LIE EXTENSIONS OF NONLINEAR CONTROL SYSTEMS

ANDREY V. SARYCHEV

Abstract. We survey some classical geometric control techniques for studying controllability of finite- and infinite-dimensional nonlinear control systems.

]

1. Brief introduction into nonlinear control theory

1.1. Basic definitions, see [20, 3].

Definition 1.1.1. *Dynamical control system, or C^∞ or C^ω dynamical polysystem, is a family of C^∞ (correspondingly C^ω) vector fields parameterized by* control parameter *u:*

$$\mathcal{F} = \{f(\cdot, u)|\ u \in U\};$$

U is arbitrary subset of $\mathbb{R}^r$.

The value of u changes with time. Typical choice is measurable bounded dependence $u(t)$: $u(\cdot) \in L_\infty([0,T], U)$. For most of our purposes piecewise-continuous or even piecewise-constant controls will suffice. This latter choice results in a class of piecewise smooth trajectories, where each piece is driven by a vector field $f(\cdot, u^0)$ with u^0 fixed. We will denote the corresponding flow by $e^{tf_{u^0}}$. The flow corresponding to a piecewise-constant control has form

$$e^{t_1 f_1} \circ e^{t_2 f_2} \circ \cdots e^{t_N f_N},$$

where $f_j = f_{u^j}$, $u^j \in U$. For a general not necessarily piecewise-constant control we obtain a flow generated by the ODE

$$\dot{x} = X_t(x), \text{ where } X_t(\cdot) = f(\cdot, u(t)).$$

We will be interested in the controllability issue which is closely related to the notion of attainability and attainable sets.

Definition 1.1.2. *A point $\tilde{x}$ is attainable from $\hat{x}$ in time T (corr. in time $\leq T$) for the system $\dot{x} = f(x,u)$ if for some admissible control $\tilde{u}(\cdot)$ the corresponding trajectory, which starts at $\hat{x}$ at $t = 0$, attains $\tilde{x}$ at $t = T$ (at some $t \leq T$). A point $\tilde{x}$ is attainable from $\hat{x}$ if it is attainable from $\hat{x}$ in some time $T \geq 0$. The set of points attainable from $\hat{x}$ in time T (in time $\leq T$) is called time-T (time-$\leq T$) attainable set from $\hat{x}$ and is denoted by $\mathcal{A}^T_{\mathcal{F}}(\hat{x})$ (resp. $\mathcal{A}^{\leq T}_{\mathcal{F}}(\hat{x})$. The set of points attainable from $\hat{x}$ is called attainable set from $\hat{x}$ and is denoted by $\mathcal{A}_{\mathcal{F}}(\hat{x})$. We say that the system is globally controllable (globally*

Key words and phrases. nonlinear systems, controllability, relaxation, Lie extension,controlled PDE.
Lectures given at Trimester on Dynamical and Control Systems, SISSA-ICTP, Trieste, Fall 2003.

controllable in time T or in time $\leq T$) from $\hat{x}$ if its attainable set $\mathcal{A}_{\mathcal{F}}(\hat{x})$ (attainable set $\mathcal{A}^{T}_{\mathcal{F}}(\hat{x})$, or resp. $\mathcal{A}^{\leq T}_{\mathcal{F}}(\hat{x})$)) coincide with the whole state space. $\square$

Now we introduce the notions of global controllability.

Definition 1.1.3. *We say that the system is globally controllable (globally controllable in time T or in time $\leq T$) from $\hat{x}$ if its attainable set $\mathcal{A}_{\mathcal{F}}(\hat{x})$ (attainable set $\mathcal{A}^{T}_{\mathcal{F}}(\hat{x})$, or resp. $\mathcal{A}^{\leq T}_{\mathcal{F}}(\hat{x})$) coincide with the whole state space.* $\square$

It is convenient to represent the attainable sets as images of some maps related to control system.

Definition 1.1.4. *Let us fix the initial condition for trajectories of the control system.*

The correspondence between admissible controls - $u(\cdot)$ and the corresponding trajectories of the system is established by input/trajectory map *($\mathcal{IT}$-map).*

If the there is an output

$$y = h(x)$$

attributed to the system then the correspondence between the control $u(\cdot)$ and the output function $h(x(t))$ is established by nput/output map *(IO-map).*

If the dynamics of the system is restricted to an interval $[0, T]$, then the map

$$\mathcal{IT}_T\,(u(\cdot)) \mapsto x(T)$$

is called end-point map. $\square$

Remark 1.1.5. *Evidently time-T global controllability of the NS system in observed projection is the same as surjectiveness of the end-point map EP_T.* $\square$

Another useful notion tightly related with the issue of optimality is the one of local controllability along a reference trajectory.

Definition 1.1.6. *Consider a reference trajectory $\tilde{x}(\cdot)$ of our control system driven by some admissible control $\tilde{u}(\dot{})$. The system is locally controllable along this trajectory in time T if the end-point map E/P_T is locally onto.* $\square$

To define local controllability properly we have to introduce a metric in the space of admissible controls. In the future it will be metric either generated by L_∞-norm, or L_1-norm or a weaker metric, such as metric of relaxed controls.

1.2. **Elements of chronological calculus, see** [1, 2, 3]**.** Chronological calculus is a formalism for representation and asymptotic analysis of solutions of time-variant differential equations. It has been developed by A.A.Agrachev and R.V.Gamkrelidze at the end of 70's.

Let us consider a time-variant differential equation in $\mathbb{R}^N$:

$$\dot{x} = X_t(x).$$

If this vector field is complete, i.e. all the solutions are defined $\forall t \in \mathbb{R}$ then one says that the ODE defines a flow $P_t,\ P_0 = Id$.

It will be convenient to introduce the "operator notation" $(P \circ X)(x) = X(P(x))$.

Then the differential equation (1) can be written as

$$\frac{d}{dt}P_t(x) = P_t \circ X_t(x),$$

or after suppressing x:

$$\frac{d}{dt}P_t = P_t \circ X_t,\ P_0 = I. \tag{1}$$

Definition 1.2.7. *The flow defined by the ODE (1) is called right chronological exponential and is denoted by* $\overrightarrow{\exp}\int_0^t X_\tau d\tau$. $\square$

Remark 1.2.8. *Left chronological exponential* $\overleftarrow{\exp}\int_0^t X_\tau d\tau$ *denotes the flow defined by the equation*

$$(d/dt)Q_t = X_t \circ Q_t,\ Q_0 = I.\ \square$$

The right chronological exponential admits a series expansion. Indeed let us write Volterra integral equation

$$P_t = I + \int_0^t P_{\tau_1} \circ X_{\tau_1} d\tau_1$$

and "iterate" it obtaining

$$P_t = I + \int_0^t \left(I + \int_0^{\tau_1} P_{\tau_2} \circ X_{\tau_2} d\tau_2 \right) \circ X_{\tau_1} d\tau_1 =$$

$$= I + \int_0^t \int_0^{\tau_1} X_{\tau_2} d\tau_2 \circ X_{\tau_1} d\tau_1 + \int_0^t \int_0^{\tau_1} P_{\tau_2} \circ X_{\tau_2} d\tau_2 \circ X_{\tau_1} d\tau_1.$$

At the end we obtain so-called Volterra expansion for right chronological exponential.

Definition 1.2.9. *Volterra expansion or Volterra series for the chronological exponential is (see [AG78,ASkv]):*

$$\overrightarrow{\exp}\int_0^t X_\tau d\tau \asymp I + \sum_{i=1}^{\infty} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \dots \int_0^{\tau_{i-1}} d\tau_i (X_{\tau_i} \circ \dots \circ X_{\tau_1}).\ \square \tag{2}$$

We will consider the Whitney topology in the space of functions $\varphi(x) \in C^\infty(\mathbb{R}^N)$ defined by a family of seminorms $\|\cdot\|_{s,K}$, where $s \geq 0$, $K \subset \mathbb{R}^N$ is compact:

$$\|\varphi(x)\|_{s,K} = \sup\left\{ \left| \frac{\partial \varphi}{\partial x^\alpha}(x) \right| \middle|\ x \in K,\ |\alpha| \leq s \right\}.$$

For a vector field X one can define seminorms componentwise but more elegant definition is

$$\|X\|_{s,K} = \sup\{\|X\varphi\|_{s,K} | \|\varphi\|_{s+1,K} = 1\}.$$

If a time-variant vector field X_t is bounded, analytic , then the series (2) converges provided that $\int_0^t \|X_\tau\| d\tau$ is sufficiently small ([1]).

If a time-variant vector field X_t is bounded, analytic (and admits an analytic continuation onto a neighborhood of the time interval $[0, T]$ in the complex plane $\mathbb{C}$, then the series (2) converges provided that $\int_0^t \|X_\tau\| d\tau$ is sufficiently small ([1]). The norm $\|X_\tau\|$ is the norm of the analytic continuation.

In C^∞-case the Volterra expansion provides asymptotics for the chronological exponential

Proposition 1.2.10 ([1]). *Let $P_t = \overrightarrow{\exp} \int_0^t X_\tau d\tau$ be the solution of the equation (1) and X_τ is locally integrable: $\int_0^T \|X_\tau\|_{s,K} d\tau \leq \rho_s < +\infty$. Then*

$$\|P_t \circ \varphi\|_{s,K} \leq C_1 e^{C_2 \int_0^t \|X_\tau\|_s d\tau}; \tag{3}$$

$$\left\| \left(P_t - \left(I + \sum_{i=1}^{m-1} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \dots \int_0^{\tau_{i-1}} d\tau_i (X_{\tau_i} \circ \cdots X_{\tau_1}) \right) \right) \varphi \right\|_{s,K} \leq$$

$$\leq C_1 e^{C_2 \int_0^t \|X_\tau\|_s d\tau} (1/m!) \left| \int_0^t \|X_\tau\|_{s+m-1} d\tau \right|^m \|\varphi\|_{s+m,M}, \tag{4}$$

where C_1, C_2 depend on s, K, ρ and M is ρ-neighborhood of K. □

The formula (3) indicates that there must be continuity of solutions with respect to the right-hand sides X_t evaluated in $L_t^1 C_x^s$ norms $\int_0^t \|X_\tau\|_s\| d\tau$. In fact a much stronger fact holds: there is continuity with respect to a norm of relaxed controls. This will be referred to later.

1.3. **Variational formula, see** [1, 2, 3]**.** Assume that we deal with a "perturbed" ODE:

$$\dot{x} = (X_t + Y_t)(x) \text{ or } \frac{d}{dt} P_t = P_t \circ (X_t + Y_t), \ P_0 = I. \tag{5}$$

We would like to represent the corresponding flow $\overrightarrow{\exp} \int_0^t (X_\tau + Y_\tau) d\tau$ as a multiplicative variation of the non-perturbed flow $\overrightarrow{\exp} \int_0^t X_\tau d\tau$, namely as a composition of this latter with a perturbation flow C_t:

$$\overrightarrow{\exp} \int_0^t (X_\tau + Y_\tau) d\tau = \overrightarrow{\exp} \int_0^t X_\tau d\tau \circ R_t, \text{ or} \tag{6}$$

$$\overrightarrow{\exp} \int_0^t (X_\tau + Y_\tau) d\tau = L_t \circ \overrightarrow{\exp} \int_0^t X_\tau d\tau. \tag{7}$$

To derive the equation for the perturbation flow L_t in (7) we differentiate this equality. To simplify the notation denote $\overrightarrow{\exp} \int_0^t X_\tau d\tau$ by R_t and $\overrightarrow{\exp} \int_0^t (X_\tau + Y_\tau) d\tau$ by P_t. To derive the equation for the perturbation flow L_t in (7) we differentiate the equality $P_t = L_t \circ R_t$, obtaining

$$dP/dt = dL_t/dt \circ R_t + L_t \circ dR_t/dt$$

or

$$P_t \circ (X_t + Y_t) = \dot{L}_t \circ R_t + L_t \circ R_t \circ X_t.$$

Substituting the $L_t \circ R_t$ instead of P_t in the latter formula we conclude

$$L_t \circ R_t \circ Y_t = \dot{L}_t \circ R_t,$$

from where we obtain

$$\dot{L}_t = L_t \circ (R_t \circ Y_t \circ R_t^{-1}).$$

If R is a diffeomorphism $\mathrm{Ad}R$ denotes adjoint action over the Lie algebra of vector fields

$$\mathrm{Ad}R[X, Y] = [\mathrm{Ad}RX, \mathrm{Ad}RY],$$

or the group of diffeomorphisms: $\mathrm{Ad}R(P \circ Q) = (\mathrm{Ad}RP) \circ (\mathrm{Ad}R)Q$.

$\mathrm{Ad}RY$ can be also seen as a pull-back of the vector field Y by the diffeomorphism R^{-1}: $(R_*^{-1}Y)|_x = DR^{-1}|_{R(x)}Y|_{R(x)}$.

The equation (8) can be written down as

$$\dot{L}_t = L_t \circ (\mathrm{Ad}R_tY_t), \tag{8}$$

from where $L_t = \overrightarrow{\exp} \int_0^t \mathrm{Ad}\left(\overrightarrow{\exp} \int_0^\tau X_\xi d\xi\right) Y_\tau d\tau$.

