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Analysis of the Kohn Laplacian on quadratic CR manifolds[☆]

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Abstract

We study the Kohn Laplacian $\Box_b^{(q)}$ acting on $(0, q)$-forms on quadratic CR manifolds. We characterize the operators $\Box_b^{(q)}$ that are locally solvable and hypoelliptic, respectively, in terms of the signatures of the scalar components of the Levi form.

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0. Introduction

Let V be an n-dimensional complex vector space, W an m-dimensional real vector space, $W^{\mathbb{C}}$ the complexification of W, and

$$\Phi : V \times V \to W^{\mathbb{C}}$$

a Hermitean map (i.e. $\Phi(z, z') = \overline{\Phi(z', z)}$ for every $z, z' \in V$, where complex conjugation in $W^{\mathbb{C}}$ is referred to the real form W).

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We consider the associated *quadratic manifold*

$$S = \{(z, t + iu) \in V \times W^{\mathbb{C}} : u = \Phi(z, z)\} \tag{1}$$

in $n + m$ complex dimensions. Then S is a CR manifold of CR-dimension n and real codimension m.

We consider the $\bar{\partial}_b$-complex on S, its adjoint $\bar{\partial}_b^*$ (with respect to the Lebesgue measure $dz\,dt$ on S and to a fixed Hermitean inner product on V), and the Kohn Laplacians

$$\Box_b^{(q)} = \bar{\partial}_b \bar{\partial}_b^* g + \bar{\partial}_b^* \bar{\partial}_b$$

acting on $(0, q)$-forms on S.

We address the problem of determining under which assumptions on Φ and q the operator $\Box_b^{(q)}$ satisfies either of the following properties:

(a) it is solvable, in the sense that, given any smooth $(0, q)$-form ϕ on S with compact support, there exists a $(0, q)$-current ω on S such that $\Box_b^{(q)}\omega = \phi$;

(b) it is hypoelliptic, i.e. any $(0, q)$-current ω on S such that $\Box_b^{(q)}\omega$ is smooth on an open set U is also smooth on U.

CR manifolds appear in connection with different problems in complex analysis, such as extension theorems for CR functions or boundary behavior of holomorphic functions. Questions about solvability or hypoellipticity of (systems of) differential operators with multiple characteristics naturally arise in this context. We refer the reader to the monographs [AK,B,ChSh] for accounts on these matters.

Analysis of the $\bar{\partial}_b$-complex on quadratic CR manifolds appears in [RoV], see also [T2] for a recent overview on this topic.

The form Φ can be identified with the (vector-valued) Levi form on S, and most of the properties of S have been recognized to depend on the signatures of the scalar-valued forms

$$\Phi^\lambda(z, z') = \lambda(\Phi(z, z')),$$

depending on $\lambda \in W^*$. For a given $\lambda \in W^*$, let $n^+(\lambda)$, resp. $n^-(\lambda)$, the number of positive, resp. negative, eigenvalues of Φ^λ. In [RoV] it was proved that, under the assumption that Φ^λ is generically non-degenerate, the CR-equation $\bar{\partial}_b u = f$ is solvable for any smooth $\bar{\partial}_b$-closed $(0, q)$-form f if and only if there exists no $\lambda \in W^*$ such that $n^+(\lambda) = n - q$ and $n^-(\lambda) = q$. The "only if" part of this statement was extended to general CR manifolds in [AFN].

Another relevant part of the literature concerns subelliptic estimates for the Kohn Laplacian. In [K] the so-called condition $Y(q)$ was given as a sufficient condition for the subellipticity of the Kohn Laplacian on CR manifolds of codimension 1 (see also [FK,RtS]). The condition stated in Theorem 2 below is equivalent to a natural extension of condition $Y(q)$ to the present setting (see condition (v) in Theorem 7.1

and the remark that follows).[1] Solvability of $\Box_b^{(q)}$ in absence of hypoellipticity does not seem to have been considered so far.

We prove that the signatures of the scalar forms Φ^λ, as λ varies in W^*, completely determine both solvability and hypoellipticity of $\Box_b^{(q)}$. One of the novelties of our results lies in the fact that we can include the case where Φ^λ is degenerate for every λ. Our main results are the following.

Theorem 1. *Let $n^+(\lambda)$, resp. $n^-(\lambda)$, the number of positive, resp. negative, eigenvalues of Φ^λ. Then $\Box_b^{(q)}$ is solvable if and only if there is no $\lambda \in W^*$ such that $n^+(\lambda) = q$ and $n^-(\lambda) = n - q$.*

Theorem 2. *Let $n^+(\lambda)$, resp. $n^-(\lambda)$ be as in Theorem 1. Then $\Box_b^{(q)}$ is hypoelliptic if and only if there is no $\lambda \in W^* \setminus \{0\}$ such that $n^+(\lambda) \leqslant q$ and $n^-(\lambda) \leqslant n - q$.*

We also prove that:

(i) property (a) is equivalent to the existence of a tempered fundamental solution for $\Box_b^{(q)}$, and also to the property that the L^2-null-space of $\Box_b^{(q)}$ is trivial;

(ii) when $\Box_b^{(q)}$ is not solvable, the orthogonal projection onto its L^2-null-space is given by convolution on G_Φ with an operator-valued distribution $\mathscr{C}_q$ for which we give an explicit formula;

(iii) property (b) is equivalent to the fact that $\Box_b^{(q)}$ satisfies non-isotropic subelliptic estimates of order 2.

The precise statements require further notation and they can be found as Theorems 4.4, 5.2. 6.1, 6.5, and 7.1.

It is worth mentioning that there are non-trivial cases in which all the Φ^λ are degenerate (see the remark in Section 3a). Theorem 1 has the following consequence.

Corollary 3. *Suppose that the Hermitean forms Φ^λ are degenerate for all λ. Then the operator $\Box_b^{(q)}$ is solvable for any q.*

Theorem 1 contains some of the results in [NRS], namely Theorems 7.2.1 and 7.3.1, in the particular case where Φ is "diagonal", i.e.

$$\Phi(z, z') = \sum_{j=1}^{n} z_j \overline{z'_j} w_j,$$

in an appropriate coordinate system on V, with $w_j \in W$.

[1] After writing this paper, we were informed of recent results of S.-C. Chen and M.-C. Shaw extending the $Y(q)$ condition and the relative sufficiency theorem to generic CR manifolds of higher codimension. These results seem to overlap with part of our Theorem 7.1.

In the diagonal case the operator $\Box_b^{(q)}$ diagonalizes in the basis of the elementary $(0,q)$-forms $d\bar{z}^I$, in the sense that

$$\Box_b^{(q)}\left(\sum_{|I|=q} f_I d\bar{z}^I\right) = \sum_{|I|=q} \Box_b^{(I)} f_I d\bar{z}^I,$$

where each $\Box_b^{(I)}$ acts on scalar-valued functions. This fact reduces the analysis of $\Box_b^{(q)}$ to the study of each individual $\Box_b^{(I)}$.

This reduction is not possible in the general case. We use the fact that a similar decoupling is possible after taking Fourier transform in the W-variables. However, this can be done, for each individual $\lambda \in W^*$, in a coordinate system on V that depends on λ (in fact a system that diagonalizes Φ^λ).

Our proofs involve the identification of S with a step-2 nilpotent group G_Φ, the Fourier inversion formula on G_Φ and the analysis of the image of $\Box_b^{(q)}$, realized as a system of harmonic oscillators, under the irreducible unitary representations of G_Φ.

In certain cases S coincides with the Šilov boundary of a Siegel domain of type II. This happens when the form Φ is positive w.r. to a proper cone in W. In fact this is equivalent to saying that there exists $\lambda \in W^*$ such that $n^+(\lambda) = n$ and $n^-(\lambda) = 0$. Under this assumption, the basic representation theory of G_Φ was established in [OV]. In Section 3, we give a self-contained presentation of the Fourier analysis on G_Φ in the general case. We note in passing that, w.r. to [OV], we prefer to privilege the Schrödinger model of the representations versus the Bargmann model.

This work has been motivated in part by the above-mentioned results in [NRS]. Some of the techniques for constructing fundamental solutions and related operators are derived from [MR]; the construction of a non-smooth solution of the equation $\Box_b^{(q)}\omega = 0$ in the proof of Theorem 7.1 has an analogue in [RtS].

We finally remark that, from Theorem 1, one can deduce the results in [RoV] on solvability of the CR-equation $\bar{\partial}_b u = f$, and extend them to the case where Φ^λ is always degenerate. We address these matters elsewhere [PR].

1. The nilpotent group associated to a quadratic manifold

Let S be the quadratic manifold defined by the equation

$$\operatorname{Im} w = \Phi(z,z),$$

with $z \in V$ and $w \in W^{\mathbb{C}}$. For elements $w \in W^{\mathbb{C}}$ the expressions $\operatorname{Re} w, \operatorname{Im} w, \bar{w}$ have the obvious meaning. For $(z',w') \in S$ the complex-affine transformation of $V \times W^{\mathbb{C}}$

$$\tau_{(z',w')}(z,w) = (z+z', w+w'+2i\Phi(z,z'))$$

maps S onto itself, and

$$\tau_{(z',w')}\tau_{(z'',w'')} = \tau_{(z'+z'',w'+w''+2i\Phi(z',z''))}$$

$$\tau^{-1}_{(z',w')} = \tau_{(-z',-w'+2i\Phi(z',z'))}.$$

Under the identification of $\tau_{(z',w')}$ with $(z',w')\in S$, this composition law defines a Lie group structure on S. As customary, we introduce coordinates $(z,t)\in V\times W$ to denote the element $(z,t+i\Phi(z,z))\in S$. Once pulled back to $V\times W$, the group multiplication takes the form

$$(z,t)(z',t') = (z+z', t+t'+2\,\mathrm{Im}\,\Phi(z,z')).$$

We call G_Φ this group and $\mathfrak{g}_\Phi$ its Lie algebra, that we now describe in detail.

For $v\in V$, denote by $\partial_v f$ the directional derivative of a function f on $V\times W$ in the direction v, and let X_v be the left-invariant vector field on G_Φ that coincides with ∂_v at the origin. It is easy to check that

$$X_v f\,(z,t) = \partial_v f(z,t) + 2\,\mathrm{Im}\,\Phi(z,v)\cdot\nabla_t f(z,t).$$

As we are going to introduce complex vector fields on G_Φ, it is convenient to adopt the notation Jv (instead of iv) for the complex structure on V. We then define $Z_v, \bar{Z}_v\in\mathfrak{g}_\Phi^{\mathbb{C}}$ as

$$Z_v = \frac{1}{2}(X_v - iX_{Jv}) = \frac{1}{2}(\partial_v - i\partial_{Jv}) + i\overline{\Phi(z,v)}\cdot\nabla_t,$$

$$\bar{Z}_v = \frac{1}{2}(X_v + iX_{Jv}) = \frac{1}{2}(\partial_v + i\partial_{Jv}) - i\Phi(z,v)\cdot\nabla_t.$$

The commutation rules are

$$[X_v, X_{v'}] = 4\,\mathrm{Im}\,\Phi(v,v')\cdot\nabla_t,$$

$$[Z_v, Z_{v'}] = [\bar{Z}_v, \bar{Z}_{v'}] = 0,$$

$$[Z_v, \bar{Z}_{v'}] = -2i\Phi(v,v')\cdot\nabla_t. \tag{2}$$

Hence, $\mathfrak{g}_\Phi$ is 2-step nilpotent and, under its identification with $V\times W$,

$$[\mathfrak{g}_\Phi, \mathfrak{g}_\Phi]\subseteq\{0\}\times W\subseteq\mathfrak{z}_\Phi,$$

where $\mathfrak{z}_\Phi$ denotes the center of $\mathfrak{g}_\Phi$.

2. The Kohn Laplacian on G_Φ

A $(0,q)$-form on S is a section of the vector bundle $\Lambda^{0,q}(T^*S)$, whose fiber at each point can be identified with the exterior product $\Lambda_q = \Lambda^{0,q}(V^*)$. As every vector bundle on S is trivial, we regard $(0,q)$-forms as vector-valued functions on G_Φ with values in Λ_q.

Let $\{v_1, \dots, v_n\}$ be any orthonormal basis of V with respect to the given inner product. Let $(z_1, \dots, z_n)$ denote the coordinates on V with respect to this basis. As customary, we write

$$Z_j = \frac{1}{2}(X_{v_j} - iX_{Jv_j}), \quad \bar{Z}_j = \frac{1}{2}(X_{v_j} + iX_{Jv_j}), \quad j = 1, \dots, n.$$

The $\bar{\partial}_b$ complex is defined as follows.

