_l$ when $\chi_m = \chi_k \otimes \chi_l$ marks a vertex v of valency 5 or 6 (excluding three straight lines meeting at a point);
  - $\mathcal{R}_l \otimes \mathcal{R}_m = \mathcal{R}_i \otimes \mathcal{R}_j \otimes \mathcal{R}_k$ when the pair of characters χ_l and χ_m satisfying $\chi_l \otimes \chi_m = \chi_i \otimes \chi_j \otimes \chi_k$ marks the intersection point v of three straight lines.

As usual we apply these considerations to our case $X = \mathbb{Z}_6$-Hilb($\mathbb{C}^6_3$) and we summarize them in the decorated fan of figure 2.10.

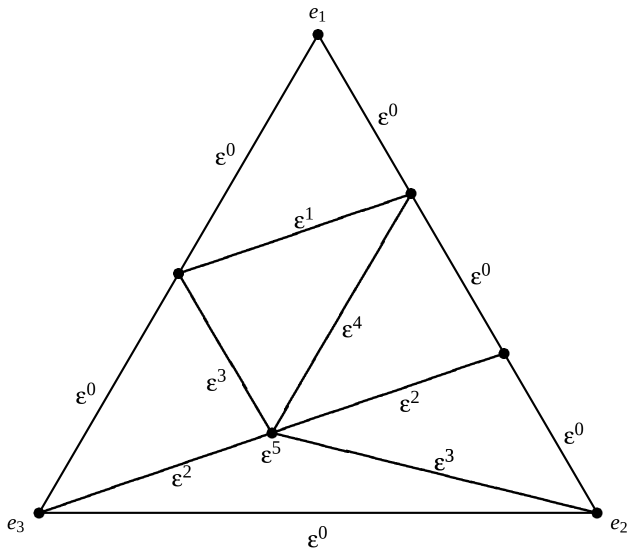

Figure 2.10: Fan for $\mathbb{Z}_6$-Hilb(C^3) decorated with Reid's recipe

The resulting tautological line bundles, that give a $\mathbb{Z}$-basis of $K(X)$, are:

$$\begin{aligned}&\mathcal{R}_0 = \mathcal{O}_X, \quad \mathcal{R}_1 = \mathcal{O}_X(D_1), \quad \mathcal{R}_2 = \mathcal{O}_X(D_2), \quad \mathcal{R}_3 = \mathcal{O}_X(D_3),\\ &\mathcal{R}_4 = \mathcal{O}_X(-D_1 + D_2 + D_3 - D_4), \quad \mathcal{R}_5 = \mathcal{R}_2 \otimes \mathcal{R}_3 = \mathcal{O}_X(D_2 + D_3).\end{aligned} \tag{2.5.4}$$

2.5.3 $K(X)$ and $K^c(X)$

Chosen a base of generators for $K(X)$ we can find the dual basis for the compact K-theory $K^c(X)$ as in [Ito-Nak] using the perfect pairing

$$(\ |\ ) : K(X) \times K^c(X) \longrightarrow \mathbb{Z}, \quad (\mathcal{R}, \mathcal{S}) \longmapsto (\mathcal{R}|\mathcal{S}) = \int_X \mathrm{ch}(\mathcal{R})\mathrm{ch}^c(\mathcal{S})\mathrm{td}(X)\ ,$$

so that

$$(\mathcal{R}_i|\mathcal{S}_j) = \delta_{ij}. \tag{2.5.5}$$

As usual, the integral is by definition the coefficient of the point class. Using this fact, the standard computations of Chern and Todd characters and the intersection product table 2.2, from condition (2.5.5) we find

$$\begin{aligned}
\mathrm{ch}^c(\mathcal{S}_0) &= D_7^c - \left(C_{46}^c + C_{57}^c + \frac{3}{2}C_{67}^c + 2C_{47}^c\right) + \frac{7}{6}P^c\ ,\\
\mathrm{ch}^c(\mathcal{S}_1) &= C_{46}^c\ ,\\
\mathrm{ch}^c(\mathcal{S}_2) &= -D_7^c + \left(C_{57}^c + \frac{1}{2}C_{67}^c + C_{47}^c\right) - \frac{1}{6}P^c\ ,\\
\mathrm{ch}^c(\mathcal{S}_3) &= -D_7^c + \left(\frac{3}{2}C_{67}^c + C_{47}^c\right) - \frac{1}{6}P^c\ ,\\
\mathrm{ch}^c(\mathcal{S}_4) &= C_{47}^c\ ,\\
\mathrm{ch}^c(\mathcal{S}_5) &= D_7^c - \left(\frac{1}{2}C_{67}^c + C_{47}^c\right) + \frac{1}{6}P^c\ .
\end{aligned} \tag{2.5.6}$$

In the spirit of the paper [Hos1] we now express the elements $\mathcal{S}_i$ in terms of a symplectic D-brane basis of $K^c(X)$. Such basis can be constructed starting from the generators of the compact Chow ring. We choose

$$B_0 := \mathcal{O}_P^c\ ; \quad B_a := \mathcal{O}_{C_a^c}(-T_a)\ ; \quad B_5 := \mathcal{O}_{D_7^c}(-T_2 - T_3)\ , \tag{2.5.7}$$

with $a = 1, \dots, 4$ and $\mathcal{O}_{C_a^c}(-T_a) := \mathcal{O}_{C_a^c} \otimes \mathcal{O}_X(-T_a)$, $\mathcal{O}_{D_7^c}(-T_2 - T_3) := \mathcal{O}_{D_7^c} \otimes \mathcal{O}_X(-T_2 - T_3)$. To express the basis $\mathcal{S}_i$ in terms of B_j we can compare their compact Chern characters. Using (2.5.2) and the multiplicative property of Chern character we find

$$\mathrm{ch}^c(B_0) = P^c \quad \mathrm{ch}^c(B_a) = C_a^c \quad \mathrm{ch}^c(B_5) = D_7^c - \left(\tfrac{1}{2}C_{67}^c + C_{47}^c\right) + \tfrac{1}{6}P^c \tag{2.5.8}$$

and then

$$\begin{aligned}
\mathcal{S}_0 &= B_0 - B_1 - B_2 - B_3 - B_4 + B_5\ ,\\
\mathcal{S}_1 &= B_1\ ,\\
\mathcal{S}_2 &= B_2 - B_5\ ,\\
\mathcal{S}_3 &= B_3 - B_5\ ,\\
\mathcal{S}_4 &= B_4\ ,\\
\mathcal{S}_5 &= B_5\ ,
\end{aligned} \tag{2.5.9}$$

Finally we write the B_i basis of $K^c(X)$ in terms of the $\mathcal{S}_i$ and its dual basis Φ_i of $K(X)$ in term of $\mathcal{R}_i$:

$$\begin{aligned}
B_0 &= \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_5\ , & \Phi_0 &= \mathcal{R}_0\ ,\\
B_1 &= \mathcal{S}_1\ , & \Phi_1 &= -\mathcal{R}_0 + \mathcal{R}_1\ ,\\
B_2 &= \mathcal{S}_2 + \mathcal{S}_5\ , & \Phi_2 &= -\mathcal{R}_0 + \mathcal{R}_2\ ,\\
B_3 &= \mathcal{S}_3 + \mathcal{S}_5\ , & \Phi_3 &= -\mathcal{R}_0 + \mathcal{R}_3\ ,\\
B_4 &= \mathcal{S}_4\ , & \Phi_4 &= -\mathcal{R}_0 + \mathcal{R}_4\ ,\\
B_5 &= \mathcal{S}_5\ . & \Phi_5 &= \mathcal{R}_0 - \mathcal{R}_2 - \mathcal{R}_3 + \mathcal{R}_5\ .
\end{aligned} \tag{2.5.10}$$

2.6 The Hosono conjecture

We will restate shortly here the Hosono conjecture [Hos1], for convenience. The main point is that the periods for the mirror manifold are solutions of a set of Picard-Fuchs equations, and the general solution can be expressed in terms of an hypergeometric function with value in the cohomology of X:

$$w = w\left(x_1,\ldots,x_4;\frac{J_1}{2\pi i},\ldots,\frac{J_4}{2\pi i}\right)$$

Then the conjecture (adapted to our case) states as follows.

2.6.1 Hosono conjecture

Define the basis for $H^*(X,\mathbb{Q})$

$$Q_i := \mathrm{ch}(\Phi_i)\ , \quad i = 0,\ldots,5$$

and expand the cohomology-valued hypergeometric series w with respect to this basis:

$$w\left(x_1,\ldots,x_4;\frac{J_1}{2\pi i},\ldots,\frac{J_4}{2\pi i}\right) = \sum_{i=0}^{5} w_i(x_1,\ldots,x_4)Q_i\ .$$

Thus

1. the coefficient hypergeometric series $w_i(x_1,\ldots,x_4)$ may be identified with the period integrals over the cycles $mir(B_i)$,

$$w_i(x_1,\ldots,x_4) = \int_{mir(B_i)} \Omega(Y_x)\ ;$$

2. the monodromy of the hypergeometric series is integral and symplectic with respect to the symplectic form defined in $K^c(X)$

$$\chi(B_i,B_j) = \int_X \mathrm{ch}(B_i^\vee)\mathrm{ch}(B_j)\mathrm{td}(X)\ ;$$

3. the central charge of an element $F \in K^c(X)$ is expressed in terms of the cohomology valued hypergeometric w as

$$Z(F) = \int_X \mathrm{ch}(F) w\left(x_1,\ldots,x_4;\frac{J_1}{2\pi i},\ldots,\frac{J_4}{2\pi i}\right)\mathrm{td}(X)\ .$$

The symplectic form of point 2 can be easily computed with respect to the basis $\mathcal{S}_i^-$ following the paper of $Ito-Nakajima$ [Ito-Nak]. Let Q be the 3-dimensional representation given by the inclusion $G \subset SL(3,\mathbb{C})$ and $\{\rho_i\}_{i=0}^r$ be the irreducible representations. The decomposition

$$Q \otimes \rho_j = \bigoplus_k a_{ij}\rho_i$$

is related to the symplectic form by

$$\chi(\mathcal{S}_i,\mathcal{S}_j) = a_{ji} - a_{ij}\ .$$

In our example

$$\chi(\mathcal{S}_i,\mathcal{S}_j)=\begin{pmatrix} 0 & 1 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}, \tag{2.6.1}$$

and then, for our chosen basis,

$$\chi(B_i,B_j)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 & 0 \end{pmatrix}. \tag{2.6.2}$$

This matrix gives a symplectic correspondence between the space $H^4(X,\mathbb{Q})$ and a one dimensional subspace of $H^2(X,\mathbb{Q})$. There is an obvious ambiguity in such a correspondence, but we will turn back to it later.

2.6.2 The cohomological hypergeometric series

The vectors ℓ_a, $a=1,\dots,4$ are given by the intersection numbers between the Mori cone generators and the invariant divisors of X, so that we find

$$\begin{aligned} C_1 &: \quad \ell_1=(1,0,0,-1,0,-1,1)\ , \\ C_2 &: \quad \ell_2=(0,1,0,1,-2,0,0)\ , \\ C_3 &: \quad \ell_3=(0,0,1,1,0,-1,-1)\ , \\ C_4 &: \quad \ell_4=(0,0,0,-1,1,1,-1)\ . \end{aligned} \tag{2.6.3}$$

The hypergeometric series is then

$$w=\sum_{\vec{m}\in\mathbb{Z}^4_{\geq 0}} \frac{x_1^{m_1+\rho_1}x_2^{m_2+\rho_2}x_3^{m_3+\rho_3}x_4^{m_4+\rho_4}}{\prod_{i=1}^7\Gamma_i(\vec{m}+\vec{\rho})}\Bigg|_{\vec{\rho}=\frac{\vec{J}}{2\pi i}} \tag{2.6.4}$$

$$\begin{aligned} \Gamma_1(\vec{m}) &= \Gamma(1+m_1)\ , \\ \Gamma_2(\vec{m}) &= \Gamma(1+m_2)\ , \\ \Gamma_3(\vec{m}) &= \Gamma(1+m_3)\ , \\ \Gamma_4(\vec{m}) &= \Gamma(1-m_1+m_2+m_3-m_4)\ , \\ \Gamma_5(\vec{m}) &= \Gamma(1-2m_2+m_4)\ , \\ \Gamma_6(\vec{m}) &= \Gamma(1-m_1-m_3+m_4)\ , \\ \Gamma_7(\vec{m}) &= \Gamma(1+m_1-m_3-m_4)\ . \end{aligned} \tag{2.6.5}$$

We need to expand this function in power series in $\vec{J}$. Because of the ring relations for $H^*(X,\mathbb{Q})$, we see that the expansion stops at order two. The coefficient functions, with respect to the basis $\{1,\vec{J},C\}$ of $H^*(X,\mathbb{Q})$, are computed in the appendix.

However we chosen the basis B_i in $K^c(X)$ so that we need to rewrite the expansion in terms of the dual basis $Q_i = \mathrm{ch}(\Phi_i)$:

$$Q_0 = 1,\ Q_1 = J_1,\ Q_2 = J_2,\ Q_3 = J_3 + \tfrac{1}{2}C,\ Q_4 = J_4 + C,\ Q_5 = C. \tag{2.6.6}$$

If we make this change of basis and use the mirror symmetry identification

$$w\left(\vec{x}, \frac{\vec{J}}{2\pi i}\right) = Q_0 1 + \sum_{a=1}^{4} Q_a t_a + Q_5 g(t_1, \dots, t_4)\ , \tag{2.6.7}$$

then we find

$$\begin{aligned}
2\pi i t_1 &= \log x_1 + \Psi(x_1x_3) + \Phi(x_2, x_1x_4) + \aleph(\vec{x})\ ,\\
2\pi i t_2 &= \log x_2 - \Phi(x_2, x_1x_4) + 2\Phi(x_1x_4, x_2)\ ,\\
2\pi i t_3 &= \log x_3 - \Phi(x_2, x_1x_4) + \Psi(x_1x_3) - \aleph(\vec{x})\ ,\\
2\pi i t_4 &= \log x_4 - \Phi(x_1x_4, x_2) + \Phi(x_2, x_1x_4) - \Psi(x_1x_3) - \aleph(\vec{x})\ ,
\end{aligned} \tag{2.6.8}$$

and

$$\begin{aligned}
(2\pi i)^2 g(\vec{t}) = &-\frac{\pi^2}{3} - \pi i(\log x_3 - \Phi(x_2, x_1x_4) + \Psi(x_1x_3) - \aleph(\vec{x}))\\
&-2\pi i(\log x_4 - \Phi(x_1x_4, x_2) + \Phi(x_2, x_1x_4) - \Psi(x_1x_3) - \aleph(\vec{x}))\\
&+7\aleph^{(1)}(\vec{x}) - 3\aleph^{(2)}(\vec{x}) - 2\aleph^{(3)}(\vec{x}) - \aleph^{(4)}(\vec{x}) - \aleph^{(5)}(\vec{x})\\
&-\Psi_1(x_2, x_1x_4) + \Psi_2(x_2, x_1x_4) - \Psi_1(x_1x_4, x_2) + \Psi_2(x_1x_4, x_2)\\
&-\Psi_1(x_1x_4, x_2) + \Psi_3(x_1x_4, x_2) - \Psi_4(x_1x_3) + \Psi_5(x_1x_3) + \Psi_6(x_2, x_4)\\
&+\Lambda_1(\vec{x}) - \Lambda_2(\vec{x}) - \Lambda_3(\vec{x})\\
&+\frac{1}{2}(\log x_3)^2 + \log x_3[\Psi(x_1x_3) - \Phi(x_2, x_1x_4) - \aleph(\vec{x})]\\
&+(\log x_4)^2 + 2\log x_4[\Phi(x_2, x_1x_4) - \Phi(x_1x_4, x_2) - \Psi(x_1x_3) - \aleph(\vec{x})]\\
&+\log x_2 \log x_3 + \log x_2[\Psi(x_1x_3) - \Phi(x_2, x_1x_4) - \aleph(\vec{x})]\\
&+\log x_3[2\Phi(x_1x_4, x_2) - \Phi(x_2, x_1x_4)] + \log x_2 \log x_4\\
&+\log x_2[\Phi(x_2, x_1x_4) - \Phi(x_1x_4, x_2) - \Psi(x_1x_3) - \aleph(\vec{x})]\\
&+\log x_4[2\Phi(x_1x_4, x_2) - \Phi(x_2, x_1x_4)] + 2\log x_3 \log x_4\\
&+2\log x_3[\Phi(x_2, x_1x_4) - \Phi(x_1x_4, x_2) - \Psi(x_1x_3) - \aleph(\vec{x})]\\
&+2\log x_4[\Psi(x_1x_3) - \Phi(x_2, x_1x_4) - \aleph(\vec{x})]\ .
\end{aligned} \tag{2.6.9}$$

Using the above expressions we find

$$g(\vec{t}) = P_2(\vec{t}) + \frac{1}{(2\pi i)^2}\phi(\vec{t})\ , \tag{2.6.10}$$

where P_2 is the degree two polynomial part

$$P_2(\vec{t}) = \frac{1}{12} - \frac{1}{2}t_3 - t_4 + \frac{1}{2}t_3^2 + t_4^2 + t_2t_3 + t_2t_4 + 2t_3t_4\ , \tag{2.6.11}$$

and

$$\begin{aligned}
\phi(\vec{t}) = &\,7\aleph^{(1)}(\vec{x}) - 3\aleph^{(2)}(\vec{x}) - 2\aleph^{(3)}(\vec{x}) - \aleph^{(4)}(\vec{x}) - \aleph^{(5)}(\vec{x})\\
&-\Psi_1(x_2, x_1x_4) + \Psi_2(x_2, x_1x_4) - \Psi_1(x_1x_4, x_2) + \Psi_2(x_1x_4, x_2)
\end{aligned}$$

$$
\begin{aligned}
&-\Psi_1(x_1x_4,x_2)+\Psi_3(x_1x_4,x_2)-\Psi_4(x_1x_3)+\Psi_5(x_1x_3)+\Psi_6(x_2,x_4)\\
&+\Lambda_1(\vec{x})-\Lambda_2(\vec{x})-\Lambda_3(\vec{x})\\
&+\frac{1}{2}\Psi^2(x_1x_3)+\frac{1}{2}\Phi^2(x_2,x_1x_4)+\Phi^2(x_1x_4,x_2)-\frac{7}{2}\aleph^2(\vec{x})\\
&-\Psi(x_1x_3)\aleph(\vec{x})-\Phi(x_2,x_1x_4)\aleph(\vec{x})\\
&-\Phi(x_2,x_1x_4)\Psi(x_1x_3)-\Phi(x_2,x_1x_4)\Phi(x_1x_4,x_2)\ ,
\end{aligned} \tag{2.6.12}
$$

with $\vec{x}$ expressed as a function of $\vec{t}$ by inverting system (2.6.8), is the part corresponding to instantonic contributions. Following Hosono and using (2.6.2) we find

$$
(\partial_{t_1}-\partial_{t_3}-\partial_{t_4})F(\vec{t})=g(\vec{t})\ , \tag{2.6.13}
$$

where F is the prepotential. To integrate this equation we must expect for the prepotential to be as usual the sum of a classical term, a cubic polynomial in $\vec{t}$ and a quantum instantonic contribution. Setting

$$
q_k := e^{2\pi i t_k}\ , \tag{2.6.14}
$$

we then find

$$
\begin{aligned}
F(\vec{t}) = &-\frac{t_4}{12}+\frac{t_4^2}{4}+\frac{t_3t_4}{2}-\frac{t_4^3}{6}-\frac{t_3t_4}{2}(t_3+t_4+2t_2)\\
&+F_{\text{inst}}(\vec{q})+P_{\text{class}}(t_2,t_1+t_3,t_1+t_4)+Q_{\text{inst}}(q_2,q_1q_3,q_1q_4)\ ,
\end{aligned} \tag{2.6.15}
$$

where F_{inst} are the instantonic corrections, obtained integrating ϕ, P_{class} is an arbitrary cubic polynomial of three variables and Q_{inst} an arbitrary function of three variables which we assume analytic in $(0,0,0)$. P_{class} and Q_{inst} represent the contributions which are undetermined by the equation (2.6.13).

As an example, let us compute the Gromov-Witten (GW) invariants up to degree six. Here we actually list the integral Gopakumar-Vafa (G-V) invariants for rational curves, in place of the original fractional GW–invariants.We will use $[d_1,d_2,d_3,d_4]$ to indicate the homology class of the curves in the Mori cone, corresponding to the generators $J_1,\dots,J_4$. Thus we consider the curves with $d_1+d_2+d_3+d_4\leq 6$. The curves in the integer cone generated by $[0,1,0,0]$, $[1,0,1,0]$, $[1,0,0,1]$ must be excluded, because corresponding to the undetermined part of the prepotential. The only nonvanishing invariants in the considered range are

$$
\begin{aligned}
&GV_{[0,0,0,1]}=GV_{[0,0,1,0]}=GV_{[1,0,0,0]}=\\
&GV_{[1,0,1,1]}=GV_{[0,1,0,1]}=GV_{[1,1,1,1]}=1;\\
&GV_{[0,0,1,1]}=GV_{[0,1,1,1]}=GV_{[1,1,1,2]}=-2;\\
&GV_{[0,1,1,2]}=GV_{[1,1,2,2]}=3;\\
&GV_{[0,1,2,2]}=-4;\qquad GV_{[0,1,2,3]}=5.
\end{aligned} \tag{2.6.16}
$$

Chapter 3

The flopped resolutions

In this section we study the remaining four crepant resolutions X of the orbifold $\mathbb{C}^3_6$.

3.1 Intersection theory

For any resolution we have

$$\begin{aligned} A^0(X) &= \mathbb{Z}\cdot X \ , \\ A^1(X) &= \bigoplus_{i=1}^{4} \mathbb{Z}\cdot D_i \simeq \mathbb{Z}^4 \ , \\ A^2(X) &= \mathbb{Z}\cdot C \ , \\ A^3(X) &= 0 \ . \end{aligned} \tag{3.1.1}$$

In $A_2(X)$ the divisors $D_5,\ D_6,\ D_7$ can be expressed in terms of the others

$$D_5 = -D_1 - D_2 + D_3 - D_4 \ , \tag{3.1.2}$$

$$D_6 = -2D_1 + D_2 - D_4 \ , \tag{3.1.3}$$

$$D_7 = 2D_1 - D_2 - 2D_3 + D_4 \ . \tag{3.1.4}$$

The curve C depends on the resolution, as well as the intersection product in $A^*(X)$. If we call R the set of relations given by the intersection product, we have

$$A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R \ . \tag{3.1.5}$$

$A^c_*(X)$ is an $A^*(X)$-module, it is generated as group by the compact divisor D^c_7, the four compact curves depending on the resolution and the point class P^c. Finally the intersection pairing $A^*(X) \otimes A^c_*(X) \to A^c_*(X)$ depends on the resolution.