Proposition 1.3.11 (variational formula)**.** *The formulae (7)-(6) hold with*

$$L_t = \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_0^\tau (adX_\xi) d\xi\right) Y_\tau d\tau, R_t = \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_t^\tau (adX_\xi) d\xi\right) Y_\tau d\tau. \tag{9}$$

Corollary 1.3.12. *For time-invariant vector field $X_t \equiv X$ we obtain the formulae*

$$L_t = \overrightarrow{\exp} \int_0^t e^{\tau\, adX} Y_\tau d\tau, R_t = \overrightarrow{\exp} \int_0^t e^{(\tau - t)\, adX} Y_\tau d\tau. \tag{10}$$

By analogy with the Proposition 1.2.10 one can obtain the following estimate for the r.-h. of the equation for the perturbation flow.

Proposition 1.3.13. *Let $P_t = \overrightarrow{\exp} \int_0^t X_\tau d\tau$ be the flow corresponding to a locally integrable vector field X_τ: $\int_0^T \|X_\tau\|_{s,K} d\tau \leq \rho < +\infty$. Then*

$$\begin{aligned} &\Bigg\| \Bigg(Ad\left(\overrightarrow{\exp} \int_0^t X_\tau d\tau\right) - \\ &- \left(I + \sum_{i=1}^{m-1} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{i-1}} d\tau_i (adX_{\tau_i} \circ \cdots \circ adX_{\tau_1})\right)\Bigg) Y \Bigg\|_{s,K} \\ &\leq C_1 e^{C_2 \int_0^t \|X_\tau\|_{s+1} d\tau} (1/m!) \left(\int_0^t \|X_\tau\|_{s+m} d\tau\right)^m \|Y\|_{s+m,M}, \end{aligned} \tag{11}$$

where C_1, C_2 depend on s, K, ρ and M is ρ-neighborhood of K. □

2. Nonlinear controllability

Though mainly we will study controllability of nonlinear systems we start with famous controllability criterion for linear systems and with linearization principle for controllability of nonlinear systems.

2.1. **Linear controllability.** The controllability of linear time-invariant system

$$\dot{x} = Ax + Bu,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^r,\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times r}, \tag{1}$$

is verified by the following Kalman criterion.

Proposition 2.1.1. *If for some $T > 0$ the system (1) is globally time-T controllable then*

$$rank\left(B \mid AB \mid \cdots \mid A^{n-1}B\right) = n. \tag{2}$$

If (2) holds then the system (1) is globally time-T controllable for each $T > 0$. $\square$

Question. What one can say about global (for all times) controllability of the system (1)?

For time-variant linear system

$$\dot{x} = A(t)x + B(t)u, \tag{3}$$

there is an established controllability criterion, but its verification is more difficult.

To formulate the criterion consider the fundamental matrix $\Phi(t)$ of the homogeneous linear system

$$\dot{X} = A(t)X,\ X(0) = I.$$

The each trajectory of the linear system (3) can be represented as

$$x(T) = \Phi(T)\left(x^0 + \int_0^T \Phi^{-1}(s)B(s)u(s)ds\right).$$

As long as $\Phi(T)$ is invertible matrix this map is onto iff the map

$$u(\cdot) \to \int_0^T \Phi^{-1}(s)B(s)u(s)ds$$

is onto. The last happens iff

$$\mathrm{span}\{\Phi(T)\Phi^{-1}(s)b_j(s)\mid\ s \in [0,T],\ j = 1,\dots,r\} = \mathbb{R}^n.$$

Differentiating $\Phi^{-1}(t)$ w.r.t. t we obtain

$$(d/ds)\Phi^{-1}(s) = -\Phi^{-1}(s)A(s)\Phi(s)\Phi^{-1}(s) = \Phi^{-1}(s)(-A(s)),$$

and therefore $\Phi^{-1}(t) = \overrightarrow{\exp} \int_0^t(-A_\tau)d\tau$. If $A(t)$ is C^∞ or C^ω in t we obtain

$$(d/ds)(\Phi^{-1}(s)B(s)) = B^1(s) = \Phi^{-1}(s)\left(-A(s) + d/ds\right)B(s),$$

and in general

$$(d^k/ds^k)(\Phi^{-1}(s)B(s)) = B^k(s) = \Phi^{-1}(s)\left(-A(s) + d/ds\right)^k B(s),\ k \geq 0.$$

Therefore if at some point s_0:

$$\mathrm{span}\{b_j^k(s_0)|j = 1,\dots,r;\ k \leq N\} = \mathbb{R}^n,$$

then the system is time-T controllable for any $T > s_0$. In C^ω case this condition is necessary for time-T controllability.

2.2. **Linearization principle for controllability.** A nonlinear system can be in some aspects satisfactory approximated locally by its linearization, therefore one can conclude local controllability of the nonlinear system from controllability of its linearization.

Linearization principles for controllability and observability are related to the fact that (in nonsingular case) linearization (Frechet differential) of the end-point map exists and determines local properties of this map. Moreover this linearization is calculated via some linear control system which is natural to call the linearization of the original control system.

Proposition 2.2.2. *Let the r.-h. side of the control system $\dot{x} = f(x,u)$ be C^1-smooth. Consider $L_\infty([0,T],U)$ as the set of admissible controls with the corresponding metric. Then the end-point map $EP_{x^0}^T$ is differentiable at any $\bar{u} \in L_\infty$. The input/trajectory map is differentiable if one provides the space of trajectories with C^0-metric. The differential of the latter metric is a correspondence $v(\cdot) \mapsto y(\cdot)$ defined by the linearization of the system $\dot{x} = f(x,u)$ at $\tilde{u}(\cdot)$:*

$$\dot{y} = A(t)y + B(t)v, \ A = (\partial f/\partial x)|_{\bar{u}(t),\bar{x}(t)}, \ B(t) = (\partial f/\partial u)|_{\bar{u}(t),\bar{x}(t)}.$$

The linearization of the end-point map at $\bar{u}(\cdot)$ is then defined by the correspondence

$$v(\cdot) \mapsto y(T).$$

By the Cauchy formula these differentials can be calculated as

$$v(\cdot) \mapsto x(T) = \Phi(t)\left(x^0 + \int_0^t \Phi^{-1}(s)B(s)u(s)ds\right),$$

where Φ is the fundamental matrix of the homogeneous linear system $\dot{\Phi} = A(t)\Phi, \ \Phi(0) = I$. $\square$

Proposition 2.2.3 (Linearization principle for controllability)**.** *If under the assumptions of the previous proposition the linearization is controllable, then the original system is locally controllable along the trajectory $\bar{x}(t)$ driven by the control $\bar{u}(t)$.* $\square$

2.3. **Beyond the linearization principle: differential-geometric methods.** There are many cases where the linearization principle fails to predict controllability correctly. Often if linearization is noncontrollable nonlinear terms manage to provide controllability.

Example 1. Controlled rotation of a satellite

Example 2. Control-linear system

$$\dot{x} = \sum_{j=1}^{r} X^j(x)u_j(t),$$

along zero control.

2.4. **Symmetric systems; orbit theorem.** Typical system with noncontrollable linearization is control-linear system

$$\dot{x} = \sum_{i=1}^{r} f^i(x)u_i, \ x \in \mathbb{R}^n, \ u = (u_1, \ldots, u_r) \in \mathbb{R}^r. \tag{4}$$

Linearization at point $x^0, \ \bar{u} \equiv 0$ is $\dot{\xi} = Bv$, where the rank of $B = \left(f^1(x^0) \cdots f^r(x^0)\right)$ is $\leq r < n$.

Still in "many" cases this system is controllable, even more generic control-linear system (with more than one control) is controllable. This is a consequence of Rashevsky -Chow theorem, whose generalization - so-called Orbit Theorem - we are going to formulate.

It will be clear from our presentation below that controllability of (4) by means of measurable bounded controls is equivalent to controllability by means of piecewise-constant controls. Moreover one can take piecewise-constant controls with only one nonvanishing component $u = (0, \ldots, \pm 1, \ldots 0)$. Then the corresponding flows of the system (4) are the compositions

$$P = e^{t_1 f^{j_1}} \circ \cdots \circ e^{t_N f^{j_N}} \tag{5}$$

where $j_k \in \{1, \ldots, r\}$ and $t_j \in \mathbb{R}$. The attainable set of this system from point x^0 is called an *orbit* $\mathcal{O}_{x^0}$ of the family of vector fields $\mathcal{F} = \{f^1, \ldots, f^r\}$.

Proposition 2.4.4 (Orbit Theorem; Nagano-Stefan-Sussmann). *An orbit of the family $\mathcal{F} = \{f^1, \ldots, f^r\}$ is an immersed submanifold of $\mathbb{R}^n$; the tangent space to the orbit at a point $\hat{x}$ is spanned by the evaluated at $\hat{x}$ vector fields $AdPf^j$, where P are arbitrary diffeomorphisms defined by (5), $j = 1, \ldots, r$. In the C^ω-case the tangent space to the orbit at a point $\hat{x}$ is spanned by the evaluated at $\hat{x}$ iterated Lie brackets of the vector fields $f^j, \ j = 1, \ldots, r$. In the C^∞-case the values of the Lie brackets are contained in the tangent space to the orbit.* $\square$

The following classical result is an immediate corollary of the Orbit Theorem.

Proposition 2.4.5 (Rashevsky-Chow Theorem). *If for each $\hat{x} \in \mathbb{R}^n$ the iterated Lie brackets of the vector fields from $\mathcal{F}$ evaluated at $\hat{x}$ span $\mathbb{R}^n$, then the orbit coincides with $\mathbb{R}^n$.* $\square$

2.5. **Positive orbit: nonvoidness of interior and its consequences.** The Orbit Theorem resolves the issue of controllability for the system (4) which possesses an important property of symmetry: $f \in \mathcal{F} \Rightarrow (-f) \in \mathcal{F}$. This allows to involve "negative time-durations" in (5); indeed a motion in negative time direction along a vector field $f \in \mathcal{F}$ is the same as motion in positive time direction along a vector field $-f \in \mathcal{F}$.

If a control system is nonsymmetric, e.g. if it is control-affine system of the form:

$$\dot{x} = f^0(x) + \sum_{i=1}^{r} f^i(x)u_i,$$

then controllability is related to the notion of *positive orbit* of the system.

Definition 2.5.6. *Positive orbit* $\mathcal{O}^+_{x^0}$ *of the system* $\mathcal{F}$ *is the set of points attained from* x^0 *by means of the compositions of diffeomorphisms* $e^{t_1 f^{j_1}} \circ \cdots \circ e^{t_N f^{j_N}}$, *where* $j_k \in \{1, \ldots, r\}$ *and* $t_j \in \mathbb{R}_+$. □

Positive orbits are very far from being immersed submanifolds. Nevertheless they possess some important properties.

Proposition 2.5.7 (Krener theorem). *Interior of positive orbit is nonvoid; moreover this interior is dense in the positive orbit.* □

Proof. see [20, 3] □

Remark 2.5.8. *Let us note that the interior point constructed in this proof is* normally or regularly achieved, *i.e. it is attainable by a regular control along which the linearization is controllable.* □

The argument involved in the proof of these theorem has many applications. Thus by this argument we can derive property of global normal controllability from global controllability.

Definition 2.5.9. *A system is normally globally controllable from* $\tilde{x}$ *if the corresponding end-point map is surjective and besides each point of* $\mathbb{R}^n$ *is normally attainable.* □

Proposition 2.5.10. *Global controllability* $\Rightarrow$ *global normal controllability.*

Another consequence is the following result.

Proposition 2.5.11. *If a system satisfies the Lie rank necessary condition and its attainable set is dense in* $\mathbb{R}^N$, *then this attainable set coincides with* $\mathbb{R}^N$. □

3. Lie extension (saturation) of nonlinear control systems

3.1. Some types of Lie extensions.

Definition 3.1.1 (Lie saturation; [20]). *Let* $\mathcal{F}$ *be an analytic (or Lie determined) system. Strong Lie saturate of* $\mathcal{F}$ *is the maximal set* $\hat{\mathcal{F}} \subseteq Lie(\mathcal{F})$ *such that*

$$clos\mathcal{A}^{\leq T}_{\hat{\mathcal{F}}}(\hat{x}) \subseteq clos\mathcal{A}^{\leq T}_{\mathcal{F}}(\hat{x}). \tag{1}$$

The Lie saturate is the maximal set $\tilde{\mathcal{F}} \subseteq Lie(\mathcal{F})$ *such that*

$$clos\mathcal{A}_{\tilde{\mathcal{F}}}(\hat{x}) \subseteq clos\mathcal{A}_{\mathcal{F}}(\hat{x}). \ \square \tag{2}$$

Remark 3.1.2. *For a symmetric system the Lie saturation of* $\mathcal{F}$ *coincides with* $Lie(\mathcal{F})$. *It is very difficult to construct in general the Lie saturation.*

Definition 3.1.3. *A (not necessarily maximal) set* $\hat{\mathcal{F}}$ *which satisfies (1) is called Lie extension.* □

Let us mention some types of Lie extensions.

First one can take a closure of $\mathcal{F}$ in the topology defined by seminorms introduced in the subsection 1.2.