We denote by $d\bar{z}^I$ the $(0,q)$-form $d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}$, where $I = (i_1, \dots, i_q)$ is a strictly increasing multi-index. Given a $(0,q)$-form $\phi = \sum_{|I|=q} \phi_I d\bar{z}^I$ with smooth coefficients, we set

$$\bar{\partial}_b \phi = \sum_{|I|=q} \sum_{k=1}^{n} \bar{Z}_k(\phi_I) d\bar{z}_k \wedge d\bar{z}^I = \sum_{|J|=q+1} \sum_{k,|I|=q} \varepsilon^J_{kI} \bar{Z}_k(\phi_I) d\bar{z}^J. \tag{3}$$

Here $\varepsilon^J_{kI} = 0$ if $J \neq \{k\} \cup I$ as sets, and it equals the parity of the permutation that rearranges $(k, i_1, \dots, i_q)$ in increasing order if $J = \{k\} \cup I$.

The inner product on V induces a Hermitean product $(\cdot, \cdot)$ on each Λ_q in such a way that the elements $d\bar{z}^I$ form an orthonormal system.

Let $dz\,dt$ denote the left-invariant Haar measure on G_Φ. On the space $L^2(G_\Phi) \otimes \Lambda_q$ of $(0,q)$-forms with coefficients in $L^2(G_\Phi)$ we consider the inner product

$$\langle \phi, \psi \rangle = \int_{G_\Phi} (\phi(z,t), \psi(z,t))\, dz\, dt.$$

The formal adjoint $\bar{\partial}_b^*$ of $\bar{\partial}_b$ can be easily computed to yield

$$\bar{\partial}_b^* \left(\sum_{|I|=q} \phi_I d\bar{z}^I \right) = \sum_{|J|=q-1} \left(-\sum_{k,|I|=q} \varepsilon^I_{kJ} Z_k \phi_I \right) d\bar{z}^J. \tag{4}$$

We now compute the Kohn Laplacian $\Box_b^{(q)} = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$.

Given two multi-indices K and L such that $|K| = |L| = q$ and $|\{K \cap L\}| = q-1$, we set

$$\varepsilon(K, L) = (-1)^m, \tag{5}$$

where m is the number of elements in $K \cap L$ between the unique element $k \in K \setminus L$ and the unique element $\ell \in L \setminus K$.

Proposition 2.1. *With respect to any fixed orthonormal basis on V, the operator $\Box_b^{(q)}$ is represented by a matrix $(\Box_{LK})$ of scalar left-invariant differential operators on G_Φ as*

$$\Box_b^{(q)}\left(\sum_K \phi_K d\bar{z}^K\right) = \sum_L\left(\sum_K \Box_{LK}\phi_K\right) d\bar{z}^L.$$

Then,

$$\Box_{LK} = -\delta_{LK}\mathscr{L} + M_{LK}$$

where δ_{LK} is the Kronecker delta, $\mathscr{L} = \frac{1}{2}\sum_{k=1}^n (\bar{Z}_k Z_k + Z_k\bar{Z}_k)$ and

$$M_{LK} = \begin{cases} \frac{1}{2}\left(\sum_{k\in K}[Z_k, \bar{Z}_k] - \sum_{k\notin K}[Z_k, \bar{Z}_k]\right) & \textit{if } K = L, \\ \varepsilon(K, L)[Z_k, \bar{Z}_\ell] & \textit{if } |\{K\cap L\}| = q-1, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. By (3) and (4) we have

$$\begin{aligned} \bar{\partial}_b(\bar{\partial}_b^*\phi) &= \bar{\partial}_b\left(-\sum_{|J|=q-1}\left(\sum_{k,|K|=q}\varepsilon_{kJ}^K Z_k\phi_K\right) d\bar{z}^J\right) \\ &= -\sum_{|L|=q}\left(\sum_{k,\ell,|J|=q-1,|K|=q}\varepsilon_{kJ}^K\varepsilon_{\ell J}^L\bar{Z}_\ell Z_k\phi_K\right) d\bar{z}^L. \end{aligned}$$

On the other hand,

$$\begin{aligned} \bar{\partial}_b^*(\bar{\partial}_b\phi) &= \bar{\partial}_b^*\left(\sum_{|H|=q+1}\left(\sum_{j,|K|=q}\varepsilon_{jK}^H\bar{Z}_j\phi_K\right) d\bar{z}^H\right) \\ &= -\sum_{|L|=q}\left(\sum_{i,j,|H|=q+1,|K|=q}\varepsilon_{jK}^H\varepsilon_{iL}^H Z_i\bar{Z}_j\phi_K\right) d\bar{z}^L. \end{aligned}$$

Hence,

$$\begin{aligned} \Box_b^{(q)}(\phi) = -\sum_{|L|=q}\sum_{|K|=q}\Bigg(&\sum_{\ell,k,|J|=q-1}\varepsilon_{kJ}^K\varepsilon_{\ell J}^L\bar{Z}_\ell Z_k \\ &+ \sum_{i,j,|H|=q+1}\varepsilon_{jK}^H\varepsilon_{iL}^H Z_i\bar{Z}_j\Bigg)\phi_K\, d\bar{z}^L. \end{aligned}$$

Then,

$$\square_{LK} = -\sum_{\ell,k,|J|=q-1} \varepsilon^K_{kJ}\varepsilon^L_{\ell J}\bar{Z}_\ell Z_k - \sum_{i,j,|H|=q+1} \varepsilon^H_{jK}\varepsilon^H_{iL} Z_i\bar{Z}_j. \tag{6}$$

When $K = L$ the indices k and ℓ are forced to be equal, as well as i and j. Hence,

$$\begin{aligned}\square_{LL} &= -\left(\sum_{k\in L} \bar{Z}_k Z_k + \sum_{j\notin L} Z_j\bar{Z}_j\right)\\ &= -\frac{1}{2}\sum_{k=1}^{n}(\bar{Z}_k Z_k + Z_k\bar{Z}_k) - \frac{1}{2}\left(\sum_{k\in L}[\bar{Z}_k, Z_k] + \sum_{k\notin L}[Z_k,\bar{Z}_k]\right).\end{aligned}$$

This proves the statement for the terms along the diagonal.

On the other hand, when $K\neq L$, the coefficient $\varepsilon^K_{kJ}\varepsilon^L_{\ell J}$ is different from 0 if only if $K = J\cup\{k\}$ and $L = J\cup\{\ell\}$. Notice that, given K and L such that $|\{K\cap L\}| = q-1$, they uniquely determine J, k and ℓ. Analogously, $\varepsilon^H_{jK}\varepsilon^H_{iL}\neq 0$ if and only if $H = K\cup\{j\} = L\cup\{i\}$. Then, necessarily, $|\{K\cap L\}| = q-1$ as before, and if k and ℓ are as above, $j=\ell$ and $i=k$.

It follows that $\square_{LK} = 0$ unless $|\{K\cap L\}| = q-1$. In this case, each of the sums in (6) reduces to one single term, and

$$\square_{LK} = -\varepsilon^K_{kJ}\varepsilon^L_{\ell J}\bar{Z}_\ell Z_k - \varepsilon^H_{\ell K}\varepsilon^H_{kL} Z_k\bar{Z}_\ell,$$

with $J = K\cap L$ and $H = K\cup L$. Furthermore,

$$\varepsilon^K_{kJ}\varepsilon^L_{\ell J} = -\varepsilon^H_{\ell K}\varepsilon^H_{kL} = \varepsilon(K,L),$$

where $\varepsilon(K,L)$ is defined in (5).

Thus,

$$\square_{LK} = \varepsilon(K,L)[Z_k,\bar{Z}_\ell],$$

which proves the proposition. $\square$

3. Fourier analysis on G_Φ

3.1. Representations and Plancherel formula

The irreducible unitary representations of G_Φ can be described as follows.

By Schur's lemma, if π is an irreducible unitary representation of G_Φ, there is $\lambda\in W^*$ such that $\pi(0,t) = e^{i\lambda(t)}$. Then, by (2),

$$d\pi([Z_v,\bar{Z}_{v'}]) = 2\lambda(\Phi(v,v'))I = 2\Phi^\lambda(v,v')I. \tag{7}$$

We diagonalize Φ^λ with respect to an orthonormal basis $\{v_1^\lambda, \dots, v_n^\lambda\}$ of V, in such a way that

$$\Phi^\lambda(v_j^\lambda, v_k^\lambda) = \delta_{jk}\mu_j(\lambda),$$

with $\mu_j = \mu_j(\lambda) \neq 0$ for $j \leqslant v(\lambda)$ and $\mu_j = 0$ for $j > v(\lambda)$, where $0 \leqslant v(\lambda) = \operatorname{rank} \Phi^\lambda \leqslant n$. We call

$$X_j^\lambda = X_{v_j}^\lambda, \quad Y_j^\lambda = X_{Jv_j}^\lambda, \quad Z_j^\lambda = \frac{1}{2}(X_j^\lambda - iY_j^\lambda), \quad \bar{Z}_j^\lambda = \frac{1}{2}(X_j^\lambda + iY_j^\lambda).$$

Then

$$\begin{aligned} d\pi([X_j^\lambda, X_k^\lambda]) &= d\pi([Y_j^\lambda, Y_k^\lambda]) = 0, \\ d\pi([X_j^\lambda, Y_k^\lambda]) &= -4i\mu_j\delta_{jk}I, \end{aligned}$$

for every j, k. It follows from the Stone–von Neumann theorem that there is $\eta = a + ib \in \mathbb{C}^{n-v(\lambda)}$ such that π is unitarily equivalent to the representation $\pi_{\lambda,\eta}$ of G_ϕ on $L^2(\mathbb{R}^{v(\lambda)})$ such that

$$\left.\begin{aligned} d\pi_{\lambda,\eta}(X_j^\lambda) &= 2\partial_{\xi_j} \\ d\pi_{\lambda,\eta}(Y_j^\lambda) &= -2i\mu_j\xi_j \end{aligned}\right\} j \leqslant v(\lambda), \\ \left.\begin{aligned} d\pi_{\lambda,\eta}(X_j^\lambda) &= 2ia_{j-v(\lambda)} \\ d\pi_{\lambda,\eta}(Y_j^\lambda) &= 2ib_{j-v(\lambda)} \end{aligned}\right\} j > v(\lambda). \tag{8}$$

Given λ, let $(z_1^\lambda, \dots, z_n^\lambda)$ be the coordinates on V induced by the basis $\{v_j^\lambda\}$, with $z_j^\lambda = x_j^\lambda + iy_j^\lambda$. In order to simplify the notation, we set

$$x^\lambda = (x_1^\lambda, \dots, x_n^\lambda), \quad x' = (x_1^\lambda, \dots, x_{v(\lambda)}^\lambda), \quad x'' = (x_{v(\lambda)+1}^\lambda, \dots, x_n^\lambda),$$

and similarly for y^λ, y', y''. We also set $z'' = x'' + iy''$. In doing so, we must remember that x', x'', etc. are components that depend on λ.

The integrated form of $\pi_{\lambda,\eta}$ is, because of (8),

$$(\pi_{\lambda,\eta}(x, y, t)\phi)(\xi) = e^{i(\lambda(t)+2\,\mathrm{Re}\langle z'',\eta\rangle)}e^{-2i\sum_1^{v(\lambda)} \mu_j y_j^\lambda(\xi_j + x_j^\lambda)}\phi(\xi + 2x'). \tag{9}$$

It must be observed that these formulas depend on the choice of the (ordered) basis of V that diagonalizes Φ^λ. However, different choices of the basis lead to equivalent representations.

For a function f on G_Φ, we define

$$\pi_{\lambda,\eta}(f) = \int f(z, t)\pi_{\lambda,\eta}(z, t)^{-1}\, dz\, dt. \tag{10}$$

This definition has the effect that $\pi_{\lambda,\eta}(f * g) = \pi_{\lambda,\eta}(g)\pi_{\lambda,\eta}(f)$. The disadvantage of producing an inversion in the order of the two factors is compensated by a more natural formalism when dealing with vector-valued functions.

Observe that if $\mathscr{L}$ is a left-invariant differential operator, then

$$\pi_{\lambda,\eta}(\mathscr{L}f) = d\pi_{\lambda,\eta}(\mathscr{L})\pi_{\lambda,\eta}(f).$$

Definition 3.1. Let $v = \max_{\lambda\in W^*} v(\lambda)$. We call Ω the Zariski-open set $\Omega \subseteq W^*$ such that $v(\lambda) = v$ for $\lambda \in \Omega$. For $\lambda \in \Omega$, we set

$$D(\lambda) = \prod_{j=1}^{v} |\mu_j|.$$

If $v = n$, then $D(\lambda) = |\det \Phi^\lambda|$.