Resolution $X = \mathrm{R}_2{-}\mathbb{C}^3_6$

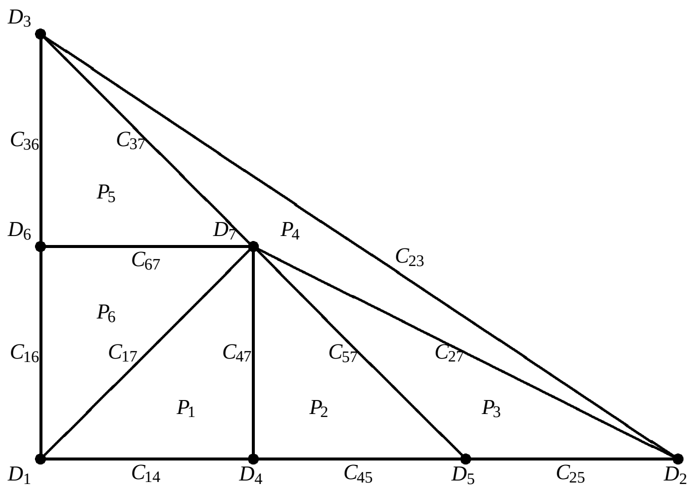

Figure 3.1:

This resolution differs from the G-Hilb by the flop

$$C_{46} \longrightarrow C_{17} \ . \tag{3.1.6}$$

We define $C = [C_{14}]$ and we report the relations between any other toric curve and C in the decorated fan of figure 3.2.

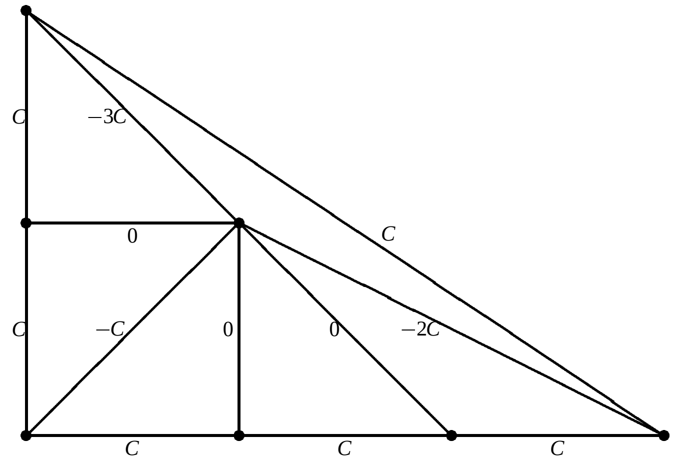

Figure 3.2:

In table 3.1 we summarize the intersection products, which give the relations R in the Chow ring $A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R$.

	X	D_1	D_2	D_3	D_4	C
X	X	D_1	D_2	D_3	D_4	C
D_1	D_1	C	0	0	$-C$	0
D_2	D_2	0	0	$-C$	0	0
D_3	D_3	0	$-C$	$-C$	0	0
D_4	D_4	$-C$	0	0	$2C$	0
C	C	0	0	0	0	0

Table 3.1: Intersection product in $A^*(X)$

The group of compact subvarieties of X is

$$A^c_*(X) = \mathbb{Z}\cdot D^c_7 \oplus \mathbb{Z}\cdot C^c_{17} \oplus \mathbb{Z}\cdot C^c_{57} \oplus \mathbb{Z}\cdot C^c_{67} \oplus \mathbb{Z}\cdot C^c_{47} \oplus \mathbb{Z}\cdot P^c \ , \tag{3.1.7}$$

with relations to other compact curves

$$C^c_{27} = 2C_{17} + C^c_{67} + C^c_{47} \ , \qquad C^c_{37} = 3C_{17} + C^c_{57} + C^c_{67} + 2C^c_{47} \ . \tag{3.1.8}$$

In table 3.2 we summarize the intersection pairing.

	D^c_7	C^c_{17}	C^c_{57}	C^c_{67}	C^c_{47}	P^c
X	D^c_7	C^c_{17}	C^c_{57}	C^c_{67}	C^c_{47}	P^c
D_1	C^c_{17}	$-P^c$	0	P^c	P^c	0
D_2	C^c_{27}	0	P^c	0	0	0
D_3	C^c_{37}	0	0	P^c	0	0
D_4	C^c_{47}	P^c	P^c	0	$-2P^c$	0
C	P^c	0	0	0	0	0

Table 3.2: Intersection pairing $A^*(X) \otimes A^c_*(X) \to A^c_*(X)$

The Mori cone generators are C_a, $a = 1, \dots, 4$, with

$$C_1 = C_{17} \ , \qquad C_2 = C_{57} \ , \qquad C_3 = C_{67} \ , \qquad C_4 = C_{47} \ . \tag{3.1.9}$$

The Kähler cone is generated by the dual elements T_a, $a = 1, \dots, 4$ with

$$T_1 = -2D_1 + D_2 + 2D_3 - D_4 \ , \qquad T_2 = D_2 \ , \qquad T_3 = D_3 \ , \qquad T_4 = -D_1 + D_2 + D_3 - D_4 \ . \tag{3.1.10}$$

	T_1	T_2	T_3	T_4
T_1	$6C$	$2C$	$3C$	$4C$
T_2	$2C$	0	C	C
T_3	$3C$	C	C	$2C$
T_4	$4C$	C	$2C$	$2C$

Table 3.3: Intersection between Kähler generators

If we call J_a the Kähler generators in $H^2(X,\mathbb{Q})$ corresponding to the T_a then the cohomology ring is

$$H^*(X,\mathbb{Q}) = \mathbb{Q}[J_1, \dots, J_4]/\sim \ , \tag{3.1.11}$$

with $\sim$ given by table 3.3.

Resolution $X = \mathrm{R}_3\text{–}\mathbb{C}^3_6$

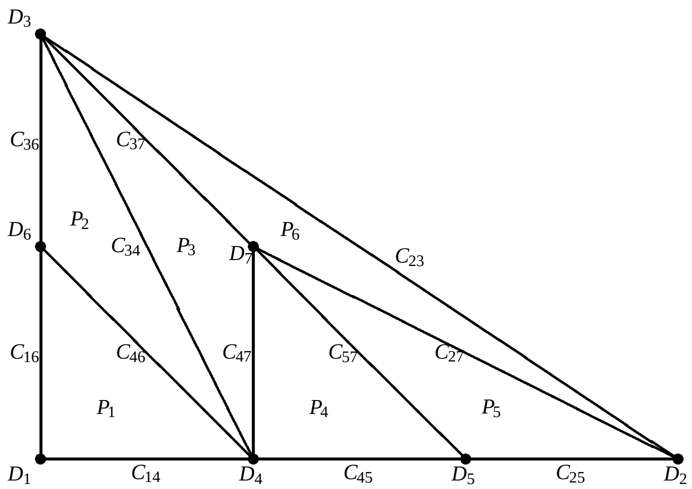

Figure 3.3:

This resolution differs from the G-Hilb by the flop

$$C_{67} \longrightarrow C_{34}\ . \tag{3.1.12}$$

We set $C = [C_{34}]$ and we report the relations between any other toric curve and C in the decorated fan of figure 3.4.

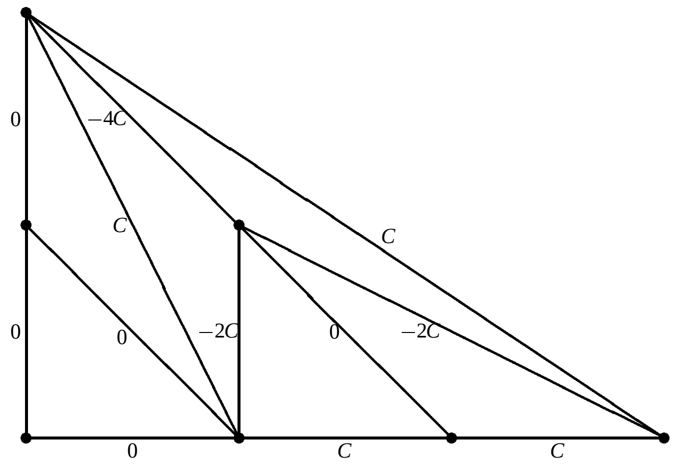

Figure 3.4:

In table 3.4 we summarize the intersection products, which give the relations R in the Chow ring $A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R$.

	X	D_1	D_2	D_3	D_4	C
X	X	D_1	D_2	D_3	D_4	C
D_1	D_1	0	0	0	0	0
D_2	D_2	0	0	C	0	0
D_3	D_3	0	C	$2C$	C	0
D_4	D_4	0	0	C	0	0
C	C	0	0	0	0	0

Table 3.4: Intersection product in $A^*(X)$

The group of compact subvarieties of X is

$$A^c_*(X) = \mathbb{Z}\cdot D^c_7 \oplus \mathbb{Z}\cdot C^c_{46} \oplus \mathbb{Z}\cdot C^c_{57} \oplus \mathbb{Z}\cdot C^c_{67} \oplus \mathbb{Z}\cdot C^c_{47} \oplus \mathbb{Z}\cdot P^c \ , \tag{3.1.13}$$

with relations to other compact curves

$$C^c_{27} = C^c_{47} \ , \qquad C^c_{37} = C_{57} + 2C^c_{47} \ . \tag{3.1.14}$$

In table 3.5 we summarize the intersection pairing.

	D^c_7	C^c_{46}	C^c_{57}	C^c_{34}	C^c_{47}	P^c
X	D^c_7	C^c_{46}	C^c_{57}	C^c_{34}	C^c_{47}	P^c
D_1	0	P^c	0	0	0	0
D_2	C^c_{27}	0	P^c	0	0	0
D_3	C^c_{37}	P^c	0	$-P^c$	P^c	0
D_4	C^c_{47}	0	P^c	$-P^c$	0	0
C	P^c	0	0	0	0	0

Table 3.5: Intersection pairing $A^*(X) \otimes A^c_*(X) \to A^c_*(X)$

The Mori cone generators are C_a, $a = 1,\dots,4$, with

$$C_1 = C_{46} \ , \qquad C_2 = C_{57} \ , \qquad C_3 = C_{34} \ , \qquad C_4 = C_{47} \ . \tag{3.1.15}$$

The Kähler cone is generated by the dual elements T_a, $a = 1,\dots,4$ with

$$T_1 = D_1 \ , \qquad T_2 = D_2 \ , \qquad T_3 = D_2 - D_4 \ , \qquad T_4 = -D_1 + D_2 + D_3 - D_4 \ . \tag{3.1.16}$$

	T_1	T_2	T_3	T_4
T_1	0	0	0	0
T_2	0	0	0	C
T_3	0	0	0	0
T_4	0	C	0	$2C$

Table 3.6: Intersection between Kähler generators

If J_a are the Kähler generators in $H^2(X,\mathbb{Q})$ corresponding to the T_a then the cohomology ring is

$$H^*(X,\mathbb{Q}) = \mathbb{Q}[J_1,\dots,J_4]/\sim \ , \tag{3.1.17}$$

with $\sim$ given by table 3.6.

Resolution $X = \mathrm{R}_4 - \mathbb{C}^3_6$

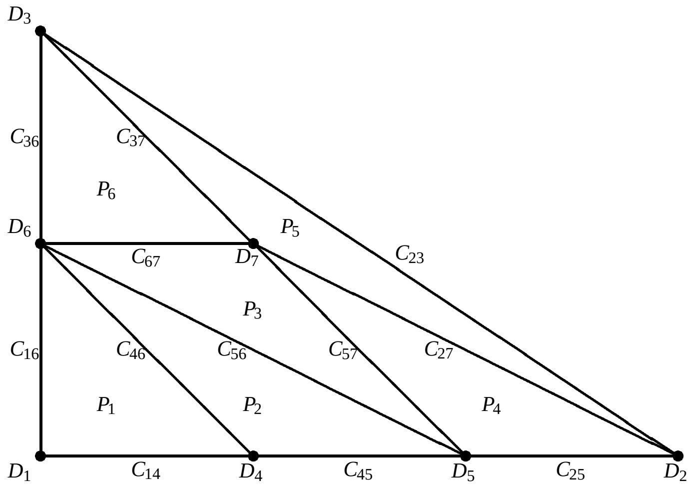

Figure 3.5:

This resolution differs from the G-Hilb by the flop

$$C_{47} \longrightarrow C_{56} \ . \tag{3.1.18}$$

We set $C = [C_{56}]$ and we report the relations between any other toric curve and C in the decorated fan of figure 3.6.

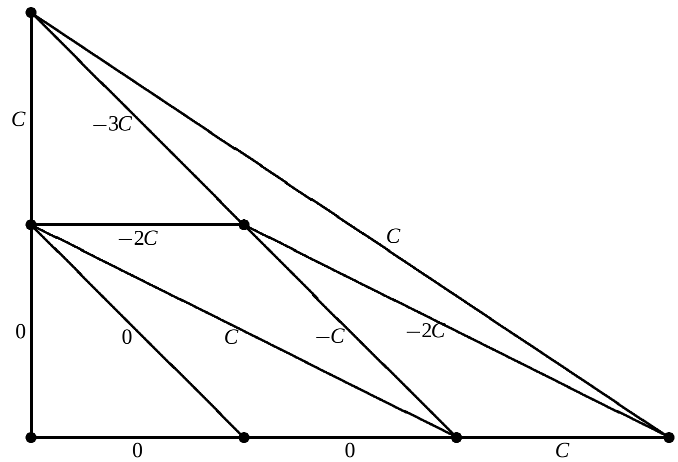

Figure 3.6:

In table 3.4 we summarize the intersection products, which give the relations R in the Chow ring $A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R$.

	X	D_1	D_2	D_3	D_4	C
X	X	D_1	D_2	D_3	D_4	C
D_1	D_1	0	0	0	0	0
D_2	D_2	0	0	C	0	0
D_3	D_3	0	C	C	0	0
D_4	D_4	0	0	0	0	0
C	C	0	0	0	0	0

Table 3.7: Intersection product in $A^*(X)$

The group of compact subvarieties of X is

$$A^c_*(X) = \mathbb{Z}\cdot D^c_7 \oplus \mathbb{Z}\cdot C^c_{46} \oplus \mathbb{Z}\cdot C^c_{57} \oplus \mathbb{Z}\cdot C^c_{67} \oplus \mathbb{Z}\cdot C^c_{56} \oplus \mathbb{Z}\cdot P^c\ , \tag{3.1.19}$$

with relations to other compact curves

$$C^c_{27} = C^c_{67}\ , \qquad C^c_{37} = C^c_{57} + C^c_{67}\ . \tag{3.1.20}$$

In table 3.8 we summarize the intersection pairing.

	D^c_7	C^c_{46}	C^c_{57}	C^c_{67}	C^c_{56}	P^c
X	D^c_7	C^c_{46}	C^c_{57}	C^c_{67}	C^c_{56}	P^c
D_1	0	P^c	0	0	0	0
D_2	C^c_{27}	0	P^c	0	0	0
D_3	C^c_{37}	0	0	P^c	0	0
D_4	0	$-2P^c$	0	0	P^c	0
C	P^c	0	0	0	0	0

(3.1.21)

Table 3.8: Intersection pairing $A^*(X) \otimes A^c_*(X) \to A^c_*(X)$

The Mori cone generators are C_a, $a = 1, \ldots, 4$, with

$$C_1 = C_{46}\ , \qquad C_2 = C_{57}\ , \qquad C_3 = C_{67}\ , \qquad C_4 = C_{56}\ . \tag{3.1.22}$$

The Kähler cone is generated by the dual elements T_a, $a = 1, \ldots, 4$ with

$$T_1 = D_1\ , \qquad T_2 = D_2\ , \qquad T_3 = D_3\ , \qquad T_4 = 2D_1 + D_4\ . \tag{3.1.23}$$

	T_1	T_2	T_3	T_4
T_1	0	0	0	0
T_2	0	0	C	0
T_3	0	C	C	0
T_4	0	0	0	0

Table 3.9: Intersection between Kähler generators

If J_a are the Kähler generators in $H^2(X,\mathbb{Q})$ corresponding to the T_a then the cohomology ring is

$$H^*(X,\mathbb{Q}) = \mathbb{Q}[J_1, \ldots, J_4]/\sim\ , \tag{3.1.24}$$

with $\sim$ given by table 3.9.

Resolution $X = \mathrm{R}_5 - \mathbb{C}^3_6$

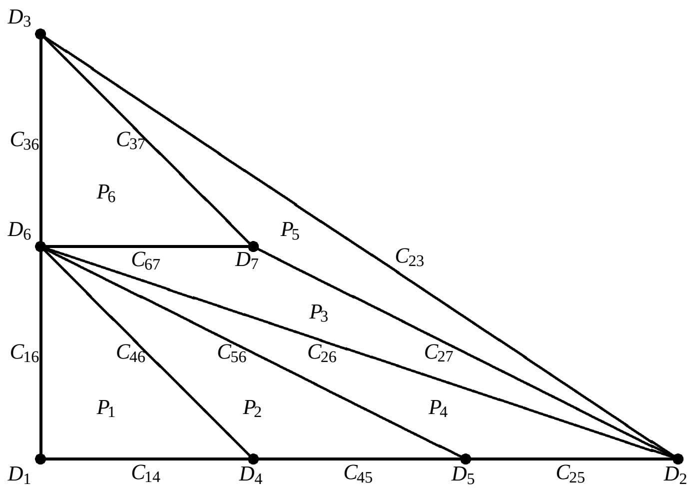

Figure 3.7:

This resolution differs from the G-Hilb by the flops

$$C_{47} \longrightarrow C_{56} \,, \qquad\qquad C_{57} \longrightarrow C_{26} \,. \tag{3.1.25}$$

We set $C = [C_{56}]$ and we report the relations between any other toric curve and C in the decorated fan of figure 3.4.

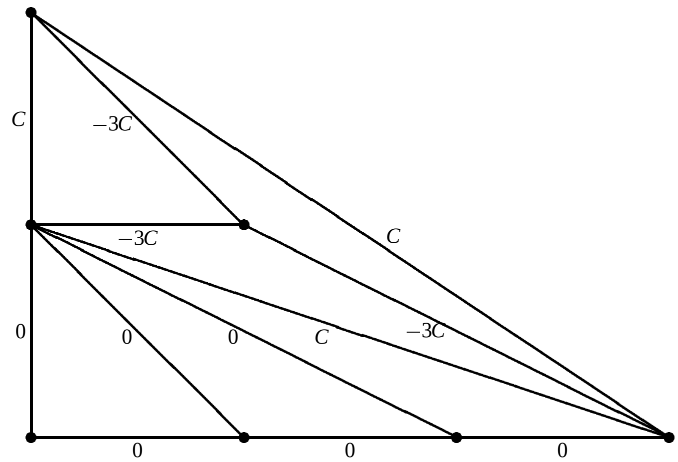

Figure 3.8:

In table 3.10 we summarize the intersection products, which give the relations R in the Chow ring $A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R$.

	X	D_1	D_2	D_3	D_4	C
X	X	D_1	D_2	D_3	D_4	C
D_1	D_1	0	0	0	0	0
D_2	D_2	0	C	C	0	0
D_3	D_3	0	C	C	0	0
D_4	D_4	0	0	0	0	0
C	C	0	0	0	0	0

(3.1.26)

Table 3.10: Intersection product in $A^*(X)$

The group of compact subvarieties of X is

$$A_*^c(X) = \mathbb{Z} \cdot D_7^c \oplus \mathbb{Z} \cdot C_{46}^c \oplus \mathbb{Z} \cdot C_{26}^c \oplus \mathbb{Z} \cdot C_{67}^c \oplus \mathbb{Z} \cdot C_{56}^c \oplus \mathbb{Z} \cdot P^c , \qquad (3.1.27)$$

with relations to other compact curves

$$C_{27}^c = C_{67}^c , \qquad C_{37}^c = C_{67}^c . \qquad (3.1.28)$$

In table 3.8 we summarize the intersection pairing.

	D_7^c	C_{46}^c	C_{26}^c	C_{67}^c	C_{56}^c	P^c
X	D_7^c	C_{46}^c	C_{26}^c	C_{67}^c	C_{56}^c	P^c
D_1	0	P^c	0	0	0	0
D_2	C_{27}^c	0	$-P^c$	P^c	P^c	0
D_3	C_{37}^c	0	0	P^c	0	0
D_4	0	$-2P^c$	0	0	P^c	0
C	P^c	0	0	0	0	0

(3.1.29)

Table 3.11: Intersection pairing $A^*(X) \otimes A_*^c(X) \to A_*^c(X)$

The Mori cone generators are C_a, $a = 1, \ldots, 4$, with

$$C_1 = C_{46} , \qquad C_2 = C_{26} , \qquad C_3 = C_{67} , \qquad C_4 = C_{56} . \qquad (3.1.30)$$

The Kähler cone is generated by the dual elements T_a, $a = 1, \ldots, 4$ with

$$T_1 = D_1 , \qquad T_2 = 2D_1 - D_2 + D_3 + D_4 , \qquad T_3 = D_3 , \qquad T_4 = 2D_1 + D_4 . \qquad (3.1.31)$$

	T_1	T_2	T_3	T_4
T_1	0	0	0	0
T_2	0	0	0	0
T_3	0	0	C	0
T_4	0	0	0	0

(3.1.32)

Table 3.12: Intersection between Kähler generators

If we call J_a the Kähler generators in $H^2(X, \mathbb{Q})$ corresponding to the T_a then the cohomology ring is

$$H^*(X, \mathbb{Q}) = \mathbb{Q}[J_1, \ldots, J_4] / \sim , \qquad (3.1.33)$$

with $\sim$ given by table 3.12.

