Scuola estiva su campi vettoriali di Hörmander,
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Geometric Methods for Control in Rigid and Fluid Mechanics: $2\frac{1}{2}$ Examples

Andrey Sarychev

Dipartimento di Matematica per le Decisioni, Università di Firenze

Geometric Theory for controllability of nonlinear systems

nonlinear dynamical control system = dynamical polysystem = =collection of vector fields on a (finite-dimensional) manifold

Geometric Control Theory - properties of the control system via structure of the Lie algebra generated by this set of vector fields

Simple application of this idea to controllability - Rashevsky-Chow theorem * (1938, 1940), which establishes controllability of a of symmetric (driftless) dynamical polysystems, under bracket generating (Hörmander) condition

We **extend** the polysystem by Lie brackets of its vector fields

*P.K.Rashevsky worked in "nonholonomic geometry"; W.L.Chow based on a previous work of C.Caratheodory, 1909

Symmetric vs. nonsymmetric systems

Hörmander condition is NOT sufficient for controllability if the system is nonsymmetric. *

Asymmetry occurs "often", e.g. if the controls involved are "one-sided" or the system has a "drift".

*Symmetry (in a strict sense) of a dynamic polysystem means that $-f$ belongs to a polysystem iff f does

Basic $\frac{1}{2}$Example

$$\dot{x} = u, \ \dot{y} = x^2, \ u \in \mathbb{R}. \tag{1}$$

This is a particular case of control-affine system

$$\dot{q} = f(q) + g(q)u, \ u \in \mathbb{R}.$$

Evidently $f = x^2\partial/\partial y, \ g = \partial/\partial x$ and by direct computation

$$[g, [g, f]] = 2\partial/\partial y,$$

Span$\{g, [g, [g, f]]\} = \mathbb{R}^2$ - one has bracket generating property.

BUT Obviously starting from the origin we **ONLY** achieve points with $y \geq 0$.

Analysis of the basic example $\dot{x} = u,\ \dot{y} = x^2$; obstructions for controllability

The Lie bracket $[g, [g, f]]$ is an obstruction, or bad, or one-sided Lie bracket.

cf. H.J.Sussmann, H.Hermes, G.Stefani ...

The velocity of the motion in the direction of $[g, [g, f]] = 2\partial/\partial y$ equals $x^2 > 0$ - the SQUARE of the primitive of $u(\cdot)$.

Attainable set for the basic example

"Theorem" *The attainable set of* $\dot{x} = u,\ \dot{y} = x^2$ *from the origin* O *is the upper half-plane*

$$\{O\} + \{y > 0\}.$$

The proof consists of two steps.

Proposition. *The attainable set is dense in the upper half-plane.*

Krener's theorem. *Under bracket generating property an attainable set possesses nonvoid interior and is contained in the closure of this latter.*

Lie extension or Lie saturation of a control system(cf. V.Jurdjevic, I.Kupka, H.J.Sussmann)

is a set-theoretic extension of the polysystem, under which the closures of attainable sets $\mathcal{A}_{\hat{x}}$ persist.

Remark. Evidently this definition is nonconstructive.

If proceeding with a series of extensions one arrives to a controllable system, then the original system is "almost controllable" - it attainable set is dense

To complete the argument apply Krener's theorem

Some examples of Lie extensions

- The closure of a polysystem in Whitney topology is a Lie extension;

- convexification or conification of a polysystem $\mathcal{F}$

$$\text{conv}\mathcal{F} = \left\{ \sum_{j=1}^{N} \alpha_j(q) f^j | \alpha_j \in C^\infty, f^j \in \mathcal{F},\ \alpha_j \geq 0, \sum_{j=1}^{N} \alpha_j = 1 \right\}.$$

  is Lie extension (theory of relaxed controls, homogenization,etc.);

- extension by an adjoint action of a normalizer* P: $\mathcal{F} \bigcup \text{Ad}P\mathcal{F}$ (Lie algebraic control theory).