Proposition 3.2. *The Plancherel formula for G_Φ is*

$$||f||_2^2 = \int_\Omega \int_{\mathbb{C}^{n-v}} ||\pi_{\lambda,\eta}(f)||_{\mathrm{HS}}^2 D(\lambda)\, d\eta\, d\lambda \tag{11}$$

for an appropriate normalization of the Lebesgue measure $d\lambda$ on W^*, and the inversion formula takes the form

$$f(z,t) = \int_\Omega \int_{\mathbb{C}^{n-v}} \mathrm{tr}(\pi_{\lambda,\eta}(f)\pi_{\lambda,\eta}(z,t))D(\lambda)\, d\eta\, d\lambda.$$

Proof. It follows from (7) that, for $\lambda \in \Omega$,

$$\begin{aligned}(\pi_{\lambda,\eta}(f)\phi)(\xi) &= \int_{\mathbb{R}^{2v}} \mathscr{F}_{x'',y'',t} f(x', y', 2\eta, \lambda) e^{2i\sum_1^v \mu_j y_j(\xi_j - x_j)} \phi(\xi - 2x')\, dx'\, dy' \\ &= \int_{\mathbb{R}^v} K_{\lambda,\eta}(\xi, \xi')\phi(\xi')\, d\xi',\end{aligned}$$

with

$$K_{\lambda,\eta}(\xi,\xi') = \mathscr{F}_{x'',y^\lambda,t} f\left(\frac{\xi_1 - \xi_1'}{2}, \dots, \frac{\xi_v - \xi_v'}{2}, -\mu_1 \frac{\xi_1 + \xi_1'}{2}, \dots, -\mu_v \frac{\xi_v + \xi_v'}{2}, 2\eta, \lambda\right).$$

The conclusion follows from the fact that $||\pi_{\lambda,\eta}(f)||_{\mathrm{HS}}^2 = \int |K_{\lambda,\eta}(\xi,\xi')|^2\, d\xi\, d\xi'$ and from the Euclidean Plancherel formula. $\square$

When $v = n$, i.e. when there exists $\lambda \in W^*$ such that Φ^λ is non-degenerate, the Plancherel formula takes the simpler form

$$||f||_2^2 = \int_\Omega ||\pi_\lambda(f)||_{\mathrm{HS}}^2 |\det \Phi^\lambda|\, d\lambda.$$

Remark. It must be noticed that it is quite possible that all the Φ^λ are degenerate, even though there is no common radical that can be factored out to decompose G_Φ as the product of a nilpotent and an abelian group. An example is obtained by taking $V = \mathbb{C}^3$, $W = \mathbb{R}^2$, $\Phi = (\Phi_1, \Phi_2)$, with $\Phi_j(z, z') = z'^* A_j z$ and

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We also observe that G_Φ is stratified (i.e. the vector fields Z_v and $\bar{Z}_v$ generate the full complex Lie algebra) if and only if there is no $\lambda \neq 0$ such that $\Phi^\lambda = 0$. This remark will be recalled in Section 7.

3.2. Hermite bases

In dealing with the representation $\pi_{\lambda,\eta}$ we privilege a particular orthonormal basis of $L^2(\mathbb{R}^{v(\lambda)})$ that depends on λ.

Denote by h_j the jth Hermite function on the real line:

$$h_j(t) = (2^j \sqrt{\pi} j!)^{-1/2} (-1)^j e^{t^2/2} \frac{d^j}{dt^j} e^{-t^2}. \tag{12}$$

Given a multi-index $m \in \mathbb{N}^{v(\lambda)}$, we set

$$h_m^\lambda(\xi) = \prod_{j=1}^{v(\lambda)} |\mu_j|^{1/4} h_{m_j}(|\mu_j|^{1/2} \xi_j). \tag{13}$$

As a further simplification in the notation, for $\xi \in \mathbb{R}^{v(\lambda)}$ we set

$$R_\lambda \xi = (|\mu_1|^{1/2} \xi_1, \dots, |\mu_{v(\lambda)}|^{1/2} \xi_{v(\lambda)}).$$

Lemma 3.3. *Let* $\chi_{m,m'}^{\lambda,\eta}(x^\lambda, y^\lambda, t)$ *be the matrix entry* $\langle \pi_{\lambda,\eta}(x^\lambda, y^\lambda, t) h_m^\lambda, h_{m'}^\lambda \rangle$. *There exist Schwartz functions* $\psi_{m,m'}^\varepsilon$ *on* $\mathbb{R}^{2v(\lambda)}$ *depending only on* m, m' *and on the signatures* $\varepsilon_j = \mu_j / |\mu_j|$ *such that*

$$\chi_{m,m'}^{\lambda,\eta}(x^\lambda, y^\lambda, t) = e^{i(\lambda(t) + 2\,\mathrm{Re}\langle z'', \eta \rangle)} \psi_{m,m'}^\varepsilon(R_\lambda x^\lambda, R_\lambda y^\lambda).$$

Proof. Write

$$h_m^\lambda(\xi) = \left(\prod_{j=1}^{v(\lambda)} |\mu_j|^{1/4} \right) e^{-\frac{1}{2} \sum_{j=1}^{v(\lambda)} |\mu_j| \xi_j^2} P_m(R_\lambda \xi),$$

with P_m a real polynomial containing only monomials ξ^α with $\alpha \leqslant m$. Then

$$\begin{aligned}
&\chi_{m,\,m'}^{\lambda,\eta}(x^\lambda, y^\lambda, t)\\
&= e^{i(\lambda(t)+2\,\mathrm{Re}\langle z'',\eta\rangle)} \left(\prod_{j=1}^{v(\lambda)} |\mu_j|^{1/2} \right) \int_{\mathbb{R}^{v(\lambda)}} e^{-2i\sum_{j=1}^{v(\lambda)} \mu_j y_j^\lambda (\xi_j + x_j^\lambda)} e^{-\frac{1}{2} \sum_{j=1}^{v(\lambda)} |\mu_j| (\xi_j + 2x_j^\lambda)^2}\\
&\quad \times e^{-\frac{1}{2} \sum_{j=1}^{v(\lambda)} |\mu_j| \xi_j^2} P_m(R_\lambda(\xi + 2x^\lambda)) P_{m'}(R_\lambda \xi)\, d\xi\\
&= e^{i(\lambda(t)+2\,\mathrm{Re}\langle z'',\eta\rangle)} \left(\prod_{j=1}^{v(\lambda)} |\mu_j|^{1/2} \right) \int_{\mathbb{R}^{v(\lambda)}} e^{-2i\sum_{j=1}^{v(\lambda)} \mu_j y_j^\lambda \xi_j} e^{-\frac{1}{2} \sum_{j=1}^{v(\lambda)} |\mu_j| (\xi_j + x_j^\lambda)^2}\\
&\quad \times e^{-\frac{1}{2} \sum_{j=1}^{v(\lambda)} |\mu_j| (\xi_j - x_j^\lambda)^2} P_m(R_\lambda(\xi + x^\lambda)) P_{m'}(R_\lambda(\xi - x^\lambda))\, d\xi\\
&= e^{i(\lambda(t)+2\,\mathrm{Re}\langle z'',\eta\rangle)} \left(\prod_{j=1}^{v(\lambda)} |\mu_j|^{1/2} \right) e^{-\sum_{j=1}^{v(\lambda)} |\mu_j| x_j^{\lambda 2}} \sum_{\alpha+\beta \leqslant m+m'} c_{m,\,m',\alpha,\beta} (R_\lambda x^\lambda)^\alpha\\
&\quad \times \int_{\mathbb{R}^{v(\lambda)}} e^{-2i\sum_{j=1}^{v(\lambda)} \mu_j y_j^\lambda \xi_j} e^{-\sum_{j=1}^{v(\lambda)} |\mu_j| \xi_j^2} (R_\lambda \xi)^\beta\, d\xi\\
&= e^{i(\lambda(t)+2\,\mathrm{Re}\langle z'',\eta\rangle)} e^{-\sum_{j=1}^{v(\lambda)} |\mu_j| x_j^\lambda 2}\\
&\quad \times \sum_{\alpha+\beta \leqslant m+m'} c_{m,\,m',\alpha,\beta} (R_\lambda x^\lambda)^\alpha \prod_{j=1}^{v(\lambda)} (\mathrm{sgn}\,\mu_j)^{\beta_j} \mathscr{F}(e^{-|\xi|^2} \xi^\beta)(2R_\lambda y_\eta).
\end{aligned}$$

The conclusion follows from the fact that the Fourier transform of a monomial times $e^{-|\xi|^2}$ equals $e^{-|\cdot|^2/4}$ times a polynomial. □

Remark. As on the Heisenberg group, the functions $\psi_{m,\,m'}^{\varepsilon}$ can be expressed in terms of Laguerre functions (see e.g. [F]). However, we shall not need their explicit expression, except for the case $m = m' = 0$. The proof of Lemma 3.3 shows that

$$\chi_{0,0}^{\lambda,\eta}(z, t) = e^{i(\lambda(t)+2\,\mathrm{Re}\langle z'',\eta\rangle)} e^{-\sum_{j=1}^{v(\lambda)} |\mu_j| |z_j^\lambda|^2}. \tag{14}$$

3.3. Smoothly varying frames on V and Schwartz functions on the group

Among the elements of Ω we select those λ for which the number of distinct eigenvalues of Φ^λ is maximum. These elements form a subset Ω' which is Zariski-open, and therefore it carries the full Plancherel measure.

Fix $\lambda_0 \in \Omega'$, and let $\mu_1, \ldots, \mu_\ell$ be the distinct eigenvalues of Φ^{λ_0}, with multiplicities $m_1, \ldots, m_\ell$, respectively. By the implicit function theorem, there is a connected neighborhood U of λ_0 in Ω' on which one can define real-analytic functions $\mu_i(\lambda)$ for $1 \leqslant i \leqslant \ell$, such that $\mu_i(\lambda_0) = \mu_i$ and $\mu_i(\lambda)$ is an eigenvalue of Φ^λ with multiplicity m_i. Also, $\mu_i(\lambda) \leqslant 0$ for $\lambda \in U$, except for at most one i (in case $v < n$), for which $\mu_i(\lambda)$ is identically 0 on U.

For each i and each $\lambda \in U$, we can also find an orthonormal basis of the $\mu_i(\lambda)$-eigenspace of Φ^λ, in such a way that the kth basis element depends analytically on λ for every k.

At this point, we relabel the eigenvalues, allowing repetitions according to their multiplicity, and ordering them in such a way that $\mu_{v+1}(\lambda) = \cdots = \mu_n(\lambda) = 0$.

Hence, for each $\lambda \in U$ we have an orthonormal basis $\{v_1^\lambda, \ldots, v_n^\lambda\}$ of V, such that v_j^λ depends analytically on λ and

$$\Phi^\lambda(v_j^\lambda, v_k^\lambda) = \delta_{jk}\mu_j(\lambda).$$

The corresponding coordinate functions $z_j^\lambda = x_j^\lambda + iy_j^\lambda$ are then real-analytic in λ for $\lambda \in U$.

Define the representations $\pi_{\lambda,\eta}$ for $(\lambda, \eta) \in U \times \mathbb{C}^{n-v}$ according to this choice of the coordinates. If $m, m' \in \mathbb{N}^v$, we set

$$\widehat{f}(\lambda, \eta; m, m') = \langle \pi_{\lambda,\eta}(f) h_m^\lambda, h_{m'}^\lambda \rangle = \int f(x^\lambda, y^\lambda, t) \overline{\chi_{m',m}^{\lambda,\eta}(x^\lambda, y^\lambda, t)}\, dx^\lambda\, dy^\lambda\, dt. \quad (15)$$

Lemma 3.4. *Let $\phi(\lambda, \eta)$ be a C^∞ function with compact support in $U \times \mathbb{C}^{n-v}$, and let $m, m' \in \mathbb{N}^v$. There is a function $f \in \mathscr{S}(G_\Phi)$ such that*

(i) $\pi_{\lambda,\eta}(f) = 0$ *for* $\lambda \notin U$;
(ii) $\widehat{f}(\lambda, \eta; m, m') = \phi(\lambda, \eta)$ *for* $(\lambda, \eta) \in U \times \mathbb{C}^{n-v}$;
(iii) $\widehat{f}(\lambda, \eta; p, p') = 0$ *for* $(p, p') \neq (m, m')$ *and* $(\lambda, \eta) \in U \times \mathbb{C}^{n-v}$.

Proof. Define

$$f(z, t) = \int_{U \times \mathbb{C}^{n-v}} \phi(\lambda, \eta) \chi_{m',m}^{\lambda,\eta}(x^\lambda, y^\lambda, t) D(\lambda)\, d\lambda\, d\eta,$$

where x^λ, y^λ are the real coordinates of $z \in V$ in the basis $\{v_j^\lambda\}$. As U is connected and contained in Ω', the signatures ε_j of the eigenvalues $\mu_j(\lambda)$ are constant on U. Therefore,

$$f(z,t) = \int_{U \times \mathbb{C}^{n-\nu}} \phi(\lambda,\eta) e^{i(\lambda(t)+2\,\mathrm{Re}\langle z'',\eta\rangle)} \psi^\varepsilon_{m',m}(R_\lambda x^\lambda, R_\lambda y^\lambda) D(\lambda)\, d\lambda\, d\eta,$$

with $\psi^\varepsilon_{m',m}$ as in Lemma 3.3 and ε fixed.

The fact that f is a Schwartz function can be easily deduced from the smoothness of the functions $x_j^\lambda, y_j^\lambda, \mu_j(\lambda)$ and the fact that the $\mu_j(\lambda)$ are bounded away from zero on the support of ϕ.