Proposition 3.1.4. *A closure $clos(\mathcal{F})$ of $\mathcal{F}$ in Whitney topology is a Lie extension.* □

Proof based on classical results on continuous dependence of the solutions of ODE on initial data and the r.-h. side.

Another kind of Lie extension is extension by convexification.

Proposition 3.1.5. *For a control system $\mathcal{F}$ its convexification*

$$conv(\mathcal{F}) = \left\{ \sum_{j=1}^{N} \alpha_j f^j \mid \right.$$

$$\left. \alpha_j \in C^\infty(\mathbb{R}^n), f^j \in \mathcal{F},\ N \in \mathbb{N},\ \sum_{j=1}^{N} \alpha_j = 1,\ \alpha_j \geq 0,\ j = 1, \ldots, N, \right\}$$

is a strong Lie extension. Its conic hull

$$conv(\mathcal{F}) = \left\{ \sum_{j=1}^{N} \alpha_j f^j \mid \alpha_j \in C^\infty(\mathbb{R}^n), f^j \in \mathcal{F},\ N \in \mathbb{N},\ \alpha_j \geq 0,\ j = 1, \ldots, N \right\},$$

is a Lie extension. □

This is a very important kind of extension which underlies a powerful theory of relaxed or sliding mode controls. We talk about it in the next subsection.

Another type of Lie extension is extension by an adjoint action of normalizer.

Definition 3.1.6 (see [20]). *A diffeomorphism P is a strong normalizer for the control system $\mathcal{F}$ if*

$$P\left(\mathcal{A}_{\mathcal{F}}^{\leq T}(P^{-1}(\hat{x}))\right) \subseteq clos\mathcal{A}_{\mathcal{F}}^{\leq T}(\hat{x}),$$

$\forall \hat{x},\ \forall T > 0$*;*

A diffeomorphism P is a normalizer for the control system $\mathcal{F}$ if

$$P\left(\mathcal{A}_{\mathcal{F}}(P^{-1}(\hat{x}))\right) \subseteq clos\mathcal{A}_{\mathcal{F}}(\hat{x}),$$

$\forall \hat{x}$. □

In practice one uses the following sufficient criterion for searching normalizers.

Lemma 3.1.7 (see [20]). *A diffeomorphism P is a strong normalizer for the control system $\mathcal{F}$ if both $P(\hat{x})$ and $P^{-1}(\hat{x})$ belong to $clos\mathcal{A}_{\mathcal{F}}^{\leq T}(\hat{x}),\ \forall \hat{x},\ \forall T > 0$.*

A diffeomorphism P is a normalizer for the control system $\mathcal{F}$ if both $P(\hat{x})$ and $P^{-1}(\hat{x})$ belong to $clos\mathcal{A}_{\mathcal{F}}(\hat{x}),\ \forall \hat{x}$. □

Proposition 3.1.8. *The set*

$$\tilde{\mathcal{F}} = \{AdPf \mid f \in \mathcal{F}, P \text{ - strong normalizer of } \mathcal{F}\}$$

is strong Lie extension of $\mathcal{F}$;

The set

$$\tilde{\mathcal{F}} = \{AdPf \mid f \in \mathcal{F}, P \text{ - normalizer of } \mathcal{F}\}$$

is Lie extension of $\mathcal{F}$. □

Later on we will consider in details a extension by adjoint action of a particular kind of normalizer; this extension arises from so-called "reduction formula" (see [4]).

4. Extension by convexification and by time rescaling. Relaxed (sliding mode) controls

Example. Zig-zag motion. The tajectories of

$$\dot{x} = u, \ u \in \{-1, 1\}, \tag{1}$$

can approximate arbitrarily well in C^0-metric any curve lying in the cone $|x| \leq t$. That means that we can approximately follow any dynamics which is a convex combination of original dynamics.

Let us consider a slightly more complex dynamics.

$$\dot{x} = f(x, u), \ u \in \{u^-, u^+\} \subset \mathbb{R}^r, \tag{2}$$

i.e. at each point one can move either along vector field $f^-(x) = f(x, u^-)$ or $f^+(x) = f(x, u^+)$. Even in this case we can approximate convex combinations of f^- and f^+.

Divide interval $[0, 1]$ into N intervals of equal lengths N^{-1} and each of the subintervals I_j $(j = 1, \ldots, N)$ into two subintervals I_j^+, I_j^- of length $N^{-1}/2$. Consider piecewise constant control $\bar{u}(t)$ equal to u^+ on all intervals I_j^+ and u^- on all intervals I_j^-. It is plausible that, if N is large, then trajectory driven by this control is close to the trajectory of the vector field $f^0(x) = (f^+(x) + f^-(x))/2$.

Let observe that

$$\int_0^t f(x, \bar{u}(t))dt - tf^0(x) \to 0$$

uniformly with respect to $t \in [0, 1]$ as $N \to +\infty$. It looks like beings a correct convergence notion for the r.h. sides of ODE.

This is one of the ideas underlying theory of relaxed controls.

4.1. Continuous dependence of solutions of ODE on the r.-h. side in the metric of relaxed controls.

Consider a time-variant ODE

$$\dot{x} = X(t, x), \tag{3}$$

in $\mathbb{R}^n$.

Definition 4.1.1 ([16]). *The relaxation pseudometric in the space of time-variant vector fields $X(t, x)$ is defined by the seminorms*

$$\|X(t, x)\|_K^{rx} = \max_{t, t' \in \mathbb{R}} \left\{ \left\| \int_t^{t'} X(\tau, x) d\tau \right\|_{0,K} \right\}.$$

The relaxation metric is obtained by identification of the vector fields whose difference vanishes for almost all $\tau \in [0, T]$. □

We can also introduce the norms

$$\|X(t,x)\|^{rx}_{s,K} = \max_{t,t'\in\mathbb{R}} \left\{ \left\| \int_t^{t'} X(\tau,x)d\tau \right\|_{s,K} \right\}.$$

If the initial moment is chosen fixed $t = 0$ we can define the relaxation seminorm as

$$\|X(t,x)\|^{rx}_{s,K} = \max_{t'\in\mathbb{R}} \left\{ \left\| \int_0^{t'} X(\tau,x)d\tau \right\|_{s,K} \right\}.$$

Example. Fast-oscillating vector field. Consider a vector field of the form $X_t(x) = \cos\omega t Y(x)$. Its relaxation seminorm is computed as

$$\max_{t,t'} \left\| \int_t^{t'} \cos\omega\tau d\tau Y(x) \right\| = \omega^{-1} \max_{t,t',x} |\sin\omega(t'-t)| \|Y\| \leq 2\omega^{-1}\|Y\| \to 0, \text{ as } \omega \to +\infty.$$

Below all our vector fields $X(t,x)$ will vanish for t outside some finite interval $[a,b]$, besides they and their derivatives w.r.t. x are bounded by integrable functions:

$$\|X(t,x)\| + \left\| \frac{\partial X}{\partial x}(t,x) \right\| \leq L_X(t), \ \forall t, x. \tag{4}$$

A family of vector fields is called uniformly Lipschitzian if for each of them (4) is satisfied and there exists uniform bound $C \geq \int_{\mathbb{R}} L_X(t)dt$.

The following result concerning continuous dependence of solutions of ODE on the r.-h. side holds:

Theorem 4.1.2. *Consider ODE's (3) with r.-h. sides $X(t,x)$ from a uniformly Lipschitzian family. Then the solutions depend continuously in uniform C^0-metric from the r.-h. sides varying continuously in relaxation metric $\|\cdot\|^{rx}_0$.* $\square$

see [3]. We prove this fact under stronger assumption of convergence in $\|\cdot\|^{rx}_2$. Let $X^n_t(x) = X_t(x) + Y^n_t(x)$ and $\|\|Y^n_t(x)\|^{rx}_1 \to 0$.

Consider $\overrightarrow{\exp}\int_0^t (X_\tau(x) + Y^n_\tau(x))d\tau$ which by variational formula can be represented as

$$\overrightarrow{\exp}\int_0^t (X_\tau + Y^n_\tau d\tau = \overrightarrow{\exp}\int_0^t (\overrightarrow{\exp}\int_0^\tau \mathrm{ad}X_\theta d\theta)Y^n_\tau d\tau \circ \overrightarrow{\exp}\int_0^t X_\tau d\tau.$$

Let us prove that $\|Z^n_t\|^{rx}_0 = \|(\overrightarrow{\exp}\int_0^t \mathrm{ad}X_\theta d\theta)Y^n_t\|^{rx}_0 \to 0$. Indeed

$$\int_0^t (\overrightarrow{\exp}\int_0^\tau \mathrm{ad}X_\theta d\theta)Y^n_\tau d\tau =$$

$$= (\overrightarrow{\exp}\int_0^t \mathrm{ad}X_\theta d\theta)Y^n_t - \int_0^t (\overrightarrow{\exp}\int_0^\tau \mathrm{ad}X_\theta d\theta)[X_\tau, \int_0^\tau Y^n_\theta d\theta]d\tau \to 0.$$

The equation for the perturbation flow L_t is

$$dL_t/dt = L_t \circ Z^n_t, \ L_0 = I.$$

Then

$$L_t = I + \int_0^t L_\tau \circ Z_\tau^n d\tau = I + L_t^n \circ \int_0^t Z_\tau^n d\tau - \int_0^t L_\tau \circ Z_\tau^n \circ \int_0^\tau Z_\theta^n d\theta d\tau,$$

whart proves that $L_t \to I$. □

□

4.2. **Sliding modes or relaxed controls.** Consider probabilistic Radon measures on the space of control parameters $\mathbb{R}^r$. (Recall that Radon measures μ are linear continuous functionals on the space of continuous functions with compact supports.) Being probabilistic means that they are nonnegative and that for $\zeta(u) \equiv 1$ there holds $\langle \mu, \zeta \rangle = 1$.

In our case these measure will act not on functions which merely depend on u, but on the r.-h. sides of control systems which are either functions $f(x,u)$ or $f(t,x,u)$. Besides we will involve as controls time-dependent families of measures $t \mapsto \mu_t$.

Definition 4.2.3. *A family $t \mapsto \mu_t$ is weakly measurable in t if for each continuous function $g(t,u)$ with compact support in u for each t the function*

$$\gamma(t) = \langle \mu_t, g(t,u) \rangle = \int g(t,u) d\mu_t(u)$$

is Lebesgue-measurable. □

In future we will assume that all the measures are supported in a bounded set $N \subset \mathbb{R}^r$.

Example. 1) The family $\mu_t = \delta_{u(t)}$ is an ordinary or nonrelaxed control: $\langle \mu_t, g(t,u) \rangle = g(t,u(t))$.

2) The family $\mu_t = \left(\delta_{u^+(t)} + \delta_{u^-(t)}\right)/2$ is a relaxed control:

$$\langle \mu_t, g(t,u) \rangle = \left((g(t,u^-(t)) + g(t,u^+(t))\right)/2.$$

Control system

$$\dot{x} = f(x,u)$$

driven by a relaxed control μ_t is a differential equation

$$\dot{x} = \langle \mu_t, f(t,x,u) \rangle. \tag{5}$$

Consider the set

$$F_{co}(t,x) = \{\langle \mu_t, f(t,x,u) \rangle |\ \mu \text{ are all probability measures}\}.$$

Proposition 4.2.4. *The set $F(t,x)$ coincides with the convex hull of the set $F(t,x) = \{f(t,x,u)|\ u \in U\}$.* □

Definition 4.2.5. *A sequence ν^j of probability measures converges weakly to a measure ν if for each continuous $g(u)$ with compact support*

$$\langle \nu^j, g(u) \rangle \to \langle \nu, g(u) \rangle, \ \textit{as } j \to \infty.$$

A sequence μ_t^j of relaxed controls converges weakly to a relaxed control $\tilde{\mu}_t$ if for each continuous $g(t,u)$ with compact support

$$\int_{\mathbb{R}} \langle \mu_t^j, g(t,u)\rangle dt \to \int_{\mathbb{R}} \langle \tilde{\mu}_t, g(t,u)\rangle, \ \textit{as } j \to \infty. \ \square$$

Ordinary controls considered as relaxed controls may converge weakly to a relaxed control; this convergence is not the same as weak convergence of functions.

Let us define also a strong convergence of relaxed controls. Strong convergence of Radon measures μ is defined by a norm:

$$\|\mu\|_s = \mathrm{Var}(\mu) = \sup\{\langle \mu, g(u)\rangle : \ \|g(u)\|_{C^0} \leq 1\}.$$

Strong convergence of relaxed controls μ_t^j to the relaxed control $\tilde{\mu}_t$ means:

$$\int_{\mathbb{R}} \|\mu_t^j - \tilde{\mu}_t\|_s \to 0, \ \text{as } j \to \infty.$$

What is strong convergence for "ordinary controls" seen as generalized controls??

A very important fact is that weak convergence of relaxed controls implies convergence of the r.-h. sides (which result from substitution of these controls into control system (5)) in the relaxation metric.