*Diffeomorphism P is a normalizer for $\mathcal{F}$ if $\forall \hat{q}:\ P(\hat{q}), P^{-1}(\hat{q}) \in \text{clos}(\mathcal{A}_{\hat{q}})$

"Reduction formula" for control-affine systems

For a control-affine system (with no a priori bounds on controls)

$$\dot{q} = f(q) + G(q)u = f(x) + \sum_{j=1}^{r} g^j(q)u_j$$

the diffeomorphisms $e^{G(q)v}$ with fixed $v \in \mathbb{R}^r$ are normalizers.

If in addition $[g^j, g^k] = 0, \; j,k = 1,\ldots,r$ one can extend the system by $\mathrm{Ad}\left(e^{G(q)v(t)}\right) f$ with $v(t)$ - time-variant

("Reduction formula", cf. A.Agrachev, A.S., 1986,Math. USSR Sbornik)

Remark. For constant vector field $G(q) \equiv G$:

$$\mathrm{Ad}\left(e^{Gv(t)}\right) f(q) = f(q + Gv(t)).$$

Example 1: Control of a rotational motion of a satellite (rigid body)

The equations of the rotational motion of a satellite (cf. A.Bloch's lectures) are

$$\dot{Q} = Q\hat{\Omega}, \;\; \dot{M} = M \times \Omega = M \times JM, \tag{2}$$

where $Q \in SO(3)$ is a position of the body, $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$ is its angular velocity, $M \in \mathbb{R}^3$ - a momentum, symmetric (3×3)-matrix J - inverse of the tensor of inertia of the body, '$\times$' - vector product in $\mathbb{R}^3$;

$$\hat{\Omega} = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix} \in so(3).$$

Control of a rotational motion of a satellite (ctn.)

Assume that a pair of torques are applied to create the forcing momentum along an axis L. The corresponding equations of controlled motion are:

$$\dot{Q} = Q\widehat{JM}, \ \dot{M} = M \times JM + Lu(t), \ u \in \mathbb{R}, \ L \in \mathbb{R}^3.$$

There may exist bounds on control: $\|u(t)\| \leq b$.

6-dimensional system with 1-dimensional control.

Controllability - ?

Reduction formula for the controlled dynamic equation of satellite

Consider just the dynamic equation

$$\dot{M} = f(M) + Lu = M \times JM + Lu$$

Applying the reduction formula we obtain

$$\text{Ad}\left(e^{Lv}\right) f = f(M + Lv) = (M + Lv) \times J(M + Lv) =$$
$$= M \times JM + (M \times JL + L \times JM)v + (L \times JL)v^2.$$

Note that the v-quadratic term multiplies the Lie bracket $[L, [L, f]]$ which equals to a constant vector field L'.

The v-linear term can be killed by convexification, while the v-quadratic term is sign-definite - it is an obstruction(!)

Recurrence property of the drift and its inversion

What "saves" the controllability is the recurrence property of the drift $f = M \times JM$ - almost all of its trajectories are periodic.

For the "big" 6-dimensional system almost all points are Poisson stable. Does not hold for satellite subject to damping.

This recurrence allows to extend the system by the field $-f$ (Lobry-Bonnard theorem) and after a reduction obtain the vector field $-L' = -[L, [L, f]] = -(L \times JL)$.

Repeating once more the reduction we obtain another constant vector field $L'' = (L' \times JL')$. For generic J, L the vectors L, L', L'' are linearly independent and the dynamic equation is controllable. Controllability of 6-dimensional case follows easily.

Example 2: Infinite-dimensional systems: 2D and 3D Navier-Stokes equations controlled by forcing in few low modes

$$\partial u/\partial t + (u \cdot \nabla)u + \nabla p = \nu \Delta u + V(t,x),$$
$$\nabla \cdot u = 0.$$

$u(t,x)$ - velocity of the fluid at instant t at point x

p - pressure;

$\nu \Delta u$ - "dissipative term"

$V(t,x)$ - forcing term, taken as a control.