3.2 K-theory generators

We have seen that G-Hilb is the moduli space of G-cluster in $\mathbb{C}^3$. The natural generalization of G-cluster is G-constellation. For a finite group $G \subset GL(n,\mathbb{C})$, a G-constellation is a G-equivariant coherent sheaf F on C^n with global sections $H^0(F)$ isomorphic as a $C[G]$-module to the regular representation R of G. Set

$$\Theta := \{\theta \in \mathrm{Hom}_{\mathbb{Z}}(R(G),\mathbb{Q}) | \theta(R) = 0\} ,$$

where $R(G)$ is the representation ring of G. This is an hyperplane in $\mathbb{Q}^r$, where r is the order of G. For $\theta \in \Theta$, a G-constellation F is said to be θ-stable (or θ-semistable) if every proper G-equivariant coherent subsheaf $0 \subset E \subset F$ satisfies $\theta(E) > 0$ (or $\theta(E) \geq 0$). The moduli space $\mathcal{M}_\theta$ of θ-stable constellation is constructed using GIT (cf. [SI]). The space Θ is subdivided into polyhedral convex cones C called GIT chamber. Given θ and θ' in the same chamber C the moduli spaces $\mathcal{M}_\theta$ and $\mathcal{M}_{\theta'}$ are isomorphic, so we write $\mathcal{M}_C$ in place of $\mathcal{M}_\theta$ for any $\theta \in C$. Ito and Nakajima [Ito-Nak] observed that G-Hilb= $\mathcal{M}_{C_0}$ for some chamber $C_0 \subset \Theta$ and more generally the method of [BKR] shows that for any chamber $C \subset \Theta$ there is a crepant resolution $\tau : \mathcal{M}_C \to \mathbb{C}^3/G$ and an equivalence of $\Phi_C : D(\mathcal{M}_C) \to D^G(\mathbb{C}^3)$ between derived categories of coherent sheaves on $\mathcal{M}_C$ and derived categories of G-equivariant sheaves on $\mathbb{C}^3$. Craw and Ishii [CI] proved that in the Abelian case every crepant resolution may be realized as a moduli space $\mathcal{M}_C$ for some chamber. Moreover they uncovered explicit equivalence between the derived categories of moduli $\mathcal{M}_\theta$ for parameters lying in adjacent GIT chambers. Therefore starting from G-Hilb and by analyzing the chamber structure of Θ, we can define the tautological bundles $\mathcal{R}_\rho$ that generate $K(X)$ on every flopped resolution X and that are the Fourier-Mukay transforms of the original tautological bundles on G-Hilb.

Here we summarize how calculate the chamber structure of Θ and the transformation induced by crossing the walls W of the chambers C. We refer to [CI, Deg] for detailed explanations. The derived equivalence Φ_C induces a $\mathbb{Z}$-linear isomorphism

$$\varphi_C : K^c(\mathcal{M}_C) \to R(G) \qquad \sum a_i \mathcal{S}_i \longmapsto \bigoplus a_i \rho_i \tag{3.2.1}$$

where as usual $\mathcal{S}_i$ is the element of the basis of $K^c(\mathcal{M}_C)$ dual of the tautological bundle $\mathcal{R}_i$ and ρ_i is the irreducible representation of character i. Let $C \subset \Theta$ be a chamber. Then $\theta \in C$ if and only if

- for every exceptional curve ℓ, we have $\theta(\varphi_C(\mathcal{O}_\ell)) = \sum_i \theta(\rho_i) \deg(\mathcal{R}_{\rho_i}|_\ell) > 0$; [1]
- for every compact reduced divisor D [2] and irreducible representation ρ, we have

  $$\theta(\varphi_C(\mathcal{R}_\rho^{-1} \otimes \omega_D)) < 0 \qquad \text{and} \qquad \theta(\varphi_C(\mathcal{R}_\rho^{-1}|_D)) > 0 ,$$

  where ω_D is the canonical bundle of D. [3]

These inequalities determine the walls of the chamber C, but we have to pay attention that some of them may be redundant.

Let $\theta \in \Theta$ be a generic parameter, C the chamber containing it and θ_0 a parameter on its wall W. The wall is said to be of type 0, I, II or III as follows:

- type 0 if $\mathcal{M}_{\theta_0}$ isomorphic to $\mathcal{M}_\theta$,

[1] Recall that if $\mathcal{R}_\rho = \mathcal{O}_X(D')$ then $\deg(\mathcal{R}_\rho|_\ell) = D' \cdot \ell$

[2] I.e. $D = \sum a_i D_i$ where D_i are compact invariant divisors and the coefficient $a_i \in \{0, 1, -1\}$.

[3] If $\mathcal{R}_\rho = \mathcal{O}_X(D')$ and $\omega_D = \mathcal{O}_D(K_D)$ then $\mathcal{R}_\rho^{-1} \otimes \omega_D = \mathcal{O}_D(-D' \cdot D + K_D)$ and $\mathcal{R}_\rho^{-1}|_D = \mathcal{O}_D(-D' \cdot D)$. Then we calculate the inequalities with the help of (2.5.2) and (3.2.1).

- type I if $\mathcal{M}_{\theta_0}$ is obtained from $\mathcal{M}_\theta$ by the contraction of a curve to a point,
- type II if $\mathcal{M}_{\theta_0}$ is obtained from $\mathcal{M}_\theta$ by the contraction of a divisor to a point,
- type III if $\mathcal{M}_{\theta_0}$ is obtained from $\mathcal{M}_\theta$ by the contraction of a divisor to a curve.

The inequalities coming from curves determine walls of type I or III, while the others determine walls of type 0. There are no walls of type II.
If C' is the chamber behind the wall W, the relation between $\mathcal{M}_{C'}$ and $\mathcal{M}_C$ and their tautological bundles depends on the type of the wall.

- W of type 0: $\mathcal{M}_{\theta'}$ is isomorphic to $\mathcal{M}_\theta$; the wall $W \subset \Theta$ is the zero locus of an equation of the form $R(\theta_0(\rho_1),\dots,\theta_0(\rho_r)) = a_1\theta_0(\rho_1) + \dots + a_r\theta_0(\rho_r) = 0$ and, if D is the divisor defining the wall, the tautological bundles $\mathcal{R}_i$ and $\mathcal{R}_i'$ are related as follows:
  - Case $+$: if $R(\theta(\rho_1),\dots,\theta(\rho_r)) > 0$ then
  $$\mathcal{R}_i' = \begin{cases} \mathcal{R}_i & \text{if } a_i = 0, \\ \mathcal{R}_i \otimes \mathcal{O}_{\mathcal{M}_{\theta'}}(D) & \text{if } a_i \neq 0; \end{cases}$$
  - Case $-$: if $R(\theta(\rho_1),\dots,\theta(\rho_r)) < 0$ then
  $$\mathcal{R}_i' = \begin{cases} \mathcal{R}_i & \text{if } a_i = 0, \\ \mathcal{R}_i \otimes \mathcal{O}_{\mathcal{M}_{\theta'}}(-D) & \text{if } a_i \neq 0. \end{cases}$$
- W of type I: $\mathcal{M}_{\theta'}$ is the variety obtained from $\mathcal{M}_\theta$ by the flop of the curve ℓ determining the wall; the tautological bundles $\mathcal{R}_i'$ are the proper transform of $\mathcal{R}_i$.
- W of type III: $\mathcal{M}_{\theta'}$ is isomorphic to $\mathcal{M}_\theta$; if D is the divisor contracted in $\mathcal{M}_{\theta_0}$, the tautological bundles $\mathcal{R}_i$ and $\mathcal{R}_i'$ are related as follows:
  - Case $+$: if $\{\deg(\mathcal{R}_i|_\ell)\} = \{0,1\}$ then
  $$\mathcal{R}_i' = \begin{cases} \mathcal{R}_i & \text{if } \deg(\mathcal{R}_i|_\ell) = 0, \\ \mathcal{R}_i \otimes \mathcal{O}_{\mathcal{M}_{\theta'}}(D) & \text{if } \deg(\mathcal{R}_i|_\ell) = 1; \end{cases}$$
  - Case $-$: if $\{\deg(\mathcal{R}_i|_\ell)\} = \{0,-1\}$ then
  $$\mathcal{R}_i' = \begin{cases} \mathcal{R}_i & \text{if } \deg(\mathcal{R}_i|_\ell) = 0, \\ \mathcal{R}_i \otimes \mathcal{O}_{\mathcal{M}_{\theta'}}(-D) & \text{if } \deg(\mathcal{R}_i|_\ell) = -1. \end{cases}$$

Thus, crossing walls of type I induces flops, while walls of type 0 and III induce self-equivalence of the derived category of the resolved variety. One can start from the chamber of the G-Hilb resolution, follow the change of the tautological bundles crossing the walls and reconstruct the chamber structure of Θ.

In our example the tautological bundles for the $\mathbb{Z}_6$-Hilb are

$$\begin{aligned} &\mathcal{R}_0 = \mathcal{O}_X, \quad \mathcal{R}_1 = \mathcal{O}_X(D_1), \quad \mathcal{R}_2 = \mathcal{O}_X(D_2), \quad \mathcal{R}_3 = \mathcal{O}_X(D_3), \\ &\mathcal{R}_4 = \mathcal{O}_X(-D_1 + D_2 + D_3 - D_4), \quad \mathcal{R}_5 = \mathcal{R}_2 \otimes \mathcal{R}_3 = \mathcal{O}_X(D_2 + D_3). \end{aligned} \tag{3.2.2}$$

We write parameters θ as $(\theta_0,\dots,\theta_5)$, where $\theta_i := (\theta(\rho_i))$. The inequalities defining the $\mathbb{Z}_6$-Hilb chamber are

$\theta_1 > 0$	wall of type I related to the flop of the curve C_1;	(3.2.3)
$\theta_2+\theta_5 > 0$	wall of type III+ related to the contraction of the divisor D_5;	(3.2.4)
$\theta_3+\theta_5 > 0$	wall of type I related to the flop of the curve C_3;	(3.2.5)
$\theta_4 > 0$	wall of type I related to the flop of the curve C_4;	(3.2.6)
$\theta_5 > 0$	wall of type 0+ defined by $\theta(\varphi_C(\mathcal{R}_5^{-1}\|_{D_7})) > 0$;	(3.2.7)
$\theta_2+\theta_3+\theta_4+\theta_5 > 0$	wall of type 0+ defined by $\theta(\varphi_C(\mathcal{R}_5^{-1}\otimes\omega_{D_7})) < 0$.	(3.2.8)

Any other inequality is redundant. As it is proved in section 9 of [CI], the flop of any single curve in the G-Hilb is achieved by crossing a wall of the chamber (generally if we are in a chamber different from the G-Hilb's it may be necessary first cross a type 0 wall to realize a flop).

Resolution $\mathrm{R}_2-\mathbb{C}^3_6$

Starting from the G-Hilb chamber we obtain this resolution by crossing the wall (3.2.3). The tautological bundles are again

$$\begin{aligned}&\mathcal{R}_0=\mathcal{O}_X,\quad \mathcal{R}_1=\mathcal{O}_X(D_1),\quad \mathcal{R}_2=\mathcal{O}_X(D_2),\quad \mathcal{R}_3=\mathcal{O}_X(D_3),\\ &\mathcal{R}_4=\mathcal{O}_X(-D_1+D_2+D_3-D_4),\quad \mathcal{R}_5=\mathcal{R}_2\otimes\mathcal{R}_3=\mathcal{O}_X(D_2+D_3).\end{aligned} \tag{3.2.9}$$

while the pure D-brane basis is

$$B_0:=\mathcal{O}_p\ ;\quad B_a:=\mathcal{O}_{C_a}(-T_a)\ ;\quad B_5:=\mathcal{O}_{D_7}(-T_2-T_3)\ , \tag{3.2.10}$$

with $a=1,\dots,4$. In terms of the $\mathcal{R}_i$ and their duals $\mathcal{S}_i$, the B_i-basis of $K(X)$ and its dual Φ-basis of $K^c(X)$ are thus:

$$\begin{aligned}
B_0&=\mathcal{S}_0+\mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3+\mathcal{S}_4+\mathcal{S}_5\ , & \Phi_0&=\mathcal{R}_0\ ,\\
B_1&=-\mathcal{S}_1\ , & \Phi_1&=-\mathcal{R}_0-\mathcal{R}_1+\mathcal{R}_3+\mathcal{R}_4\ ,\\
B_2&=\mathcal{S}_2+\mathcal{S}_5\ , & \Phi_2&=-\mathcal{R}_0+\mathcal{R}_2\ ,\\
B_3&=\mathcal{S}_1+\mathcal{S}_3+\mathcal{S}_5\ , & \Phi_3&=-\mathcal{R}_0+\mathcal{R}_3\ ,\\
B_4&=\mathcal{S}_1+\mathcal{S}_4\ , & \Phi_4&=-\mathcal{R}_0+\mathcal{R}_4\ ,\\
B_5&=\mathcal{S}_5\ . & \Phi_5&=\mathcal{R}_0-\mathcal{R}_2-\mathcal{R}_3+\mathcal{R}_5\ .
\end{aligned} \tag{3.2.11}$$

The symplectic form in the selected basis is

$$\chi(B_i,B_j)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}\ . \tag{3.2.12}$$

Resolution $\mathrm{R}_3-\mathbb{C}^3_6$

Starting from the G-Hilb chamber we obtain this resolution by crossing the wall (3.2.5). The tautological bundles are again

$$\begin{aligned}&\mathcal{R}_0 = \mathcal{O}_X, \quad \mathcal{R}_1 = \mathcal{O}_X(D_1), \quad \mathcal{R}_2 = \mathcal{O}_X(D_2), \quad \mathcal{R}_3 = \mathcal{O}_X(D_3),\\ &\mathcal{R}_4 = \mathcal{O}_X(-D_1+D_2+D_3-D_4), \quad \mathcal{R}_5 = \mathcal{R}_2\otimes\mathcal{R}_3 = \mathcal{O}_X(D_2+D_3).\end{aligned} \tag{3.2.13}$$

while the pure D-brane basis is

$$B_0 := \mathcal{O}_p \,; \quad B_a := \mathcal{O}_{C_a}(-T_a) \,; \quad B_5 := \mathcal{O}_{D_7}(-T_1-T_2+T_3-T_4) \,, \tag{3.2.14}$$

with $a=1,\ldots,4$. In terms of the $\mathcal{R}_i$ and their duals $\mathcal{S}_i$, the B_i-basis of $K(X)$ and its dual Φ-basis of $K^c(X)$ are thus:

$$\begin{aligned}
B_0 &= \mathcal{S}_0+\mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3+\mathcal{S}_4+\mathcal{S}_5 \,, & \Phi_0 &= \mathcal{R}_0 \,,\\
B_1 &= \mathcal{S}_1+\mathcal{S}_3+\mathcal{S}_5 \,, & \Phi_1 &= -\mathcal{R}_0+\mathcal{R}_1 \,,\\
B_2 &= \mathcal{S}_2+\mathcal{S}_5 \,, & \Phi_2 &= -\mathcal{R}_0+\mathcal{R}_2 \,,\\
B_3 &= -\mathcal{S}_3-\mathcal{S}_5 \,, & \Phi_3 &= -\mathcal{R}_0+\mathcal{R}_1-\mathcal{R}_3+\mathcal{R}_4 \,,\\
B_4 &= \mathcal{S}_3+\mathcal{S}_4+\mathcal{S}_5, & \Phi_4 &= -\mathcal{R}_0+\mathcal{R}_4 \,,\\
B_5 &= \mathcal{S}_5 \,. & \Phi_5 &= \mathcal{R}_0-\mathcal{R}_2-\mathcal{R}_3+\mathcal{R}_5 \,.
\end{aligned} \tag{3.2.15}$$

The symplectic form in the selected basis is

$$\chi(B_i,B_j) = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&-2\\ 0&0&0&-1&2&0 \end{pmatrix} . \tag{3.2.16}$$

Resolution $\mathrm{R}_4-\mathbb{C}^3_6$

Starting from the G-Hilb chamber we obtain this resolution by crossing the wall (3.2.6). The tautological bundles are again

$$\begin{aligned}&\mathcal{R}_0 = \mathcal{O}_X, \quad \mathcal{R}_1 = \mathcal{O}_X(D_1), \quad \mathcal{R}_2 = \mathcal{O}_X(D_2), \quad \mathcal{R}_3 = \mathcal{O}_X(D_3),\\ &\mathcal{R}_4 = \mathcal{O}_X(-D_1+D_2+D_3-D_4), \quad \mathcal{R}_5 = \mathcal{R}_2\otimes\mathcal{R}_3 = \mathcal{O}_X(D_2+D_3).\end{aligned} \tag{3.2.17}$$

while the pure D-brane basis is

$$B_0 := \mathcal{O}_p \,; \quad B_a := \mathcal{O}_{C_a}(-T_a) \,; \quad B_5 := \mathcal{O}_{D_7}(-T_2-T_3) \,, \tag{3.2.18}$$

with $a=1,\ldots,4$. In terms of the $\mathcal{R}_i$ and their duals $\mathcal{S}_i$, the B_i-basis of $K(X)$ and its dual Φ-basis of $K^c(X)$ are thus:

$$\begin{aligned}
B_0 &= \mathcal{S}_0+\mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3+\mathcal{S}_4+\mathcal{S}_5 \,, & \Phi_0 &= \mathcal{R}_0 \,,\\
B_1 &= \mathcal{S}_1+\mathcal{S}_4 \,, & \Phi_1 &= -\mathcal{R}_0+\mathcal{R}_1 \,,\\
B_2 &= \mathcal{S}_2+\mathcal{S}_4+\mathcal{S}_5 \,, & \Phi_2 &= -\mathcal{R}_0+\mathcal{R}_2 \,,\\
B_3 &= \mathcal{S}_3+\mathcal{S}_4+\mathcal{S}_5 \,, & \Phi_3 &= -\mathcal{R}_0+\mathcal{R}_3 \,,\\
B_4 &= -\mathcal{S}_4 \,, & \Phi_4 &= -\mathcal{R}_0+\mathcal{R}_1-\mathcal{R}_4+\mathcal{R}_5 \,,\\
B_5 &= \mathcal{S}_4+\mathcal{S}_5 \,. & \Phi_5 &= \mathcal{R}_0-\mathcal{R}_2-\mathcal{R}_3+\mathcal{R}_5 \,.
\end{aligned} \tag{3.2.19}$$

The symplectic form in the selected basis is

$$\chi(B_i,B_j)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&0&-2\\0&0&0&0&0&1\\0&0&1&2&-1&0\end{pmatrix}. \tag{3.2.20}$$

Resolution $\mathrm{R}_5-\mathbb{C}^3_6$

Starting from the above chamber of the resolution $\mathrm{R}_4-\mathbb{C}^3_6$ we obtain this resolution by crossing a single wall of type I. The tautological bundles are again

$$\begin{aligned}&\mathcal{R}_0=\mathcal{O}_X, \quad \mathcal{R}_1=\mathcal{O}_X(D_1), \quad \mathcal{R}_2=\mathcal{O}_X(D_2), \quad \mathcal{R}_3=\mathcal{O}_X(D_3),\\ &\mathcal{R}_4=\mathcal{O}_X(-D_1+D_2+D_3-D_4), \quad \mathcal{R}_5=\mathcal{R}_2\otimes\mathcal{R}_3=\mathcal{O}_X(D_2+D_3).\end{aligned} \tag{3.2.21}$$

while the pure D-brane basis is

$$B_0:=\mathcal{O}_p\ ; \quad B_a:=\mathcal{O}_{C_a}(-T_a)\ ; \quad B_5:=\mathcal{O}_{D_7}(T_2-2T_3-T_4)\ , \tag{3.2.22}$$

with $a=1,\dots,4$. In terms of the $\mathcal{R}_i$ and their duals $\mathcal{S}_i$, the B_i-basis of $K(X)$ and its dual Φ-basis of $K^c(X)$ are thus:

$$\begin{aligned}
&B_0=\mathcal{S}_0+\mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3+\mathcal{S}_4+\mathcal{S}_5\ , && \Phi_0=\mathcal{R}_0\ ,\\
&B_1=\mathcal{S}_1+\mathcal{S}_4\ , && \Phi_1=-\mathcal{R}_0+\mathcal{R}_1\ ,\\
&B_2=-\mathcal{S}_2-\mathcal{S}_4-\mathcal{S}_5\ , && \Phi_2=-\mathcal{R}_0+\mathcal{R}_1-\mathcal{R}_2+\mathcal{R}_3-\mathcal{R}_4+\mathcal{R}_5,\\
&B_3=\mathcal{S}_2+\mathcal{S}_3+2\mathcal{S}_4+2\mathcal{S}_5\ , && \Phi_3=-\mathcal{R}_0+\mathcal{R}_3\ ,\\
&B_4=\mathcal{S}_2+\mathcal{S}_5\ , && \Phi_4=-\mathcal{R}_0+\mathcal{R}_1-\mathcal{R}_4+\mathcal{R}_5\ ,\\
&B_5=\mathcal{S}_4+\mathcal{S}_5\ . && \Phi_5=\mathcal{R}_0-\mathcal{R}_2-\mathcal{R}_3+\mathcal{R}_5\ .
\end{aligned} \tag{3.2.23}$$

The symplectic form in the selected basis is

$$\chi(B_i,B_j)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&-3\\0&0&0&0&0&0\\0&0&-1&3&0&0\end{pmatrix}. \tag{3.2.24}$$

3.3 The cohomological hypergeometric series and GW–invariants

The hypergeometric series are specified by the ℓ vectors corresponding to large Kähler parameters and the hypergeometric coefficients are determined expanding them with respect to the basis $Q_i=\mathrm{ch}(\Phi_i)$ of $H^*(X,\mathbb{Q})$.