Taking Fourier transform in t, we find that

$$\int f(z,t) e^{-i\lambda(t)}\, dt = 0$$

identically for $\lambda \notin U$, which implies that $\pi_{\lambda,\eta}(f) = 0$ for $\lambda \notin U$.

From the definition of $\chi^{\lambda,\eta}_{m',m}$, we have that

$$\begin{aligned} f(z,t) &= \int_{U \times \mathbb{C}^{n-\nu}} \phi(\lambda,\eta) \langle \pi_{\lambda,\eta}(x^\lambda, y^\lambda, t) h^\lambda_{m'}, h^\lambda_m \rangle D(\lambda)\, d\lambda\, d\eta \\ &= \int_{U \times \mathbb{C}^{n-\nu}} \mathrm{tr}(\pi_{\lambda,\eta}(x^\lambda, y^\lambda, t) A^{\lambda,\eta}_{m,m'}) D(\lambda)\, d\lambda\, d\eta, \end{aligned}$$

where $A^{\lambda,\eta}_{m,m'} h^\lambda_m = \phi(\lambda,\eta) h^\lambda_{m'}$ and $A^{\lambda,\eta}_{m,m'} h^\lambda_p = 0$ if $p \neq m$.

By uniqueness of the Fourier transform, it follows that $\pi_{\lambda,\eta}(f) = A^{\lambda,\eta}_{m,m'}$ for $(\lambda,\eta) \in U \times \mathbb{C}^{n-\nu}$. Hence

$$\widehat{f}(\lambda,\eta;p,p') = \langle A^{\lambda,\eta}_{m,m'} h^\lambda_p, h^\lambda_{p'} \rangle,$$

and the conclusion follows. □

3.4. *Fourier transform of vector-valued functions*

Let f be a function on G_Φ taking values in a finite-dimensional complex space E. Following (10), we define

$$\pi_{\lambda,\eta}(f) = \int_{G_\Phi} \pi_{\lambda,\eta}(z,t)^{-1} \otimes f(z,t)\, dz\, dt \in \mathrm{End}(L^2(\mathbb{R}^{\nu(\lambda)})) \otimes E.$$

Let K be a function on G_Φ with values in $\mathrm{Hom}\,(E,F)$, with E and F finite-dimensional spaces. Then the convolution operator

$$f \mapsto f * K(z,t) = \int_{G_\Phi} K((w,u)^{-1}(z,t)) f(w,u)\, dw\, du$$

maps E-valued functions into F-valued functions and it is left-invariant. We have

$$\pi_{\lambda,\eta}(f * K) = \pi_{\lambda,\eta}(K)\pi_{\lambda,\eta}(f),$$

if we understand that the composition of $T \otimes A \in \mathrm{End}(L^2(\mathbb{R}^{\nu(\lambda)})) \otimes \mathrm{Hom}(E,F)$ with $U \otimes v \in \mathrm{End}(L^2(\mathbb{R}^{\nu(\lambda)})) \otimes E$ is $TU \otimes Av \in \mathrm{End}(L^2(\mathbb{R}^{\nu(\lambda)})) \otimes F$.

Let now $(\cdot,\cdot)$ be a Hermitean inner product on E and let

$$\langle f, g\rangle = \int_{G_\Phi} (f(z,t), g(z,t))\, dz\, dt$$

be the induced inner product on $L^2(G_\Phi) \otimes E$.

Introducing an orthonormal basis on E, one can easily express this pairing in terms of the Fourier transform of f and g, using the polarized form of the Plancherel formula. In order to obtain a coordinate-free formula, consider the inner product $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ on $\mathrm{HS}(L^2(\mathbb{R}^\nu)) \otimes E$ such that

$$\langle\!\langle T \otimes v, U \otimes w \rangle\!\rangle = \mathrm{tr}(TU^*)(v,w), \tag{16}$$

where T, U are Hilbert–Schmidt operators on $L^2(\mathbb{R}^\nu)$, $v, w \in E$. We then have

$$\langle f, g\rangle = \int_\Omega \int_{\mathbb{C}^{n-\nu}} \langle\!\langle \pi_{\lambda,\eta}(f), \pi_{\lambda,\eta}(g) \rangle\!\rangle D(\lambda)\, d\eta\, d\lambda. \tag{17}$$

We shall use this formula to define vector-valued distributions on G_Φ. In doing so, we adopt the convention that the pairing $\langle\!\langle u, f \rangle\!\rangle$ between a distribution u and a test function f is linear in u and anti-linear in f.

3.5. The Fourier transform of $\Box_b^{(q)}$

We shall be primarily concerned with the situation where $E = F = \Lambda_q = \Lambda_V^{(0,q)}$, with the inner product naturally inherited from the inner product on V. If ϕ is a Schwartz $(0,q)$-form on G_Φ, then $\pi_{\lambda,\eta}(\phi) \in \mathrm{End}(L^2(\mathbb{R}^{\nu(\lambda)})) \otimes \Lambda_q$.

We want to describe the image of $\Box_b^{(q)}$ under $\pi_{\lambda,\eta}$. Observe that $d\pi_{\lambda,\eta}(\Box_b^{(q)}) \in \mathrm{End}(L^2(\mathbb{R}^{\nu(\lambda)})) \otimes \mathrm{End}(\Lambda_q)$.

Proposition 3.5. *Let $\{v_1^\lambda, \dots, v_n^\lambda\}$ be an orthonormal basis of V that diagonalizes Φ^λ, and let $(z_1^\lambda, \dots, z_n^\lambda)$ be the corresponding coordinates on V. For a strictly increasing multi-index L with $|L| = q$, denote by ω_L^λ the elementary form $d\bar{z}^{\lambda^L}$. Then, for $\phi = \sum_{|L|=q} \phi_L \otimes \omega_L^\lambda \in \mathscr{S}(\mathbb{R}^{\nu(\lambda)}) \otimes \Lambda_q$, we have*

$$d\pi_{\lambda,\eta}(\Box_b^{(q)}) \left(\sum_{|L|=q} \phi_L \otimes \omega_L^\lambda \right) = \sum_{|L|=q} (\mathscr{H}_{\lambda,\eta} + \alpha_L^\lambda) \phi_L \otimes \omega_L^\lambda,$$

where

$$\mathscr{H}_{\lambda,\eta} = -\sum_{j=1}^{\nu(\lambda)} (\partial_{\xi_j}^2 - \mu_j^2 \xi_j^2) + |\eta|^2,$$

and

$$\alpha_L^\lambda = \sum_{k \in L} \mu_k - \sum_{k \notin L} \mu_k. \tag{18}$$

In particular, $d\pi_{\lambda,\eta}(\Box_b^{(q)})$ *acts diagonally with respect to the basis* $\{h_m^\lambda \otimes \omega_L^\lambda\}$ *of* $L^2(\mathbb{R}^{\nu(\lambda)}) \otimes \Lambda_q$. *Precisely,*

$$d\pi_{\lambda,\eta}(\Box_b^{(q)})(h_m^\lambda \otimes \omega_L^\lambda) = \left(\sum_{j=1}^{\nu(\lambda)} |\mu_j|(1 + 2m_j) + |\eta|^2 + \alpha_L^\lambda\right) h_m^\lambda \otimes \omega_L^\lambda. \tag{19}$$

Proof. For the given orthonormal basis we write $Z_j^\lambda, \bar{Z}_j^\lambda$ as in (2). From (7) we have

$$d\pi_{\lambda,\eta}([Z_j^\lambda, \bar{Z}_k^\lambda]) = 2\delta_{jk}\mu_k.$$

Notice that $d\pi_{\lambda,\eta}(\mathscr{L}) = \sum_{j=1}^{\nu(\lambda)} \partial_{\xi_j}^2 - \mu_j^2 \xi_j^2 - |\eta|^2 = -\mathscr{H}$. The result now follows from Proposition 2.1 and from the fact that the Hermite function $h_j(t)$ on the real line is an eigenfunction of the Hermite operator $-(d/dt)^2 + t^2$ with eigenvalue $2j + 1$. $\Box$

The next result will be needed in Section 6. When $f \in \mathscr{S}(G_\Phi) \otimes E$, we still denote by $\widehat{f}$ the E-valued function

$$\widehat{f}(\lambda, \eta; m, m') = \int f(x^\lambda, y^\lambda, t) \overline{\chi_{m',m}^{\lambda,\eta}(x^\lambda, y^\lambda, t)}\, dx^\lambda\, dy^\lambda\, dt.$$

With an abuse of notation, we write

$$\widehat{f}(\lambda, \eta; m, m') = \langle \pi_{\lambda,\eta}(f) h_m^\lambda, h_{m'}^\lambda \rangle,$$

keeping in mind that the inner product on the right-hand side is vector-valued.

We also denote by $|\cdot|$ the norm on E.

Lemma 3.6. *For each positive integer* N, *there exist a Sobolev norm* $||\cdot||_{N'}$ *and a constant* $c_N > 0$ *such that for all* $f \in \mathscr{S}(G_\Phi) \otimes E$ *we have*

$$|\widehat{f}(\lambda, \eta; m, m)| \leqslant c_N \frac{||f||_{N'}}{(1 + |\eta|^2)^N (1 + |\lambda|)^N (1 + \sum_{j=1}^{\nu(\lambda)} (1 + 2m_j)|\mu_j|)^N}.$$

Proof. Consider the operator $\mathscr{L}\otimes I$ acting on $\mathscr{S}(G_\Phi)\otimes E$, where I denotes the identity map on E. Then

$$((\mathscr{L}\otimes I)f)^\wedge(\lambda,\eta;m,\, m) = \langle \pi_{\lambda,\eta}((\mathscr{L}\otimes I)f)h_m^\lambda, h_m^\lambda\rangle$$
$$= \langle d\pi_{\lambda,\eta}(\mathscr{L}\otimes I)\pi_{\lambda,\eta}(f)h_m^\lambda, h_m^\lambda\rangle$$
$$= \langle \pi_{\lambda,\eta}(f)h_m^\lambda, d\pi_{\lambda,\eta}(\mathscr{L}\otimes I)h_m^\lambda\rangle.$$

(The fact that $d\pi_{\lambda,\eta}(\mathscr{L}\otimes I)$ is self-adjoint on $L^2(\mathbb{R}^\nu)\otimes E$ follows from the polarized form of the Plancherel formula, see (17).)

Then,

$$((\mathscr{L}\otimes I)f)^\wedge(\lambda,\eta;m,\, m) = \left(|\eta|^2 + \sum_{j=1}^{\nu}(1+2m_j)|\mu_j|\right)\langle \pi_{\lambda,\eta}(f)h_m^\lambda, h_m^\lambda\rangle$$
$$= \left(|\eta|^2 + \sum_{j=1}^{\nu}(1+2m_j)|\mu_j|\right)\widehat{f}(\lambda,\eta;m,\, m).$$

The conclusion follows easily, once we observe that, from (15) and Lemma 3.3,

$$(1+|\eta|^2)(1+|\lambda|^2)\widehat{f}(\lambda,\eta;m,\, m) = \int f(x^\lambda, y^\lambda, t)\overline{\mathscr{P}_{t,z''}\chi^{\lambda,\eta}_{m',m}(x^\lambda, y^\lambda, t)}\, dx^\lambda\, dy^\lambda\, dt,$$

for a constant coefficient differential operator $\mathscr{P}_{t,z''}$ in t and z''. □

4. Non-solvability of $\square_b^{(q)}$

In this section we prove the negative result in Theorem 1. In fact we prove the stronger statement that, under the given assumption, the operator $\square_b^{(q)}$ is not even locally solvable.[2]

We will use the following necessary criterion for local solvability, which is the vector-valued extension of the corresponding version for scalar operators, due to Corwin and Rothschild [CoRt].

Lemma 4.1. *Let $\mathscr{M}$ be a homogeneous left-invariant differential operator from $\mathscr{S}(G_\Phi)\otimes E$ to $\mathscr{S}(G_\Phi)\otimes F$, and let $\mathscr{M}^*:\mathscr{S}(G_\Phi)\otimes F'\to\mathscr{S}(G_\Phi)\otimes E'$ be the adjoint operator. Suppose that there exists a non-trivial $\phi\in\mathscr{S}(G_\Phi)\otimes F'$ such that $\mathscr{M}^*\phi=0$. Then $\mathscr{M}$ is not locally solvable.*

[2] A differential operator P is said to be locally solvable at a point x_0 if there exists an open neighborhood U of x_0 such that for any test function f with support contained in U there exists a distribution u such that $Pu=f$ on U. For translation invariant operators, local solvability does not depend on x_0.

Proof. We argue by contradiction. By Hörmander's condition [Hö], $\mathscr{M}$ is locally solvable at a point $(z_0, t_0) \in G_\Phi$ if and only if there exist a neighborhood U of (z_0, t_0), $k \in \mathbb{N}$, and a constant $c > 0$ such that

$$||g||_{-k} \leqslant c||\mathscr{M}^* g||_k$$

for all $g \in C_0^\infty(U) \otimes F'$, where $|| \cdot ||_r$ denotes the Sobolev norm.