2D and 3D Navier-Stokes equations controlled by forcing ctn.

$$\partial u/\partial t + (u \cdot \nabla)u + \nabla p = \nu \Delta u + V(t,x), \ \nabla \cdot u = 0.$$

boundary conditions periodic w.r.t. x : $u(t,x)$ evolves on a torus $\mathbb{T}^2$ or $\mathbb{T}^3$

the controlled forcing $V(t,x)$ is degenerate: only few low modes (harmonics) are forced;

There are no a priori constraints on the magnitudes of controls.

Spectral representation for 2D and 3D NS system

Introduce the vorticity $w = \nabla^\perp \cdot u$, put $v = \nabla^\perp \cdot V$. Then

$$\partial w/\partial t + (u \cdot \nabla)w = \nu \Delta w + v(t,x).$$

Spectral method for periodic b.c. $=$ take Fourier expansions:

$$w(t,x) = \sum_{k \in \mathbb{Z}^2} q_k(t) e^{ikx}, \; v(t,x) = \sum_{k \in \mathbb{Z}^2} v_k(t) e^{ikx} :$$

Infinite system of ODE("infinite-dimensional rigid body"; cf. V.I.Arnold):

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k + v_k(t), \; k,m,n \in \mathbb{Z}^2.$$

For $3D$ **NS system** we obtain (cf. G.Gallavotti, "Foundations of Fluid Mechanics")

$$\dot{\underline{q}}_k = -i \sum_{m+n=k} (\underline{q}_m \cdot n) \Pi_k \underline{q}_n - \nu |k|^2 \underline{q}_k + \underline{v}_k,$$

Here $\underline{q}_{\mathbf{k}} \perp \mathbf{k}$ and $\mathbf{\Pi_k}$ is the orthogonal projection of $\mathbb{R}^{\mathbf{3}}$ onto $\mathbf{k}^{\perp}$.

Degenerate controlled forcing - finite set $\mathcal{K}^1 \subset \mathbb{Z}^{2,3}$ of controlled modes;

$\mathbf{v_k(t)}$ with $k \in \mathcal{K}^1$ - measurable, essentially bounded controls;

$\mathbf{v_k} \equiv 0, \forall k \notin \mathcal{K}^1$

Are the 2D and 3D NS systems controllable?

Global controllability for the NS systems: various problem settings

Controllability

Due to smoothing properties of the NS system hard to be expected. (cf. A.Fursikov, Y.Immanuilov for alternative definitions)

Approximate controllability

Is attainable set from a given (initial) point dense in the space of u's?

Controllability in finite-dimensional projections

We select a finite set $\mathcal{K}^{\mathbf{obs}} \subset \mathbb{Z}^{\mathbf{2,3}}$ of observed modes and follow its dynamics according to complete NS system.

Controllability in observed projection: can one start from any (initial) point and attain (in fixed time $\mathbf{T}$) any preassigned observed projection?

Finite-dimensional Galerkin approximations of NS systems

We truncate the (2D or 3D) NS system, putting all the unobserved modes equal to zero, obtaining finite-dimensional control-affine system - the Galerkin approximation.

Is the finite-dimensional Galerkin approximation of NS system globally controllable?

Controllability for Navier-Stokes systems: answers provided

cf. A.Agrachev, A.Sarychev, Doklady Mathematical Sciences, v. 69, N.1/2,2004,pp.112-115.

&

Journal of Mathematical Fluid Mechanics, v. (2004), 45 pp.

Controllability of finite-dimensional Galerkin approximations (= finite-dimensional truncations) of the 2D and 3D NS systems - **YES**

Controllability of finite-dimensional projections of trajectories of complete 2D NS systems - **YES**

Approximate controllability of 2D NS system - **YES**

Controllability for NS and Euler systems: work in progress

Controllability of 2D Euler equation for incompressible ideal fluid

Lie algebraic methods and accessibility property for NS system

Control of viscous incompressible fluid under various boundary conditions

Finite-dimensional treatment: controllability of Galerkin approximations of (2D and 3D) NS systems (GANS)

The Galerkin Approximation of (2D and 3D) Navier-Stokes system is a particular type of control-affine system, actually an multidimensional version of Euler (satellite) equation

$$\dot{q}_j = f_j(q) + u_j, \ j \in \mathcal{K}^1,$$

$$\dot{q}_j = f_j(q), j \in \mathcal{K}^{obs} \setminus \mathcal{K}^1,$$

$$f_j(q) = \sum_{m+n=j} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_j, \ j, m, n \in \mathcal{K}^{obs}$$

Controllability of GANS - ctd.