Invariants for $\mathrm{R}_2\!-\!\mathbb{C}^3_6$

The vectors ℓ_a, $a = 1,\dots,4$ are

$$\begin{aligned} C_1 &: \quad \ell_1 = (-1,0,0,1,0,1,-1)\ , \\ C_2 &: \quad \ell_2 = (0,1,0,1,-2,0,0)\ , \\ C_3 &: \quad \ell_3 = (1,0,1,0,0,-2,0)\ , \\ C_4 &: \quad \ell_4 = (1,0,0,-2,1,0,0)\ . \end{aligned} \tag{3.3.1}$$

The selected basis of the cohomology is

$$\begin{aligned} &Q_0 = 1\ , \quad Q_1 = J_1 - 2C\ , \quad Q_2 = J_2\ , \\ &Q_3 = J_3 - \frac{1}{2}C\ , \quad Q_4 = J_4 - C\ , \quad Q_5 = -C. \end{aligned} \tag{3.3.2}$$

If we make this change of basis and use the mirror symmetry identification

$$w\left(\vec{x},\frac{\vec{J}}{2\pi i}\right) = Q_0 1 + \sum_{a=1}^{4} Q_a t_a + Q_5 g(t_1,\dots,t_4)\ , \tag{3.3.3}$$

then we find

$$\begin{aligned} 2\pi i t_1 &= \log x_1 - \Psi(x_3) + \Phi(x_2,x_4) - \aleph(\vec{x})\ , \\ 2\pi i t_2 &= \log x_2 + \Phi(x_2,x_4) - 2\Phi(x_4,x_2)\ , \\ 2\pi i t_3 &= \log x_3 + 2\Psi(x_3)\ , \\ 2\pi i t_4 &= \log x_4 - 2\Phi(x_2,x_4) + \Phi(x_4,x_2)\ , \end{aligned} \tag{3.3.4}$$

and

$$g(\vec{t}) = P_2(\vec{t}) + \frac{1}{(2\pi i)^2}\phi(\vec{t})\ , \tag{3.3.5}$$

where P_2 is the degree two polynomial part

$$\begin{aligned} P_2(\vec{t}) = &-2t_1 - \frac{1}{2}t_3 - t_4 + 3t_1^2 + \frac{1}{2}t_3^2 + t_4^2 \\ &+2t_1t_2 + 3t_1t_3 + 4t_1t_4 + t_2t_3 + t_2t_4 + 2t_3t_4\ , \end{aligned} \tag{3.3.6}$$

and

$$\begin{aligned} \phi(\vec{t}) = &\ 6\aleph^{(1)}(\vec{x}) - 3\aleph^{(2)}(\vec{x}) - 2\aleph^{(3)}(\vec{x}) - \aleph^{(6)}(\vec{x}) - \Lambda_4(\vec{x}) - \Lambda_5(\vec{x}) \\ &+\Psi_6(x_2,x_4) - 2\Psi_4(x_3) + 2\Psi_5(x_3) + 2\Psi_1(x_2,x_4) + 2\Psi_1(x_4,x_2) \\ &-\Psi_2(x_2,x_4) - \Psi_2(x_4,x_2) - \Psi_3(x_2,x_4) - \Psi_3(x_4,x_2) \\ &-3\aleph^2(\vec{x}) + \Psi^2(x_3) + \Phi^2(x_2,x_4) + \Phi^2(x_4,x_2) - \Phi(x_2,x_4)\Phi(x_4,x_2), \end{aligned} \tag{3.3.7}$$

with $\vec{x}$ expressed as a function of $\vec{t}$ by inverting system (3.3.4), is the part corresponding to instantonic contributions. Following Hosono and using (3.2.12) we find

$$(-\partial_{t_1})F(\vec{t}) = g(\vec{t})\ , \tag{3.3.8}$$

where F is the prepotential. Setting

$$q_k := e^{2\pi i t_k}\ , \tag{3.3.9}$$

we then find

$$F(\vec{t}) = t_1^2 + \frac{1}{2}t_1t_3 - t_1t_4 - t_1^3 - \frac{1}{2}t_1t_3^2 - t_1t_4^2 - t_1^2t_2 - \frac{3}{2}t_1^2t_3 - 2t_1^2t_4$$
$$-t_1t_2t_3 - t_1t_2t_4 - 2t_1t_3t_4 + F_{\text{inst}}(\vec{q}) + P_{\text{class}}(t_2,t_3,t_4) + Q_{\text{inst}}(q_2,q_3,q_4) \ . \qquad (3.3.10)$$

P and Q are the undetermined parts. We list the Gopakumar-Vafa invariants for rational curves up to degree six. The curves in the integer cone generated by $[0,1,0,0]$, $[0,0,1,0]$, $[0,0,0,1]$ must be excluded, because corresponding to the undetermined part of the prepotential. The only nonvanishing invariants in the considered range are

$$\begin{array}{l} GV_{[1,0,0,0]} = GV_{[1,0,1,0]} = GV_{[1,0,0,1]} = \\ GV_{[1,0,1,1]} = GV_{[1,1,0,1]} = GV_{[1,1,1,1]} = 1; \\ GV_{[2,0,1,1]} = GV_{[2,1,1,1]} = GV_{[2,1,1,2]} = -2. \end{array} \qquad (3.3.11)$$

Invariants for $\mathrm{R}_3 - \mathbb{C}^3_6$

The vectors ℓ_a, $a = 1, \ldots, 4$ are

$$\begin{array}{lcl} C_1 & : & \ell_1 = (1,0,1,0,0,-2,0) \ , \\ C_2 & : & \ell_2 = (0,1,0,1,-2,0,0) \ , \\ C_3 & : & \ell_3 = (0,0,-1,-1,0,1,1) \ , \\ C_4 & : & \ell_4 = (0,0,1,0,1,0,-2) \ . \end{array} \qquad (3.3.12)$$

The selected basis of the cohomology is

$$Q_0 = 1 \ , \quad Q_1 = J_1 \ , \quad Q_2 = J_2 \ , \quad Q_3 = J_3 \ , \quad Q_4 = J_4 + C \ , \quad Q_5 = C. \qquad (3.3.13)$$

Making use of the mirror symmetry identification

$$w\left(\vec{x}, \frac{\vec{J}}{2\pi i}\right) = Q_0 1 + \sum_{a=1}^{4} Q_a t_a + Q_5 g(t_1, \ldots, t_4) \ , \qquad (3.3.14)$$

then we find

$$\begin{array}{l} 2\pi i t_1 = \log x_1 + 2\Psi(x_1) \ , \\ 2\pi i t_2 = \log x_2 - \Phi(x_2, x_1x_3^2x_4) + 2\Phi(x_1x_3^2x_4, x_2) \ , \\ 2\pi i t_3 = \log x_3 - \Psi(x_1) + \aleph(\vec{x}) \ , \\ 2\pi i t_4 = \log x_4 - \Phi(x_1x_3^2x_4, x_2) - 2\aleph(\vec{x}) \ , \end{array} \qquad (3.3.15)$$

and

$$g(\vec{t}) = P_2(\vec{t}) + \frac{1}{(2\pi i)^2}\phi(\vec{t}) \ , \qquad (3.3.16)$$

where P_2 is the degree two polynomial part

$$P_2(\vec{t}) = \frac{1}{6} - t_4 + t_4^2 + t_2t_4 \ , \qquad (3.3.17)$$

and

$$\phi(\vec{t}) = 8\aleph^{(1)}(\vec{x}) - 4\aleph^{(2)}(\vec{x}) - 2\aleph^{(3)}(\vec{x}) - \aleph^{(4)}(\vec{x}) + \Lambda_6(\vec{x}) - 2\Lambda_7(\vec{x}) + \Lambda_8(\vec{x})$$

$$
\begin{aligned}
&+\Psi_3(x_2,x_1x_3^2x_4)+\Psi_2(x_1x_3^2x_4,x_2)+\Psi_3(x_1x_3^2x_4,x_2)-2\Psi_1(x_1x_3^2x_4,x_2)\\
&-4\aleph^2(\vec{x})-2\aleph(\vec{x})\Phi(x_2,x_1x_3^2x_4)-\Phi(x_2,x_1x_3^2x_4)\Phi(x_1x_3^2x_4,x_2)\\
&+\Phi^2(x_1x_3^2x_4,x_2),
\end{aligned}
\tag{3.3.18}
$$

with $\vec{x}$ expressed as a function of $\vec{t}$ by inverting system (3.3.15), is the part corresponding to instantonic contributions. Using (3.2.16) we find

$$
(\partial_{t_3}-2\partial_{t_4})F(\vec{t})=g(\vec{t})\ , \tag{3.3.19}
$$

where F is the prepotential. Setting

$$
q_k:=e^{2\pi i t_k}\ , \tag{3.3.20}
$$

we then find

$$
\begin{aligned}
F(\vec{t})=&\frac{1}{6}t_3+\frac{1}{4}t_4^2-\frac{1}{6}t_4^3-\frac{1}{4}t_2t_4^2\\
&+F_{\rm inst}(\vec{q})+P_{\rm class}(t_1,t_2,2t_3+t_4)+Q_{\rm inst}(q_1,q_2,q_3^2q_4)\ .
\end{aligned}
\tag{3.3.21}
$$

P and Q are the undetermined parts. We list the GV–invariants up to degree six. The curves in the integer cone generated by $[1,0,0,0]$, $[0,1,0,0]$, $[0,0,2,1]$ must be excluded, because corresponding to the undetermined part of the prepotential. The only nonvanishing invariants in the considered range are

$$
\begin{aligned}
&GV_{[0,0,1,0]}=GV_{[0,0,1,1]}=GV_{[0,1,1,1]}=\\
&GV_{[1,0,1,0]}=GV_{[1,0,1,1]}=GV_{[1,1,1,1]}=1;\\
&GV_{[0,0,0,1]}=GV_{[0,1,0,1]}=GV_{[1,1,2,2]}=-2;\quad GV_{[0,1,1,2]}=GV_{[1,1,1,2]}=3;\\
&GV_{[0,1,0,2]}=-4;\quad GV_{[0,1,1,3]}=GV_{[1,1,1,3]}=GV_{[0,2,1,3]}=5;\\
&GV_{[0,1,0,3]}=GV_{[0,2,0,3]}=-6;\quad GV_{[0,1,1,4]}=7;\quad GV_{[0,1,0,4]}=-8;\\
&GV_{[0,1,0,5]}=-10;\quad GV_{[0,2,0,4]}=-32.
\end{aligned}
\tag{3.3.22}
$$

Invariants for R_4–$\mathbb{C}^3_6$

The vectors ℓ_a, $a=1,\dots,4$ are

$$
\begin{aligned}
C_1 &: \quad \ell_1=(1,0,0,-2,1,0,0)\ ,\\
C_2 &: \quad \ell_2=(0,1,0,0,-1,1,-1)\ ,\\
C_3 &: \quad \ell_3=(0,0,1,0,1,0,-2)\ ,\\
C_4 &: \quad \ell_4=(0,0,0,1,-1,-1,1)\ .
\end{aligned}
\tag{3.3.23}
$$

The selected basis of the cohomology is

$$
\begin{aligned}
&Q_0=1\ ,\quad Q_1=J_1\ ,\quad Q_2=J_2\ ,\\
&Q_3=J_3+\frac{1}{2}C\ ,\quad Q_4=J_4\ ,\quad Q_5=C.
\end{aligned}
\tag{3.3.24}
$$

Via the mirror symmetry identification

$$
w\left(\vec{x},\frac{\vec{J}}{2\pi i}\right)=Q_0 1+\sum_{a=1}^{4}Q_a t_a+Q_5 g(t_1,\dots,t_4)\ , \tag{3.3.25}
$$

we get

$$
\begin{aligned}
2\pi i t_1 &= \log x_1 + 2\Phi(x_1, x_2x_4) - \Phi(x_2x_4, x_1)\ ,\\
2\pi i t_2 &= \log x_2 + \Phi(x_2x_4, x_1) - \Psi(x_1x_3x_4^2) - \aleph(\vec{x})\ ,\\
2\pi i t_3 &= \log x_3 - \Phi(x_2x_4, x_1) - 2\aleph(\vec{x})\ ,\\
2\pi i t_4 &= \log x_4 + \Psi(x_1x_3x_4^2) - \Phi(x_1, x_2x_4) + \Phi(x_2x_4, x_1) + \aleph(\vec{x})\ ,
\end{aligned}
\tag{3.3.26}
$$

and

$$g(\vec{t}) = P_2(\vec{t}) + \frac{1}{(2\pi i)^2}\phi(\vec{t})\ , \tag{3.3.27}$$

where P_2 is the degree two polynomial part

$$P_2(\vec{t}) = \frac{1}{6} - \frac{1}{2}t_3 + \frac{1}{2}t_3^2 + t_2t_3\ , \tag{3.3.28}$$

and

$$
\begin{aligned}
\phi(\vec{t}) = {} & 8\aleph^{(1)}(\vec{x}) - 3\aleph^{(2)}(\vec{x}) - 2\aleph^{(3)}(\vec{x}) - 2\aleph^{(5)}(\vec{x}) - \aleph^{(7)}(\vec{x})\\
& -\Lambda_9(\vec{x}) + \Lambda_{10}(\vec{x}) - 2\Lambda_{11}(\vec{x})\\
& -\Psi_1(x_2x_4, x_1) + \Psi_2(x_2x_4, x_1) + \Psi_5(x_1x_3x_4^2)\\
& -4\aleph^2(\vec{x}) - 2\aleph(\vec{x})\Psi(x_1x_3x_4^2) - \aleph(\vec{x})\Phi(x_2x_4, x_1)\\
& -\Psi(x_1x_3x_4^2)\Phi(x_2x_4, x_1) + \frac{1}{2}\Phi^2(x_2x_4, x_1)
\end{aligned}
\tag{3.3.29}
$$

with $\vec{x}$ expressed as a function of $\vec{t}$ by inverting system (3.3.26), is the part corresponding to instantonic contributions. Using (3.2.20) we find

$$(\partial_{t_4} - \partial_{t_2} - 2\partial_{t_3})F(\vec{t}) = g(\vec{t})\ , \tag{3.3.30}$$

where F is the prepotential. Setting

$$q_k := e^{2\pi i t_k}\ , \tag{3.3.31}$$

we then find

$$
\begin{aligned}
F(\vec{t}) = {} & \frac{1}{6}t_4 + \frac{1}{8}t_3^2 - \frac{1}{12}t_3^3 - \frac{1}{2}t_2^2t_3 + \frac{1}{3}t_2^3\\
& + F_{\text{inst}}(\vec{q}) + P_{\text{class}}(t_1, t_4 + t_2, 2t_4 + t_3) + Q_{\text{inst}}(q_1, q_4q_2, q_4^2q_3)\ ,
\end{aligned}
\tag{3.3.32}
$$

P and Q being the undetermined parts. We list the GV–invariants up to degree six. The curves in the integer cone generated by $[0,1,0,1]$, $[0,0,1,2]$, $[1,0,0,0]$ must be excluded, because corresponding to the undetermined part of the prepotential. The only nonvanishing invariants in the considered range are

$$
\begin{aligned}
& GV_{[0,0,0,1]} = GV_{[0,0,1,1]} = GV_{[0,1,0,0]} =\\
& GV_{[1,0,0,1]} = GV_{[1,0,1,1]} = GV_{[1,1,1,2]} = 1;\\
& GV_{[0,0,1,0]} = GV_{[0,1,1,1]} = GV_{[1,1,1,1]} = -2; \quad GV_{[0,1,1,0]} = GV_{[1,1,2,2]} = 3;\\
& GV_{[0,1,2,1]} = GV_{[1,1,2,1]} = -4; \quad GV_{[0,1,2,0]} = GV_{[0,2,2,1]} = GV_{[1,2,2,1]} = 5;\\
& GV_{[0,1,3,1]} = GV_{[0,2,2,0]} = GV_{[1,1,3,1]} = -6; \quad GV_{[0,1,3,0]} = 7;\\
& GV_{[0,1,4,1]} = -8; \quad GV_{[0,1,4,0]} = 9; \quad GV_{[0,1,5,0]} = 11; \quad GV_{[0,3,3,0]} = 27;\\
& GV_{[0,2,3,0]} = -32; \quad GV_{[0,2,3,1]} = 35; \quad GV_{[0,2,4,0]} = -110.
\end{aligned}
\tag{3.3.33}
$$

Invariants for $\mathrm{R}_5 - \mathbb{C}^3_6$

The vectors ℓ_a, $a = 1, \ldots, 4$ are

$$\begin{array}{lcl} C_1 & : & \ell_1 = (1,0,0,-2,1,0,0) \ , \\ C_2 & : & \ell_2 = (0,-1,0,0,1,-1,1) \ , \\ C_3 & : & \ell_3 = (0,1,1,0,0,1,-3) \ , \\ C_4 & : & \ell_4 = (0,1,0,1,-2,0,0) \ . \end{array} \tag{3.3.34}$$

The selected basis of the cohomology is

$$\begin{array}{l} Q_0 = 1 \ , \quad Q_1 = J_1 \ , \quad Q_2 = J_2 \ , \\ Q_3 = J_3 + \frac{1}{2}C \ , \quad Q_4 = J_4 \ , \quad Q_5 = C . \end{array} \tag{3.3.35}$$

If we make this change of basis and use the mirror symmetry identification

$$w\left(\vec{x}, \frac{\vec{J}}{2\pi i}\right) = Q_0 1 + \sum_{a=1}^{4} Q_a t_a + Q_5 g(t_1, \ldots, t_4) \ , \tag{3.3.36}$$

we get

$$\begin{array}{l} 2\pi i t_1 = \log x_1 + 2\Phi(x_1,x_4) - \Phi(x_4,x_1) \ , \\ 2\pi i t_2 = \log x_2 + \Psi(x_1 x_2^3 x_3 x_4^2) - \Phi(x_4,x_1) - \aleph(\vec{x}) \ , \\ 2\pi i t_3 = \log x_3 - \Psi(x_1 x_2^3 x_3 x_4^2) + 3\aleph(\vec{x}) \ , \\ 2\pi i t_4 = \log x_4 - \Phi(x_1,x_4) + 2\Phi(x_4,x_1) \ , \end{array} \tag{3.3.37}$$

and

$$g(\vec{t}) = P_2(\vec{t}) + \frac{1}{(2\pi i)^2}\phi(\vec{t}) \ , \tag{3.3.38}$$

where P_2 is the degree two polynomial part

$$P_2(\vec{t}) = \frac{1}{4} - \frac{1}{2}t_3 + \frac{1}{2}t_3^2 \ , \tag{3.3.39}$$

and

$$\begin{aligned} \phi(\vec{t}) = {} & 9\aleph^{(1)}(\vec{x}) - 3\aleph^{(2)}(\vec{x}) - 3\aleph^{(3)}(\vec{x}) - 3\aleph^{(5)}(\vec{x}) + \Lambda_{12}(\vec{x}) + \Lambda_{13}(\vec{x}) \\ & + \Psi_4(x_1 x_2^3 x_3 x_4^2) + \Psi_5(x_1 x_2^3 x_3 x_4^2) \\ & - \frac{1}{2}\Psi^2(x_1 x_2^3 x_3 x_4^2) - \frac{9}{2}\aleph^2(\vec{x}) + 3\Psi(x_1 x_2^3 x_3 x_4^2)\,\aleph(\vec{x}) \end{aligned} \tag{3.3.40}$$

with $\vec{x}$ expressed as a function of $\vec{t}$ by inverting system (3.3.37), is the part corresponding to instantonic contributions. Using (3.2.24) we find

$$(\partial_{t_1} - 3\partial_{t_3})F(\vec{t}) = g(\vec{t}) \ , \tag{3.3.41}$$

where F is the prepotential. Setting

$$q_k := e^{2\pi i t_k} \ , \tag{3.3.42}$$

we then find

$$F(\vec{t}) = \frac{1}{4}t_1 + \frac{1}{12}t_3^2 - \frac{1}{18}t_3^3 \\ +F_{\rm inst}(\vec{q}) + P_{\rm class}(3t_1+t_3,t_2,t_4) + Q_{\rm inst}(q_1^3q_3,q_2,q_4)\ . \qquad (3.3.43)$$

We list the GV–invariants up to degree six. The curves in the integer cone generated by $[0,3,1,0]$, $[1,0,0,0]$, $[0,0,0,1]$ must be excluded, because corresponding to the undetermined part of the prepotential. The only nonvanishing invariants in the considered range are

$$\begin{array}{l} GV_{[0,1,0,0]} = GV_{[0,1,0,1]} = GV_{[1,2,1,1]} = GV_{[1,2,1,2]} = GV_{[1,1,0,1]} = 1; \\ GV_{[0,1,1,0]} = GV_{[1,1,1,1]} = -2; \quad GV_{[0,0,1,0]} = 3; \\ GV_{[0,2,2,1]} = GV_{[1,2,2,1]} = -4; \quad GV_{[0,1,2,0]} = GV_{[0,1,2,1]} = GV_{[1,1,2,1]} = 5; \\ GV_{[0,0,2,0]} = GV_{[0,1,2,0]} = -6; \quad GV_{[0,2,3,0]} = 7; \quad GV_{[0,0,3,0]} = 27; \\ GV_{[0,1,3,0]} = GV_{[0,1,4,1]} = GV_{[0,1,4,0]} = -32; \quad GV_{[0,2,3,1]} = 35; \\ GV_{[0,2,4,0]} = -110; \quad GV_{[0,0,4,0]} = -192; \quad GV_{[0,1,4,0]} = GV_{[0,1,4,1]} = 286; \\ GV_{[0,0,5,0]} = 1695; \quad GV_{[0,1,5,1]} = 3038; \quad GV_{[0,0,6,0]} = -17064. \end{array} \qquad (3.3.44)$$

Chapter 4

Conclusions

This part of the thesis should be meant as the first step of a detailed analysis of aspects of local (homological) mirror symmetry in relation to its physical meaning. We apply local mirror symmetry to the construction of a prepotential accounting for the lower genus Gopakumar-Vafa invariants. In particular, for this purpose, we have adopted a particularly elegant way introduced by Hosono in [Hos1] and which we dubbed the "Hosono conjecture". Our results can be thus interpreted also as a positive (partial) check of the Hosono conjecture for the case of an orbifold with multiple resolutions.

Indeed, we applied the Hosono conjecture to an orbifold model admitting five distinct crepant resolutions, showing that, for each resolution, it partially determines a prepotential encoding information about the Gromov-Witten invariants. As we seen, not all GW–invariant are determined. Indeed it is not even clear how they could be defined as some ambiguity is introduced by noncompactness of the varieties considered. However, we can note that for all resolutions, the only non computable invariants are the one associated to curves having zero intersection with the compact divisor D_7. Actually this seems reasonable because the local mirror symmetry focuses on the Calabi-Yau geometry near compact divisors in a Calabi-Yau manifold [CKYZ].

Curves having negative intersection with D_7 cannot deform out of D_7. When they have non negative intersection with all the other (non compact) divisors, then we expect for the invariant numbers to count the number of deformations in D_7. However, when the intersection with some of the noncompact divisors is negative, then the deformations are constrained on the intersection between the divisors and we expect for the GV numbers to vanish.

The invariants predicted by the prepotential agree with the ones computed directly by means of the methods described in [CKYZ]. In place of repeating such computations here, we will simply compare some of the invariants of our examples with the ones provided in [CKYZ]. Let us start with resolution five. It contains a $\mathbb{P}^2$ associated the compact divisor D_7 and the curves C_{27}, C_{37} and C_{67}, all equivalent. Then, let us fix $b := C_3 \equiv C_{67}$. It has intersection -3 with D_7 so that it is the null section of the normal bundle of D_7 in X. It also has intersection numbers 1 with D_3, D_5, D_6 and 0 with the other noncompact divisors. Thus it freely deforms out from all the noncompact divisors, in the sense discussed above, and then one expects that the number of its deformations is just the number of deformations inside $\mathbb{P}^2$. From the list (3.3.44) we see that, up to degree six, the corresponding numbers are $GV[d] = GV[0,0,d,0]$ with

$$\begin{aligned} &GV[1] = 3, \quad GV[2] = -6, \quad GV[3] = 27, \\ &GV[4] = -192, \quad GV[5] = 1695, \quad GV[6] = -17064, \end{aligned} \tag{4.0.1}$$

which indeed coincide with the GV–numbers of $O(-3) \to \mathbb{P}^2$, see table 1 in [CKYZ].