Theorem 4.2.6. *Assume that*

$$\mu_t^j \stackrel{weak}{\to} \tilde{\mu}_t,$$

as $j \to \infty$, then

$$\langle \mu_t^j, f(t,x,u)\rangle \to \langle \tilde{\mu}_t, f(t,x,u)\rangle$$

in the relaxation metric, i.e.

$$\sup_{t_0,t_1,x} \left| \int_{t_0}^{t_1} \langle \mu_t^j - \tilde{\mu}_t, f(t,x,u)\rangle dt \right| \to 0 \ \textit{as } j \to \infty. \ \square$$

4.3. **Approximation of relaxed controls by ordinary controls.** We have already established the following sequence of facts:

$$\begin{array}{c} \text{weak convergence of relaxed controls} \\ \Downarrow \ \text{Theorem 4.2.6} \\ \text{convergence of r.h. sides in relaxation metric} \\ \Downarrow \ \text{Theorem 4.1.2} \\ \text{uniform convergence of the trajectories} \end{array}$$

What lacks for proving that relaxation is a particular type of Lie extension is the fact that sets of points attainable by relaxed controls are close to the ones attainable by ordinary controls. In fact a stronger fact is true: as we saw in examples trajectories generated by relaxed controls can be uniformly approximated by the ones corresponding to ordinary controls. Due to the previous diagram it suffices to prove that the relaxed controls can be weakly approximated by ordinary controls.

This fact is the contents of the following

Theorem 4.3.7 (Approximation lemma; [16])**.** *Let Σ be a metric space. Let $\sigma \mapsto \mu_t(\sigma)$ be a family of relaxed controls which is continuous with respect to $\sigma \in \Sigma$ in topology of strong convergence. Let the supports of all $\mu_t(\sigma)$ be contained in a bounded set $B \subset \mathbb{R}^r$. Then there exists a family of piecewise-constant ordinary controls $u^j(t;\sigma),\ j = 1, 2, \dots$ with values in B such that the sequence $\delta_{u^j(t;\sigma)}$ converges weakly to $\mu_t(\sigma)$ uniformly w.r.t. $\sigma in \Sigma$ as $j \to +\infty$.* $\square$

5. Reduction formula and applications

5.1. **Control-affine systems: adjoint action of control flow.** Consider control-affine nonlinear system:

$$\dot{q} = f(q) + G(q)v(t),\ q \in \mathbb{R}^N,\ v \in \mathbb{R}^r, \tag{1}$$

where $G(q) = \left(g^1(q), \dots, g^r(q)\right)$, and $f(q), g^1(q), \dots, g^r(q)$ are complete real-analytic vector fields in $\mathbb{R}^N$; $v(t) = (v_1(t), \dots, v_r(t))$ is a control.

We will use the notation the notation of chronological calculus introduced above. The following result is a useful consequence of the varational formula introduced in the Subsection 1.3.

Proposition 5.1.1. *Assume that the vector fields $g^1(q), \dots, g^r(q)$, are mutually commuting: $\left[g^i, g^j\right] = 0,\ \forall i, j$. Then the flow of the system (13) can be represented as a composition of flows:*

$$\overrightarrow{\exp} \int_0^t (f + Gv(\tau))\, d\tau = \overrightarrow{\exp} \int_0^t (e^{-GV(\tau)})_* f d\tau \circ e^{GV(t)}, \tag{2}$$

where $V(t) = \int_0^t v(s)ds$. $\square$

The following result will be instrumental in our reasoning. It is based on the formula (2) on one side and on the results on continuous dependence of flows on the right-hand side of ODE's (see [4, Propositions 1 and 1']).

It says that one can reduce the study of controllability of the system 13 to the study of controllability of the reduced control system

$$\dot{x} = \left(\left(e^{-GV(\tau)}\right)_* f\right)(x), \tag{3}$$

on the quotient space $\mathbb{R}^N/\mathcal{G}$, where $\mathcal{G}$ is the linear span of the values of the constant vector fields $g^1, \dots, g^r$.

Denote by $\mathcal{F}$ the family of vector fields $\{f(q) + G(q)v|\ v \in \mathbb{R}^r\}$. Denote by $\mathcal{F}'$ the family of vector fields $\{\left(e^{-GV}\right)_* f|\ V \in \mathbb{R}^r\}$.

Theorem 5.1.2. *(see* [4]*) Let $\pi_{\mathcal{G}}$ be the canonical projection of the quotient space $\mathbb{R}^N \to \mathbb{R}^N/\mathcal{G}$ and $\mathcal{A}_{\mathcal{F}'}(\pi_{\mathcal{G}}(\tilde{x}))$ be the attainable set of the reduced system (15). Then the closures of the sets $\mathcal{A}_{\mathcal{F}}(\tilde{x})$ and $\pi_{\mathcal{G}}^{-1}(\mathcal{A}_{\mathcal{F}'}(\pi_{\mathcal{G}}(\tilde{x})))$ in $\mathbb{R}^N$ coincide, as well as coincide the closures of the sets $\mathcal{A}^T_{\mathcal{F}}(\tilde{x})$ and $\pi_{\mathcal{G}}^{-1}\left(\mathcal{A}^T_{\mathcal{F}'}(\pi_{\mathcal{G}}(\tilde{x}))\right)$.* $\square$

Evidently the fact of system being control-affine is important for the validity of the formula (2) and therefore of the previous Theorem.

One can derive various controllability results from the Theorem 7.4.9. We refer the readers to [4] for their formulation.

6. Control-affine systems with impulsive and distribution-like inputs

2.1. Introduction. In this section we will work with control affine nonlinear systems of the form:

$$\dot{x}(\tau) = f_\tau(x(\tau)) + G_\tau\left(x(\tau)\right) u(\tau),\ x \in R^n,\ u \in R^r,\ G_\tau(x) \in R^{n\times r} \tag{1}$$

where $G_\tau(x) = \left(g_\tau^1(x) \cdots g_\tau^r(x)\right)$ and $f_\tau(x), g_\tau^i(x)$ $(i = 1, \ldots, r)$ are time-variant vector fields in R^n. We develop a formalism for dealing with distribution-like inputs for the system (1).

There are various difficulties arising when one tries to define trajectory corresponding to a distribution-like input u. For example if the input u of a system $\dot{x}(\tau) = g_\tau\left(x(\tau)\right) u(\tau),\ x(0) = x_0$, is a Dirac measure $\delta(\tau - \tau_0)$, then it is natural to expect that the corresponding trajectory $x(\cdot)$ will 'jump' at τ_0. Transforming the differential equation into integral one $x(t) = x_0 + \int_0^t g_\tau(x(\tau))u(\tau)d\tau$ we encounter a necessity to integrate an apparently discontinuous function $g_\tau(x(\tau))$ with respect to a measure $\delta(\tau - \tau_0)$, which contains an atom exactly at the point of discontinuity. Such an integration is not defined properly.

Here we describe an approach to a construction of generalized trajectories for the system (1). The idea (which is close to the one represented in [21, 24]) amounts to furnishing the space of 'ordinary', say ,integrable, inputs $u(\cdot)$ (say $\mathcal{U} = L_1^r[0,T]$) and of trajectories $x(\cdot)$ with weak topologies for which the input-trajectory map $u(\cdot) \mapsto x(\cdot)$ is still (uniformly) continuous. In this case one can extend this map by continuity onto a completion of the space of inputs, which may contain distributions.

The core issue of this approach is proving the continuity of the input/trajectory map. It is convenient to introduce topology in the space of inputs as an induced one by a topology in the space of their primitives. Note that the integrable inputs their primitives and also the corresponding trajectories belong to $W_{1,1}[0,T]$ - the space of absolutely continuous functions.

Let us survey briefly the existing results. In the early 70's M.A.Krasnosel'sky and A.V.Pokrovsky ([21]) considered C^0-metric in the space $W_{1,1}[0,T]$ of the primitives and of the trajectories, and established continuity (called by them vibrocorrectness) of the input/trajectory map. They proved the extensibility of the input-trajectory map onto the space of continuous measures - generalized derivatives of continuous (but not absolutely continuous) functions. Yu.V.Orlov ([24]) used similar method to prove extensibility of the input-trajectory map to the space of Radon measures (the generalized derivatives of the functions of bounded variation). Our method ([27, 28, 29]) allows not only to extend the input-trajectory map onto a larger space $W_{-1,\infty}$ of generalized derivatives of measurable essentially bounded functions, but also to obtain a representation of the generalized trajectories via the generalized primitives of the inputs. About the same time A.Bressan proved ([7]) extensibility of the input-trajectory map on the same space.

The key tool of our approach is a class of representation formulae for the trajectories of the system (1). These formulae are multiplicative analogies of the classical integration by parts formula. They allow to represent the (generalized) trajectories via solutions of ODE, involving the (generalized) primitives of the (generalized) inputs.

There is another approach to the construction of generalized trajectories corresponding to the distribution-like inputs - the one based on completion and reparametrization of graphs of discontinuous functions. It allows to deal with the systems for which the 'commutativity assumption' fails, but also the continuity of the input/trajectory map is not maintained. We refer to the publications of B.Miller ([23]), A.Bressan, F.Rampazzo ([8]) and to the bibliography therein for the detailed description of this approach.

6.1. **Multiplicative Analogy of Integration by Parts Formula.** First consider control-linear (without a drift term) system

$$\dot{x}(\tau) = Y_\tau(x(\tau))u(\tau),\ \tau \in [0,T],\ x \in R^n,\ u \in R. \tag{2}$$

For a moment assume the control $u \in R$ to be scalar-valued, $u(\cdot) \in L_1[0,T]$. Let the right-hand side Y_τ be differentiable with respect to x and C^1 with respect to τ. If for a given $u(\cdot)$ the solution (the flow) generated by the equation (2) exists for $\tau \in [0,T]$, we will denote it (following [1]) by $P_t = \overrightarrow{\exp} \int_0^t Y_\tau u(\tau)d\tau,\ t \in [0,T]$ and call it right chronological exponential. The following proposition provides an expression for P_t in terms of the primitive $v(\cdot) = \int_0^{\cdot} u(\xi)d\xi$ of $u(\cdot)$.

We make an agreement concerning the notation. If a composition of diffeomorphisms $P \circ Q$ is applied to a point x^0 this means that first P and then Q is applied. In general a result of application of a diffeomorphism P to a point x^0 will be denoted by $x^0 \circ P$.

Proposition 6.1. *If the solution of the equation (2) and the diffeomorphisms $e^{Y_t v(t)}$ exist for all $t \in [0,T]$ then the following equality holds:*

$$P_t = \overrightarrow{\exp} \int_0^t Y_\tau u(\tau)d\tau = \overrightarrow{\exp} \int_0^t \left(- \int_0^1 \left(e^{-\xi Y_\tau v(\tau)}\right)_* \dot{Y}_\tau v(\tau) d\xi \right) d\tau \circ e^{Y_t v(t)}. \tag{3}$$

Remark. The diffeomorphism $e^{Y_t v(t)} = \overrightarrow{\exp} \int_0^1 Y_t v(t) d\xi$ in the formula (3) is generated by the time-invariant vector field $Y_t v(t)$ with t *fixed*. The notation $\left(e^{-\xi Y_\tau v(\tau)}\right)_* \dot{Y}_\tau v(\tau)$ stays for the pullback of the vector field $\dot{Y}_\tau v(\tau)$ by the differential of the diffeomorphism $e^{-\xi Y_\tau v(\tau)}$ with τ fixed.

We relate (3) to the integration by parts formula due to the following reason. If for all $\tau \in [0,t]$ the vector fields Y_τ and $\dot{Y}_\tau$ commute, then $e^{\xi \mathrm{ad} Y_\tau v(\tau)} \dot{Y}_\tau = \dot{Y}_\tau,\ \forall \tau \in [0,t]$, and the formula (3) takes form

$$\overrightarrow{\exp} \int_0^t Y_\tau u(\tau)d\tau = \overrightarrow{\exp} \int_0^t (-\dot{Y}_\tau v(\tau))d\tau \circ e^{Y_t v(t)}, \tag{4}$$

becoming a multiplicative analogy of the integration by parts formula

$$\int_0^t Y_\tau u(\tau)d\tau = \int_0^t Y_\tau dv(\tau) = - \int_0^t \dot{Y}_\tau v(\tau)d\tau + Y_t v(t).$$

The result of the Proposition 6.1 can be reformulated for the multi-input system with $Y_\tau(\tau)=\sum_{i=1}^r Y_\tau^i u_i(\tau)d\tau$ under one crucial additional assumption.

Commutativity assumption. The vector fields $Y_\tau^1,\dots Y_\tau^r$ are pairwise commuting for each τ: $\left[Y_\tau^i, Y_\tau^j\right]=0, \forall i,j=1,\dots,r; \forall\tau\in[0,T]$.

M.A.Krasnoselsky and A.V.Pokrovsky proved ([21]), that this condition is necessary for vibrocorrectness, or in other words for the extensibility by continuity of the input/trajectory map.

This condition is equivalent to the Frobenius integrability condition for the differential (Pfaffian) systems span$\{Y_\tau^i : i=1,\dots,r\}$ with arbitrary fixed τ. These systems are called 'distributions' in differential geometry and global analysis; we keep the name 'distribution' for the generalized inputs.