Suppose that $\mathscr{M}$ is locally solvable. Using the homogeneity of $\mathscr{M}$, the proof of Lemma 1 in [CoRt] goes through without changes to the case of vector-valued functions to imply that there exists $k \in \mathbb{N}$ such that the following holds. For each $\psi \in C_0^\infty(G_\Phi) \otimes F$ there exists $\{f_m\} \subseteq C_0^\infty(G_\Phi) \otimes E$ such that: (i) $\operatorname{supp} f_m \subseteq \{|(z,t)| \leqslant m+1\}$; (ii) $\mathscr{M} f_m = \psi$ on $\{|(z,t)| \leqslant m\}$; (iii) $|\mathscr{M} f_m(z,t)| \leqslant m^k$.

Given ϕ as in the statement, let $\psi \in C_0^\infty(G_\Phi) \otimes F$. Then

$$\begin{aligned}
&\left| \int_{|(z,t)| \leqslant m+1} \langle \phi(z,t), \psi(z,t) \rangle \, dz\, dt \right| \\
&\quad = \left| \int_{|(z,t)| \leqslant m+1} \langle \phi(z,t), \psi(z,t) - \mathscr{M} f_m(z,t) \rangle \, dz\, dt \right| \\
&\quad = \left| \int_{m \leqslant |(z,t)| \leqslant m+1} \langle \phi(z,t), \psi(z,t) - \mathscr{M} f_m(z,t) \rangle \, dz\, dt \right| \\
&\quad \leqslant c_\psi \int_{m \leqslant |(z,t)| \leqslant m+1} |\phi(z,t)| m^k \, dz\, dt,
\end{aligned}$$

which tends to 0 as $m \to +\infty$. Then $\phi = 0$, a contradiction. Hence, $\mathscr{M}$ is not locally solvable. $\square$

We state for future reference a lemma whose proof is essentially contained in the last part of Section 3.

Lemma 4.2. *Given the partial differential equation* $(\mathscr{H}_{\lambda,\eta} + \alpha_L^\lambda) f = 0$, *the following conditions are equivalent*:

(i) *there exists a non-trivial solution* $f \in \mathscr{S}(\mathbb{R}^{\nu(\lambda)})$;
(ii) $\eta = 0$ *and the multi-index* L *is such that* $\mu_k \leqslant 0$ *for* $k \in L$ *and* $\mu_k \geqslant 0$ *for* $k \notin L$.

Recall that, given $\lambda \in W^*$, we denote by $n^+(\lambda)$ the number of positive eigenvalues of the form Φ^λ, and by $n^-(\lambda)$ the number of negative eigenvalues.

Definition 4.3. We define Ω_q to be the cone

$$\Omega_q = \{\lambda : n^+(\lambda) = q, n^-(\lambda) = n - q\}.$$

Therefore, Theorem 1 can be restated by saying that $\Box_b^{(q)}$ is (locally) solvable if and only if Ω_q is empty (or equivalently if and only if $\Omega_{n-q} = -\Omega_q$ is empty).

Theorem 4.4. *Assume that Ω_q is non-empty. Then there is a non-trivial $\omega \in \mathcal{S}(G_\Phi) \otimes \Lambda_q$ such that $\Box_b^{(q)}\omega = 0$.*

Proof. Under the given assumptions, $\Omega'_{n-q} = \Omega_{n-q} \cap \Omega'$ is non-empty. As $v = n$, there is no η in the parameters for the generic irreducible representations of G_Φ.

Let $\lambda_0 \in U \subset \Omega'_{n-q}$ be as in Section 3. Let $z_j^\lambda = x_j^\lambda + iy_j^\lambda$ be the coordinates adapted to a corresponding smoothly varying frame on V, and let $\omega_L^\lambda = d\bar{z}^{\lambda^L}$, as in Section 3. Then ω_L^λ varies smoothly with λ.

Let $\bar{L}$ be the multi-index of length q formed by those k for which $\mu_k(\lambda) < 0$ on U. Slightly modifying the construction in the proof of Lemma 3.4 we take a C^∞-function $\phi(\lambda)$ with compact support in U and set

$$\omega(z,t) = \int_U \phi(\lambda)\chi_{0,0}^\lambda(z,t)D(\lambda)\omega_{\bar{L}}^\lambda \, d\lambda.$$

It follows easily from (14) that $\omega \in \mathcal{S}(G_\Phi) \otimes \Lambda_q$. As in the proof of Lemma 3.4, it is easily shown that the only irreducible unitary representations of G_Φ for which $\pi_\lambda(\omega) \neq 0$ are those with λ in the support of ϕ. For these λ we have

$$\pi_\lambda(\omega) = \phi(\lambda)A_{0,0}^\lambda \otimes \omega_{\bar{L}}^\lambda,$$

where $A_{0,0}$ is the orthogonal projection onto the one-dimensional subspace of $L^2(\mathbb{R}^n)$ spanned by h_0^λ.

It follows from Proposition 3.5 that, for λ in the support of ϕ,

$$\pi_\lambda(\Box_b^{(q)}\omega) = \left(\sum_{j=1}^n |\mu_j| + \alpha_{\bar{L}}^\lambda \right) \phi(\lambda)A_{0,0}^\lambda \otimes \omega_{\bar{L}}^\lambda = 0,$$

because

$$\alpha_{\bar{L}}^\lambda = \sum_{k \in \bar{L}} \mu_k(\lambda) - \sum_{k \notin \bar{L}} \mu_k(\lambda) = -\sum_{k=1}^n |\mu_k(\lambda)|.$$

By uniqueness of the Fourier transform, $\Box_b^{(q)}\omega = 0$. $\Box$

5. The orthogonal projection on the null space of $\Box_b^{(q)}$

Assume that Ω_q is non-empty. It follows from Theorem 4.4 that the null space of $\Box_b^{(q)}$ is non-trivial in the space of Schwartz $(0, q)$-forms. We shall determine the null space in $L^2(G_\Phi) \otimes \Lambda_q$ and obtain an expression for the corresponding orthogonal projector involving a kind of Laplace transform.

Let $\{U_j\}$ be a locally finite open covering of Ω'_{n-q} such that each U_j is relatively compact in Ω'_{n-q} and for each $\lambda \in U_j$ there is an orthonormal coordinate system $(z_1^\lambda, \dots, z_n^\lambda)$ on V that varies smoothly with λ and diagonalizes Φ^λ as $\Phi^\lambda(z, z) = \sum_{k=1}^n \mu_k |z_k^\lambda|^2$. Let $\bar{L}$ be the multi-index of length q containing those k for which $\mu_k < 0$.

Let also $\{\rho_j\}$ be a smooth partition of unity on Ω'_{n-q} subordinated to the given covering.

Lemma 5.1. *Let $\omega \in L^2(G_\Phi) \otimes \Lambda_q$. The following are equivalent*:

(i) *ω is in the null space of $\Box_b^{(q)}$*;

(ii) *$\pi_\lambda(\omega) = 0$ a.e. outside of Ω_{n-q} and, a.e. on each U_j, $\pi_\lambda(\omega) = T_j^\lambda \otimes \omega_{\bar{L}}^\lambda$, where T_j^λ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$, with range in the linear span of h_0^λ.*

Proof. A form ω in $L^2(G_\Phi) \otimes \Lambda_q$ is in the null space of $\Box_b^{(q)}$ if and only if, for every $\tau \in \mathscr{S}(G_\Phi) \otimes \Lambda_q$,

$$
\begin{aligned}
\langle \Box_b^{(q)} \omega, \tau \rangle &= \langle \omega, \Box_b^{(q)} \tau \rangle \\
&= \int_\Omega \langle \pi_\lambda(\omega), d\pi_\lambda(\Box_b^{(q)}) \pi_\lambda(\tau) \rangle D(\lambda)\, d\lambda \\
&= 0.
\end{aligned}
\tag{20}
$$

Assume that ω satisfies (ii). Then

$$
\begin{aligned}
\langle \Box_b^{(q)} \omega, \tau \rangle &= \int_{\Omega_{n-q}} \langle \pi_\lambda(\omega), d\pi_\lambda(\Box_b^{(q)}) \pi_\lambda(\tau) \rangle D(\lambda)\, d\lambda \\
&= \sum_j \int_{U_j} \rho_j(\lambda) \langle T_j^\lambda \otimes \omega_{\bar{L}}^\lambda, d\pi_\lambda(\Box_b^{(q)}) \pi_\lambda(\tau) \rangle D(\lambda)\, d\lambda \\
&= \sum_j \int_{U_j} \rho_j(\lambda) \sum_{m \in \mathbb{N}^n} \langle T_j^\lambda h_m^\lambda \otimes \omega_{\bar{L}}^\lambda, d\pi_\lambda(\Box_b^{(q)}) \pi_\lambda(\tau) h_m^\lambda \otimes \omega_{\bar{L}}^\lambda \rangle D(\lambda)\, d\lambda \\
&= \sum_j \int_{U_j} \rho_j(\lambda) \sum_{m \in \mathbb{N}^n} \langle d\pi_\lambda(\Box_b^{(q)})(T_j^\lambda h_m^\lambda \otimes \omega_{\bar{L}}^\lambda), \pi_\lambda(\tau) h_m^\lambda \otimes \omega_{\bar{L}}^\lambda \rangle D(\lambda)\, d\lambda \\
&= 0,
\end{aligned}
$$

by Proposition 3.5, because $T_j^\lambda h_m^\lambda \otimes \omega_{\bar{L}}^\lambda$ is a scalar multiple of $h_0^\lambda \otimes \omega_{\bar{L}}^\lambda$ and $\alpha_{\bar{L}}^\lambda = -\sum_{k=1}^n |\mu_k|$. Hence (ii) implies (i).

Assume now that (i) holds, i.e. that (20) is satisfied for every Schwartz form τ.

Take $\lambda_0 \in \Omega'$ and let U be a neighborhood of λ_0 allowing a smoothly varying frame with coordinates $(z_1^\lambda, \dots, z_n^\lambda)$ of V for $\lambda \in U$. Let ϕ be a smooth function with compact support in U, $m, m' \in \mathbb{N}^n$ and L a multi-index of length q. We set

$$\tau(z,t) = \int_U \phi(\lambda)\chi_{m',m}^\lambda(x^\lambda, y^\lambda, t)\omega_L^\lambda D(\lambda)\,d\lambda.$$

As in the proof of Lemma 3.4, we find that, for $\lambda \in U$, $\pi_\lambda(\tau) = \phi(\lambda)A_{m,m'}^\lambda \otimes \omega_L^\lambda$ for $\lambda \in U$, where $A_{m,m'}^\lambda h_p^\lambda = \delta_{m,p} h_{m'}^\lambda$, and 0 otherwise.

Therefore,

$$\pi_\lambda(\Box_b^{(q)}\tau) = \left(\sum_{j=1}^n |\mu_j|(1+2m_j') + \alpha_L^\lambda \right) \phi(\lambda) A_{m,m'}^\lambda \otimes \omega_L^\lambda$$

for $\lambda \in U$ and 0 otherwise.

Since (20) holds,

$$\int_U \left(\sum_{j=1}^n |\mu_j|(1+2m_j') + \alpha_L^\lambda \right) \phi(\lambda) \langle \pi_\lambda(\omega), A_{m,m'}^\lambda \otimes \omega_L^\lambda \rangle D(\lambda)\,d\lambda = 0$$

for every ϕ. So, either $\sum_{j=1}^n |\mu_j|(1+2m_j') + \alpha_L^\lambda = 0$, or $\langle \pi_\lambda(\omega), A_{m,m'}^\lambda \otimes \omega_L^\lambda \rangle = 0$ for a.e. $\lambda \in U$.

The first condition is satisfied if and only if $m' = 0$, $U \subset \Omega'_{n-q}$ and $L = \bar{L}$. This concludes the proof. $\Box$

In order to describe the projection operator, observe that, by translation invariance, it must have the form

$$\omega \mapsto \omega * \mathscr{C}_q,$$

where $\mathscr{C}_q$ is a distribution taking values in $\mathrm{End}(\Lambda_q)$. It is important at this point to make the following remark.

As we have already observed, each point in Ω'_{n-q} has a neighborhood U on which we can define a smooth function $\lambda \mapsto \omega_{\bar{L}}^\lambda$ with values in Λ_q and where the multi-index $\bar{L}$ consists of the indices j such that $\mu_j < 0$.

In general, this function cannot be extended to all of Ω'_{n-q}.[3] If two neighborhoods U and U' intersect, then the two corresponding choices of $\omega^\lambda_{\bar{L}}$ differ by a scalar factor of absolute value 1.

This implies however that, at each $\lambda \in U \cap U'$, the two corresponding orthogonal projections of Λ_q onto the linear span of $\omega^\lambda_{\bar{L}}$ coincide. This orthogonal projection, that we call P^λ_-, is hence well defined and smooth on all of Ω'_{n-q}.