Let us move to the fourth resolution. It contains the Hirzebruch surface $\mathbb{F}_1$ associated to D_7 and the curves C_{27},

C_{37}, C_{57}, C_{67}. The independent curves are $b := C_2 \equiv C_{57}$ and $f := C_3 \equiv C_{67}$ which define the base and the fiber of the Hirzebruch fibration. Note that b has intersection -1 with D_5 so that we expect for its eventual deformations in $\mathbb{F}_1$ to be constrained. This does not happens for f or for all combinations $[d_B, d_F] \equiv [0, d_B, d_F, 0]$ with $d_F \geq d_B$ which have negative intersections with D_7 only. Thus we again expect for the GV–invariants corresponding to $[d_B, d_F]$ to be the same as in F_1. Indeed from table 10 in [CKYZ] we see that deformations appears for $K_{\mathbb{F}_1}$ only for $d_F \geq d_B$ (apart from the case $[1,0]$). We can see that our results, as listed in (3.3.33), are in perfect agreement

$$\begin{aligned} &GV[1,0] = 1, \quad GV[0,1] = -2, \quad GV[1,1] = 3, \quad GV[1,2] = 5, \\ &GV[2,2] = -6, \quad GV[1,3] = 7, \quad GV[1,4] = 9, \quad GV[1,5] = 9, \\ &GV[3,3] = 27, \quad GV[2,3] = -32, \quad GV[2,4] = -110, \end{aligned} \tag{4.0.2}$$

and $GV[i, j] = GV[0, k] = 0$ for $j < i$, $i > 2$, $i + j \leq 6$ and $k = 2, 3, 4, 5$.
A similar comparison can be done for resolution three. In that case, D_7 and the curves C_{27}, C_{37}, C_{47} and C_{57} define an Hirzebruch surface $\mathbb{F}_2$ with base $b := C_2 \equiv C_{57}$ and fiber $f := C_4 \equiv C_{47}$. Note that b has intersection 0 with D_7 so that curves $[d_B, 0]$ are not countable. These correspond to the first column of table 11 in [CKYZ]. Next curves $[d_B, d_F] = [0, d_B, 0, d_F]$ with positive d_F are computable in D_7, but only for $d_F > d_B$ their intersections with D_i are negative only when they intersect the compact divisor. We then expect for the curves $[d_B, d_B + 1 + k]$ to determine the same numbers as for $K_{\mathbb{F}_2}$, whereas for $d_F \leq d_B$ they can be constrained by the fact they have negative intersection with D_5 also. However, as follows from table 11 in [CKYZ] all such numbers vanish (excluding the case $[d_B, d_F] = [1, 1]$) and again our results, collected in (3.3.22), agree with the numbers of $K_{\mathbb{F}_2}$

$$\begin{aligned} &GV[1,1] = -2, \quad GV[1,2] = -4, \quad GV[1,3] = GV[2,3] = -6, \\ &GV[1,4] = -8, \quad GV[1,5] = -10, \quad GV[2,4] = -32, \end{aligned} \tag{4.0.3}$$

and $GV[i, j] = 0$ for all the remaining ones up to degree 6 (and with $j \neq 0$).

All these are only a part of the numbers predicted by means of the Hosono construction. Indeed, these are the ones corresponding to curves having negative intersection number with the compact divisor and thus admitting a representative contained in it. However, as yet remarked, the constructed potential results to determine much more numbers, all the ones related to the curves that intersect transversally the compact divisor. Actually the true meaning of this fact and the non computability of the curves having null intersection with the compact divisor is not completely clear and deserve a deeper analysis. One way to proceed in such a direction is to search for an extended GKZ system whose solutions permit to extend the computation of the invariants to all curves, as proposed for example in [FoJi1],[FoJi2]. This also should provide a slight improvement of the Hosono conjecture.

Furthermore, Hosono conjecture goes beyond the determination of the prepotential (or the central charge), involving the monodromy properties of the hypergeometric components and a concrete determination of the mirror map at list at the K–theoretical level, and partial information on the homological mirror map Mir. The multiple resolutions of our example, corresponding to a single mirror family, are related by flop transformations and must be related by Fourier–Mukay transforms at the level of derived categories (see section 3.2). In this contest it could be helpful to find the solutions of our GKZ system in the full B-moduli space using the approach of [ABK, BrTa]. This is the deepest aspect of the conjecture and the most interesting development of the present work.

Appendix A

The cohomology valued hypergeometric series

Here we compute the coefficient hypergeometric series.

A.1 Computation of the coefficients

First note that $J_i^3 = 0$ so that we need the terms up to order two. Also at order zero is survives only the term with $\vec{m} = \vec{0}$, because for non positive integer argument the Γ function diverges. Thus, at order zero $w = 1$.

A.1.1 Some properties of the Gamma function

The Euler Gamma function has integral representation

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \ , \qquad Re(0) > 0 \ , \tag{A.1.1}$$

and admits analytical continuation to the whole complex plane excluding the non positive integers. Indeed it admit the very useful Weierstrass representation

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \ , \tag{A.1.2}$$

where

$$\gamma = \lim_{N\to\infty} \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \tfrac{1}{N} - \log(N+1)\right) \sim 0.5772156649\ldots \tag{A.1.3}$$

is the Euler-Mascheroni constant.
From (A.1.2) it follows easily the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma(z+1/2) \ . \tag{A.1.4}$$

Another useful function is the Psi function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \ . \tag{A.1.5}$$

From (A.1.2)

$$\psi(z) = -\gamma - \tfrac{1}{z} + \textstyle\sum_{n=1}^{\infty} \tfrac{z}{n(n+z)} \qquad \Rightarrow \qquad \psi'(z) = \textstyle\sum_{n=0}^{\infty} \tfrac{1}{(z+n)^2} = \zeta_z(2) \ . \tag{A.1.6}$$

Here $\zeta_a(z) = \zeta(a;z)$ is the usual Hurwitz Zeta function. In particular $\psi'(1) = \pi^2/6$ and, if N is a non negative integer, using these relations we have

1.
$$\frac{1}{\Gamma(N+1)} = \frac{1}{N!} \ , \qquad \frac{1}{\Gamma(-N)} = 0 \ ;$$

2.
$$\partial_\rho \frac{1}{\Gamma(1+N+a\rho)}\bigg|_{\rho=0} = \frac{a}{N!}\psi(N+1) = \frac{a}{N!}\left(\gamma - 1 - \frac{1}{2} - \ldots - \frac{1}{N}\right)$$

   if $N \neq 0$, and

$$\partial_\rho \frac{1}{\Gamma(1+a\rho)}\bigg|_{\rho=0} = a\gamma \ ;$$

   also

$$\partial_\rho \frac{1}{\Gamma(-N+\rho)}\bigg|_{\rho=0} = (-1)^N N! \ .$$

3.
$$\partial_\rho\partial_\sigma \frac{1}{\Gamma(1+N+a\rho+b\sigma)}\bigg|_{\rho=\sigma=0} = \frac{ab}{N!}(\psi(N+1)^2 - \psi'(N+1)) \ ;$$

   In particular for $N=0$

$$\partial_\rho\partial_\sigma \frac{1}{\Gamma(1+a\rho+b\sigma)}\bigg|_{\rho=\sigma=0} = ab\left(\gamma^2 - \frac{\pi^2}{6}\right) \ ;$$

4.
$$\partial_\rho\partial_\sigma \frac{1}{\Gamma(-N+a\rho+b\sigma)}\bigg|_{\rho=\sigma=0} = -2ab(-1)^N N! \psi(N+1) \ .$$

A.1.2 Order one

To compute the coefficients at order one, we can distinguish three cases:

- the derivative acts on the numerator. This gives a term $\log x$ only, because the sum contributes only with the term $\vec{m} = 0$;

- the derivative acts on a factor of the form $1/\Gamma(N+1)$. Then the remaining factors force again $\vec{m}$ to zero and by (2) we see that it contribute with a factor $a\gamma$. There is one such factor for any Gamma factor, and all sum up to zero. This is due to the fact that for any fixed ℓ the sum of its components vanishes;

- the main contributions come out when the derivative act on a factor $1/\Gamma(-N+1)$. In this case such factor does not contribute to limiting the allowed values for $\vec{m}$, which is no more constrained to zero.

In conclusion, all the results can be expressed in terms of the following functions

$$\Psi(x) = \sum_{n=1}^{\infty} \frac{(2n-1)!}{(n!)^2} x^n \ , \tag{A.1.7}$$

$$\Phi(x,y) = \sum_{\substack{(m,k)\in\mathbb{Z}_{\geq} \\ (m,k)\neq(0,0)}} \frac{(2k+3m-1)!}{m!(k+2m)!k!} (-x)^m y^{k+2m} \ , \tag{A.1.8}$$

$$\aleph(\vec{x}) = -\sum_{\substack{\vec{n}\in\mathbb{Z}^4_{\geq} \\ \vec{n}\neq 0}} \frac{(6n_1+4n_2+2n_3+3n_4-1)!}{n_1!n_2!n_3!n_4!(2n_1+n_2+n_4)!(3n_1+2n_2+n_3+n_4)!} \cdot \\ \cdot (x_1x_2^2x_3^3x_4^4)^{n_1}(x_2x_3^2x_4^2)^{n_2}(x_3x_4)^{n_3}(-x_2x_3x_4^2)^{n_4}. \tag{A.1.9}$$

A.1.3 Order two

The second order term is obtained applying the second order operator

$$O_2 = \frac{1}{2}\partial^2_{\rho_3} + \partial^2_{\rho_4} + \partial^2_{\rho_2\rho_3} + \partial^2_{\rho_2\rho_4} + 2\partial^2_{\rho_3\rho_4} \tag{A.1.10}$$

at $\vec{\rho} = 0$. In this case there are several contributions[1]:

- both derivatives acts on the numerator in the terms of the series. This gives rise to terms of the form
$$(\log x_i)^2 \ , \qquad \log x_i \log x_j \ ;$$
- one derivative acts on the numerator and the other one acts on the Gamma factors. This gives terms of the form
$$\log x_i w_j^{(1)} \ ,$$
where $w_j^{(1)}$ is one of the first order terms computed before;
- both derivatives acts on two regular Gamma factors. These can be two distinct factors or the same factor. In both the cases it contributes only the $\vec{m} = 0$ term. For two distinct factors we will find a contribution proportional to γ^2 and for the same factor one finds a term proportional to $\gamma^2 - \pi^2/6$. A simple argumentation similar to the first order case shows that the terms in γ^2 sum up to zero. Thus we expect only a term proportional to π^2;
- one derivative acts on a regular term and the other one acts on a singular term. This gives rise to a contribution very similar to (A.1.7), (A.1.8), where the terms of the series are corrected by a multiplicative factor of the form $\psi(N+1)$;
- both derivatives acts on the same singular term. This gives a contribution very similar to the previous point;
- the derivatives acts on two distinct singular Gamma terms. These give the more complicated series, because there are minimal constrictions for the range of $\vec{m}$ in the sums.

[1] for simplicity we will call *regular* the Gamma factors with positive argument, and *singular* the Gamma factors with negative argument;

A.1.4 Second order functions

Here we collect the functions which appear in the second order terms of the cohomological hypergeometric functions. When the ranges $\vec{m} \in \mathbb{Z}^4_>$ are intended to be restricted to the subsets where all factorials and psi functions are well defined. The results can then be expressed in terms of the following twenty six functions

$$\Psi_1(x,y) = \sum_{\substack{(m,k)\in\mathbb{Z}_\geq \\ (m,k)\neq(0,0)}} \frac{(2k+3m-1)!}{m!(k+2m)!k!}[\psi(2k+3m)-\psi(1)](-x)^m y^{k+2m}\ ; \tag{A.1.11}$$

$$\Psi_2(x,y) = \sum_{\substack{(m,k)\in\mathbb{Z}_\geq \\ (m,k)\neq(0,0)}} \frac{(2k+3m-1)!}{m!(k+2m)!k!}[\psi(1+k+2m)-\psi(1)](-x)^m y^{k+2m}\ ; \tag{A.1.12}$$

$$\Psi_3(x,y) = \sum_{\substack{(m,k)\in\mathbb{Z}_\geq \\ (m,k)\neq(0,0)}} \frac{(2k+3m-1)!}{m!(k+2m)!k!}[\psi(1+k)-\psi(1)](-x)^m y^{k+2m}\ ; \tag{A.1.13}$$

$$\Psi_4(x) = \sum_{m=1}^{\infty} \frac{(2m-1)!}{(m!)^2}[\psi(2m)-\psi(1)]x^m\ ; \tag{A.1.14}$$

$$\Psi_5(x) = \sum_{m=1}^{\infty} \frac{(2m-1)!}{(m!)^2}[\psi(m+1)-\psi(1)]x^m\ ; \tag{A.1.15}$$

$$\Psi_6(x,y) = \sum_{\substack{(m,n)\in\mathbb{Z}^2_\geq \\ 2n-m>0 \\ 2m-n>0}} \frac{(2n-m-1)!(2m-n-1)!}{m!n!}(-x)^m(-y)^n\ ; \tag{A.1.16}$$

$$\aleph^{(i)}(\vec{x}) = \sum_{\substack{\vec{n}\in\mathbb{Z}^4_\geq \\ \vec{n}\neq 0}} \frac{(x_1x_2^2x_3^3x_4^4)^{n_1}(x_2x_3^2x_4^2)^{n_2}(x_3x_4)^{n_3}(-x_2x_3x_4^2)^{n_4}}{n_1!n_2!n_3!n_4!(2n_1+n_2+n_4)!(3n_1+2n_2+n_3+n_4)!}\chi^{(i)}_{\vec{n}}\ , \tag{A.1.17}$$

$i = 1,\dots,7;$

$$\chi^{(1)}_{\vec{n}} = (6n_1+4n_2+2n_3+3n_4-1)![\psi(6n_1+4n_2+2n_3+3n_4)-\psi(1)]; \tag{A.1.18}$$

$$\chi^{(2)}_{\vec{n}} = (6n_1+4n_2+2n_3+3n_4-1)![\psi(1+3n_1+2n_2+n_3+n_4)-\psi(1)]; \tag{A.1.19}$$

$$\chi^{(3)}_{\vec{n}} = (6n_1+4n_2+2n_3+3n_4-1)![\psi(1+2n_1+n_2+n_4)-\psi(1)]; \tag{A.1.20}$$

$$\chi^{(4)}_{\vec{n}} = (6n_1+4n_2+2n_3+3n_4-1)![\psi(1+n_2)-\psi(1)]; \tag{A.1.21}$$

$$\chi^{(5)}_{\vec{n}} = (6n_1+4n_2+2n_3+3n_4-1)![\psi(1+n_4)-\psi(1)]; \tag{A.1.22}$$

$$\chi^{(6)}_{\vec{n}} = (6n_1+4n_2+2n_3+3n_4-1)![\psi(1+n_1)-\psi(1)]; \tag{A.1.23}$$

$$\chi^{(7)}_{\vec{n}} = (6n_1+4n_2+2n_3+3n_4-1)![\psi(1+n_3)-\psi(1)]; \tag{A.1.24}$$

$$\Lambda_1(\vec{x}) = \sum_{\substack{\vec{n} \in \mathbb{Z}^4_{\geq} \\ n_1+n_3 \neq 0}} \frac{n_1!n_2!n_3!n_4!(n_1+2n_2+n_3+n_4)!}{(n_1+3n_2+2n_4)!(n_1+2n_3)!x_1^{n_1}} \cdot \\ \cdot (-x_1^2x_2x_4^2)^{n_2}(x_1x_3)^{n_3}(x_1x_4)^{n_4} ; \tag{A.1.25}$$

$$\Lambda_2(\vec{x}) = \sum_{\vec{m}\in\mathbb{Z}^4_{>}} \frac{(m_1-m_2-m_3+m_4-1)!(m_3+m_4-m_1-1)!}{m_1!m_2!m_3!(m_4-2m_2)!(m_1-m_3-m_4)!} \cdot \\ \cdot x_1^{m_1}(-x_2)^{m_2}x_3^{m_3}x_4^{m_4} ; \tag{A.1.26}$$

$$\Lambda_3(\vec{x}) = \sum_{\vec{m}\in\mathbb{Z}^4_{>}} \frac{(m_1+m_3-m_4-1)!(m_3+m_4-m_1-1)!}{m_1!m_2!m_3!(m_2+m_3-m_1-m_4)!(m_4-2m_2)!} \cdot \\ \cdot x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}; \tag{A.1.27}$$

$$\Lambda_4(\vec{x}) = \sum_{\vec{m}\in\mathbb{Z}^4_{>}} \frac{(m_1-m_3-m_4-1)!(m_1-1)!}{m_2!m_3!(m_1+m_2-2m_4)!(m_4-2m_2)!(m_1-2m_3)!} \cdot \\ \cdot x_1^{m_1}x_2^{m_2}(-x_3)^{m_3}(-x_4)^{m_4}; \tag{A.1.28}$$

$$\Lambda_5(\vec{x}) = \sum_{\vec{m}\in\mathbb{Z}^3_{>}} \frac{(2m_3-m_1-1)!(2m_2-1)!}{m_1!m_2!(m_2+m_3)!(m_3-2m_1)!} \cdot \\ \cdot (-x_2)^{m_1}x_3^{m_2}(-x_4)^{m_3}; \tag{A.1.29}$$

$$\Lambda_6(\vec{x}) = \sum_{\vec{m}\in\mathbb{Z}^4_{>}} \frac{(m_3-m_4-m_1-1)!(m_3-m_2-1)!}{m_1!m_2!(m_4-2m_2)!(m_3-2m_2)!(m_3-2m_4)!} \cdot \\ \cdot (-x_1)^{m_1}(-x_2)^{m_2}x_3^{m_3}(-x_4)^{m_4}; \tag{A.1.30}$$

$$\Lambda_7(\vec{x}) = \sum_{\vec{m}\in\mathbb{Z}^4_{>}} \frac{(m_3-m_2-1)!(2m_4-m_3-1)!}{m_1!m_2!(m_1-m_3+m_4)!(m_4-2m_2)!(m_3-2m_2)!} \cdot \\ \cdot x_1^{m_1}(-x_2)^{m_2}x_3^{m_3}x_4^{m_4}; \tag{A.1.31}$$

$$\Lambda_8(\vec{x}) = \sum_{\vec{m}\in\mathbb{Z}^4_{>}} \frac{(m_3-m_2-1)!(2m_2-m_4-1)!}{m_1!m_2!(m_1-m_3+m_4)!(m_3-2m_2)!(m_3-2m_4)!} \cdot \\ \cdot x_1^{m_1}(-x_2)^{m_2}(-x_3)^{m_3}(-x_4)^{m_4}; \tag{A.1.32}$$

$$\Lambda_9(\vec{x}) = \sum_{\vec{m}\in\mathbb{Z}^4_{>}} \frac{(m_4-m_3+m_2-m_1-1)!(2m_3+m_2-m_4-1)!}{m_1!m_2!m_3!(m_4-2m_1)!(m_2-m_4)!} \cdot \\ \cdot (-x_1)^{m_1}x_2^{m_2}(-x_3)^{m_3}x_4^{m_4}; \tag{A.1.33}$$

$$\Lambda_{10}(\vec{x}) = \sum_{\vec{m}\in\mathbb{Z}^4_{>}} \frac{(m_4-m_3+m_2-m_1-1)!(m_4-m_2-1)!}{m_1!m_2!m_3!(m_4-2m_1)!(m_4-2m_3-m_2)!} \cdot \\ \cdot (-x_1)^{m_1}x_2^{m_2}(-x_3)^{m_3}x_4^{m_4}; \tag{A.1.34}$$

$$\Lambda_{11}(\vec{x}) = \sum_{\vec{m}\in\mathbb{Z}^4_>} \frac{(m_4-m_2-1)!(2m_3+m_2-m_4-1)!}{m_1!m_2!m_3!(m_4-2m_1)!(m_1+m_3-m_2-m_4)!} \cdot x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}; \tag{A.1.35}$$

$$\Lambda_{12}(\vec{x}) = \sum_{\vec{m}\in\mathbb{Z}^4_>} \frac{(m_2-m_3-m_4-1)!(m_2-m_3-1)!}{m_1!m_3!(m_4-2m_1)!(m_1+m_2-2m_4)!(m_2-3m_3)!} \cdot x_1^{m_1}x_2^{m_2}x_3^{m_3}(-x_4)^{m_4}; \tag{A.1.36}$$

$$\Lambda_{13}(\vec{x}) = \sum_{\vec{m}\in\mathbb{Z}^4_>} \frac{(m_2-m_3-1)!(3m_3-m_2-1)!}{m_1!m_3!(m_3+m_4-m_2)!(m_4-2m_1)!(m_1+m_2-2m_4)!} \cdot x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}. \tag{A.1.37}$$

Part II

The world on a D3 brane

Chapter 6

The Verlinde and Wijnholt model

6.1 Bottom-up string phenomenology

String theory is the most famous theoretical attempt to unify quantum mechanics and general relativity furnishing a predictive and testable theory of particle physics. To meet this dual challenge, string phenomenology traditionally adopts a "top-down" point of view, which aims to construct realistic compactification scenarios starting from the full 10-dimensional closed string theory, possibly augmented with one or more D-branes [GSW][Po]. By simultaneously controlling and scanning both the string scale geometry and the low energy field theory, one hopes to isolate realistic backgrounds that meet all consistency requirements at both ends. Actually the vast collection of potentially stabilized string vacua, has strengthened the belief that such consistent backgrounds indeed exist [KKLT], possibly even in abundance [Dou2]. Finding a single one of them, however, still seems a too far challenging task at present.

Recently the better understanding of string dualities and D-brane physics supported theoretical physicists to develop tools for building 4-d supersymmetric field theories in the open string theory contest. In particular, it is now realized that by taking a judicious low energy limit of the world-volume theory on N D3-branes, one recovers a purely 3+1-dimensional gauge theory, decoupled from gravity and higher dimensional dynamics [Mald]. In this decoupling limit, the closed string background gets frozen into a set of non-dynamical, and thus largely tunable, gauge invariant couplings. By placing one or more D-branes near various types of geometric singularities, realizations of large classes of gauge theories have been uncovered. In this contest are included the models which we studied in the first part of this thesis. Now the question to reproduce the spectrum and couplings within phenomenological bounds in the open string theory landscape is less ambitious than finding a fully realistic closed string compactification but it is easier to answer and it could be a first step in the study of the closed string theory landscape.

The exponential separation between the TeV scale of particle physics and the Planck scale of quantum gravity guides the bottom-up approach to string phenomenology. Warped compactification [GKP, DG] provide a geometric implementation of the gauge hierarchy, via the assumption that the whole low energy physics takes place in a highly red-shifted region of the internal geometry. This geometric viewpoint thus naturally places the Standard Model on a world-brane near the apex of a warped throat. In the low energy worldbrane physics one can isolate the high energy closed string dynamics, by taking a decoupling limit in which the Planck scale is sent off to infinity. In geometric terms, this limit replaces the finite warped throat region by an infinite, non-compact Calabi-Yau singularity.