Proposition 6.2. *If the commutativity assumption is verified then the formula (3) holds for the flow* $\overrightarrow{\exp}\int_0^t Y_\tau u(\tau)d\tau$ *generated by the multiinput system* $\dot{x}=Y_\tau(x)u(\tau)$. $\square$

In [28] more versions of the integration by parts formula can be found. The following result provides representation formula for the flow generated by control-affine nonlinear systems.

Teorema 6.3. *Let vector fields* f_τ, g_τ^i $(i=1,\dots,r)$ *be differentiable with respect to* x, *continuous with respect to* τ *and* g_τ^i *be* C^1 *with respect to* τ. *Let the vector fields* g_τ^i $(i=1,\dots,r)$ *satisfy the commutativity assumption for all* $\tau\in[0,t]$. *If for the input* $u(\cdot)\in L_1^r[0,t]$ *the solution* $\overrightarrow{\exp}\int_0^t(f_\tau+G_\tau u(\tau))d\tau$ *of the equation (1) and the diffeomorphisms* $e^{G_t v(t)}$ *exist for all* $t\in[0,T]$, *then*

(5)
$$\overrightarrow{\exp}\int_0^t (f_\tau+G_\tau u(\tau))d\tau = \overrightarrow{\exp}\int_0^t\left(e^{adG_\tau v(\tau)}f_\tau-\int_0^1 e^{\xi adG_\tau v(\tau)}\dot{G}_\tau v(\tau)d\xi\right)d\tau\circ e^{G_t v(t)},$$

where $v(\cdot)=\int_0^{\cdot}u(\eta)d\eta$. $\square$

Up to the end of this section we assume the commutativity assumption to hold for the vector fields g_τ^i $(i=1,\dots,r)$.

6.2. **Continuity of the Input-Trajectory Map. Generalized inputs and trajectories.** As long as we have obtained the formulae for trajectories in terms of the primitives of the inputs, the extensibility of the input/trajectory map follows rather easily from standard results on continuous dependence of solutions of ODE on the right-hand side.

Let us fix the initial point x_0 of our trajectories (we do it for the sake of simplicity of presentation; in [27] it is done for flows). Consider an input $u(\cdot)\in L_1^r[0,T]$, its primitive $v(\cdot)=\int_0^{\cdot}u(\xi)d\xi$ and the vector field

$$F_\tau(v)=e^{\mathrm{ad}G_\tau v}f_\tau-\int_0^1 e^{\xi\mathrm{ad}G_\tau v(\tau)}\dot{G}_\tau v(\tau)d\xi. \tag{6}$$

According to (5), the trajectory corresponding to the input $u(\cdot)$ can be represented as

$$P_t(u(\cdot)) = Q_t(v(\cdot)) = x_0 \circ \overrightarrow{\exp} \int_0^t F_\tau(v(\tau))d\tau \circ e^{G_t v(t)}.$$

Consider the triple of maps $u(\cdot) \overset{J}{\mapsto} v(\cdot) \mapsto Q_t(v(\cdot))$, where $u(\cdot) \in L_1^r[0,T]$, $v(\cdot) \in W_{1,1}^r[0,T]$. Introduce L_1^r-norm in the space $W_{1,1}^r[0,T]$ of $v(\cdot)$'s. The induced norm in the space of inputs will be denoted by DL_1:

$$\|u(\cdot)\|_{DL_1} = \left\| \int_0^{\cdot} u(\eta)d\eta \right\|_{L_1}.$$

As long as $W_{1,1}^r[0,T]$ is dense subspace of $L_1^r[0,T]$, then the completion of the space $L_1^r[0,T]$ of inputs with respect to the DL_1-norm coincides with the space of distributions, which are generalized derivatives of the functions from $L_1^r[0,T]$. With some abuse of notation we denote this space of distributions by $W_{-1,1}^r[0,T]$. (Recall that the smaller space of generalized derivatives of the square-integrable functions is Sobolev space denoted by $W_{-1,2}^r[0,T]$ or $H_{-1}^r[0,T]$.)

We will need another space of generalized inputs - the one adjoint to $W_{1,1}^r[0,T]$. Recall that any linear continuous functional on $W_{1,1}^r[0,T]$ can be defined by the formula:

$$\forall z(\cdot) \in W_{1,1}^r[0,T]: \ z(\cdot) \mapsto v_0 z(0) - \int_0^T v(\tau)\dot{z}(\tau)d\tau, \ v_0 \in R^r, \ v(\cdot) \in L_\infty^r.$$

This adjoint space will be denoted by $W_{-1,\infty}^r$; it can be identified with $R^r \times L_\infty$. The subspace of $W_{-1,\infty}^r$ identified with $\{(0, v(\cdot)\}$ is denoted by $\overset{o}{W}{}^r{}_{-1,\infty}$; the function $v(\cdot) \in L_\infty$ will be called generalized primitive of the corresponding element from $\overset{o}{W}{}^r{}_{-1,\infty}$.

What for the space of trajectories (which are absolutely continuous) then we furnish it with the L_1-norm.

Let us take any $\alpha > 0$ and consider the set $\mathcal{U}_\alpha$ of the inputs from $L_1^r[0,T]$ whose primitives are uniformly bounded by α on $[0,T]$. The DL_1-completion of $\mathcal{U}_\alpha$ coincides with the α-ball in the space $W_{-1,\infty}^r$.

We proved in [27], that the input-trajectory map is uniformly continuous on $\mathcal{U}_\alpha$, furnished with DL_1-norm, if the space of trajectories is furnished with the L_1 norm. This means that we can (as long as $\alpha > 0$ is arbitrary) extend the input-trajectory map onto the set of generalized inputs $W_{-1,\infty}^r$. The corresponding generalized trajectories will be the functions from $L_1^n[0,T]$. They can be computed via the generalized primitives of the inputs by means of the equation (5). This gives us the following result.

Theorem 6.4. *Consider control-affine nonlinear system (1)*

$$\dot{x}(\tau) = f_\tau(x(\tau)) + \sum_{i=1}^r g_\tau^i(x(\tau))u(\tau), \ q \in R^n, \ u \in R^r.$$

Let f_τ, g_τ^i $(i = 1, \ldots, r)$ be time-variant vector fields which are infinitely differentiable with respect to x, continuous with respect to τ. Let g_τ^i $(i = 1, \ldots, r)$ be C^1 with respect to τ and satisfy the commutativity assumption. Then for each generalized input from

$\overset{o}{W}{}^{r}{}_{-1,\infty}$ *with the generalized primitive* $v(\cdot)$ *the formula (5) defines the* DL_1*-continuous extension of the input/trajectory of this system. The extension coincides with the classical input/trajectory map on the space of ordinary inputs and is continuous with respect to* DL_1*-norm of the space* $\overset{o}{W}{}^{r}{}_{-1,\infty}$ *and* L_1*-norm in the space of trajectories.* $\square$

6.3. **Example: impulsive controls.** To illustrate the previous result let us compute the trajectory of the control-affine system (1)

$$\dot{x}(\tau) = f_\tau(x(\tau)) + G_\tau(x(\tau))u(\tau)$$

driven by the impulsive control $u = \sum_{i=1}^{N} u_i\delta(\tau - \tau_i)$ - a linear combination of Dirac measures located on the time-axis. In principle N can be finite or infinite; in the latter case we assume the series $\sum_{i=1}^{\infty} u_i$ to be convergent.

Let N be finite and $0 = \tau_0 < \tau_1 < \cdots < \tau_N \leq T$. The primitive of u is $v(\tau) = \sum_{i=1}^{N} u_i h(\tau - \tau_i),\ v(0) = 0$, with $h(\tau)$ being Heavyside function: $h(\tau) = 0,$ for $\tau < 0,\ h(\tau) = 1,$ for $\tau \geq 0)$. The function $v(\tau)$ is piecewise constant and equals $v_m = \sum_{i=1}^{m} u_i$, on the interval $[\tau_m, \tau_{m+1})$, while $v(0) = 0$. The expression (5) can be splitted into the product

$$Q_t = \prod_{i=1}^{N} \overrightarrow{\exp} \int_{\tau_{i-1}}^{\tau_i} \left(e^{\mathrm{ad} G_\tau v_{i-1}} f_\tau - \int_0^1 e^{\xi \mathrm{ad} G_\tau v_{i-1}} \dot{G}_\tau v_{i-1} d\xi \right) d\tau \circ$$

$$\overrightarrow{\exp} \int_{\tau_m}^{t} \left(e^{\mathrm{ad} G_\tau v_m} f_\tau - \int_0^1 e^{\xi \mathrm{ad} G_\tau v_m} \dot{G}_\tau v_m d\xi \right) d\tau \circ e^{G_t v(t)}, \text{para } \tau_m \leq t < \tau_{m+1}. \tag{7}$$

The following equality is established in [28].

$$\overrightarrow{\exp} \int_{\eta}^{\zeta} \left(e^{\mathrm{ad} G_\tau v} f_\tau - \int_0^1 e^{\xi \mathrm{ad} G_\tau v} \dot{G}_\tau v_{i-1} d\xi \right) d\tau = e^{G_\eta v} \circ \overrightarrow{\exp} \int_{\eta}^{\zeta} f_\xi d\xi \circ e^{-G_\zeta v}.$$

Applying it to the product (7) we obtain

$$Q_t = \left(\prod_{i=1}^{N} \left(\overrightarrow{\exp} \int_{\tau_{i-1}}^{\tau_i} f_\tau d\tau \circ e^{G_{\tau_i} u_i} \right) \right) \circ \overrightarrow{\exp} \int_{\tau_m}^{t} f_\tau d\tau. \tag{8}$$

From (8) one derives the following facts for the trajectories generated by the impulsive controls: i) they are piecewise continuous functions; ii) their continuous parts are pieces of the trajectories of the vector field f_τ; iii) their jumps occur at the instances τ_i $(i = 1, \ldots, N)$ are along the trajectories of the time-invariant vector field $G_{\tau_i} u_i$ and correspond to the time-duration 1.

If N is infinite and $u = \sum_{i=1}^{\infty} u_i\delta(\tau - \tau_i)$, with $\tau_i < \tau_{i+1}$ $(i = 0, 1, \ldots),\ \lim_{i\to\infty} \tau_i = \bar{\tau} \leq T$, then for $t < \bar{\tau}$, we proceed as in the previous example (only finite number of impulses occur before t). If $t \geq \bar{\tau}$, then Q_t is defined by (8) with $N = \infty$.

We proved in [27], that if $\sum_{i=1}^{\infty} u_i < \infty$ then this infinite product can be computed as a limit of partial finite products $\prod_{i=1}^{m},\ m \to \infty$.

6.4. **Time-Optimality of Generalized Controls.** In this subsection we will use the representation of the generalized trajectories for studying optimal control problems with generalized controls. We will formulate first-order optimality condition for these problems in the Hamiltonian form. An alternative approach and many results regarding optimality of generalized controls can be found in the book of Yu.Orlov ([25]).

Let us start with the definition of attainability for generalized controls. As long as the generalized trajectories are measurable functions, their values at a given instant t are not properly defined, and we define the attainability in approximative sense.

Definition 6.5. *Given a system (1), a point $\bar{x}$ is attainable from the point x_0 on the interval $[0,t]$ by means of a generalized control $\bar{u} \in \overset{o}{W}{}^r{}_{-1,\infty}$, if there exists a sequence of controls $u^m(\cdot) \in L_1^r[0,T]$, which converges to the control $\bar{u}$ in DL_1-norm, such that the points $x^m(t)$ of the corresponding trajectories $x^m(\cdot)$, (starting at $x(0) = x_0$) converge to $\bar{x}$.* $\square$

According to (5) the set $\mathcal{A}_{x_0}(u;[0,t])$ of points attainable from x_0 on the interval $[0,t]$ by means of the control $u \in \overset{o}{W}{}^r{}_{-1,\infty}$ is contained in the integral manifold $\mathcal{O}_{\tilde{x}}(\hat{G}_t)$ of the integrable (by virtue of the commutativity assumption) differential system $\hat{G}_t = \mathrm{span}\{g_t^1,\dots,g_t^r\}$ (with t fixed). This manifold passes through the point

$$\tilde{x} = x_0 \circ \overrightarrow{\exp} \int_0^t \left(e^{\mathrm{ad} G_\tau v(\tau)} f_\tau - \int_0^1 e^{\xi \mathrm{ad} G_\tau v(\tau)} \dot{G}_\tau v(\tau) d\xi \right) d\tau.$$

Here again $v(\cdot)$ is the generalized primitive of the control u.

In [4] we proved that the attainable set $\mathcal{A}_{x_0}(u;[0,t])$ *coincides* with this integral manifold. Therefore there is an r-dimensional manifold of points attainable from given x_0 on a given time interval $[0,t]$ by means of a given (!) generalized control $u \in \overset{o}{W}{}^r{}_{-1,\infty}$.

If $\mathcal{U}$ is a set of generalized controls, then the set of points attainable from x_0 on the time interval $[0,t]$ by means of some control from $\mathcal{U}$ will be denoted by $\mathcal{A}_{x_0}(\mathcal{U};[0,t])$.

