$$J(u+h) - J(u) = \int_{\Omega} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} h \rangle \,\mathrm{d}x + J(h),$$

for every $h \in C_0^\infty(\Omega, \mathbb{R})$. We call *critical point* of J any function $u \in C^\infty(\Omega, \mathbb{R})$ such that

$$\textstyle\int_{\Omega} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} h \rangle \,\mathrm{d}x, \quad \forall\, h \in C_0^\infty(\Omega, \mathbb{R}).$$

Then, given $u \in C^\infty(\Omega, \mathbb{R})$, we have: *$u$ is a critical point of J if and only if $\Delta_{\mathbb{G}} u = 0$ in Ω.* Indeed, since $Z_j^* = -Z_j$, an integration by parts gives

$$\begin{aligned}\textstyle\int_{\Omega} \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} h \rangle \,\mathrm{d}x &= \textstyle\sum_{j=1}^{N_1} \int_{\Omega} Z_j u \, Z_j h \,\mathrm{d}x = - \sum_{j=1}^{N_1} \int_{\Omega} (Z_j^2 u)\, h \,\mathrm{d}x \\ &= - \textstyle\int_{\Omega} (\Delta_{\mathbb{G}} u)\, h \,\mathrm{d}x,\end{aligned}$$

for every $u \in C^\infty(\Omega, \mathbb{R})$ and $h \in C_0^\infty(\Omega, \mathbb{R})$.

We end this section with the following remarkable result.

Proposition 1.4.6. *Let Ω be an open connected subset of the Carnot group $\mathbb{G}$. Then a function $u \in C^1(\Omega, \mathbb{R})$ is constant in Ω if and only if its canonical intrinsic gradient $\nabla_{\mathbb{G}} u$ vanishes identically on Ω.*

Proof. Suppose $Z_1 u, \ldots, Z_{N_1} u$ vanish identically on Ω. Since the Lie algebra of $\mathbb{G}$ is given by $\mathrm{Lie}\{Z_1, \ldots Z_{N_1}\}$ (see (1.46)), then for every vector field Z_j of the Jacobian basis, we have $Z_j u \equiv 0$. We end by applying Proposition 1.2.13. □

1.5 Examples

1.5.1 Euclidean group

The additive group $(\mathbb{R}^N, +)$ is a homogeneous group with respect to the dilations

$$\delta_\lambda(x) = \lambda\, x, \quad \lambda > 0.$$

We call $\mathbb{E} = (\mathbb{R}^N, +, \delta_\lambda)$ the *Euclidean group*. $\mathbb{E}$ is a Carnot group of step 1. Its generators are $\partial_{x_1}, \ldots, \partial_{x_N}$. Thus, the canonical sub-Laplacian on $\mathbb{E}$ is the classical Laplace operator

$$\Delta = \textstyle\sum_{j=1}^N \partial_{x_j}^2 .$$

We want to stress that $\mathbb{E}$ is the only Carnot group of step 1 and N generators.

1.5.2 Heisenberg-Weyl group

Let us consider in $\mathbb{R}^{2n+1}$ the following composition law

$$(z,t) \circ (z',t') = (z + z', t + t' + 2\,\mathrm{Im}(z \cdot \overline{z'})). \tag{1.59}$$

Hereafter we agree to identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and to use the following notation to denote the point of $\mathbb{R}^{2n+1} \equiv \mathbb{C}^n \times \mathbb{R}$:

$$(z,t) = (x,y,t) = (x_1,\dots,x_n,y_1,\dots,y_n,t)$$

with $x_j, y_j, t \in \mathbb{R}$, $z_j = x_j + iy_j$, $z = (z_1,\dots,z_n)$. In (1.59), $z \cdot \overline{z'}$ denotes the usual hermitian inner product in $\mathbb{C}^n$, i.e., $z \cdot \overline{z'} = \sum_{j=1}^n (x_j + iy_j)(x_j' - iy_j')$. Then, the composition law $\circ$ can be written as

$$(z,t) \circ (z',t') = (x + x', y + y', t + t' + 2\,(yx' - xy')).$$

It is quite easy to verify that $(\mathbb{R}^{2n+1}, \circ)$ is a Lie group whose zero is the origin and where the inverse is given by $(z,t)^{-1} = (-z,-t)$. Let us now consider the dilations

$$\delta_\lambda : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}, \quad \delta_\lambda(z,t) = (\lambda z, \lambda^2 t).$$