Due to the interaction with the ambient geometry, a D3-brane on a CY singularity breaks up into various fractional branes. As a result, its world-volume theory takes the non-trivial form of a quiver gauge theory: it has

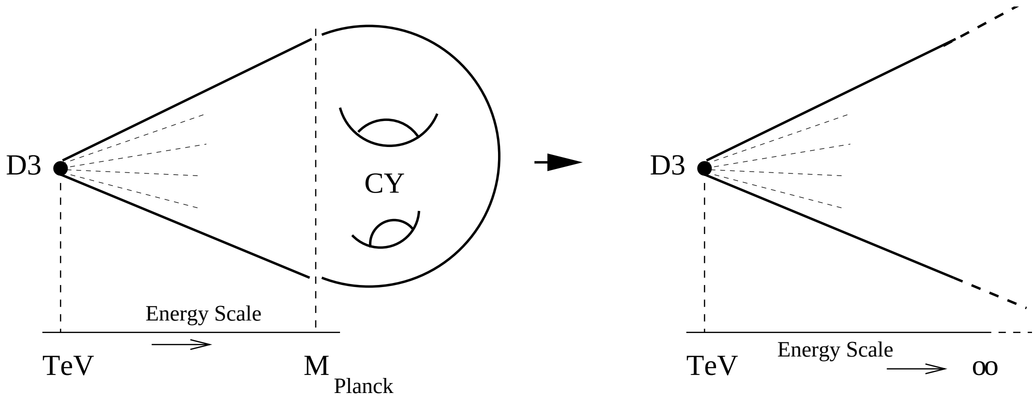

Figure 6.1: The bottom-up approach to string phenomenology assumes that the Standard Model is localized on a D3-brane at the apex of a highly warped throat. The D3-brane theory is accurately described via a decoupling limit, in which the Planck scale is sent off to infinity, leaving behind a non-compact Calabi-Yau singularity.

one $U(n_i)$ gauge multiplet for each constituent fractional brane and bifundamental chiral matter associated with each brane intersection (see for example [DoMo, Asp]). In the decoupling limit, the gauge invariant coupling constants of this quiver gauge theory are determined by non-dynamical asymptotic boundary conditions on the closed string fields, and can thus be viewed as continuously tunable parameters.

As we said realizations of large classes of such quiver gauge theories have been uncovered. At the moment the better explicit realization of the supersymmetric Standard Model as the world-volume theory on D3-branes has been supplied by Verlinde and Wijnholt in their celebrated work [VeWi].

6.2 Building the Standard Model on a $D3$-brane

6.2.1 The minimal quiver extension of the Standard Model

A quiver gauge theory is a quantum field theory in which many relevant informations on matter and interactions are encoded in a quiver, a oriented graph satisfying few simple rules (see [VeWi, MaVe]). In figure 6.2 we report the quiver defining the minimal quiver extension of the minimal supersymmetric Standard Model (MQSSM) proposed in [VeWi]:

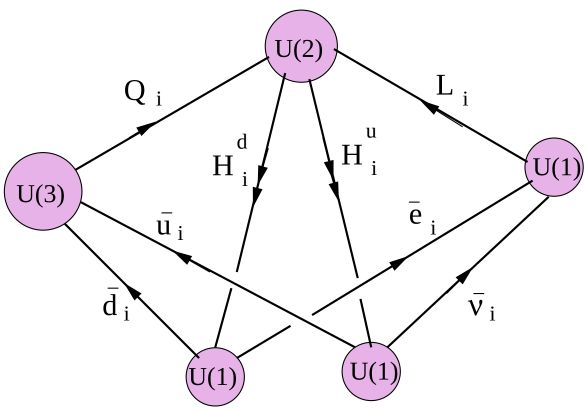

Figure 6.2: The minimal quiver extension of the Standard Model.

Each node of the quiver represents a gauge multiplet with $U(k)$ gauge symmetry, while each oriented line between two node represents a bi-fundamental chiral multiplet. The MQSSM gauge theory has the Standard Model gauge group, $SU(3)_C \times SU(2)_L \times U(1)_Y$, and matter content, three families of quarks and leptons with all the right charges, plus a somewhat extended Higgs sector. The presence of extra $U(1)$ factors as well as extra Higgs fields is characteristic of many string theoretic models. Both are acceptable extensions of the Standard Model, provided the masses and couplings are tuned to satisfy the appropriate phenomenological bounds.

6.2.2 D3-brane on dP_8 singularities and on $\mathbb{C}^3/\Delta_{27}$ orbifold

In [VeWi] the authors try to find the geometric dual of the above quiver gauge theory. They start identifying a class of Calabi-Yau singularities such that the probe D3-brane theory is just large enough to contain the MQSSM and that has a space of couplings and vacua rich enough to allow the necessary tuning. They concentrate on the del Pezzo 8 singularities, the local Calabi-Yau varieties obtained as the complex cones over the compact surfaces dP_8, i.e. the projective plane $\mathbb{P}^2$ blown up into 8 points (we recall some general properties of del Pezzo surfaces in chapter 9). Let X be one such del Pezzo cone and consider D3-brane configurations that fill the 3+1 flat directions, and therefore are localized at a point in X. In the strongly curved background at the tip of the cone, the D3-brane will typically split into several fractional branes that wrap vanishing cycles in X.

As far as the F-terms is concerned, one may blow up the vanishing cycles and perform computations in the large volume limit. From a large volume perspective, the geometric characterization of a fractional brane is as a sheaf $\mathbb{F}_i$, which one can think of as a bundle supported on the collapsing del Pezzo surface. The RR-charges of a sheaf $\mathbb{F}_i$ are combined in the charge vector

$$\mathrm{ch}(\mathbb{F}_i) = (\mathrm{rk}(\mathbb{F}_i), c_1(\mathbb{F}_i), \mathrm{ch}_2(\mathbb{F}_i)),$$

which specifies the (D7,D5,D3) charge of $\mathbb{F}_i$. The D3-brane itself is naturally represented as a sky-scraper sheaf O_p localized at a single point p. It splits up in a collection of fractional branes $\mathbb{F}_i$, each with integer multiplicities n_i, such that the charge vectors of all fractional branes add up to that of a single D3-brane

$$\sum_i n_i \,\mathrm{ch}(\mathbb{F}_i) = \mathrm{ch}(O_p) = (0, 0, 1)$$

Each type of fractional brane, with multiplicity n_i, contributes to the total gauge group of the world-volume theory with a $U(|n_i|)$ factor. The corresponding $\mathcal{N} = 1$ gauge multiplet is furnished by the lightest modes of open strings with end-points on the same type of fractional brane. The massless spectrum of open strings that stretch between two different types of fractional branes $\mathbb{F}_i$ and $\mathbb{F}_j$ represent chiral multiplets that transform in the bi-fundamental representation of the corresponding $U(|n_i|) \times U(|n_j|)$ gauge group. In the case when the branes are space filling, i.e. have support on the whole Calabi-Yau, the massless modes correspond to elements of the cohomology of the Dolbault operator acting on the space of bi-fundamental valued anti-holomorphic forms, $\Omega^{(0,\cdot)}(\mathbb{F}_i^*, \mathbb{F}_j)$. The number of bi-fundamental fields is therefore counted by the proper generalization to sheaves of the cohomology group $H^{(0,\cdot)}(\mathbb{F}_i^* \otimes \mathbb{F}_j)$, the Ext groups $\mathrm{Ext}^k(\mathbb{F}_i, \mathbb{F}_j)$. Since our fractional branes are not space-filling, we instead need to distinguish between a sheaf $\mathbb{F}$, living on the del Pezzo 4-cycle $\mathcal{B}$, and the associated push-forward $i_*\mathbb{F}$ on the Calabi-Yau $\mathcal{X}$, which can be thought of as F extended by zero on $\mathcal{X}$. One concludes that for each generator of $\mathrm{Ext}^p_{\mathcal{B}}(\mathbb{F}_j, \mathbb{F}_k)$, one has exactly one chiral field in four dimensions. These are the rules for obtaining the quiver data for a given basis of fractional branes.

The quiver gauge theory for a dP_8 singularity has been constructed for a so called exceptional collection basis [KaNo]. The resulting quiver diagram is given in figure 6.3.

It has been observed in [VeWi] that this quiver is identical to that of the D3-brane theory on the $\mathbf{C}^3/\Delta_{27}$

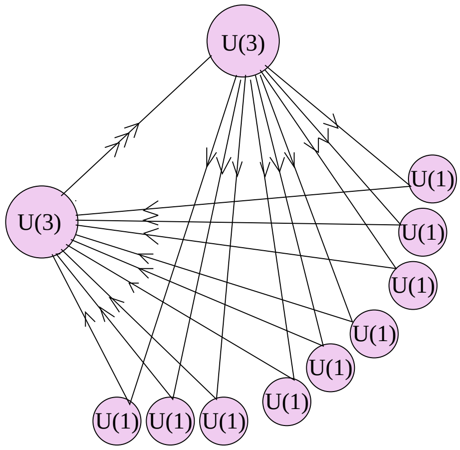

Figure 6.3: Quiver diagram of the D3-brane gauge theory on a del Pezzo 8 singularity.

orbifold singularity [BJL]. The authors conjecture that actually the dP_8 singularities should be viewed as a deformation of this orbifold. A support to their hypothesis is the following. The discrete group Δ_{27} is the non-abelian subgroup of $SL(3,\mathbb{C})$ generated by two elements:

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} , \qquad g_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} ,$$

with $\omega = e^{\frac{2\pi i}{3}}$. The following combinations of coordinates

$$\begin{aligned} x &= XYZ \\ y &= (X^3 + \omega Y^3 + \omega^2 Z^3)(X^3 + \omega^2 Y^3 + \omega Z^3) \\ z &= X^3 + Y^3 + Z^3 \\ w &= (X^3 + \omega Y^3 + \omega^2 Z^3)^3 \end{aligned} \tag{6.2.1}$$

are all invariant under the action of Δ_{27}. Actually they define a map from the orbifold to the tautological bundle over the weighted projective space $\mathbb{P}_{1,1,2,3}$. The coordinates (x,y,z,w) defined in (6.2.1) satisfy the homogeneous equation

$$w^2 + y^3 - 27wx^3 + wz^3 - 3wyz = 0.$$

Thus the orbifold $\mathbb{C}^3/\Delta_{27}$ is isomorphic to the cone over such singular hypersurface in $\mathbb{P}_{1,1,2,3}$. Since it is well known that all smooth hypersurface in $\mathbb{P}_{1,1,2,3}$ are isomorphic to a dP_8 surface, this is a strong evidence for the conjecture. As we anticipate in the introduction of the thesis, in the following chapters we clarify the details of this problem giving the exact correspondence between the two geometries.

6.2.3 The symmetry breaking process toward the MQSSM

The quiver gauge theory of figure 6.3 is quite different from the MQSSM of figure 6.2, but now it is possible to choose a suitable symmetry breaking process such that at low energy they look very similar. For completeness we briefly resume the principal steps of this procedure defined in [VeWi, BMMVW].

Preliminarily we have to change the quiver gauge theory by a duality process. For a given geometrical singularity, there are in principle many different exceptional collections of fractional branes. The allowed choices are typically inequivalent, and in particular lead to different world-volume gauge theories on the probe D3-brane. There exists a simple transitive set of transformations on the space of exceptional collections, known as mutations. A useful subclass of mutations has the physical interpretation of Seiberg duality [Se, CFIKV]: the $\mathcal{N}=1$ supersymmetric gauge theories corresponding to the original and mutated set of fractional branes are each others Seiberg dual. For a given singularity, the question of which of the dual descriptions is most appropriate is determined by the value of the geometric moduli that determine the gauge theory couplings.
Geometrically, such transformation can be recognized as a Picard-Lefschetz monodromy around a conifold point. The resulting quiver theory lives at a locus in Kähler moduli space where the del Pezzo surface has shrunk to zero size, but where string perturbation theory is still applicable. There are other places in Kähler moduli space where some cycle has shrunk to zero size and string perturbation theory breaks down – these are generalized conifold points. We could imagine to draw a loop in moduli space starting from the point where the conformal quiver theory lives, and going around a conifold point, where the central charge of a given fractional brane $\mathbb{F}_i$ vanishes. From the point of view of the worldvolume theory, the change in Kähler parameters translates into a change in the gauge coupling of the $U(|n_i|)$ gauge group. As we go around the loop, this gauge coupling is pushed through strong coupling, and we have to do a Seiberg duality on the i-th node.
In [VeWi] the authors apply a particular Seiberg duality on the quiver theory of figure 6.3 which changes one of the $U(3)$ gauge groups to a $U(6)$ group, and then they use such dual quiver theory as the starting point in the symmetry breaking process.

To effectuate the symmetry breaking to MQSSM, while preserving $\mathcal{N}=1$ supersymmetry, it is necessary to turn on a suitable set of F–I parameters and tune the superpotential W. The D-term and F-term equations can then both be solved, while dictating expectation values that result in the desired symmetry breaking pattern. As first discussed in [MoPl] (in the context of $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold), when a Calabi–Yau singularity is not isolated, the moduli space of D-branes on that Calabi–Yau has more than one branch. From a non-isolated singularity, several curves Γ_i of singularities can meet, each one having a generic singularity of ADE type R_i. In this non-isolated case, on one of the branches of the moduli space, the branes move freely on the Calabi–Yau or its (partial) resolution, and the F–I parameters are identified with “blowup modes” which specify how much blowing up is done. But there are additional branches of the moduli space associated with each Γ_i: on such a branch, the F–I parameters which would normally be used to blow up the ADE singularity R_i are frozen to zero, and new parameters arise which correspond to positions of R_i-fractional branes along Γ_i. That is, on this new branch, some of the R_i fractional branes have moved out along the curve Γ_i and their positions give new parameters.
The strategy deligned in [VeWi] to obtain a semirealistic gauge theory is the following: by appropriately tuning the superpotential (i.e., varying the complex structure) one should find a Calabi–Yau with a non-isolated singularity—a curve Γ of A_2 singular points—such that two particular curves have been blown down to an A_2 singularity on the generalized del Pezzo surface where it meets the singular locus. The symmetry-breaking involves moving onto the Γ branch in the moduli space, where the fractional brane classes wrapped to the blown down curves are free to move along the curve Γ of A_2 singularities. In particular, these branes can be taken to be very far from the primary singular point of interest, and become part of the bulk theory: any effect which they have on the physics will occur at very high energy like the rest of the bulk theory. Making this choice removes such branes from the original brane spectrum, and replaces other branes in the spectrum by bound states which are independent of them. The remaining bound state basis of the fractional branes gives a theory with the gauge group $U(3)\times U(2)\times U(1)^t$.

Finally this last theory has an excess of Higgs fields and several extra $U(1)$-factors. In [BMMVW] it has been proposed a geometrical way to give a so called Stückelberg mass to this factors and therefore to eliminate them from the low energy spectrum. We remember that in the bottom-up approach one works in a local contest, where a non compact variety X gives a description of the compact Calaby Yau threefold Y only near the singularity where D-branes live. Now we note that the harmonic forms on the compact CY manifold Y, when restricted to base S of the singularity X, in general do not span the full cohomology of S. For instance, the 2-cohomology of Y may have fewer generators than that of S, in which case there must be one or more 2-cycles that are non-trivial within S but trivial within Y. Conversely, Y may have non-trivial cohomology elements that restrict to trivial elements on S. This incomplete overlap between the two cohomologies has immediate repercussions for the D-brane gauge theory, since it implies that the compact embedding typically reduces the space of gauge invariant couplings. The couplings are all period integrals of certain harmonic forms, and any reduction of the associated cohomology spaces reduces the number of allowed deformations of the gauge theory. As a particular result one can show that such topological obstruction gives a Stückelberg mass to all $U(1)$ vector bosons, except for the ones that correspond to fractional branes that wrap 2-cycles of S that are trivial within Y.

This insight has been applied in [BMMVW] to eliminate the six extra $U(1)$ gauge sector of our model. It emerges that the singularity X needs a compactification Y in which all the two cycles of the del Pezzo surface remain non-trivial except a particular one.

Chapter 7

The Heisenberg group Γ and its action on $\mathbb{C}^3$

7.1 The Heisenberg group Γ

The matrices group $\Gamma := \Delta_{27} \subset SL(3,\mathbb{C})$ has two generators:

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

with $\omega = e^{\frac{2\pi i}{3}}$. They satisfy the relations:

$$g_1^3 = g_2^3 = I\,, \qquad g_2 g_1 = \omega^2 g_1 g_2.$$

The center of Γ, its maximal abelian subgroup, is

$$C = \{I, \omega I, \omega^2 I\}.$$

The abelianization of Γ is $\tilde{\Gamma} := \Gamma/C \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$. Thus Γ is the Heisenberg group of order 27, i.e. the non abelian central extension of the group $\mathbb{Z}_3 \times \mathbb{Z}_3$.

7.2 The quotient $\mathbb{P}^2/\tilde{\Gamma}$ and its relation with $\mathbb{C}^3/\Gamma$

The group $\tilde{\Gamma}$ is also the image of Γ in $PGL(3,\mathbb{C})$, the group of automorphisms of the projective plane. It has a natural action on $\mathbb{P}^2$ which is strictly related to the one of Γ on $\mathbb{C}^3$. $\tilde{\Gamma}$ has four cyclic subgroup isomorphic to

$\mathbb{Z}_3$:

$$< g_1 >= \{g_1, g_1^2, I\} \qquad g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

$$< g_2 >= \{g_2, g_2^2, I\} \qquad g_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$< g_1g_2 >= \{g_1g_2, g_1^2g_2^2, I\} \qquad g_1g_2 = \begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix},$$

$$< g_1^2g_2 >= \{g_1^2g_2, g_1g_2^2, I\} \qquad g_1^2g_2 = \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}.$$

Each subgroup is abelian and therefore its elements have common eigenspaces in $\mathbb{C}^3$.

$$g_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad g_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \omega \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad g_1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \omega^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$g_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad g_2 \cdot \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} = \omega \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}, \qquad g_2 \cdot \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} = \omega^2 \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix},$$

$$g_1g_2 \cdot \begin{pmatrix} 1 \\ \omega \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \omega \\ 1 \end{pmatrix}, \qquad g_1g_2 \cdot \begin{pmatrix} 1 \\ 1 \\ \omega \end{pmatrix} = \omega \begin{pmatrix} 1 \\ 1 \\ \omega \end{pmatrix}, \qquad g_1g_2 \cdot \begin{pmatrix} \omega \\ 1 \\ 1 \end{pmatrix} = \omega^2 \begin{pmatrix} \omega \\ 1 \\ 1 \end{pmatrix},$$

$$g_1^2g_2 \cdot \begin{pmatrix} 1 \\ \omega^2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \omega^2 \\ 1 \end{pmatrix}, \qquad g_1^2g_2 \cdot \begin{pmatrix} \omega^2 \\ 1 \\ 1 \end{pmatrix} = \omega \begin{pmatrix} \omega^2 \\ 1 \\ 1 \end{pmatrix}, \qquad g_1^2g_2 \cdot \begin{pmatrix} 1 \\ 1 \\ \omega^2 \end{pmatrix} = \omega^2 \begin{pmatrix} 1 \\ 1 \\ \omega^2 \end{pmatrix}.$$

To any eigenspace it corresponds a fixed point of $\tilde{\Gamma}$ in $\mathbb{P}^2$. These are the only points in the projective plane with non-trivial stabilizer in $\tilde{\Gamma}$.

$$\begin{array}{llll} \mathrm{Fix}(< g_1 >): & (1:0:0), & (0:1:0), & (0:0:1), \\ \mathrm{Fix}(< g_2 >): & (1:1:1), & (1:\omega^2:\omega), & (1:\omega:\omega^2), \\ \mathrm{Fix}(< g_1g_2 >): & (1:\omega:1), & (1:1:\omega), & (\omega:1:1), \\ \mathrm{Fix}(< g_1^2g_2 >): & (1:\omega^2:1), & (\omega^2:1:1), & (1:1:\omega^2). \end{array}$$

Locally at each fixed point the action of the stabilizer subgroup is given by $diag(\omega, \omega^2)$. This is very easy to see for the fixed points of g_1. For example near $(0:0:1)$ we can use the local chart:

$$\begin{aligned} \mathbb{P}^2 &\longrightarrow \mathbb{C}^2 \\ (X:Y:Z) &\longmapsto \left(u = \frac{X}{Z}, v = \frac{Y}{Z}\right). \end{aligned}$$

Thus the local action of g_1 is

$$g_1 \cdot (u,v) = (\omega u, \omega^2 v).$$

It follows that the singular locus of the quotient surface $\mathbb{P}^2/\tilde{\Gamma}$ consists of four singular points of type A_2, i.e. near the singular points the surface is isomorphic to the orbifold $\mathbb{C}^2/\mathbb{Z}_3$.

The original orbifold $\mathbb{C}^3/\Gamma$ is the cone over the singular surface $\mathbb{P}^2/\tilde{\Gamma}$ and in particular the four singular points correspond to four lines passing through the origin, which in parametric form are:

$$\begin{aligned}
\mathcal{C}_1: &\quad (t,0,0) \sim (0,t,0) \sim (0,0,t),\\
\mathcal{C}_2: &\quad (t,t,t) \sim (t,\omega^2 t,\omega t) \sim (t,\omega t,\omega^2 t),\\
\mathcal{C}_3: &\quad (t,\omega t,t) \sim (t,t,\omega t) \sim (\omega t,t,t),\\
\mathcal{C}_4: &\quad (t,\omega^2 t,t) \sim (\omega^2 t,t,t) \sim (t,t,\omega^2 t).
\end{aligned}$$

7.3 The Verlinde-Wijnholt isomorphism

In this section we prove that the map 6.2.1 is an isomorphism of algebraic varieties. At this purpose we show that this map defines a ring isomorphism:

$$\mathbb{C}[X,Y,Z]^\Gamma \simeq \frac{\mathbb{C}[x,z,y,w]}{(w^2+y^3-27wx^3-3wyz+wz^3)}.$$

We take

$$\phi^*: \quad \mathbb{C}[x,z,y,w] \longrightarrow \mathbb{C}[X,Y,Z]^\Gamma$$

the ring homomorphism associates to 6.2.1:

$$\phi^*: \quad (x,z,y,w) \longrightarrow (XYZ, X^3+Y^3+Z^3, (X^3+\omega Y^3+\omega^2 Z^3)(X^3+\omega^2 Y^3+\omega Z^3), (X^3+\omega Y^3+\omega^2 Z^3)^3)$$

As observed in [VeWi]

$$(w^2+y^3-27wx^3-3wyz+wz^3) \subset \operatorname{Ker}\phi^*.$$

Thus we have the ring homomorphism (that we call again ϕ^*)

$$\phi^*: \quad \frac{\mathbb{C}[x,z,y,w]}{(w^2+y^3-27wx^3-3wyz+wz^3)} \longmapsto \mathbb{C}[X,Y,Z]^\Gamma \tag{7.3.1}$$

This map is injective. In fact the geometrical map

$$\begin{aligned}
\phi_R: \quad \mathbb{C}^3 &\longrightarrow \mathbb{C}^3\\
(X,Y,Z) &\longmapsto (x,z,y)
\end{aligned}$$

is surjective. It is easy to see that any point of coordinates (x,z,y) with $x \neq 0$ has preimages having coordinates (X,Y,Z) the solutions of the following algebraic equations:

$$\begin{aligned}
&Y^9 - zY^6 + \frac{z^2-y}{3}Y^3 - x^3 = 0,\\
&Z^6 + (Y^3-z)Z^3 + \left(Y^6 - zY^3 + \frac{z^2-y}{3}\right) = 0,\\
&X - \frac{x}{YZ} = 0.
\end{aligned}$$

The case $x=0$ is similar. The surjectivity of ϕ_R implies that the map ϕ_R^* between the rings is injective and there are not relations involving x,z,y. Moreover the image of $\mathbb{C}^3/\Gamma$ under the geometrical map ϕ is a hypersurface in $\mathbb{C}^4$. It is contained in $(w^2+y^3-27wx^3-3wyz+wz^3)=0$ that is an irreducible hypersurface, thus the image of ϕ coincides with it. Hence ϕ^* in 7.3.1 is injective.