Let us consider time-optimal problem for the system (1) with generalized controls $u \in \overset{o}{W}{}^r{}_{-1,\infty}$:

$$t \to \min, \tag{9}$$

$$\dot{x}(\tau) = f_\tau(x(\tau)) + G_\tau(x(\tau))u,\ x(0) = x_0,\ x \in R^n,\ u \in \overset{o}{W}{}^r{}_{-1,\infty}, \tag{10}$$

$$x_1 \in \mathcal{A}_{x_0}(u,[0,t]). \tag{11}$$

Note that the condition (11) corresponds to the fixed end-point condition in the classical problem of time optimality.

Definition 6.6. *The generalized control $\tilde{u} \in \overset{o}{W}{}^r{}_{-1,\infty}$ is locally optimal for the problem (9)-(11), if for some δ-neighborhood $\mathcal{U}_\delta$ of $\tilde{u}$ in DL_1-metric*

$$\forall \tau < t:\ \mathcal{A}_{x_0}(\tilde{u},[0,t]) \cap \mathcal{A}_{x_0}(\mathcal{U}_\delta,[0,\tau]) = \emptyset.\ \square$$

From the representation formula (5) it is easy to conclude that the generalized time-optimal control problem (9)-(11) can be reduced to the following classical time-optimal control problem with variable end-point condition

$$x(t) \in \mathcal{O}_{x_1}(\hat{G}_t). \tag{12}$$

for the system

$$\dot{x}(\tau) = \left(e^{\mathrm{ad} G_\tau v(\tau)} f_\tau - \int_0^1 e^{\xi \mathrm{ad} G_\tau v(\tau)} \dot{G}_\tau v(\tau) d\xi \right) (x(\tau)), \; x(0) = x_0, \tag{13}$$

with admissible controls $v(\cdot) \in L^r_\infty[0,T]$. Here $\mathcal{O}_{x_1}(\hat{G}_t)$ is the integral manifold of the differential system $\hat{G}_t$ passing through the point x_1.

Proposition 6.7. *A pair* $(\tilde{t}, \tilde{u}) \in R_+ \times \overset{o}{W}{}^r{}_{-1,\infty}$ *is locally optimal for the problem (9)-(11) if and only if for* $\tilde{v}(\cdot)$ *being the generalized primitive of the control* $\tilde{u}$ *the corresponding pair* $(\tilde{t}, \tilde{v}(\cdot)) \in L^r_\infty[0,T]$ *is* L_1*-locally optimal for the time-optimal problem (9),(13),(12).* $\square$

By virtue of the Proposition 6.7 a first-order necessary optimality conditions for the problem (9)-(11) can be derived from (in fact is equivalent to) the corresponding necessary condition for the reduced problem (9),(13),(12).

If the end-point condition (12) were admitting an explicit form $\Omega_t(x(t)) = 0$, then the first-order optimality condition is well known and looks as follows.

Proposition 6.1 ([26])**.** *If the control* $\tilde{v}(\cdot)$ *is a* L_1*-local minimizer for the problem (9), (13) with the variable end-point condition* $\Omega_t(x(t)) = 0$*, then there exists an absolutely continuous covector-function* $\tilde{\psi}(\cdot)$ *and a covector* $\tilde{\nu} \in R^{d^*}$, $(\tilde{\psi}(\cdot), \tilde{\nu}) \neq 0$*, such that the quadruple* $\left(\tilde{x}(\cdot), \tilde{v}(\cdot), \tilde{\psi}(\cdot), \tilde{\nu}\right)$ *satisfies:*
i) (pseudo)-Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial \psi}(x, \psi, v, \tau), \; \dot{\psi} = -\frac{\partial H}{\partial x}(x, \psi, v, \tau), \tag{14}$$

with the Hamiltonian

$$H(x, \psi, v, \tau) = \langle \psi, (e^{adG_\tau v} f_\tau - \int_0^1 e^{\xi\, adG_\tau v} \dot{G}_\tau v d\xi)(x) \rangle, \tag{15}$$

ii) the maximality condition

$$H(\tilde{x}(\tau), \tilde{\psi}(\tau), \tilde{v}(\tau), \tau) = M(\tilde{x}(\tau), \tilde{\psi}(\tau), \tau) = \sup_v H(\tilde{x}(\tau), \tilde{\psi}(\tau), v, \tau), \; a.e.; \tag{16}$$

iii) the transversality condition

$$\tilde{\psi}(t) = \partial \langle \tilde{\nu}, \Omega_t \rangle / \partial x \bigwedge M(\tilde{x}(t), \tilde{\psi}(t), t) + \partial \langle \tilde{\nu}, \Omega_t \rangle / \partial t \geq 0. \; \square \tag{17}$$

In general it is not a feasible option to put the condition (12) into an explicit form, because it means 'integrating' the differential system $\hat{G}_t$. We choose another way which allows us to avoid such an integration. This is done by introducing an auxiliary Hamiltonian $F = \langle \lambda, G_t(x)\tilde{V}\rangle$ and corresponding Hamiltonian system with boundary conditions:

$$dz/d\theta = \partial F/\partial\lambda = G_t\tilde{V},\ d\lambda/d\theta = -\partial F/\partial z = -\langle \lambda, \partial(G_t\tilde{V})/\partial z\rangle, \tag{18}$$

$$z(0) = e^{-G_t\tilde{V}}(x_1),\ z(1) = x_1,\ \lambda(0) = \tilde{\psi}(t). \tag{19}$$

Then (see [28]) the transversality conditions for the boundary condition (12) can be written as

$$\langle \lambda(1), g_t^i(x_1)\rangle = 0,\ i = 1, \ldots, r, \bigwedge \int_0^1 \langle \lambda(\eta), \dot{G}_t(z(\eta))\tilde{V} d\eta \geq M(\tilde{x}(t), \tilde{\psi}(t), t). \tag{20}$$

Theorem 6.8. *If a pair $(\tilde{t}, \tilde{u}) \in R\times \overset{o}{W}{}^r{}_{-1,\infty}$ is local minimizer for the generalized problem (9)-(11) and $\tilde{v}(\cdot) \in L^r_\infty[0, \tilde{t}]$ is the generalized primitive of the control $\tilde{u}$, then there exists a quadruple of absolutely continuous functions $(\tilde{x}(t), \tilde{\psi}(t), \tilde{z}(t), \tilde{\lambda}(t))$ such that the triple $(\tilde{x}(\cdot), \tilde{\psi}(\cdot), \tilde{v}(\cdot)$ satisfies the (pseudo)Hamiltonian system (14) with the Hamiltonian (15), the initial condition $x(0) = x_0$ and the maximality condition (16), while the solution $(\tilde{z}(t), \tilde{\lambda}(t))$ of the auxiliary Hamiltonian system (18) satisfies the boundary conditions (19) and the transversality conditions (20).* $\square$

Remark. Note that $\tilde{x}(\cdot)$ is *not* a generalized trajectory of the system (10). $\square$

6.5. **Generalized minimizers in highly-singular linear-quadratic optimal control problem.** We provide here a brief description of the results contained in PhD thesis of M.Guerra (University of Aveiro, Portugal, 2001).

One considers classical linear quadratic problem of optimal control

$$J(x(\cdot), u(\cdot)) = \int_0^T (x'Px + 2u'Qx + u'Ru)(t)dt \to \min,\ \dot{x} = Ax + Bu, \tag{21}$$
$$x(0) = \bar{x},\ x(T) = \tilde{x}.$$

Here $x \in \mathbb{R}^n, u \in \mathbb{R}^r,\ A, B, P, Q, R$ are constant matrices of suitable dimensions.

It is well known that for this problem to have finite infimum (one presumes then that $\tilde{x}$ is attainable from $\bar{x}$ for the system $\dot{x} = Ax + Bu$) the positive semidefiniteness of the matrix R is necessary. If, in particular, R is positive definite then we get a regular LQ-problem and the existence of minimizing control in L_2 is guaranteed. The extremal controls, which are candidates for minimizers, are determined by the Pontryagin maximum principle, its optimality is studied by the theory of second variation (conjugate points, Jacobi condition, Riccati differential equation, Hamilton-Jacobi equation etc.)

Much more difficult and challenging is the singular case where R is singular, i.e. has nontrivial kernel. Here minimizer may lack to exist in $L_2[0, T]$ due to noncoerciveness of the functional. The singular L-Q problems have been extensively studied over the last 30-40 years. Still the following questions remained unanswered for the LQ problem with an arbitrary singularity: i)if minimizers lack to exist in $L_2[0, T]$ can the problem be transferred to a bigger space, where generalized minimizers exist? what space could it be? ii) how to compute (describe) the generalized minimizers? iii) do these generalized minimizers admit approximation by minimizing sequences of ordinary (square integrable) controls?

These questions have been answered in the thesis of M.Guerra. The problem has additional difficulties for the vector-valued u. The detailed description of the results can be found in ([17, 18, 19]); here they are just listed.

- a series of necessary - generalized Goh and generalized Legendre-Clebsch conditions have been obtained; they guarantee that the functional J has finite infimum for some boundary data; order r of singularity for an LQ problem has been introduced;
- if the previous conditions are satisfied and the order of singularity equals r, then it is proved that the singular LQ-problem can be extended onto a set of generalized controls, which is a subspace of Sobolev space $H_{-r}[0,T]$; whenever the infimum of the problem is finite the problem possesses a generalized minimizer in this space of distributions ;
- the generalized minimizer is a sum of an analytic function and of a distribution which is a linear combination of the Dirac measures $\delta^{(j)}(t), \delta^{(j)}(t-T),\ j = 1,\ldots,r-1$ located at the boundary of the interval $[0,T]$; the corresponding generalized trajectory is a distribution (!) belonging to the Sobolev space $H_{-(r-1)}[0,T]$.
- following approximation result is established: the elements of minimizing sequence must converge to the minimizer in corresponding topology of the Sobolev space H_{-r}; this implies (for $r > 2$) high-gain oscillation behaviour of ordinary trajectories, which approximate the minimizing generalized trajectory;
- the notion of conjugate point is introduced and Jacobi-type optimality condition for the generalized control is established;
- the optimal generalized solution in a feedback form is constructed; the feedback can be computed via solution of a couple Riccati type and linear - matrix differential equations.
- Hamiltonian formalism for the generalized extremals is developed and its relation to Dirac's theory of constrained Haniltonian mechanics is established.

The most interesting aspects of this work are: i) a complete theory of existence, uniqueness, optimality, and approximative properties for minimizers of LQ-problem suffering arbitrary singularity; ii) appearance of distributions of order > 1 as generalized inputs and generalized trajectories; iii) reformulation of the Dirac theory of constrained Hamiltonian mechanics for the LQ-problem.

7. Controllability of dissipative (Navier-Stokes) PDE via Lie extension

7.1. **Introduction and preliminary material.**

We study 2- and 3- dimensional (2D and 3D) Navier-Stokes equation under controlled (nonrandom) forcing

$$\partial u/\partial t + (u\cdot\nabla)u + \nabla p = \nu\Delta u + F, \tag{1}$$

$$\nabla\cdot u = 0 \tag{2}$$

We assume the boundary conditions to be periodic, this means that $u(t,\cdot), p(t,\cdot)$ and $F(t,\cdot)$ are defined on a 2 or 3-dimensional torus $\mathbb{T}^k,\ k=2,3$.

7.1.1. *3D Navier-Stokes Equation.* Consider the $3D$ Navier-Stokes equation (1)-(2). It will be convenient for us to transform this equation into an infinite-dimensional system of ODE. We use "spectral algorithm" ([15]), which invokes the Fourier expansion of the solution $u(t,x)$ with respect to the basis of the eigenfunctions of the Laplacian operator on $\mathbb{T}^3$: $u(x,t)=\sum_k \underline{q}_k(t)e^{ik\cdot x}$. Here k is a 3-dimensional vector with integer components and $\underline{q}_k(t)$ is *vector-valued* function. For u to satisfy the incompressibility condition there must be $\forall k:\ \underline{q}_k\cdot k=0$.

Similarly we introduce the expansions for the pressure and the forcing:

$$p(x,t)=\sum_k \underline{p}_k(t)e^{ik\cdot x},\ F(x,t)=\sum_k \underline{v}_k(t)e^{ik\cdot x}.$$

We assume that the forcing has zero average ($\underline{v}_0\equiv 0$). Changing the reference frame (to the one uniformly moving with the center of mass) we may assume $\int u\ dx=0$ and therefore $\underline{q}_0=0$. It is known that the pressure term can be separated from the equations for $\underline{q}_k(t)$ which take form:

$$\begin{aligned}\dot{\underline{q}}_k=-i\sum_{m+n=k}(\underline{q}_m\cdot n)\Pi_k\underline{q}_n-\\ -\nu|k|^2\underline{q}_k+\underline{v}_k.\end{aligned}\tag{3}$$

Here Π_k stays for the orthogonal projection of $\mathbb{R}^3$ onto the plane $k^\perp$ orthogonal to k. Formally we should also take the projection $\Pi_k\underline{v}_k(t)$ of the forcing, but the k-directed component of $\underline{v}_k$ can be taken into account by the pressure term.

Since $u(x,t),F(x,t)$ are real-valued we have to assume that $\underline{q}_k=\bar{\underline{q}}_{-k}, \underline{v}_k=\bar{\underline{v}}_{-k}$.