In fact P^λ_- is well defined and smooth on all of Ω_{n-q}. In order to see this, we must regard the elements of Λ_q as multi-linear functionals on $V \otimes_{\mathbb{R}} \mathbb{C}$. The action of P^λ_- on a $(0,q)$-form is then the composition of the form itself with the projection, in each component, onto the linear span of the $(0,q)$-eigenvectors of Φ^λ with negative eigenvalues. This operation is well defined and smooth on all of Ω_{n-q}.

Theorem 5.2. *The orthogonal projection of $L^2(\mathbb{R}^n) \otimes \Lambda_q$ onto the null space of $\Box_b^{(q)}$ maps a form ω into $\omega * \mathscr{C}_q$, where $\mathscr{C}_q \in \mathscr{S}'(\mathbb{R}^n) \otimes \mathrm{End}(\Lambda_q)$ is given by*

$$\mathscr{C}_q(z,t) = \int_{\Omega_{n-q}} e^{i\lambda(t)} e^{-|\Phi^\lambda|(z,z)} P^\lambda_- \, D(\lambda)\, d\lambda,$$

where $|\Phi^\lambda|(z,z) = \sum_{k=1}^n |\mu_k||z^\lambda_k|^2$.

The formula for $\mathscr{C}_q$ must be interpreted in the sense of distributions. To be precise, if $\psi \in \mathscr{S}(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$, we have

$$\begin{aligned}
\langle \mathscr{C}_q, \psi \rangle &= \int_{G_\Phi} \int_{\Omega_{n-q}} e^{i\lambda(t)} e^{-|\Phi^\lambda|(z,z)} \,\mathrm{tr}(P^\lambda_- \psi(z,t)^*) D(\lambda)\, d\lambda\, dz\, dt \\
&= \int_V \int_{\Omega_{n-q}} e^{-|\Phi^\lambda|(z,z)} \,\mathrm{tr}(P^\lambda_- \mathscr{F}_t \psi(z,\lambda)^*) D(\lambda)\, d\lambda\, dz \\
&= \sum_j \int_V \int_{U_j} \rho_j(\lambda) e^{-|\Phi^\lambda|(z,z)} \langle \omega^\lambda_{\bar{L}}, \mathscr{F}_t \psi(z,\lambda) \omega^\lambda_{\bar{L}} \rangle D(\lambda)\, d\lambda\, dz.
\end{aligned}$$

Proof. By Lemma 5.1, the Fourier transform of $\mathscr{C}_q$ is given by $\pi_\lambda(\mathscr{C}_q) = 0$ for $\lambda \in \Omega \backslash \Omega_{n-q}$ and $\pi_\lambda(\mathscr{C}_q) = A^\lambda_{0,0} \otimes P^\lambda_-$ for $\lambda \in \Omega_{n-q}$. Therefore, if $\psi \in \mathscr{S}(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$,

$$\begin{aligned}
\langle \mathscr{C}_q, \psi \rangle &= \int_{\Omega_{n-q}} \langle A^\lambda_{0,0} \otimes P^\lambda_-, \pi_\lambda(\psi) \rangle D(\lambda)\, d\lambda \\
&= \int_{\Omega_{n-q}} \mathrm{tr}(P^\lambda_- \hat{\psi}(\lambda, 0; 0, 0)^*) D(\lambda)\, d\lambda.
\end{aligned}$$

[3] This is possible if Ω'_{n-q} is simply connected. One way to overcome the topological problems is to lift to the universal covering of Ω'_{n-q}, as in [MR]. We have chosen to avoid this by introducing partitions of unity when strictly necessary.

By (14),

$$\hat{\psi}(\lambda,0;0,0) = \int_{G_\Phi} \psi(z,t)\overline{\chi^{\lambda}_{0,0}(z,t)}\,dz\,dt = \int_V \mathscr{F}_t\psi(z,\lambda)e^{-|\Phi^\lambda|(z,z)}\,dz,$$

and this gives the proof. □

The formula for $\mathscr{C}_q$ generalizes the classical Gindikin formula for the Cauchy-Szegö kernel on the Šilov boundary of a Siegel domain of type II (see [G] Theorem 5.3 and [KS]). As it was mentioned in the Introduction, S is the Šilov boundary of such a domain if and only if Ω_n is non-empty. If this is the case, let $\Gamma \subset W$ be the conic hull of $\{\Phi(z,z) : z \in V\}$, and let

$$\mathscr{D} = \{(z,w) :\ \operatorname{Im} w - \Phi(z,z) \in \mathring{\Gamma}\}$$

be the corresponding Siegel domain. Then S is the Šilov boundary of $\mathscr{D}$. Since Ω_n is the dual open cone of Γ, according to Gindikin's formula,

$$\mathscr{C}_0(z,t) = \int_{\Omega_n} e^{i\lambda(t)}e^{-\Phi^\lambda(z,z)}D(\lambda)\,d\lambda$$

is the (scalar-valued) convolution kernel of the orthogonal projection of $L^2(G_\Phi)$ onto the Hardy space consisting of boundary values of holomorphic H^2-functions on $\mathscr{D}$ (see [OV]).

6. Fundamental solution for $\square_b^{(q)}$

In this section we prove the positive part in Theorem 1 by constructing a tempered fundamental solution $K = K_q$ for $\square_b^{(q)}$ when Ω_q is empty. Minor modifications to the formula will give a relative fundamental solution when Ω_q is non-empty.

The definition of fundamental solution requires the introduction of some more formalism.

Let $\phi \in \mathscr{S}(G_\Phi) \otimes \operatorname{Hom}(E, \Lambda_q)$, where E is a finite-dimensional space. Because of the canonical identification of $\operatorname{Hom}(E, \Lambda_q)$ with $E' \otimes \Lambda_q$, we can write

$$\phi(z,t) = \sum_j \omega_j(z,t) \otimes \psi_j,$$

where the sum is finite, $\psi_j \in E'$ and $\omega_j \in \mathscr{S}(G_\Phi) \otimes \Lambda_q$. We then set

$$\square_b^{(q)}\phi = \sum_j (\square_b^{(q)}\omega_j) \otimes \psi_j. \tag{21}$$

This is consistent with the original definition of $\Box_b^{(q)}$ on forms, because of the identification $\Lambda_q \cong \mathrm{Hom}(\mathbb{C}, \Lambda_q)$. If E has an inner product, the action of $\Box_b^{(q)}$ can be extended to distributions in $\mathscr{S}'(G_\Phi) \otimes \mathrm{Hom}(E, \Lambda_q)$.

We then say that $K \in \mathscr{S}'(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$ is a *fundamental solution* of $\Box_b^{(q)}$ if $\Box_b^{(q)} K = \delta_0 \otimes I$, i.e. if

$$\langle K, \Box_b^{(q)} \phi \rangle = \overline{\mathrm{tr}\, \phi(0)},$$

for $\phi \in \mathscr{S}(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$.

The existence of a fundamental solution implies that $\Box_b^{(q)}$ is solvable, because for $\omega \in \mathscr{S}(G_\Phi) \otimes \Lambda_q$ we have

$$\Box_b^{(q)}(\omega * K) = \omega * (\delta_0 \otimes I) = \omega.$$

In order to construct such a fundamental solution, we distinguish between the case $v = n$ and $v < n$. In the former case we must assume explicitly that Ω_q is empty. This assumption is automatically verified in the latter case.

6.1. Case $v = n$

For $\lambda \in \Omega$ we define $\mathscr{K}_q^\lambda \in \mathrm{End}\, L^2(\mathbb{R}^n) \otimes \mathrm{End}(\Lambda_q)$ as follows. Keeping the notation in Proposition 3.5, let ω_L^λ denote the elementary form $d\bar{z}^{\lambda^L}$. Then, for $\sum_{|L|=q} \psi_L \otimes \omega_L^\lambda \in L^2(\mathbb{R}^n) \otimes \Lambda_q$, we set

$$\mathscr{K}_q^\lambda \left(\sum_{|L|=q} \psi_L \otimes \omega_L^\lambda \right) = \sum_m \sum_{|L|=q} \frac{\langle \psi_L, h_m^\lambda \rangle}{\alpha_L^\lambda + \sum_{j=1}^n (1 + 2m_j)|\mu_j|} h_m^\lambda \otimes \omega_L^\lambda. \tag{22}$$

Furthermore, for $\phi \in \mathscr{S}(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$, we define K_q by setting

$$\langle K_q, \phi \rangle = \int_\Omega \langle \mathscr{K}_q^\lambda, \pi_\lambda(\phi) \rangle |\det \Phi^\lambda| \, d\lambda, \tag{23}$$

where the pairing $\langle \cdot, \cdot \rangle$ is defined in (17).

Theorem 6.1. *Let Ω_q be empty and $v = n$. Then K_q is a well-defined tempered distribution on G_Φ, that is, $K_q \in \mathscr{S}'(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$. Moreover, K_q is a global, homogeneous, fundamental solution for $\Box_b^{(q)}$.*

In the proof we will need the following result.

Lemma 6.2. *There is $N_0 \in \mathbb{N}$ such that for any $N \geqslant N_0$ there exists a constant $c_{N,n} \geqslant 0$ such that for each multi-index L we have*

$$\sum_m \frac{1}{(\alpha_L^\lambda + \sum_{j=1}^n (1+2m_j)|\mu_j|)(1+\sum_{j=1}^n (1+2m_j)|\mu_j|)^N} \leqslant c_{n,N} \frac{1+|\lambda|^n}{|\det \Phi^\lambda|}. \qquad (24)$$

Assuming the validity of the lemma we prove Theorem 6.1.

Proof of Theorem 6.1. We begin by showing that $K = K_q \in \mathscr{S}'(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$.

For fixed $\lambda \in \Omega$, consider the orthonormal basis $(d\bar{z}^{\lambda^K})^* \otimes d\bar{z}^{\lambda^L}$ of $\Lambda_q' \otimes \Lambda_q \cong \mathrm{End}(\Lambda_q)$, where we have set $v^*(w) = \langle w, v \rangle$ for v, w in any inner product space. If $\phi \in \mathscr{S}(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$, we write

$$\phi = \sum_{K,L} \phi_{KL} (d\bar{z}^{\lambda^K})^* \otimes d\bar{z}^{\lambda^L},$$

where the ϕ_{KL} are scalar-valued functions.

By (17) and Lemma 3.6 we have

$$\begin{aligned}|\langle K, \phi \rangle| &= \left| \int_\Omega \sum_L \sum_m \frac{\overline{\hat{\phi}_{LL}(\lambda; m, m)}}{\alpha_L^\lambda + \sum_{j=1}^n (1+2m_j)|\mu_j|} |\det \Phi^\lambda| \, d\lambda \right| \\ &\leqslant c\|\phi\|_{N'} \int_\Omega \sum_L S(L, \lambda) \frac{|\det \Phi^\lambda|}{(1+|\lambda|)^N} \, d\lambda,\end{aligned}$$

where $S(L, \lambda)$ denotes the left-hand side in (24).

From Lemma 6.2 it follows that for N large enough,

$$|\langle K, g \rangle| \leqslant c\|\phi\|_{N'},$$

which shows that $K \in \mathscr{S}'(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$.

We now show that K is a fundamental solution for $\Box_b^{(q)}$. For $\phi \in \mathscr{S}(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$, we have

$$\begin{aligned}\langle \Box_b^{(q)} K, \phi \rangle &= \langle K, \Box_b^{(q)} \phi \rangle \\ &= \int_\Omega \langle \pi_\lambda(K), \pi_\lambda(\Box_b^{(q)} \phi) \rangle |\det \Phi^\lambda| \, d\lambda \\ &= \int_\Omega \langle \pi_\lambda(K), d\pi_\lambda(\Box_b^{(q)}) \pi_\lambda(\phi) \rangle |\det \Phi^\lambda| \, d\lambda\end{aligned}$$

$$= \int_\Omega \sum_L \sum_m \frac{(\alpha_L^\lambda + \sum_{j=1}^n (1+2m_j)|\mu_j|)\langle h_m^\lambda, \pi_\lambda(\phi_{LL})h_m^\lambda\rangle}{(\alpha_L^\lambda + \sum_{j=1}^n (1+2m_j)|\mu_j|)} |\det \Phi^\lambda|\, d\lambda$$

$$= \overline{\mathrm{tr}\, \phi(0)}.$$

This proves the proposition. □

Proof of Lemma 6.2. We wish to estimate the left-hand side of (24).

We split the sum for $m \in \mathbb{N}^n$ as

$$\sum_{E \subseteq \{1,\dots,n\}} \left(\sum_{\substack{m_j=0 \text{ if } j\notin E \\ m_j \geqslant 1 \text{ if } j\in E}} \frac{1}{(\alpha_L^\lambda + \sum_{j=1}^n (1+2m_j)|\mu_j|)(1+\sum_{j=1}^n (1+2m_j)|\mu_j|)^N} \right)$$

and we write $|E|$ to denote the cardinality of E.