It remains to prove the surjectivity of 7.3.1. We first observe that

$$\frac{\mathbb{C}[x,z,y,w]}{(w^2+y^3-27wx^3-3wyz+wz^3)} \simeq \mathbb{C}[x,z,y]\oplus\mathbb{C}[x,z,y]w, \tag{7.3.2}$$

thus as vector space it admits the monomial basis

$$x^a z^b y^c w^d \qquad (a,b,c)\in\mathbb{Z}^3_{\geq 0} \quad d\in\{0,1\}.$$

Similarly the vector space $\mathbb{C}[X,Y,Z]$ has the natural monomial basis

$$X^A Y^B Z^C \qquad (A,B,C)\in\mathbb{Z}^3_{\geq 0}.$$

We grade the two rings in the following way:

$$\begin{aligned} deg(x^a z^b y^c w^d) &:= 3a+3b+6c+9d,\\ deg(X^A Y^B Z^C) &:= A+B+C. \end{aligned}$$

Thus the map 7.3.1 is factorisable in vector space morphisms

$$\phi_n^* : \left(\frac{\mathbb{C}[x,z,y,w]}{(w^2+y^3-27wx^3-3wyz+wz^3)}\right)_n \longmapsto \left(\mathbb{C}[X,Y,Z]^\Gamma\right)_n .$$

The map ϕ^* is surjective if ϕ_n^* is surjective for any n. $\mathrm{Ker}(\phi^*)=\varnothing$ implies $\mathrm{Ker}(\phi_n^*)=\varnothing$ for any n, hence

$$\dim \mathrm{Im}(\phi_n^*) = \dim\left(\frac{\mathbb{C}[x,z,y,w]}{(w^2+y^3-27wx^3-3wyz+wz^3)}\right)_n .$$

Therefore ϕ^* is surjective if

$$\dim\left(\frac{\mathbb{C}[x,z,y,w]}{(w^2+y^3-27wx^3-3wyz+wz^3)}\right)_n = \dim\left(\mathbb{C}[X,Y,Z]^\Gamma\right)_n$$

for any n. We prove such equivalence.

From 7.3.2 it follows that

$$\dim\left(\frac{\mathbb{C}[x,z,y,w]}{(w^2+y^3-27wx^3-3wyz+wz^3)}\right) = \dim\left(\mathbb{C}[x,z,y]\right)_n + \dim\left(\mathbb{C}[x,z,y]\right)_{n-9} .$$

Hence we have to calculate $\dim\left(\mathbb{C}[x,z,y]\right)_n$ that is the number of monomials $x^a z^b y^c$ with $3a+3b+6c=n$. Note that

$$\begin{aligned} \frac{1}{1-x^3t^3}\frac{1}{1-z^3t^3}\frac{1}{1-y^6t^6} &= (1+x^3t^3+\ldots+x^{3a}t^{3a}+\ldots)(1+\ldots+z^{3b}t^{3b}+\ldots)(1+\ldots+y^{3c}t^{3c}+\ldots)\\ &= \sum_{a,b,c\geq 0} x^{3a}z^{3b}y^{6c}t^{3a+3b+6c}\\ &= \sum_{n=0}^{\infty}\left(\sum_{3a+3b+3c=n} x^{3a}z^{3b}y^{6c}\right)t^n . \end{aligned}$$

Putting $x = z = y = 1$ we get the generating function for $\dim(\mathbb{C}[x,z,y])_n$:

$$\sum_{n=0}^{\infty} \dim(\mathbb{C}[x,z,y])_n t^n = \frac{1}{1-t^3}\frac{1}{1-t^3}\frac{1}{1-t^6} = \frac{1}{(1-t^3)^3}\frac{1}{1+t^3}\,.$$

Thus

$$\begin{aligned}\sum_{n=0}^{\infty} \dim\left(\frac{\mathbb{C}[x,z,y,w]}{(w^2+y^3-27wx^3-3wyz+wz^3)}\right)t^n &= \sum_{n=0}^{\infty} \dim(\mathbb{C}[x,z,y])_n t^n + \dim(\mathbb{C}[x,z,y])_{n-9} t^n \\ &= \frac{1}{(1-t^3)^3}\frac{1}{1+t^3} + \frac{t^9}{(1-t^3)^3}\frac{1}{1+t^3} \\ &= \frac{t^6-t^3+1}{(1-t^3)^3} \\ &= (t^6-t^3+1)\left(\sum_{n=0}^{\infty}\binom{n+2}{2}t^{3n}\right) \\ &= \sum_{n=0}^{\infty}\left(\binom{n+2}{2}-\binom{n+1}{2}+\binom{n}{2}\right)t^{3n} \\ &= \sum_{n=0}^{\infty}\left(1+\frac{n(n+1)}{2}\right)t^{3n}\,,\end{aligned}$$

where the relation $\frac{1}{(1-t^3)^3} = \sum_{n=0}^{\infty}\binom{n+2}{2}t^{3n}$ can be obtained by differentiating $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$ twice. It follows that

$$\dim\left(\frac{\mathbb{C}[x,z,y,w]}{(w^2+y^3-27wx^3-3wyz+wz^3)}\right)_n = \begin{cases} 0 & \text{if } n \neq 0 \bmod 3,\\ 1+\frac{d(d+1)}{2} & \text{if } n = 3d.\end{cases}$$

Now we have to compute the dimension of the vector space of Γ-invariant polynomials $\left(\mathbb{C}[X,Y,Z]^{\Gamma}\right)_n$. Note that $(\mathbb{C}[X,Y,Z])_n$ is a finite dimensional representation ρ_n of Γ which admits a unique decomposition in irreducible representations:

$$(\mathbb{C}[X,Y,Z])_n = \rho_n = \rho_0^{a_{0,n}} \oplus \rho_1^{a_{1,n}} \oplus \dots.$$

However the space of Γ-invariants polynomials is isomorphic to the trivial subrepresentation in ρ_n

$$\left(\mathbb{C}[X,Y,Z]^{\Gamma}\right)_n = \bigoplus_{a_{0,n}} \rho_0$$

and its dimension is the multiplicity $a_{0,n}$ of ρ_0. The representation theory of finite groups says us that if we define the character function

$$\begin{aligned}\chi_\rho : \Gamma &\longrightarrow \mathbb{C}\\ g &\longmapsto \mathrm{tr}(\rho(g))\end{aligned}$$

the multiplicity a_i of the irreducible representation ρ_i in the decomposition of ρ is

$$a_i = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} \chi_\rho(g)\,\overline{\chi_{\rho_i}(g)}\,.$$

For the trivial representation one has $\chi_{\rho_0}(g) = 1$ for any $g \in \Gamma$, hence

$$a_{0,n} = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_{\rho_n}(g) = \frac{1}{27}\sum_{g\in\Gamma}\operatorname{tr}\rho_n(g) \ .$$

We need to determinate $\operatorname{tr}\rho_n(g)$ for any $g \in \Gamma$. In case $g \notin C$ we can diagonalize it finding new monomials (X', Y', Z') such that

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} .$$

${\rho_n}' = (\mathbb{C}[X', Y', Z'])_n$ is a representation equivalent to ρ_n, thus the characters are the same. So, it suffices to compute the trace of $diag(1, \omega, \omega^2)$ on ρ_n. Note that

$$\begin{aligned}\frac{1}{1-\lambda_1 X}\frac{1}{1-\lambda_2 Y}\frac{1}{1-\lambda_3 Z} &= (1+\lambda_1 X+\ldots+\lambda_1^A X^A+\ldots)(1+\ldots+\lambda_2^B Y^B+\ldots)(1+\ldots+\lambda_3^C Z^C+\ldots)\\ &= \sum_{A,B,C\geq 0}\lambda_1^A\lambda_2^B\lambda_3^C X^A Y^B Z^C \ .\end{aligned}$$

The matrix $g = diag(\lambda_1, \lambda_2, \lambda_3)$ has trace $\sum_{A+B+C=n}\lambda_1^A\lambda_2^B\lambda_3^C$ on $(\mathbb{C}[X,Y,Z])_n$. Putting $X = Y = Z = t$ we thus get the generating function:

$$\sum_{n=0}^{\infty}\operatorname{tr}\rho_n(g)\,t^n = \sum_{n=0}^{\infty}\left(\sum_{A+B+C=n}\lambda_1^A\lambda_2^B\lambda_3^C\right)t^n = \frac{1}{1-\lambda_1 t}\frac{1}{1-\lambda_2 t}\frac{1}{1-\lambda_3 t} \ .$$

In case $g = diag(1, \omega, \omega^2)$ we obtain:

$$\sum_{n=0}^{\infty}\operatorname{tr}\rho_n(g)\,t^n = \frac{1}{1-t}\frac{1}{1-\omega t}\frac{1}{1-\omega^2 t} = \frac{1}{1-t^3} = \sum_{n=0}^{\infty}t^{3n} \ .$$

The case $g = \omega^a I \in C$ is easier:

$$\operatorname{tr}\rho_n(\omega^a I) = \omega^{an}\dim\left(\mathbb{C}[X,Y,Z]\right)_n = \omega^{an}\binom{n+2}{2} ,$$

where the last equivalence can be obtained as the one for $\dim\left(\mathbb{C}[x,z,y]\right)_n$. It follows that $\left(\mathbb{C}[X,Y,Z]^\Gamma\right)_n = 0$ if $n \neq 0 \bmod 3$, as it should appear obvious since ωI acts as multiplication by ω^n on them. Instead for $n = 3d$:

$$\dim\left(\mathbb{C}[X,Y,Z]^\Gamma\right)_{3d} = \frac{1}{27}\left(1\cdot 24 + (1+\omega^{3d}+\omega^{6d})\binom{3d+2}{2}\right) = \frac{1}{27}\frac{27d^2+27d+54}{2} = 1+\frac{d(d+1)}{2}.$$

This prove the surjectivity of ϕ^* and thus the thesis of the section.

The isomorphism ϕ^* determines the geometrical isomorphism ϕ that maps the orbifold $\mathbb{C}^3/\Gamma$ in the hypersurface

$$w^2 + y^3 - 27wx^3 - 3wyz + wz^3 = 0 \ \subset \ \mathbb{C}^4 \ .$$

In particular it sends the singular curves of the orbifold to the singular locus of the hypersurface, which by standard analysis is the union of four curves intersecting in the origin:

$$\begin{aligned} \mathcal{C}_1 &\longmapsto (0, t, t^2, t^3),\\ \mathcal{C}_2 &\longmapsto (t, 3t, 0, 0),\\ \mathcal{C}_3 &\longmapsto (t, 3\omega t, 0, 0),\\ \mathcal{C}_4 &\longmapsto (t, 3\omega^2 t, 0, 0). \end{aligned}$$

Finally, we observe that the map ϕ defines an isomorphism of surfaces between the quotient $\mathbb{P}^2/\Gamma$ and a singular hypersurface in a weighted projective space:

$$\begin{aligned} \tilde{\phi}: \quad \mathbb{P}^2/\tilde{\Gamma} &\longrightarrow \mathbb{P}_{1,1,2,3} \\ (X:Y:Z) &\longmapsto (x:z:y:w) \,. \end{aligned}$$

In particular it sends the singular points to:

$$\begin{aligned} q_1 = (1:0:0) \sim (0:1:0) \sim (0:0:1) \quad &\longmapsto \quad (0:1:1:1), \\ q_2 = (1:1:1) \sim (1:\omega^2:\omega) \sim (1:\omega:\omega^2) \quad &\longmapsto \quad (1:3:0:0), \\ q_3 = (1:\omega:1) \sim (1:1:\omega) \sim (\omega:1:1) \quad &\longmapsto \quad (1:3\omega:0:0), \\ q_4 = (1:\omega^2:1) \sim (\omega^2:1:1) \sim (1:1:\omega^2) \quad &\longmapsto \quad (1:3\omega^2:0:0). \end{aligned}$$

Chapter 8

The automorphism group of $\mathbb{P}^2/\tilde{\Gamma}$ and the Hesse pencil of cubic curves

8.1 The automorphism group of $\mathbb{P}^2/\tilde{\Gamma}$

We keep on the analysis of the orbifold by a deeper study of the properties of Γ. The normalizer $N \subset GL(3,\mathbb{C})$ of Γ is the group defined by

$$N := \{n \in GL(3,\mathbb{C}) : n\Gamma n^{-1} \subset \Gamma\}.$$

It acts naturally on $\mathbb{C}^3$ and it is the largest subgroup of $GL(3,\mathbb{C})$ which sends a Γ orbit into another one. We define

$$D := \{zI : z \in \mathbb{C}^*\}$$

the normal subgroup of N of diagonal matrices in $GL(3,\mathbb{C})$. The image of N in $PGL(3,\mathbb{C})$ is the quotient $\tilde{N} := N/D$. It acts naturally on $\mathbb{P}^2$ and it is the largest subgroup of $PGL(3,\mathbb{C})$ which sends a $\tilde{\Gamma}$ orbit into another one. $\tilde{\Gamma}$ is a normal subgroup of $\tilde{N}$ and the quotient group $\tilde{N}/\tilde{\Gamma}$ acts naturally on the quotient surface $\mathbb{P}^2/\tilde{\Gamma}$. In fact, if $\overline{x} \in \mathbb{P}^2/\tilde{\Gamma}$ has representative $x \in \mathbb{P}^2$ and if $\overline{n} \in \tilde{N}/\tilde{\Gamma}$ has representative $n \in \tilde{N}$ then $\overline{n} \cdot \overline{x} = \overline{nx}$ is well defined because if $g,h \in \tilde{\Gamma}$ we have $n(gh)n^{-1} = g' \in \tilde{\Gamma}$, so $\overline{ng} \cdot \overline{hx} = \overline{g'nx} = \overline{nx}$. We claim and then prove that this group is isomorphic to $SL(2,\mathbb{Z}_3)$ and that the group $\tilde{N}$ coincides exactly to the Hessian group.

Any $n \in N$ defines an automorphism

$$\begin{aligned} \phi_n : \quad & \Gamma \xrightarrow{\simeq} \Gamma \\ & g \longmapsto ngn^{-1} \,. \end{aligned}$$

Any elements in Γ can be written uniquely as $\omega^k g_1^a g_2^b$, with $k,a,b \in \{0,1,2\}$. For any $n \in N$ the automorphism ϕ_n is determined by its action on the generators g_1, g_2 of Γ. Therefore any $n \in N$ is determined by the elements $k,l,a,b,c,d \in \{0,1,2\}$:

$$\phi_n(g_1) = \omega^k g_1^a g_2^b \,, \qquad \phi_n(g_2) = \omega^l g_1^c g_2^d \,.$$

It is easy to verify that the map

$$\begin{aligned} \chi : \quad & N \longrightarrow GL(2,\mathbb{Z}_3) \\ & n \longmapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix} \end{aligned}$$

is a homomorphism of groups. This map defines a short exact sequence:

$$0 \longrightarrow D\cdot\Gamma \xrightarrow{\ i\ } N \xrightarrow{\ \chi\ } SL(2,\mathbb{Z}_3) \longrightarrow 0\ , \tag{8.1.1}$$

where i is the natural inclusion. To see that the image of χ is contained in $SL(2,\mathbb{Z}_3)$ we use the property

$$g_2g_1 = \omega^2 g_1g_2 \qquad \Rightarrow \qquad g_2^n g_1^m = \omega^{2nm} g_1^m g_2^n$$

and the fact that for any element $c \in C$ in the center of Γ we have $\phi_n(c) = c$. We get

$$\phi_n(g_2)\phi_n(g_1) = \omega^2\phi_n(g_1)\phi_n(g_2)$$

and as

$$\begin{cases} \phi_n(g_2)\phi_n(g_1) &= \omega^{k+l} g_1^c g_2^d g_1^a g_2^b &= \omega^{k+l+2ad} g_1^{a+c} g_2^{b+d}\ , \\ \phi_n(g_1)\phi_n(g_2) &= \omega^{k+l} g_1^a g_2^b g_1^c g_2^d &= \omega^{k+l+2bc} g_1^{a+c} g_2^{b+d}\ , \end{cases}$$

it follows that $k+l+2ad = 2+k+l+2bc$, that is $ad-bc = 1$.
To show the surjectivity of χ we find preimages in N for any generators of $SL(2,\mathbb{Z}_3)$. This group has cardinality 24. In fact for any matrices in $GL(2,\mathbb{Z}_3)$ there are $9-1=8$ choices for the first column and $9-3=6$ choices for the second column, that give 48 matrices with determinants equal to ± 1. Hence $|SL(2,\mathbb{Z}_2)| = 24$. We can choose as generators the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let

$$N_S := \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \qquad \text{then} \qquad N_Sg_1N_S^{-1} = g_2 = g_1^0g_2^1, \qquad N_Sg_2N_S^{-1} = g_1^2 = g_1^{-1}g_2^0,$$

hence $N_S \in N$ and $N_S \mapsto S$.
Let

$$N_T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{then} \qquad N_Tg_1N_T^{-1} = g_1 = g_1^1g_2^0, \qquad N_Tg_2N_T^{-1} = g_1g_2 = g_1^1g_2^1,$$

hence $N_T \in N$ and $N_T \mapsto T$.
Finally it remains to show that the kernel of χ is exactly $D\cdot\Gamma$, i.e. that any element $n \in \ker\chi$ can be seen as the product of a diagonal matrix times an elements $g \in \Gamma$. Firstly we observe that if $n \in \ker\chi$ then

$$\phi_n(g_1) = ng_1n^{-1} = \omega^k g_1 \qquad \text{and} \qquad \phi_n(g_2) = ng_2n^{-1} = \omega^l g_2.$$

Let us define

$$g := g_1^l g_2^{-k} \in \Gamma \qquad \text{and} \qquad n' := ng^{-1} \in N \qquad \Rightarrow \qquad n = n'g.$$

We know that

$$(g_1^lg_2^{-k})g_1(g_1^lg_2^{-k})^{-1} = \omega^k g_1, \qquad (g_1^lg_2^{-k})g_2(g_1^lg_2^{-k})^{-1} = \omega^l g_2.$$

Thus

$$\omega^k g_1 = ng_1n^{-1} = n'gg_1g^{-1}n'^{-1} = \omega^k n'g_1n'^{-1} \qquad \text{and} \qquad \omega^l g_2 = ng_2n^{-1} = n'gg_2g^{-1}n'^{-1} = \omega^l n'g_2n'^{-1}.$$

This means that n' has to commute with g_1, therefore is a diagonal matrix, and with g_2, which implies that all its entries are equal. Hence $n' \in D$ and it follows the thesis.

When mapped to $PGL(3,\mathbb{C})$ the sequence 8.1.1 gives a second exact sequence:

$$0 \longrightarrow \tilde{\Gamma} \xrightarrow{\tilde{i}} \tilde{N} \xrightarrow{\tilde{\chi}} SL(2,\mathbb{Z}_3) \longrightarrow 0\,.$$

This proves that $\tilde{N}/\tilde{\Gamma} \simeq SL(2,\mathbb{Z}_3)$. Moreover the group $\tilde{N}$ is a finite group of order $9 \cdot 24 = 216$, equal to the order of the Hessian group. It is generated by

$$\tilde{N} = < g_1, g_2, N_S, N_T >\,. \tag{8.1.2}$$

We will show in the next section that $\tilde{N}$ coincides with the Hessian group.

8.2 The Hesse pencil of plane cubic curves

The Hesse pencil [Hes1, Hes2] is a 1-parameter family of plane cubic curves $E_\mu \subset \mathbb{P}^2,\ \mu \in \mathbb{P}^1$, passing for 9 base points in particular position, actually any three of them are collinear. We take the following base points for the pencil and we arrange them in a square array:

$$\begin{array}{lll} p_0 = (0:1:-1), & p_1 = (1:0:-1), & p_2 = (1:-1:0), \\ p_3 = (0:1:-\omega), & p_4 = (1:0:-\omega^2), & p_5 = (1:-\omega:0), \\ p_6 = (0:1:-\omega^2), & p_7 = (1:0:-\omega), & p_8 = (1:-\omega^2:0). \end{array}$$

Thus the pencil is defined by:

$$E_\mu : \qquad x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0.$$

It is easy to see that there are twelve lines containing the points in the horizontal, vertical and diagonal rows of the above array. Each group of three lines is the support of one of the 4 singular cubic curves of the pencil:

$$\begin{array}{ll} \mu = \infty : & x_0 x_1 x_2 = 0, \\ \mu = 1 : & (x_0 + x_1 + x_2)(x_0 + \omega x_1 + \omega^2 x_2)(x_0 + \omega^2 x_1 + \omega x_2) = 0, \\ \mu = \omega : & (x_0 + \omega x_1 + x_2)(\omega x_0 + x_1 + x_2)(x_0 + x_1 + \omega x_2) = 0, \\ \mu = \omega^2 : & (x_0 + \omega^2 x_1 + x_2)(x_0 + x_1 + \omega^2 x_2)(\omega^2 x_0 + x_1 + x_2) = 0. \end{array}$$

With this choice of the base points, the lines in each singular fiber intersect in three points constituting a degenerate orbit of $\tilde{\Gamma}$:

$$\begin{array}{l} \mathrm{Sing}(E_0) = \mathrm{Fix}(< g_1 >) = \{(1:0:0), (0:1:0), (0:0:1)\}, \\ \mathrm{Sing}(E_1) = \mathrm{Fix}(< g_2 >) = \{(1:1:1), (1:\omega^2:\omega), (1:\omega:\omega^2)\}, \\ \mathrm{Sing}(E_\omega) = \mathrm{Fix}(< g_1^2 g_2 >) = \{(1:\omega^2:1), (\omega^2:1:1), (1:1:\omega^2)\}, \\ \mathrm{Sing}(E_{\omega^2}) = \mathrm{Fix}(< g_1 g_2 >) = \{(1:\omega:1), (1:1:\omega), (\omega:1:1)\}. \end{array}$$

The Hessian group [Jo, Mas, SS] G is the subgroup of $PGL(3,\mathbb{C})$, the automorphism group of $\mathbb{P}^2$, that keeps the set of the base points of the Hesse pencil invariant. It sends a curve E_μ of the pencil into another one (eventually itself). This is a group of order 216. Therefore to show that $\tilde{N}$, the automorphism group of $\tilde{\Gamma}$, coincides to the Hesse group it suffices to observe that $\tilde{N} \subset G$, namely that $\tilde{N}$ preserves the base points of the pencil. A trivial computation shows that the generators 8.1.2 of $\tilde{N}$ satisfy such requirement and this completes the proof.