7.1.2. *2D Navier-Stokes Equation.* In the 2D case the reduction to the ODE form is easier. Introducing the vorticity $w=\nabla^\perp\cdot u=\partial u_2/\partial x_1-\partial u_1/\partial x_2$ and applying the operator $\nabla^\perp$ to the equation (1) we arrive to the equation

$$\partial w/\partial t+(u\cdot\nabla)w=\nu\Delta w+f,\tag{4}$$

where $f=\nabla^\perp\cdot F$.

Remark that: i) $\nabla^\perp\cdot\nabla p=0$, ii) $\nabla^\perp$ and Δ commute as long as both are differential operators with constant coefficients; iii) $\nabla^\perp\cdot(u\cdot\nabla)u=(u\cdot\nabla)(\nabla^\perp\cdot u)+(\nabla^\perp\cdot u)(\nabla\cdot u)=(u\cdot\nabla)w$, for each u satisfying (2).

It is known that u satisfying (2) can be recovered uniquely (up to an additive constant) from w. From now on we will deal with the equation (4), which can be considered as an evolution equation in H^1.

Introduce again the Fourier expansions

$$w(t,x)=\sum_k q_k(t)e^{ik\cdot x},\ f(t,x)=\sum_k v_k(t)e^{ik\cdot x}.$$

As far as w and v are real-valued, we get $\bar{w}_n = w_{-n},\ \bar{v}_n = v_{-n}$. We assume $w_0 = 0, v_0 = 0$. Then $\partial w/\partial t = \sum_k \dot{q}_k(t)e^{ik\cdot x}$ and computing $(u\cdot\nabla)w$ we arrive to the (infinite-dimensional) system of ODE:

$$\dot{q}_k = \tag{5}$$
$$= \sum_{m+n=k}(m\wedge n)|m|^{-2}q_mq_n - \nu|k|^2q_k + v_k.$$

7.2. **Navier-Stokes (NS) system controlled by degenerate forcing. Problem setting.** From now on we assume the forcing terms $\underline{v}_k$ in (3) and v_k in (5) to be controls at our disposal. Then (3) and (5) can be seen as infinite-dimensional control-affine systems. We are going to study their controllability properties.

We will be interested in the case in which the controlled forcing is *degenerate*. This means that all but few v_k vanish identically, while these few can be chosen freely. From now on we fix a set of controlled modes $\mathcal{K}^1 \subset \mathbb{Z}^j,\ j = 2,3$, and assume $v_k \equiv 0,\ \forall k \notin \mathcal{K}^1$.

Further on we select set of *observed* modes indexed by a *finite* set $\mathcal{K}^{obs}$. We assume $\mathcal{K}^{obs} \supset \mathcal{K}^1$. As we will see nontrivial controllability issues arise only if $\mathcal{K}^1$ is a proper subset of $\mathcal{K}^{obs} \subset \mathbb{Z}^j,\ j = 2,3$. We identify the space of observed modes with $\mathbb{R}^N$ and denote by Π^{obs} the operator of projection of H^1 onto this space.

We can represent 2D NS system, controlled by degenerate forcing, in the following way:

$$\dot{q}_k(t) = \sum_{m+n=k}(m\wedge n)|m|^{-2}q_m(t)q_n(t) -$$
$$-\nu|k|^2q_k(t) + v_k(t),\ k\in\mathcal{K}^1, \tag{6}$$
$$\dot{q}_k(t) = \sum_{m+n=k}(m\wedge n)|m|^{-2}q_m(t)q_n(t)$$
$$-\nu|k|^2q_k(t),\ k\in\mathcal{K}^{obs}\setminus\mathcal{K}^1, \tag{7}$$
$$\dot{Q}(t) = B_2(q,Q)|_t - \nu A_2Q|_t. \tag{8}$$

In the latter equation $-\nu A_2 Q$ stays for the dissipative term and $B_2(q,Q)$ stays for nonlinear (bilinear) term.

Analogously 3D NS system, controlled by degenerate forcing, can be written in the form:

$$\dot{\underline{q}}_k(t) = -i\sum_{m+n=k}(\underline{q}_m(t)\cdot n)\Pi_k\underline{q}_n(t) -$$
$$-\nu|k|^2\underline{q}_k(t) + \underline{v}_k(t),\ k\in\mathcal{K}^1, \tag{9}$$
$$\dot{\underline{q}}_k(t) = -i\sum_{m+n=k}(\underline{q}_m(t)\cdot n)\Pi_k\underline{q}_n(t) -$$
$$-\nu|k|^2\underline{q}_k(t),\ k\in\mathcal{K}^{obs}\setminus\mathcal{K}^1, \tag{10}$$
$$\dot{Q}(t) = B_3(q,Q)|_t - \nu A_3Q|_t, \tag{11}$$

Let us introduce the Galerkin approximations of the 2D and 3D Navier-Stokes systems projecting these equations onto $\mathbb{R}^N$. A simple way to do it is just to eliminate the equation (8) (respectively (11)) and to put $Q = 0$ in (6)-(7) (resp. (9)-(10)). What results from this are the systems (6)-(7) (resp. (9)-(10)) under additional restriction on the modes:

$$k, m, n \in \mathcal{K}^{obs}. \tag{12}$$

The systems (6)-(7)-(12) (resp. (9)-(10)-(12)) are control-affine systems of the form

$$\dot{q} = f(q) + G(q)v(t) = f(q) + \sum_{j=1}^{r} g^j(q)v_j,\ q \in \mathbb{R}^N,\ v \in \mathbb{R}^r, \tag{13}$$

in finite-dimensional space of the observed modes.

Definition 7.2.1. *The Galerkin approximation of 2D (resp. 3D) Navier-Stokes system is time T globally controllable if for any two points $\tilde{q}, \hat{q}$ in $\mathbb{R}^N$ there exists a control which steers in time T the system (6)-(7)-(12) (resp. (9)-(10)-(12)) from $\tilde{q}$ to $\hat{q}$.* □

In the next section we formulate sufficient conditions of global controllability for the Galerkin approximations of 2D and 3D Navier-Stokes systems.

Another question we are interested in is: under what conditions the NS system is globally controllable in finite-dimensional projection?

Definition 7.2.2. *The 2D Navier-Stokes system is time T globally controllable in observed projection if for any two points $\tilde{q}, \hat{q}$ in $\mathbb{R}^N$ and any $\tilde{\varphi} \in (\Pi^{obs})^{-1}(\tilde{q})$ there exist a control which steers in time T the controlled 2D NS system from $\tilde{\varphi}$ to some $\hat{\varphi}$ with $\Pi^{obs}(\hat{\varphi}) = \hat{q}$.* □

This question is much more difficult because one has to study finite-dimensional projection of an infinite-dimensional dynamics.

The third issue we address is *approximate controllability.*

Definition 7.2.3. *The 2D Navier-Stokes system is time T globally approximately controllable, if for any two points $\tilde{\varphi}, \hat{\varphi} \in H^1$ and any $\varepsilon > 0$ there exists a control which steers in time T the controlled 2D NS system from $\tilde{\varphi}$ to the ε-neighborhood of $\hat{\varphi}$ in L_2-metric.* □

We formulate in 2D case sufficient controllability condition for controllability in observed projection and also sufficient criterion for approximate controllability.

7.3. **Main results.** In this section we formulate our controllability criteria in terms of evolution of the "sets of excited modes" $\mathcal{K}^j$ along the integer lattices $\mathbb{Z}^2$ or $\mathbb{Z}^3$ respectively.

Let $\mathcal{K}^1$ be the set of forced modes. Define the sequence of sets $\mathcal{K}^j \subset \mathbb{Z}^i$, ($i = 2$ or $3;\ j = 2, \ldots$), as:

$$\mathcal{K}^j = \{m + n \mid m, n \in \mathcal{K}^{j-1} \bigwedge \|m\| \neq \|n\| \bigwedge m \wedge n \neq 0\,\}. \tag{14}$$

Theorem 7.3.4. *Let $\mathcal{K}^1$ be the set of controlled modes. Define successively sets $\mathcal{K}^j,\ j = 2,\ldots,$ according to (14) and assume that $\bigcup_{j=1}^{M}\mathcal{K}^j$ contains all the observed modes: $\bigcup_{j=1}^{M}\mathcal{K}^j \supseteq \mathcal{K}^{obs}$. Then for any $T > 0$ the Galerkin approximations of the 2D and 3D Navier-Stokes systems are time-T globally controllable.* □

There is an extensive literature regarding controllability of the NS systems. We refer the readers to [10, 13, 14] for results and bibliography regarding controllability of NS systems by means of boundary and located control.

Results on controllability by means of degenerate forcing are scarce. We would like to mention a publication of Weinan E and J.C.Mattingly [12] on ergodicity of Navier-Stokes system under degenerate forcing. From the control-theoretic viewpoint they establish in [12] bracket generating property for finite-dimensional Galerkin approximation of the 2D NS system. This property guarantees accessibility, i.e. nonvoidness of the interior of attainable set, but in general does *not* guarantee controllability.

After the submission of the draft version of this paper we became aware of the publication [30] where sufficient controllability criterion for Galerkin approximation of 3D Navier-Stokes system has been established.

Now we formulate sufficient condition of controllability in observed projection for 2D Navier-Stokes system.

Theorem 7.3.5. *Assume the conditions of the Theorem 7.3.4 to be fulfilled for a 2D Navier-Stokes system controlled by degenerate forcing. Then this system is globally controllable in observed projection.* □

Definition 7.3.6. *A set $\mathcal{K}^1$ of controlled forcing modes in $\mathbb{Z}^2$ or $\mathbb{Z}^3$ is called saturating if for the corresponding sequence of sets $\mathcal{K}^j,\ j = 2,\ldots,$ defined by (14) and for every finite set $\mathcal{K}$ of modes there exists $M(\mathcal{K})$ such that: $\bigcup_{j=1}^{M(\mathcal{K})}\mathcal{K}^j \supset \mathcal{K}$.* □

Now we formulate sufficient condition of approximate controllability for 2D Navier-Stokes system.

Theorem 7.3.7. *Consider the 2D Navier-Stokes system controlled by degenerate forcing. Let $\mathcal{K}^1$ be the set of controlled forcing modes which is saturating. Then for any $T > 0$ the 2D NS system is time-T globally approximately controllable.* □

Examples of saturating sets in $\mathbb{Z}^2$ are provided in the following Proposition.

Proposition 7.3.8. *The subsets $\mathcal{K}^1 = \{k|\ |k_\alpha| \leq 3,\ \alpha = 1,2\}$, $\underline{\mathcal{K}}^1 = \{k = (k_1,k_2)|\ |k_1| + |k_2| \leq 2\}$ of $\mathbb{Z}^2$ are saturating.* □

7.4. **Controllability via Lie extension.** In this section we refer to some controllability criteria obtained by the technique of Lie extension.

Our idea is to proceed with a *series* of Lie extensions of the controlled NS system in order to arrive at the end to a system which is evidently controllable. We will employ two methods: relaxation and reduction formula (see Section 3).

What regards the latter then the controlled vector fields $g^1,\ldots,g^r$, of the systems (6)-(8) and (9)-(11) are constant and the commutativity assumption holds automatically.

From the Propositions 5.1.1,5.1.2 and the results of [4] it follows that one can reduce the study of the system (13) to the study of the control system

$$\dot{x} = e^{\mathrm{ad}(GV(\tau))} f(x), \tag{15}$$

on the quotient space $\mathbb{R}^N/\mathcal{G}$, where $\mathcal{G}$ is the linear span of the values of the constant vector fields $g^1, \ldots, g^r$.

The following result (see [4, Propositions 1 and 1']) is instrumental for our reasoning.

Proposition 7.4.9. *Let $\pi_{\mathcal{G}}$ be the canonical projection of the quotient space $\mathbb{R}^N \to \mathbb{R}^N/\mathcal{G}$ and $\mathcal{A}_{red}(\pi_{\mathcal{G}}(\tilde{x})$ be the attainable set of the reduced system (15). Then the closures of the sets $\mathcal{A}(\tilde{x})$ and $\pi_{\mathcal{G}}^{-1}\left(\mathcal{A}_{red}(\pi_{\mathcal{G}}(\tilde{x}))\right)$ in $\mathbb{R}^N$ coincide.* $\square$

The fact of system being control-affine is crucial for the validity of the reduction formula and of the Proposition 7.4.9.

Finally for eliminating the gap between approximate controllability and controllability of a system we invoke the Proposition 2.5.11.

7.5. Extension of the Galerkin approximations of controlled NS systems.

7.6. **2D case.** We shall use the reduction and the convexification techniques surveyed in the previous section to establish controllability.

Let us proceed with the reduction of the control-affine system (6)-(7).

Consider the set $\mathcal{K}^1$ of controlled forcing modes. The controlled vector fields $g_k = \partial/\partial q_k,\ k \in \mathcal{K}^1$ are constant. Due to it there holds for any vector field $Y(q)$:

$$\mathrm{Ad}\left(e^{V_k g_k}\right) Y = Y(q + V_k e_k),$$

where e_k is the (constant) value of g_k. Passing to the quotient space $\mathbb{R}^N/\mathcal{G}$, where $\mathcal{G} = \mathrm{span}\{g_k|\ k \in \mathcal{K}^1\}$ means that we can move freely along the directions $e_k, k \in \mathcal{K}^1$.