Observation: If $\mathcal{K}^{\mathbf{1}} = \mathcal{K}^{\mathbf{obs}}$, then

$$\dot{q}_j = f_j(q) + u_j, \;\; j \in \mathcal{K}^{obs},$$

and the controllability in observed projection can be concluded (almost) trivially.

Moreover: in this case one can design any (Lipschitzian with respect to t) evolution of the observed projection.

Proof: to design a Lipschitzian trajectory $\tilde{\mathbf{q}}(\mathbf{t})$ of the equation $\dot{q} = f(q) + v$ take

$$v(t) = \dot{\tilde{q}}(t) - f(\tilde{q}(t).$$

Controllability of GANS - ctd.

The only interesting case is:

$$\mathcal{K}^1 \subset \mathcal{K}^{obs}$$

More interesting when $\mathcal{K}^1$ can be chosen independent of $\mathcal{K}^{obs}$.

Idea: provide a series of Lie extensions in order to arrive at the end to this trivial situation

The extensions are:

- relaxation=convexification;
- extension by application of reduction formula;

Extension by reduction

1. As in "satellite example" the controlled vector fields g^j for the NS system are *constant*. They commute and the r.-h.side of the reduced system is computed as:

$$\dot{y} = f(y + GV(\tau)),$$

where $V(\cdot)$ is the primitive of the original control.

2. The drift f is *quadratic+linear*; hence this r.-h.side is polynomial of degree 2 w.r.t. V:

$$f(y + GV) = f(y) + FV + \mathcal{F}(V, V).$$

3. The linear terms w.r.t. V can be *killed* by convexification (substitute V and $-V$ and convexify).

4. Among the quadratic terms there are mixed terms $v_j v_k$, which will be our "new" or "extended" controls, which multiply constant vector fields. The values of these new controls $v_j v_k$ can take both signs.

5. Each control $v_j v_k$ "drives" the variable q_m with $m = j + k$, where $j, k \in \mathcal{K}^1$, but m may be outside $\mathcal{K}^1$. This is an **extension**. Convexifying the new controlled directions we arrive at a control-affine system.

Remark for geometric control experts: we extend our control system by good Lie brackets $[g^i[g^j, f]]$. Bad Lie brackets (obstructions) $[g^j[g^j, f]]$ vanish.

Controllability criteria for Galerkin approximations of Navier-Stokes systems (GANS)

Let $\mathcal{K}^1 \subset \mathbb{Z}^s$, $(s = 2, 3)$ be the set of controlled forcing modes. Define the sequence of sets $\mathcal{K}^\alpha \subset \mathbb{Z}^s$, $\alpha = 2, \ldots$, as:

$$\mathcal{K}^\alpha = \{j + k | \ j, k \in \mathcal{K}^{\alpha-1} \bigwedge \|j\| \neq \|k\| \bigwedge j \wedge k \neq 0\}. \square \qquad (3)$$

Theorem. *Let $\mathcal{K}^1$ be the set of controlled forcing modes. Define iteratively sequence of sets $\mathcal{K}^\alpha$, $\alpha = 2, \ldots,$ by (3) and assume that for some M : $\bigcup_{\alpha=1}^{M} \mathcal{K}^\alpha$ contains all the observed modes: $\bigcup_{\alpha=1}^{M} \mathcal{K}^\alpha \supset \mathcal{K}^{obs}$. Then for any $T > 0$ GANS are time-T globally controllable.* $\square$