8.3 The action of $\tilde{\Gamma}$ on the Hesse pencil

The group $\tilde{\Gamma} \subset \tilde{N}$ preserves the Hesse pencil structure. Actually the action of $\tilde{\Gamma}$ on any curve E_μ of the pencil is the translation by the base points of the pencil. This implies that for any curve $E_\mu/\tilde{\Gamma} \simeq E_\mu$ and it suggests a possible resolution of $\mathbb{P}^2/\tilde{\Gamma}$.

We briefly review how it is defined the group structure on a cubic curve [Har]. Let X be a non singular cubic curve in $\mathbb{P}^2$ and let $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ be the group of degree zero divisors on X. A point P_0 on X is an inflection point if the intersection multiplicity of the tangent line to X in P_0 is equal or greater than 3. A plane cubic curve has exactly 9 inflection points and any line intersecting any two of them intersects the curve in a third one. The map that to any closed point $P \in X$ associates the divisor $P - P_0 \in \mathrm{Pic}^0(X)$ is bijective and gives a group structure on the set of closed points of X, the one from $\mathrm{Pic}^0(X)$ with P_0 the identity. Similar varieties are known as group varieties. The group law has a nice geometrical interpretation. In $\mathbb{P}^2$ each couple of line L, L' are equivalent in the Picard group $\mathrm{Pic}(\mathbb{P}^2)$, therefore if $L \cap X = P, Q, R$ and $L' \cap X = P', Q', R'$ we have $P + Q + R = P' + Q' + R'$ in $\mathrm{Pic}(X)$. In particular, since P_0 is an inflection point, it follows that $P + Q + R = 3P_0$ in $\mathrm{Pic}(X)$. This implies that $P + Q + R = 0$ in the group variety X. The sum of P and Q is equal to the point T such that $P + Q - T = 0$, which means that $-T = R \in X \cap L$. However $-T \in X \cap L''$ where L'' is the line passing for T and P_0. Hence we conclude that the sum of P and Q is the third intersection point T between the curve and the line L" passing for R and the fixed inflection point P_0.

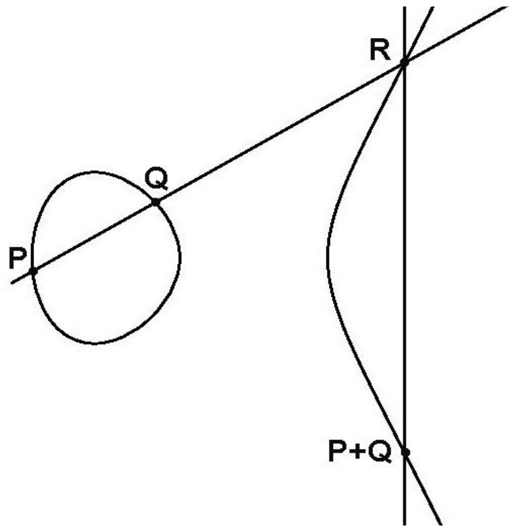

Figure 8.1: The group law on the cubic curve in $\mathbb{P}^2$ defined by $Y^2Z = X^3 - XZ^2$. The inflection point P_0 is the point at infinity.

On any smooth cubic curve of the Hesse pencil the 9 inflections points coincide with the base points of the pencil. It is simple to see that if we fix a point, for example p_0, then on any E_μ the sum $p + p_i$ of any point $p \in E_\mu$ and the base point p_i is equal to the action of an element of $\tilde{\Gamma}$ on $p \in \mathbb{P}^2$ (it is sufficient to prove it for p a base points of the pencil). Actually for any choice of the fixed point p_i we have a group isomorphism H_i between the group of inflection points (the base points of the pencil with group law the one from the group varieties E_μ) and $\tilde{\Gamma}$. For example if we fix p_0 then:

$$H_0 : (p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) \longmapsto (I, g_2, g_2^2, g_1^2 g_2, g_1 g_2, g_1 g_2^2, g_1^2 g_2^2, g_1, g_1^2).$$

Thus, the action of $\tilde{\Gamma}$ on $\mathbb{P}^2$ is the translation by the base points of the pencil, the point of order three. On any curve E_μ such translation group is called $E_\mu[3]$. Thus the image $E_\mu/\tilde{\Gamma} \subset \mathbb{P}^2/\tilde{\Gamma}$ is just $E_\mu/E_\mu[3]$ which is well known, being isomorphic to E_μ under the map:

$$\begin{array}{ccc} E_\mu & \xrightarrow{\cdot 3} & E_\mu/E_\mu[3] \\ p & \longmapsto & 3p \end{array}$$

If we exclude the base points of the Hesse pencil we can see the projective plane as a bundle of elliptic curves E_μ on $\mathbb{P}^1$. We just proved that the quotient map sends any fiber E_μ of the bundle to an elliptic curve in $\mathbb{P}^2/\tilde{\Gamma}$. Thus also $\mathbb{P}^2/\tilde{\Gamma}$ contains a natural elliptic pencil, with any fiber isomorphic to one in $\mathbb{P}^2$. However the projective plane $\mathbb{P}^2$ is not isomorphic to the singular $\mathbb{P}^2/\tilde{\Gamma}$. The pencil in $\mathbb{P}^2/\tilde{\Gamma}$ has only one base point, the image of the 9 base points of the Hesse pencil in $\mathbb{P}^2$ under the quotient. Therefore the isomorphisms on the fibers are not compatible in the base points of the Hesse pencil and $\mathbb{P}^2$ is only birational to $\mathbb{P}^2/\tilde{\Gamma}$. This suggest that if we blow up 8 of the 9 base points of the pencil we should obtain a regular morphism Ψ from a smooth surface S, actually a limit of del Pezzo surfaces, to $\mathbb{P}^2/\tilde{\Gamma}$ and hence a resolution of such variety.

$$\begin{array}{ccccc} E_\mu & \hookrightarrow & \mathbb{P}^2 & \xleftarrow{\pi} & S := Bl_{p1,\dots,p8}(\mathbb{P}^2) \\ \downarrow{\scriptstyle \cdot 3} & & \downarrow & \swarrow{\scriptstyle \Psi} & \\ E_\mu/\tilde{\Gamma} & \hookrightarrow & \mathbb{P}^2/\tilde{\Gamma} & & \end{array}$$

Chapter 9

Del Pezzo surfaces and the resolution of $\mathbb{P}^2/\tilde{\Gamma}$

9.1 Generalities on del Pezzo surfaces

For this chapter we refer to [Har, CoDo] and to the references given there. A del Pezzo surface is defined to be a surface X with ample anticanonical divisor class K_X. This means that it exists a positive integer n such that nK_X is very ample, i.e. the global sections of nK_X give an embedding of X in a projective space. The Nakai-Moishezon criterion says that a divisor D on a surface X is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curve. Let S be the surface obtained as the blow up of n points $p_1, \dots, p_n$ in $\mathbb{P}^2$. The divisor group of S is generated by H, the strict transform in S of the form defining the line in $\mathbb{P}^2$, and the exceptional divisor E_i. The intersection pairing is

$$H^2 = 1, \quad H \cdot E_i = 0 \quad \text{and} \quad E_i \cdot E_j = -\delta_{ij}.$$

The canonical divisor of the surface is $-K_S = 3H - E_1 - \dots - E_n$. It easily seen that the Nakai-Moishezon criterion implies that S is a del Pezzo surface if $n \leq 8$ and the p_i are in general position, no 3 of them are collinear and no 6 of them lie on a conic. A classical result states that every del Pezzo surfaces is either isomorphic to the blow up of $n \leq 8$ points in $\mathbb{P}^2$ or isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. From now on we specialize to the case $n = 8$.

It is a well known fact that

$$\dim H^0(S, -nK_s) = 1 + \frac{n(n+1)}{2} \qquad (n \geq 1).$$

We report the essential properties of the anticanonical maps ϕ_n, the rational maps defined by the global sections of $-nK_S$, for $n = 1, 2, 3$. We assume that the 8 points are in general position and that they are all distinct. Thus the points define a pencil of smooth cubic curves in $\mathbb{P}^2$ passing for them, all intersecting in a ninth points p_0. We note that any curve E in the pencil has a strict transform in S, again denoted by E, linearly equivalent to $-K_S$. Hence for any other E' in the pencil, $E \cdot E' = 1$ in S and p_0 is the unique points in S where they meet. The anticanonical bundle has $\dim H^0(-K_S) = 2$, therefore the cubic forms f_0, f_1 defining E and E' give a basis of $H^0(-K_S)$. The map ϕ_1 is defined by:

$$\begin{aligned} \phi_1 : \quad & S \dashrightarrow \mathbb{P}^1 \\ & q \longmapsto (f_0(q) : f_1(q)) \end{aligned}$$

This map is not defined in p_0 where $f_0(p_0) = f_1(p_0) = 0$. Blowing up p_0 the map ϕ_1 defines an elliptic fibration on $\mathbb{P}^1$, where each fiber is an elliptic curve defined by $\lambda f_0 + \mu f_1 = 0$ for some $(-\mu : \lambda) \in \mathbb{P}^1$. Note that the

exceptional divisors E_i map onto the $\mathbb{P}^1$.
Next we have $\dim H^0(-2K_S) = 4$. The polynomial $f_0^2, f_0 f_1, f_1^2 \in H^0(-2K_S)$ therefore it exists a homogeneous polynomial g of degree 6 such that $H^0(-2K_S) =< f_0^2, f_0 f_1, f_1^2, g >$. The map ϕ_2 is defined by:

$$\begin{aligned}\phi_2 : \quad & S \longrightarrow \mathbb{P}^3 \\ & q \longmapsto (X_0 : X_1 : X_2 : X_3) = (f_0(q)^2 : f_0(q) f_1(q) : f_1(q)^2 : g(q))\end{aligned}$$

Recall that $-2K_S = 6H - 2(E_1 + \ldots + E_8)$ thus the sextic curve $g = 0$ on $\mathbb{P}^2$ passes through $p_1, \ldots, p_8$ and it is singular there. Hence $p_1, \ldots, p_8$ stay in the intersection of the sextic and the cubics $\lambda f_0 + \mu f_1 = 0$ and they have multiplicity 2. There are two remaining intersection points, but it is not difficult to prove that p_0 is not one of them. This means that $g(p_0) \neq 0$ and ϕ_2 is a morphism. Any fiber of ϕ_1 is mapped $2:1$ to a $\mathbb{P}^1$:

$$\phi_2 : \lambda f_0 + \mu f_1 = 0 \longmapsto (f_0^2 : -\frac{\lambda}{\mu} f_0^2 : \frac{\lambda^2}{\mu^2} f_0^2 : g) = (1 : -\frac{\lambda}{\mu} : \frac{\lambda^2}{\mu^2} : \frac{g}{f_0^2}).$$

Hence ϕ_2 has degree two onto its image, which is the surface Q in $\mathbb{P}^3$ of equation $X_0 X_2 = X_1^2$, a quadric with unique singular point $(0:0:0:1) = \phi_2(p_0)$. The fibers of ϕ_1 map to the lines passing through the vertex of the cone.
Finally $\dim H^0(-3K_S) = 7$, thus there is a homogeneous polynomial h of degree 9 such that $H^0(-3K_S) =< f_0^3, f_0^2 f_1, f_0 f_1^2, f_1^3, g f_0, g f_1, h >$. The curve $h = 0$ in $\mathbb{P}^2$ has triple points in $p_1, \ldots, p_8$. The map ϕ_3 is defined by:

$$\begin{aligned}\phi_3 : \quad & S \longrightarrow \mathbb{P}^6 \\ & q \longmapsto (f_0^3(q) : f_0^2(q) f_1(q) : f_0(q) f_1^2(q) : f_1^3(q) : g(q) f_0(q) : g(q) f_1(q) : h(q))\end{aligned}$$

It is an embedding which sends each fiber of ϕ_1 to a smooth cubic in a $\mathbb{P}^2 \subset \mathbb{P}^6$.

Note that $\dim H^0(-6K_S) = 22$ but that $H^0(-6K_S)$ contains the 23 functions $f_0^6, f_0^5 f_1, \ldots, f_1^6, g f_0^4, \ldots, g f_1^4, g^2 f_0^2, \ldots, g^2 f_1^2, g^3, h f_0^3, \ldots, h f_1^3, h^2, f_0 g h, f_1 g h$. Thus there must be a linear relation among these functions, which is a degree two polynomial in h and its coefficients are polynomials in f_0, f_1, g, reflecting the fact that the map defined by f_0, f_1, g is $2:1$. Moreover it can be shown that this relation is unique. Thus the generators f_0, f_1 of $H^0(-K_S)$, g of $H^0(-2K_S)$ and h of $H^0(-3K_S)$ define an embedding

$$\begin{aligned}\Psi : \quad & S \longrightarrow \mathbb{P}_{1,1,2,3} \\ & q \longmapsto (f_0(q) : f_1(q) : g(q) : h(q))\end{aligned}$$

that maps S into a hypersurface of degree six of the weighted projective space.

9.2 The blow up $\mathbb{P}^2$ in 8 base points of the Hesse pencil

At the end of the section 8.3 we defined a surface S as the blow up $\mathbb{P}^2$ in the base points of the Hesse pencil $p_1, \ldots, p_8$:

$$\pi : \quad S := Bl_{p1,\ldots,p8}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2 .$$

These points are not in general position because any three of them are collinear. Take the strict transform in S of the 8 lines in the singular fibers of the pencil which do not contain p_0.

$$
\begin{aligned}
x_1 = 0: &\quad L_{147} = H - E_1 - E_4 - E_7, & x_2 = 0: &\quad L_{258} = H - E_2 - E_5 - E_8,\\
x_0 + \omega^2 x_1 + \omega x_2 = 0: &\quad L_{345} = H - E_3 - E_4 - E_5, & x_0 + \omega x_1 + \omega^2 x_2 = 0: &\quad L_{678} = H - E_6 - E_7 - E_8,\\
x_0 + \omega x_1 + x_2 = 0: &\quad L_{138} = H - E_1 - E_3 - E_8, & x_0 + x_1 + \omega x_2 = 0: &\quad L_{246} = H - E_2 - E_4 - E_6,\\
x_0 + \omega^2 x_1 + x_2 = 0: &\quad L_{156} = H - E_1 - E_5 - E_6, & x_0 + x_1 + \omega^2 x_2 = 0: &\quad L_{237} = H - E_2 - E_3 - E_7.
\end{aligned}
$$

The intersection matrix between these curves is:

	L_{147}	L_{258}	L_{345}	L_{678}	L_{138}	L_{246}	L_{156}	L_{237}
L_{147}	-2	1	0	0	0	0	0	0
L_{258}	1	-2	0	0	0	0	0	0
L_{345}	0	0	-2	1	0	0	0	0
L_{678}	0	0	1	-2	0	0	0	0
L_{138}	0	0	0	0	-2	1	0	0
L_{246}	0	0	0	0	1	-2	0	0
L_{156}	0	0	0	0	0	0	-2	1
L_{237}	0	0	0	0	0	0	1	-2

(9.2.1)

As we can see the intersection graph for each pair of curve in the singular fibers is A_2. Moreover we observe that this curve have zero intersection with the canonical divisor on S. Hence $-K_S$ do not satisfies the second requirement of the Nakai-Moishezon criterion and it is not ample. The surface S is a smooth varieties that can be seen as a degenerate limit of del Pezzo surface. In particular the anticanonical global sections are constant on the above L_{ijk}, therefore such curves get blown down by the anticanonical maps of the preceding section. In this case the map

$$
\begin{aligned}
\Psi: \quad S &\longrightarrow \mathbb{P}_{1,1,2,3}\\
q &\longmapsto (f_0(q) : f_1(q) : g(q) : h(q))
\end{aligned}
$$

is a morphism from S to a singular surface of degree six in $\mathbb{P}_{1,1,2,3}$ which sends the curve L_{ijk} to 4 singular points of type A_2.

9.3 The resolution of $\mathbb{P}^2/\tilde{\Gamma}$

The forms

$$
\begin{aligned}
f_0 &= XYZ\\
f_1 &= X^3 + Y^3 + Z^3\\
g &= X^6 + (Y^2 - YZ + Z^2)^3 + X^3(2Y^3 - 3Y^2Z - 3YZ^2 + 2Z^3)\\
h &= (X^3 + Y^3 + Z^3 + 3\omega Y^2 Z + 3\omega^2 YZ^2)(X^6 + (Y^2 - YZ + Z^2)^3 + X^3(2Y^3 - 3Y^2Z - 3YZ^2 + 2Z^3))
\end{aligned}
$$

define four curves on $\mathbb{P}^2$. The cubics $f_0 = 0$ and $f_1 = 0$ are the ones defining the Hesse pencil, hence they intersect in $p_0, \dots, p_8$. Their strict transforms in S are divisors linearly equivalent to $-K_S$. The sextic $g = 0$ has double points in $p_1, \dots, p_8$ and it does not contain p_0. Therefore it defines a divisor linearly equivalent to $-2K_S$. Finally the curve $h = 0$ has triple points in $p_1, \dots, p_8$ and it does not contain p_0. It defines a divisor linearly equivalent to $-3K_S$.

The rational map

$$\begin{aligned} \overline{\Psi}: \quad \mathbb{P}^2 & \dashrightarrow \mathbb{P}_{1,1,2,3} \\ q & \longmapsto (f_0(q) : f_1(q) : g(q) : h(q)) \end{aligned}$$

is not defined in $p_1, \dots, p_8$, but it gives a morphism

$$\Psi: \quad S \longrightarrow \mathbb{P}_{1,1,2,3} \ .$$

The image of Ψ is the surface of section 7:

$$w^2 + y^3 - 27wx^3 - 3wyz + wz^3 = 0 \subset \mathbb{P}_{1,1,2,3} \ . \tag{9.3.1}$$

Note that as it is expected the (-2)-curves L_{ijk} on S map to the singular points in 9.3.1. We proved that the quotient $\mathbb{P}^2/\tilde{\Gamma}$ is isomorphic to 9.3.1, therefore we have just proved that such quotient has a desingularization which is a quasi del Pezzo surface S.

The orbifold $\mathbb{C}^3/\Gamma$ is isomorphic to the tautological cone over 9.3.1 in $\mathbb{C}^4$. It has a partial desingularization which is the affine variety X obtained by contracting the zero section in the total space of the canonical line bundle K_S over S, with desingularization map the one from Ψ. Obviously X remains singular in the apex of the cone, the shrunk zero section of K_S.

Chapter 10

Conclusion

In the second part of the thesis we studied the geometrical properties of the orbifold $\mathbb{C}^3/\Delta_{27}$ and its relations with the geometry of cones over del Pezzo surfaces. The interest in these varieties arises quite recently in the contest of the bottom-up approach to string phenomenology. In a series of paper [VeWi, BMMVW] it has been shown that at low energy an open string theory with a D3-brane placed near the orbifold singularity is identical to the one with a D3-brane near the apex of the cone over del Pezzo surface and that this is a good starting point for a string realization of a Standard Model-like gauge theory. From a physical point of view this correspondence may be very useful, since, unlike string theory on a general del Pezzo surface, the worldsheet CFT of strings on flat space orbifolds is soluble and the D-brane boundary conditions are exactly known [DoMo, DiGo].

In our work we use some results contained in [AD] to construct a map between a cone over a quasi del Pezzo surface and $\mathbb{C}^3/\Delta_{27}$. This is a morphism that defines a desingularization of the orbifold. We arranged the work in an original way with the scope to give an understandable treatment for a physical audience. Essentially it is based on the identification of the normalizer group of Δ_{27} with the Hessian group, then on the study of the Hesse pencil behavior under the geometrical quotient.

This work presents many possible developments. Primarily it should provide a useful initial step in the concrete realization of the geometric dual of the minimal quiver extension of the minimal supersymmetric standard model. We recall that in order to obtain this result, Verlinde, Wijnholt and others in [VeWi, BMMVW, Maly] conceived a cunning symmetry breaking process for the starting quiver theory that gives many indications on the geometrical side. In particular it makes necessary the study of monodromies, the research of a partial resolution of the del Pezzo singularity with non isolated A_2 singularities and subsequently a particular Calabi-Yau compactification of this local geometry. They present a general prescription to obtain such a goal but the explicit ending variety, that should determines the actual geometrical structure of the hidden dimensions of our world, is still unknown. On such problem we only observe that, as we showed, in the orbifold side one just has non isolated A_2 singularities. Therefore we suppose that the more natural way to construct the desired partial resolution of del Pezzo singularity should pass through the orbifold geometry and our explicit map between them.

There are several other possible applications. For example we find a natural action of the Heisenberg group on the “quasi” del Pezzo geometry side of the theory. This could help to clarify the analysis of [BLZ], where such group has been studied in the role of the automorphisms group of the quiver and in the contest of AdS/CFT correspondence.

However the best known description of D-brane dynamics is in term of derived categories. It should be very interesting studying their properties in this case, also from a mathematical point of view. As a first step one should verify if the geometrical correspondence that we find between the two singular spaces implies the existence of a categorical equivalence that the quiver gauge theories identification suggests. Then one could try to use the knowledge about the homological mirror symmetry for del Pezzo surfaces [AKO] in order to extend in the non abelian case the methods valid for the abelian orbifold that we used in the first part of the thesis. For instance it should be very interesting to test the Hosono conjecture in this example.

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