A trivial computation shows that δ_λ is an automorphism of $(\mathbb{R}^{2n+1}, \circ)$ for every $\lambda > 0$. Then $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ, \delta_\lambda)$ is a homogeneous group. It is called the *Heisenberg-Weyl group* in $\mathbb{R}^{2n+1}$.

The Jacobian matrix at the origin of the left translation $\tau_{(z,t)}$ is the following block matrix

$$\mathcal{J}_{\tau_{(z,t)}}(0,0) = \begin{pmatrix} \mathbb{I}_n & 0 & 0 \\ 0 & \mathbb{I}_n & 0 \\ 2y & -2x & 1 \end{pmatrix},$$

where $\mathbb{I}_n$ denote the $n \times n$ identity matrix, while $2y$ and $-2x$ stand for the $1 \times n$ matrices $(2y_1,\dots,2y_n)$ and $(-2x_1,\dots,-2x_n)$, respectively. Then, the Jacobian basis of $\mathfrak{h}_n$, the Lie algebra of $\mathbb{H}^n$, is given by

$$X_j = \partial_{x_j} + 2y_j\,\partial_t, \quad Y_j = \partial_{y_j} - 2x_j\,\partial_t, \;\; j = 1,\dots,n, \quad T = \partial_t.$$

Then, since $[X_j, Y_j] = -\frac{1}{4}\,\partial_t$, we have

$$\begin{aligned} &\mathrm{rank}\big(\mathrm{Lie}\{X_1,\dots,X_n,Y_1,\dots,Y_n\}(0,0)\big) \\ &\quad = \dim\big(\mathrm{span}\{\partial_{x_1},\dots,\partial_{x_n},\partial_{y_1},\dots,\partial_{y_n},\partial_t\}\big) = 2n+1. \end{aligned}$$

This shows that $\mathbb{H}^n$ is a Carnot group. Its step is $r = 2$ and its generators are the vector fields X_j, Y_j, $j = 1,\dots,n$. As a consequence, the canonical sub-Laplacian on $\mathbb{H}^n$ is given

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^n \left(X_j^2 + Y_j^2\right).$$

1.5.3 B-groups

Let us consider a $N \times N$ matrix B with real entries $b_{i,j}$, $i, j = 1, \dots, N$. Let us put

$$E(t) := \exp(tB), \quad t \in \mathbb{R}.$$

In $\mathbb{R}^{1+N}$, whose point will be denoted by $z = (t, x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, let us introduce the following composition law

$$(t, x) \circ (t', x') = (t + t', x' + E(t')x).$$

One easily verifies that $\mathbb{B} = (\mathbb{R}^{1+N}, \circ)$ is a Lie group whose zero is the origin $(0, 0)$ and where the inverse is given by

$$(t, x)^{-1} = (-t, -E(-t)x).$$

The Jacobian matrix at the origin of the left translation $\tau_{(t,x)}$ is the following block matrix

$$\mathcal{J}_{\tau_{(t,x)}}(0,0) = \begin{pmatrix} 1 & 0 \\ b & \mathbb{I}_N \end{pmatrix},$$

where b stands for the $N \times 1$ matrix

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(s)x = BE(s)x\Big|_{s=0} = Bx.$$

Then, the Jacobian basis of $\mathfrak{b}$, the Lie algebra of $\mathbb{B}$, is given by

$$Y = \partial_t + \nabla_x \cdot Bx, \quad \partial_{x_1}, \dots, \partial_{x_N}. \tag{1.60}$$