Saturating sets of forcing modes

Definition. *A set $\mathcal{K}^1$ of forcing modes is called saturating if for any bounded (finite) subset $\mathcal{K}$ of $\mathbb{Z}^s$, $s = 2, 3$, there exists M such that $\mathcal{K} \subseteq \bigcup_{\alpha=1}^{M} \mathcal{K}^\alpha$, where $\mathcal{K}^\alpha$ are constructed as above.*

Lemma. *The set $\mathcal{K}^1 = \{k = (k_1, k_2) |\ |k_1| \leq 3 \wedge |k_2| \leq 3\} \subset \mathbb{Z}^2$ is saturating. The sets $\mathcal{K}^\alpha$ are growing monotonously: $\mathcal{K}^j \subset \mathcal{K}^{j+1}$, $j \geq 1$.* $\square$

Corollary. *The set $\mathcal{K}^1 = \{k = (k_1, k_2) |\ |k_1| + |k_2| \leq 2\} \subset \mathbb{Z}^2$ is saturating.* $\square$

Remark. M.Romito (U.Firenze) has proven that the set of controlled forcing modes $\mathbb{K}^1 = \{(1,0,0), (0,1,0), (0,0,1)\} \subset \mathbb{Z}^3$ is sufficient to guarantee global controllability of any finite-dimensional Galerkin approximation of 3D NS system.

Controllability in finite-dimensional observed projection for 2D NS system

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k + v_k, \ \ k \in \mathcal{K}^1, \tag{4}$$

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k, \ \ k \in \mathcal{K}^{obs} \setminus \mathcal{K}^1, \tag{5}$$

$$\dot{Q} = B(q, Q) + \nu \Delta Q. \tag{6}$$

Theorem 1.

If the set $\mathcal{K}^1$ is saturating then the system is controllable in the observed projection. □

We prove more

Theorem 2.

For any compact C in the space of observed variables there exists a family of controls $v(t, b)$ parameterized continuously in L_1-metric by a finite-dimensional compact B such that each point of C can be attained by some control from this family. $\square$

Remark. This property is stable.

Main technical difficulty: the infinite-dimensional part Q affects the evolution of the observed projection q.

One has to extend the arguments of Geometric Control Theory to take into account the infinite-dimensional part.

Relaxed controls for controlled 2D NS system

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k + v, \ \ k \in \mathcal{K}^1, v \in \Omega \text{ vs.} v \in \overline{co}\Omega$$

$$\dot{q}_k = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k, \ \ k \in \mathcal{K}^{obs} \setminus \mathcal{K}^1,$$

$$\dot{Q} = B(q, Q) + \nu \Delta Q.$$

Theorem. The trajectories of the first system are dense in the set of trajectories of the second system. □

Proof is based on Lyapunov-Schmidt-type reduction to a finite-dimensional case.

Extension of infinite-dimensional system by Lie brackets. Infinite-dimensional motion planning

Consider the system of equations

$$\dot q_1 = f_1(q) + u_1,$$

$$\dot q_2 = f_2(q) + u_2, \ f_j \text{ are quadratic+linear,}$$

$$\dot q_k = c q_1 q_2 + \sum_{m+n=k} c_{mn} q_m q_n - b_k q_k,$$

for an observed uncontrolled (=unforced) variable q_k.

How to affect the evolution of q_k?

Assume that the r.-h. side of the equation for q_k contains the term cq_1q_2. As far as we are able to design any evolution for q_1, q_2, we may treat them as controls.

Take

$$q_1 = A_1 \sin \omega t, \ q_2 = A_2 \sin \omega t, \ \text{where } A_1 A_2 c = 2,$$

and substitute it into the equation for q_k, obtaining in its r.-h. side $2 \sin^2 \omega t = 1 - \cos 2\omega t \asymp 1, \ \text{as } \omega \to \infty$.

Remark 1. $\sin \omega t \asymp 0, \ \text{as } \omega \to \infty$; this asymptotic equality is understood in the metric of relaxed controls.

Remark 2. The product q_1q_2 enters only one equation; in all other equations (including infinite-dimensional part) q_1 and q_2 enter linearly. One can prove that their effect on the variables different from q_k tends to 0 as $\omega \to \infty$.