For each L fixed, we may relabel coordinates in order to have $\alpha_L^\lambda = \sum_{j=1}^p |\mu_j| - \sum_{j=p+1}^n |\mu_j|$. Then,

$$\alpha_L^\lambda + \sum_{j=1}^n (1+2m_j)|\mu_j| = \sum_{j=1}^p |\mu_j| + 2\sum_{j=1}^n m_j|\mu_j|.$$

Notice that $p \geqslant 1$ since $v = n$ and Ω_q is empty.

Let $E = \{j_1, \dots, j_k\}$. If $|E| = k \geqslant 2$,

$$\begin{aligned}
&\sum_{\substack{m_j=0 \text{ if } j\notin E \\ m_j \geqslant 1 \text{ if } j\in E}} \frac{1}{(\alpha_L^\lambda + \sum_{j=1}^n (1+2m_j)|\mu_j|)(1+\sum_{j=1}^n (1+2m_j)|\mu_j|)^N} \\
&\leqslant \sum_{\substack{m_{j_i} \geqslant 1 \\ i=1,\dots,k}} \frac{1}{(\sum_{i=1}^k m_{j_i}|\mu_{j_i}|)(1+\sum_{i=1}^k m_{j_i}|\mu_{j_i}|)^N} \\
&\leqslant \int_0^{+\infty} \cdots \int_0^{+\infty} \frac{1}{(\sum_{i=1}^k x_{j_i}|\mu_{j_i}|)(1+\sum_{i=1}^k x_{j_i}|\mu_{j_i}|)^N}\, dx_{j_1}\cdots dx_{j_k} \\
&\leqslant c\, \frac{1}{\prod_{j_i \in E} |\mu_{j_i}|} \\
&\leqslant c\, \frac{1+|\lambda|^n}{|\det \Phi^\lambda|},
\end{aligned}$$

where the last inequality follows from estimates for the eigenvalues of a Hermitean form (see e.g. [MR, Lemma 4.2]).

Next, if $|E|=1$, the corresponding sum is bounded by a constant times

$$\sum_{m\geqslant 1}\frac{1}{m|\mu_{j_0}|(1+m|\mu_{j_0}|)^N}\leqslant\frac{1}{|\mu_{j_0}|}+\int_1^{+\infty}\frac{1}{x|\mu_{j_0}|(1+x|\mu_{j_0}|)^N}\,dx$$
$$\leqslant c\,\frac{1}{|\mu_{j_0}|},$$

and the claimed estimate follows as before.

Finally, if $|E|=0$, the corresponding term equals $1/(\sum_{j=1}^p|\mu_j|)$ for which we easily obtain the estimate. This proves the lemma. □

If Ω_q is non-empty, define $\mathscr{K}^\lambda$ by (22) if $\lambda\notin\Omega_q$, and by the same formula with the sum in L extended only to $L\neq\bar{L}$ if $\lambda\in\Omega_q$, where $\bar{L}$ is the multi-index introduced in the proof of Theorem 4.4. Then define K_{rel} according to (23).

Corollary 6.3. *If Ω_q is non-empty, K_{rel} is a relative fundamental solution of $\square_b^{(q)}$, i.e. $\square_b^{(q)}K_{\mathrm{rel}}=\delta_0\otimes I-\mathscr{C}_q$.*

We now treat the case in which the form Φ^λ is degenerate for all λ, that is the maximum rank v of Φ^λ is strictly less than n. We split this case into two subcases: when $v<n-1$ and when $v=n-1$. The former case is technically similar to the case $v=n$. Instead, the latter case requires a more involved definition of the fundamental solution. The difference between these two cases somehow resembles the difference in the formulas for the fundamental solution of the classical Laplacian in $\mathbb{R}^2$ and $\mathbb{R}^n$ with $n>2$.

6.2. Case $v<n-1$

We now assume that the form Φ^λ is degenerate for all λ and that the maximum rank v of Φ^λ is strictly less than $n-1$. As before, we denote by Ω the Zariski-open cone of the $\lambda\in W^*$ for which rank $\Phi^\lambda=v$ and by Ω' the subcone of Ω where the number of distinct eigenvalues of Φ^λ is maximum.

For $\lambda\in\Omega'$, $\eta\neq 0$ and for $\sum_{|L|=q}\psi_L\otimes\omega_L^\lambda\in L^2(\mathbb{R}^v)\otimes\Lambda_q$, we set

$$\mathscr{K}_q^{\lambda,\eta}\left(\sum_{|L|=q}\psi_L\otimes\omega_L^\lambda\right)=\sum_m\sum_{|L|=q}\frac{\langle\psi_L,h_m^\lambda\rangle}{|\eta|^2+\alpha_L^\lambda+\sum_{j=1}^n(1+2m_j)|\mu_j|}h_m^\lambda\otimes\omega_L^\lambda. \tag{25}$$

Furthermore, for $\phi \in \mathscr{S}(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$, we define K_q by setting

$$\langle K_q, \phi \rangle = \int_{\Omega'} \int_{\mathbb{C}^{\nu-n}} \langle \mathscr{K}_q^{\lambda,\eta}, \pi_\lambda(\phi) \rangle \, d\eta \, D(\lambda) \, d\lambda. \tag{26}$$

Essentially the same proof of Theorem 6.1 proves the following.

Theorem 6.4. *Let* $\nu < n-1$ *and* K_q *be defined by* (26). *Then* $K_q \in \mathscr{S}'(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$ *and it is a global, homogeneous, fundamental solution for* $\Box_b^{(q)}$.

6.3. Case $\nu = n-1$

As before, let Ω' be the subcone of Ω where the number of distinct eigenvalues of Φ^λ is maximum. We must treat with special care the values of λ for which there exists at least a multi-index L such that $\alpha_L^\lambda + \sum_{j=1}^{\nu} |\mu_j| = 0$. (The existence of such λ was excluded in the case $\nu = n$, because of the assumption $\Omega_q = \emptyset$, while in the case $\nu < n-1$ such λ do not cause any inconvenience since the function $1/|\eta|^2$ is locally integrable in $\mathbb{C}^k$ when $k > 1$.) Let Γ be the subcone of Ω' consisting of such λ.

Moreover, let

$$\mathscr{E}_\lambda = \left\{ L : |L| = q, \alpha_L^\lambda + \sum_{j=1}^{\nu} |\mu_j| = 0 \right\},$$

and

$$\mathscr{D}_\lambda = \{(L, m) : L \in \mathscr{E}_\lambda, m = 0 \in \mathbb{N}^\nu\}.$$

Let $\{U_k\}$ be an open covering of Ω' such that on each U_k a smoothly varying frame can be chosen according to Section 3.3. In particular, on each U_k we have well-defined functions $\mu_j = \mu_j(\lambda)$ parametrizing the eigenvalues of Φ^λ. We order them in such a way that $\mu_{\nu+1}(\lambda) = \cdots = \mu_n(\lambda) = 0$. Let $\{\rho_k\}$ be a smooth partition of unity subordinated to this covering.

In the present situation, we need to modify the definition of the fundamental solution of $\Box_b^{(q)}$ as follows. Let $\mathscr{U} = \{(\lambda, \eta) : \lambda \in \Gamma, \ |\eta| < 1\}$.

We set $K_q = K' + K''$ where, for $\phi \in \mathscr{S}(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$ we define

$$\langle K', \phi \rangle = \sum_k \int \int_{(\Omega' \times \mathbb{C}^{n-\nu}) \setminus \mathscr{U}} \rho_k(\lambda) \langle \mathscr{K}_q^{\lambda,\eta}, \pi_{\lambda,\eta}(\phi) \rangle \, dy \, D(\lambda) \, d\lambda, \tag{27}$$

with $\mathscr{K}_q^{\lambda,\eta}$ defined by (25), and

$$\langle K'', \phi \rangle = \sum_k \int \int_{\mathscr{U}} \rho_k(\lambda) \sum_{(L,m)\notin \mathscr{D}_\lambda} \frac{\overline{\hat{\phi}_{LL}(\lambda, \eta; m, m)}}{\alpha_L^\lambda + |\eta|^2 + \sum_{j=1}^{\nu} (1+2m_j)|\mu_j|} \, d\eta \, D(\lambda) \, d\lambda$$
$$+ \sum_k \int \int_{\mathscr{U}} \rho_k(\lambda) \sum_{L \in \mathscr{E}_\lambda} \frac{\overline{\hat{\phi}_{LL}(\lambda, \eta; 0, 0)} - \overline{\hat{\phi}_{LL}(\lambda, 0; 0, 0)}}{|\eta|^2} \, d\eta \, D(\lambda) \, d\lambda, \quad (28)$$

where $\widehat{f}(\lambda, \eta; m, m)$ is given by (15).

Theorem 6.5. *Let* $\nu = n-1$ *and let* $K_q = K' + K''$ *be defined as above. Then* $K_q \in \mathscr{S}'(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$. *Moreover,* K_q *is a fundamental solution for* $\square_b^{(q)}$.

Proof. Notice that for $(\lambda, m) \in (\Omega' \setminus \Gamma) \times \mathbb{N}^\nu$ or $\lambda \in \Gamma$ and $(L, m) \notin \mathscr{D}_\lambda$ we have

$$\alpha_L^\lambda + |\eta|^2 + \sum_{j=1}^{\nu} (1+2m_j)|\mu_j| \geqslant \sum_{j=1}^{p} |\mu_j|,$$

for some integer p, $1 \leqslant p \leqslant \nu$. Then, combining Lemma 3.6 with an argument analogous to that given in the proof of Lemma 6.2, we obtain that

$$|\langle K', \phi \rangle| \, c \sum_{|L|=q} \int_{\Omega'} \int_{\mathbb{C}^{n-\nu}} |\hat{\phi}_{LL}(\lambda, \eta; m, m)| \, d\eta (1 + |\lambda|^\nu) \, d\lambda$$
$$\leqslant c ||\phi||_{N'}.$$

The fact that $|\langle K'', \phi \rangle| \leqslant c ||\phi||_{N'}$ follows from standard arguments. This shows that $K \in \mathscr{S}'(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$.

Finally, we prove that K is a fundamental solution of $\square_b^{(q)}$. Let $\phi \in \mathscr{S}(G_\Phi) \otimes \mathrm{End}(\Lambda_q)$. By Proposition 3.5, arguing as in the proof of Lemma 3.6, we have that

$$(\square_b^{(q)} \phi)^\wedge (\lambda, \eta; m, m) = \langle \pi_{\lambda,\eta}(\square_b^{(q)} \phi) h_m^\lambda, h_m^\lambda \rangle$$
$$= \langle d\pi_{\lambda,\eta}(\square_b^{(q)}) \pi_{\lambda,\eta}(\phi) h_m^\lambda, h_m^\lambda \rangle$$
$$= \langle \pi_{\lambda,\eta}(\phi) h_m^\lambda, d\pi_{\lambda,\eta}(\square_b^{(q)}) h_m^\lambda \rangle.$$

Hence,

$$\begin{aligned}(\Box_b^{(q)}\phi)^{\wedge}_{LL}(\lambda,\eta;m,\,m) &= \left(\alpha_L^\lambda + |\eta|^2 + \sum_{j=1}^{v}\,(1+2m_j)|\mu_j|\right)\langle \pi_{\lambda,\eta}(\phi_{LL})h_m^\lambda, h_m^\lambda\rangle \\ &= \left(\alpha_L^\lambda + |\eta|^2 + \sum_{j=1}^{v}\,(1+2m_j)|\mu_j|\right)\hat{\phi}_{LL}(\lambda,\eta;m,\,m).\end{aligned}$$

From this, it also follows that, for $L\in\mathscr{E}_\lambda$, $(\Box_b^{(q)}\phi)^{\wedge}_{LL}(\lambda,\eta;0,0) = |\eta|^2\hat{\phi}_{LL}(\lambda,\eta;0,0)$.

Then, for $\phi\in\mathscr{S}(G_\Phi)\otimes\mathrm{End}(\Lambda_q)$ we have

$$\begin{aligned}\langle \Box_b^{(q)}K_q,\phi\rangle &= \langle K_q, \Box_b^{(q)}\phi\rangle \\ &= \sum_{|L|=q}\ \sum_{m\in\mathbb{N}^v}\ \int_{\Omega'}\int_{\mathbb{C}^{n-v}} \overline{\hat{\phi}_{LL}(\lambda,\eta;m,\,m)}\,D(\lambda)\,d\lambda \\ &= \overline{\mathrm{tr}\,\phi(0)},\end{aligned}$$

which is what we wished to prove. □

7. Hypoellipticity of $\Box_b^{(q)}$

We now turn to Theorem 2. We begin by noticing that if the operator $\mathscr{L}$ is hypoelliptic then $\mathrm{span}_{\mathbb{R}}\{\Phi(z,z)\} = W$. Indeed, if $\mathrm{span}_{\mathbb{R}}\{\Phi(z,z)\}$ is a proper subspace of W, $\mathscr{L}$ cannot be hypoelliptic since it is an operator on a proper subgroup of G_Φ.