The "drift" vector field f of the control-affine system (6)-(7) is polynomial:

$$f = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \nu |k|^2 q_k,$$

and the reduced system (15) takes form

$$\begin{aligned} \dot{q}_k = {} & -\nu |k|^2 \left(q_k + \chi(k) V_k\right) + \\ & + \sum_{m+n=k} \frac{m \wedge n}{|m|^2} \left(q_m + \chi(m) V_m\right)\left(q_n + \chi(n) V_n\right) \end{aligned} \tag{16}$$

where $\chi \equiv 1$ on $\mathcal{K}^1$ and vanishes outside $\mathcal{K}^1$.

The right-hand side of the reduced system (16) is polynomial with respect to (the components of) V with coefficients depending on q. Let us represent this polynomial map as $\mathcal{V}(V) = \mathcal{V}^{(0)} + \mathcal{V}^{(1)} V + \mathcal{V}^{(2)}(V)$ where $\mathcal{V}^{(0)}, \mathcal{V}^{(1)}, \mathcal{V}^{(2)}$ are the free, the linear and the quadratic terms respectively. Evidently $\mathcal{V}^{(0)}$ is the right-hand side of the projection of the unforced NS system onto the quotient space.

We are not able to apply again the reduction to the system (16) as we would wish, because it is not control-affine anymore. Still we will be able to extend (16) and then extract from this extension a control-affine subsystem which is similar to (6)-(7).

First we establish that certain constant vector fields are contained in the image of the quadratic term $\mathcal{V}^{(2)}$.

Proposition 7.6.10. *Let*

$$\mathcal{K}^2 = \{m + n \mid \\ m, n \in \mathcal{K}^1 \bigwedge \|m\| \neq \|n\| \bigwedge m \wedge n \neq 0 \} .$$

Then the image of $\mathcal{V}^{(2)}$ contains all the vectors $\pm e_k$ indexed by $k \in \mathcal{K}^{(2)}$ from the standard base . $\square$

Proof. The projection of the vector-valued quadratic form $\mathcal{V}^{(2)}(V)$ onto e_k equals (see (16))

$$\mathcal{V}_k^{(2)}(V) = \sum_{m+n=k} \frac{m \wedge n}{|m|^2} \chi(m)\chi(n) V_m V_n .$$

Grouping the coefficients of $V_m V_n$ and of $V_n V_m$ we can rewrite it as

$$\mathcal{V}_k^{(2)}(v) = \sum_{m+n=k,\ \|m\|<\|n\|,\ m,n\in\mathcal{K}^1} \gamma_{mn} V_m V_n .$$

where $\gamma_{mn} = (m \wedge n)\left(\frac{1}{\|m\|^2} - \frac{1}{\|n\|^2}\right)$. Note that for $\|m\| = \|n\|$ the corresponding coefficient γ_{mn} vanishes and the term $V_m V_n$ is lacking in the sum.

If $k \notin \mathcal{K}^{(2)}$ then there are no non-vanishing terms in the expression for $\mathcal{V}_k^{(2)}(V)$, and hence $\mathcal{V}_k^{(2)} \equiv 0$. If $k \in \mathcal{K}^{(2)}$, let us pick any $m, n \in \mathcal{K}^1$ such that $m + n = k$ and $\|m\| < \|n\|$. Construct two vectors V^+, V^- by taking $V_s^{\pm} = 0$ for $s \neq k \bigwedge s \neq m$, and then taking $V_m = V_n = 1$ for V^+ and $V_m = -V_n = 1$ for V^-.

A direct calculation shows that

$$\mathcal{V}^{(2)}(V^+) = -\mathcal{V}^{(2)}(V^-) = \\ (m \wedge n)\left(|m|^{-2} - |n|^{-2}\right) e_k . \ \square$$

Corollary 7.6.11. *The convex hull of the image of $\mathcal{V}^{(1)} + \mathcal{V}^{(2)}$ contains the (*independent of *q) linear space E^2 spanned by $\{e_k |\ k \in \mathcal{K}^{(2)}\}$.* $\square$

Proof. Indeed for each $k \in \mathcal{K}^{(2)}$ there exists v such that $\mathcal{V}^{(2)}(V) = e_k$, . Obviously $\mathcal{V}^{(2)}(-V) = e_k$, while $\mathcal{V}^{(1)}(V) = -\mathcal{V}^{(1)}(-V)$. Hence

$$\frac{1}{2}\left(\left(\mathcal{V}^{(1)} + \mathcal{V}^{(2)}\right)(V) + \left(\mathcal{V}^{(1)} + \mathcal{V}^{(2)}\right)(-V)\right) = e_k ,$$

and we come to the conclusion of the corollary. $\square$

Therefore the convex hull of the right-hand side (evaluated at q) of the reduced system (16) contains the affine space $\mathcal{V}^{(0)}(q) + E^2$. We consider this affine space as the right-hand side (evaluated at q) of a new control-affine system, which can be written as:

$$\dot q_k(t) = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n - \tag{17}$$
$$-\nu |k|^2 q_k(t) + v_k(t),\ k \in \mathcal{K}^2,$$
$$\dot q_k(t) = \sum_{m+n=k} (m \wedge n)|m|^{-2} q_m q_n -$$
$$-\nu |k|^2 q_k(t),\ k \in \mathcal{K}^{obs} \setminus \left(\mathcal{K}^1 \bigcup \mathcal{K}^2\right). \tag{18}$$

Recall that we can move freely in the directions $e_k,\ k \in \mathcal{K}^1$.

If the image of the attainable set of this latter system under the canonical projection $\mathbb{R}^N \to \mathbb{R}^N/\mathcal{G}$ coincides with $\mathbb{R}^N/\mathcal{G}$ or, in other words, the (linear) sum of this attainable set with $\mathcal{G}$ coincides with $\mathbb{R}^N$, then according to the Proposition 7.4.9 the attainable set the original system will be dense in the state space and hence by Proposition 2.5.11 will coincide with this state space.

Therefore we managed to reduce the study of controllability of the system (6)-(7) to the study of a similar system with smaller (reduced) state space or equivalently with extended set $\mathcal{K}^1 \bigcup \mathcal{K}^2$ of controlled modes.

7.7. **3D case.** Considering the control system (9)-(10) we use the same techniques as in the 2D case (see the previous subsection) to extend the set of controlled modes along the integer lattice $\mathbb{Z}^3$.

7.8. **Comments on the proofs of the main results.** For the lack of space we are not able to provide proofs; let us just give some hints for proving Theorem 7.3.4.

The proofs for the Galerkin approximations of the 2D and 3D NS systems essentially coincide. One proceeds by induction on M, where M is a number of sets $\mathcal{K}^j$ of modes appearing in the formulation of the Theorem 7.3.4.

If $M = 1$ then controllability of the Galerkin approximation is almost trivial fact. Actually we are not only able to attain arbitrary points but even to design arbitrary Lipschitzian trajectories.

By application of the arguments of the Subsections 7.6,7.7 and of the Proposition 2.5.11 we can extend the set of controlled components from $q_k(k \in \mathcal{K}^1)$ to $q_k(k \in \mathcal{K}^2)$ and hence diminish the number M of sets $\mathcal{K}^j$ to $M-1$.

What concerns the (difficult) proof of the Theorem 7.3.5 then one has to proceed with the induction steps regarding the complete (nontruncated) 2D NS system. This requires rather heavy analytic estimates. The proof is presented in the preprint [5].

After proving the Theorem 7.3.5 it is not too difficult to arrive to the conclusion of the Theorem 7.3.7. One just needs some more delicate estimates for the evolution of infinite-dimensional component.

References

[1] Agrachev A.A., R.V.Gamkrelidze, *Exponential representation of flows and chronological calculus*, Math. USSR Sbornik, v. 35(1979), 727-785.

[2] Agrachev A.A., R.V.Gamkrelidze, A.V.Sarychev, *Local Invariants of Smooth Control Systems*, Acta Aplicandae Mathematica, v.14(1989),191-237.

[3] Agrachev A.A., Yu.L.Sachkov, *Lectures on Geometric Control Theory*, SISSA, Trieste, 2001.

[4] Agrachev A.A., A.V.Sarychev, *On reduction of smooth control system*, Math. USSR Sbornik, v. 58(1987), 15-30.

[5] Agrachev A.A., A.V.Sarychev, *Navier-Stokes Equations: Controllability by Means of Low Modes Forcing*, submitted to J. Mathematical Fluid Mechanics, 2003.

[6] Babin A.V., M.I. Vishik, *Attractors of Evolution Equations*, North Holland, 1992.

[7] A.Bressan, *On Differential Systems with Impulsive Controls*, Rend. Semin. Univ. Padova, 78(1987), 227-235.

[8] A.Bressan, F.Rampazzo, *Impulsive Control Systems without Commutativity Assumptions*, J.Optimization Theory and Applications, 81(1994),435-457.

[9] Constantin P., C.Foias, *Navier-Stokes equations*, The University of Chicago Press, 1989.

[10] Coron J.-M., *Return method: some applications to flow control*, "Mathematical Control Theory", ICTP Lecture Notes Series Volume VIII -Parts 1 & 2, 2002.

[11] Dudley R.M., *Real Analysis and Probability*, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, California, 1989.

[12] E W., J.C. Mattingly, *Ergodicity for the Navier-Stokes Equation with Degenerate Random Forcing: Finite Dimensional approximation*, *Comm. Pure Appl. Math.*, v. 54, no. 11, pp. 1386-1402, 2001.

[13] Fursikov A.V., *Optimal Control of Distributed Systems. Theory and Applications*, AMS, 2000.

[14] Fursikov A.V., O.Yu. Imanuilov, *Exact controllability of the Navier-Stokes and Boussinesq equations*, Russian Mathematical Surveys, Volume 54(1999), Number 3,pp.565-618.

[15] Gallavotti G., *Foundations of Fluid Mechanics*, Springer Verlag, Berlin, Heidelberg, 2002.

[16] Gamkrelidze R.V., *Principles of Optimal Control Theory*, Plenum Press,N.Y., 1978.

[17] M.Guerra, A.Sarychev, *Generalized Optimal Controls for the Singular Linear-Quadratic Problem*, Proc. 3th Portuguese Conf. on Automatic Control, Controlo'98, Coimbra, APCA, 1998, pp. 431-436.

[18] M.Guerra, *Highly Singular L-Q Problems: Solutions in Distribution Spaces*, J. of Dynamical and Control Systems, 6(2000), 265-309.

[19] M.Guerra, *Singular L-Q Problems and the Dirac-Bergmann Theory of Constraints*, in: Nonlinear Control in the Year 2000. Isidori, A.; Lamnabhi-Lagarrigue, F.; Respondek, W. eds. Springer-Verlag, 2001, pp 409-422.

[20] Jurdjevic V., *Geometric Control Theory*, Cambridge University Press, 1997.

[21] M.A. Krasnosel'skii and A.V. Pokrovskii, *Vibrostable Differential Equations with Discontinuous Right-Hand Sides* (in Russian), Annals of Moscow Math. Society, 27(1972), 93-112.

[22] Kuksin S.B., A.Shirikian, *Ergodicity for the randomly forced 2D Navier-Stokes equations*, Mathematical Physics, Analysis and Geometry 4 (2001) pp. 147-195.

[23] B.Miller, *Nonlinear Impulsive-Control Problem*, Automation and Remote Control, 37(1976),865-873.

[24] Yu.V. Orlov, *Variational Analysis of Optimal Systems with Measure-Like Controls*, Avtomatika i Telemekhanika, 1987, No.2, pp.26-32.

[25] Yu.V.Orlov, *Theory of Optimal Systems with Generalized Controls* (in Russian), Nauka, Moscow, 1988.

[26] L.S. Pontryagin, V.G. Boltianskii, R.V. Gamkrelidze and E.F. Mischenko, *Mathematical Theory of Optimal Processes*, Pergamon Press, Oxford, 1964.

[27] A.V. Sarychev, *Integral Representation of Trajectories for Control Systems with Right-Hand Sides*, Differentsial'nye Uravnenija, 24(1988), 1551-1564. English translation: Differential Equations, 24(1988), 1021-1031.

[28] A.V. Sarychev, *Optimization for Nonlinear Time-Optimal Problem with Generalized Controls*, Differentsial'nye Uravnenija, 27(1991),788-801. English translation: Differential Equations, 27(1991),539-550.

[29] A.V. Sarychev, *Nonlinear Systems with Impulsive and Generalized Function Controls*, em: C.Y.Byrnes and A.Kurzhanskii Eds.,'Nonliear Synthesis', Proc. of IIASA Workshop held in Sopron, Hungary, June 1989, Birkhäuser, Boston, 1991, pp. 244-257.

[30] Romito M., *Ergodicity of finite-dimensional approximations of the 3D Navier-Stokes equations forced by a degenerate noise*, 2002, Preprint.

[31] Warga R.,*Optimal Control of Differential and Functional Equations*,New York, Academic Press, 1972.

DiMaD, University of Florence, v. C.Lombroso 6/17, Firenze 50134, Italy
E-mail address: asarychev@unifi.it