The fact that $\mathrm{span}_{\mathbb{R}}\{\Phi(z,z)\} = W$ is equivalent to saying that the group G_Φ is stratified, and also to saying that there is no $\lambda\neq 0$ such that $\Phi^\lambda = 0$. If this is the case and if $\{V_1,\ldots,V_{2n}\}$ is an enumeration of the vector fields $Z_1,\ldots,Z_n,\bar{Z}_1,\ldots,\bar{Z}_n$, we have

$$||V_jV_kf||_{L^2}\leqslant c||\mathscr{L}f||_{L^2}, \tag{29}$$

for each $f\in\mathscr{S}(G_\Phi)$ and $j,k=1,\ldots,2n$.

We introduce non-isotropic Sobolev norms as follows. Let $k\in\mathbb{N}$ and let $\mathscr{B}_k$ be the set of all products of the form $V_{i_1},\ldots,V_{i_j}$, where $1\leqslant i_j\leqslant 2n$ and $j\leqslant k$. For $f\in\mathscr{S}(G_\phi)$ we set

$$||f||_{(k)} = \sum_{P\in\mathscr{B}_k}||Pf||_{L^2}.$$

It is well known that, for functions with a fixed compact support, any ordinary Sobolev norm is controlled by a non-isotropic norm, see [FS].

Because of (29), for k even

$$||f||_{(k)} \approx \sum_{j=0}^{k/2} ||\mathscr{L}^j f||_{L^2}. \tag{30}$$

If we extend Sobolev norms to forms in $\mathscr{S}(G_\phi) \otimes \Lambda_q$ in the obvious way, (30) remains valid replacing $\mathscr{L}$ by $\mathscr{L} \otimes I$, where I is the identity on Λ_q.

Theorem 7.1. *The following conditions are equivalent*:

(i) $\mathrm{span}_{\mathbb{R}}\{\Phi(z,z)\} = W$ *and there exists* $C>0$ *such that for each* $\phi \in \mathscr{S}(G_\phi) \otimes \Lambda_q$

$$||(\mathscr{L} \otimes I)\phi||_{L^2} \leqslant C||\Box_b^{(q)}\phi||_{L^2};$$

(ii) $\Box_b^{(q)}$ *is hypoelliptic*;

(iii) *there exists no non-zero* $\lambda \in W^*$ *such that* $n^+(\lambda) \leqslant n - q$ *and* $n^-(\lambda) \leqslant q$;

(iv) *there exists* $\delta \in (0,1)$ *such that*, *for every* $\lambda \in \Omega$ *and every multi-index* L *with* $|L| = q$, $-\alpha_L^\lambda < (1-\delta)\sum_{j=1}^{\nu(\lambda)} |\mu_j(\lambda)|$, *where* α_L^λ *is defined in* (18);

(v) *for every* $\lambda \neq 0$, Φ^λ *has at least* $\max(q+1, n-q+1)$ *eigenvalues with the same sign*, *or at least* $\min(q+1, n-q+1)$ *pairs of eigenvalues with opposite signs.*

Remark. Condition (v) above is the natural generalization of the $Y(q)$ condition to quadratic manifolds of higher codimension mentioned in the Introduction.

Proof. We preliminary show that conditions (iii) and (v) are equivalent. The rest of the proof gives the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

It is easy to see that conditions (iii) and (v) are both equivalent to the following condition: There exists no non-zero $\lambda \in W^*$ such that

$$\begin{cases} \min(n^+(\lambda), n^-(\lambda)) \leqslant \min(q, n-q), \\ \max(n^+(\lambda), n^-(\lambda)) \leqslant \max(q, n-q). \end{cases}$$

Next, if (i) holds, formula (29) implies that

$$||\phi||_{(2)} \leqslant c(||\Box_b^{(q)}\phi||_{L^2} + ||\phi||_{L^2}),$$

for every ϕ smooth, with support in a fixed compact set. Using the fact that $\mathscr{L} \otimes I$ and $\Box_b^{(q)}$ commute, it follows by induction that

$$||\phi||_{(k+2)} \leqslant c(||\Box_b^{(q)}\phi||_{(k)} + ||\phi||_{(k)}),$$

for every k even. This implies that $\Box_b^{(q)}$ is hypoelliptic by standard arguments (see [T1, Chapter 2, Section 5]). Thus (i) implies (ii).

In order to prove that (ii) implies (iii), we show that if (iii) does not hold, we can construct a non-smooth solution of the homogeneous equation $\Box_b^{(q)} u = 0$.

Suppose then that there exists a $\lambda_0 \neq 0$ such that $n^+(\lambda_0) \leqslant n - q$ and $n^-(\lambda_0) \leqslant q$. Then there exists a multi-index L with $|L| = \nu(\lambda_0)$ such that

$$\alpha_L^{\lambda_0} + \sum_{j=1}^{\nu(\lambda_0)} |\mu_j(\lambda_0)| = 0.$$

Because the eigenvalues are homogeneous functions of λ, the same equality holds for all $\lambda = s\lambda_0$, with $s > 0$.

By Proposition 3.5,

$$d\pi_{\lambda,0}(\Box_b^{(q)}) u_\lambda = 0, \tag{31}$$

whenever $\lambda = s\lambda_0$, $s > 0$ and $u_\lambda = h_0^\lambda \otimes \omega_L^\lambda$ (see Proposition 3.5 for notation). Notice that we can take the basis $\{v_1^\lambda, \dots, v_n^\lambda\}$ that diagonalizes Φ^λ to be the same for all $\lambda = s\lambda_0$, with $s > 0$. Also, notice that $\nu(\lambda) = \nu(\lambda_0)$, and we denote this value by ν_0.

We define $u \in \mathscr{S}'(G_\Phi) \otimes \Lambda_q$ as follows. For $\phi \in \mathscr{S}(G_\Phi) \otimes \Lambda_q$ we set

$$\begin{aligned} \langle u, \phi \rangle &= \int_0^{+\infty} \langle u_{s\lambda_0}, \pi_{s\lambda_0,0}(\phi) \rangle \, ds \\ &= \int_0^{+\infty} \langle h_0^{s\lambda_0}, \pi_{s\lambda_0,0}(\phi) h_0^{s\lambda_0} \rangle \, ds \\ &= \int_0^{+\infty} \int_{\mathbb{C}^{\nu_0}} \overline{\mathscr{F}_{t,z''}\phi(z', 0, s\lambda_0)} e^{-s\sum_1^{\nu_0} |\mu_j(\lambda_0)||z_j|^2} \, dz' \, ds. \end{aligned} \tag{32}$$

We show that u is homogeneous of degree -2 with respect to the dilations $r \cdot (z,t) = (rz, r^2 t)$ on G_Φ. Making the change of variables $z' \mapsto r^{-1} z'$ and $s \mapsto r^2 s$, we have

$$\begin{aligned} \langle u, \phi(r\cdot) \rangle &= r^{-2(n+m)} \int_0^{+\infty} \int_{\mathbb{C}^{\nu_0}} \mathscr{F}_{t,z''}\phi(rz', 0, s\lambda_0/r^2) e^{-s\sum_1^{\nu_0} |\mu_j(\lambda_0)||z_j|^2} \, dz' \, ds \\ &= r^{-2(n+m)+2} \int_0^{+\infty} \int_{\mathbb{C}^{\nu_0}} \mathscr{F}_{t,z''}\phi(z', 0, s\lambda_0) e^{-s\sum_1^{\nu_0} |\mu_j(\lambda_0)|\,|z_j|^2} \, dz' \, ds \\ &= r^{-Q+2} \langle u, \phi(r\cdot) \rangle, \end{aligned}$$

where $Q = 2(n+m)$ denotes the homogeneous dimension of G_Φ.

As a distribution, u is homogeneous of degree σ if $\langle u, \phi(r\cdot) \rangle = r^{-Q-\sigma} \langle u, \phi \rangle$, for all $r > 0$ and $\phi \in \mathscr{S}(G_\Phi) \otimes \Lambda_q$. Thus, u is homogeneous of degree -2 and non-trivial, hence it cannot coincide with a smooth function.

For $\phi \in \mathscr{S}(G_\Phi) \otimes \Lambda_q$ we have

$$\begin{aligned}\langle \Box_b^{(q)} u, \phi \rangle &= \langle u, \Box_b^{(q)} \phi \rangle \\ &= \int_0^{+\infty} \langle u_{s\lambda_0}, d\pi_{s\lambda_0,0}(\Box_b^{(q)}) \pi_{s\lambda_0,0}(\phi) \rangle \, ds \\ &= 0,\end{aligned}$$

because of (31). Then u is a solution of the equation $\Box_b^{(q)} u = 0$, henceforth implying that $\Box_b^{(q)}$ is not hypoelliptic.

We prove that (iii) implies (iv). If (iii) holds, the quantity

$$A(\lambda, L) = \frac{\alpha_L^\lambda + \sum_1^\nu |\mu_j(\lambda)|}{\sum_1^\nu |\mu_j(\lambda)|}$$

is well defined for $\lambda \in \Omega$ and $|L| = q$, because the denominator does not vanish.

Proving condition (iv) is equivalent to proving that

$$\inf\{A(\lambda, L) : \lambda \in \Omega,\ |L| = q\} > 0. \tag{33}$$

Suppose then that (33) does not hold. Let $\{\lambda_k\} \subseteq \Omega$ and $\{L_k\}$ be multi-indices such that $A(\lambda_k, L_k) \to 0$ as $k \to +\infty$. Since A is homogeneous of degree 0 in λ, we may assume that $|\lambda_k| = 1$ for all k. By passing to a subsequence we may also assume that $\lambda_k \to \lambda_0$, with $|\lambda_0| = 1$.

By condition (iii), either $n^+(\lambda_0) > n - q$ or $n^-(\lambda_0) > q$. Assume for instance that $n^+(\lambda_0) > n - q$, and let $\delta > 0$ be a strict lower bound for the positive eigenvalues of Φ^{λ_0}. By Rouché's theorem, Φ^{λ_k} has at least $n - q + 1$ eigenvalues larger than δ for k large enough. Then for every L with $|L| = q$,

$$\alpha_L^{\lambda_j} + \sum_1^\nu |\mu_j(\lambda_k)| > 2\delta,$$

for k large. Since $\sum_1^\nu |\mu_j(\lambda_k)|$ remains bounded, we have a contradiction.

Finally, we show that (iv) implies (i). Condition (iv) implies that

$$\sup_{\lambda \in \Omega,\ |L|=q} \frac{\sum_1^\nu |\mu_j(\lambda)|}{\alpha_L^\lambda + \sum_1^\nu |\mu_j(\lambda)|} \leqslant \frac{1}{\delta},$$

which in turn is equivalent to

$$\sup_{\lambda \in \Omega,\ |L|=q,\ m \in \mathbb{N}^\nu,\ \eta \in \mathbb{C}^{n-\nu}} \frac{|\eta|^2 + \sum_1^\nu |\mu_j(\lambda)|(1 + 2m_j)}{\alpha_L^\lambda + |\eta|^2 + \sum_1^\nu |\mu_j(\lambda)|(1 + 2m_j)} \leqslant \frac{1}{\delta}.$$

Observe that, by Proposition 3.5, for $\lambda \in \Omega$, these quantities are precisely the eigenvalues of $d\pi_{\lambda,\eta}(\mathscr{L} \otimes I)(d\pi_{\lambda,\eta}(\Box_b^{(q)}))^{-1}$. Therefore this is a bounded operator and

$$||d\pi_{\lambda,\eta}(\mathscr{L}\otimes I)(d\pi_{\lambda,\eta}(\Box_b^{(q)}))^{-1}||\leqslant\frac{1}{\delta}$$

for every $\lambda\in\Omega$, where $||\cdot||$ denotes the operator norm.

Therefore, for $\phi\in\mathscr{S}(G_\Phi)\otimes\Lambda_q$, by (11), we have

$$\begin{aligned}||(\mathscr{L}\otimes I)\phi||_{L^2}^2 &= \int_\Omega\int_{\mathbb{C}^{n-v}}||\pi_{\lambda,\eta}((\mathscr{L}\otimes I)(\phi))||_{\text{HS}}^2 D(\lambda)\,d\eta\,d\lambda\\ &\leqslant \int_\Omega\int_{\mathbb{C}^{n-v}}||d\pi_{\lambda,\eta}(\mathscr{L}\otimes I)(d\pi_{\lambda,\eta}(\Box_b^{(q)}))^{-1}||\,||\pi_{\lambda,\eta}(\Box_b^{(q)}\phi)||_{\text{HS}}^2 D(\lambda)\,d\eta\,d\lambda\\ &\leqslant \delta||\Box_b^{(q)}\phi||_{L^2}^2.\end{aligned}$$

This proves that (iv) implies (i) and finishes the proof. $\Box$

This proves the theorem.

Remark. We have in fact proved that $\Box_b^{(q)}$ is hypoelliptic if and only if the following Rockland condition is satisfied: For every $(\lambda,\eta)\neq(0,0)$ $d\pi_{\lambda,\eta}(\Box_b^{(q)})$ is injective on $\mathscr{S}(G_\Phi)\otimes\Lambda_q$. An extension of Helffer–Nourrigat theorem [HeN] to systems of differential operators does not seem to appear in the literature.

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Further reading

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