We explicitly remark that, for a general matrix B, the group $(\mathbb{B}, \circ)$ may not be nilpotent. Indeed, an easy computation shows that, for any $j \in \{1, \dots, N\}$

$$\underbrace{[\partial_{x_j}, [\partial_{x_j}, \cdots [\partial_{x_j}}_{k \text{ times}}, Y] \cdots]] = \sum_{i=1}^{N} (B^k)_{i,j}\, \partial_{x_i}.$$

Hence, $(\mathbb{B}, \circ)$ is a nilpotent group iff B is a nilpotent matrix. For example, if $N = 1$ and $B = (1)$, the composition law is

$$(t, x) \circ (t', x') = (t + t', x' + e^{t'} x), \qquad (t, x), (t', x') \in \mathbb{R}^2,$$

and the Jacobian basis is $\partial_t + x\partial_x, \partial_x$. $\mathbb{R}^2$ equipped with the above composition law is *not* a Carnot group (and it is not even diffeomorphic to *any* Carnot group) since it is not nilpotent. In particular, the second order differential operator on $\mathbb{R}^2$ defined by

$$L = \partial_x^2 + (\partial_t + x\partial_x)^2 = (1 + x^2)\, \partial_x^2 + \partial_t^2 + 2x\, \partial_x \partial_t + x\, \partial_x$$

is a sum of squares of left-invariant vector fields on $(\mathbb{R}^2, \circ)$, it satisfies Hörmander's ipoellipticity condition

$$\operatorname{rank}\big(\operatorname{Lie}\{\partial_x, \partial_t + x\partial_x\}(t, x)\big) = 2, \qquad \forall\ (t, x) \in \mathbb{R}^2,$$

but L *is not* a sub-Laplacian on any Carnot group. Indeed, L is elliptic at any point.

1.5.4 K-type groups

Let us now suppose that the matrix B in the previous example takes the following special form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ B_1 & 0 & \cdots & 0 & 0 \\ 0 & B_2 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & B_r & 0 \end{pmatrix} \tag{1.61}$$

where B_j is a $p_j \times p_{j-1}$ block with rank p_j, $j = 1, 2, \ldots, r$. Moreover $p_0 \geq p_1 \geq \cdots \geq p_r$ and $p_0 + p_1 + \cdots + p_r = N$. We want to show that the group $\mathbb{B}$ related to this matrix is a Carnot group. It will be called a group of *Kolmogorov type* or, in short, a K-type group.

Let us split $\mathbb{R}^N$ as follows

$$\mathbb{R}^N = \mathbb{R}^{p_0} \times \cdots \times \mathbb{R}^{p_r}$$

and define, for every $\lambda > 0$,

$$D_\lambda x = D_\lambda(x^{(0)}, \ldots, x^{(r)}) = (\lambda x^{(0)}, \ldots, \lambda^{r+1} x^{(r)}), \tag{1.62}$$

where $x^{(i)} \in \mathbb{R}^{p_i}$ for $0 \leq i \leq r$. We also put $\delta_\lambda(t, x) = (\lambda t, D_\lambda x)$.

Claim 1 *For every $\lambda > 0$, the dilation δ_λ is an automorphism of $\mathbb{B}$.*

To prove this claim we need the following lemma.

Lemma 1.5.1. *For every $t \in \mathbb{R}$ and $\lambda > 0$, we have*

$$E(\lambda t)\, D_\lambda = D_\lambda\, E(t) \tag{1.63}$$

where $E(t) = \exp(tB)$, B is as in (1.61) *and D_λ is the dilation in* (1.62).

Proof. Since $B^k = 0$ for every $k \geq r + 1$, one has $E(t) = \sum_{k=0}^{r} t^k\, B^k / k!$ and (1.63) holds for every $t \in \mathbb{R}$ and $\lambda > 0$ iff

$$\lambda^k\, B^k\, D_\lambda = D_\lambda\, B^k, \qquad \forall\, k \geq 0,\ \forall\, \lambda > 0. \tag{1.64}$$

This identity holds true when $k = 0$. An easy direct computation shows that it also holds true for $k = 1$. As a consequence

$$\lambda^2 B^2 D_\lambda = \lambda B(\lambda B D_\lambda) = \lambda B(D_\lambda B) = (\lambda B D_\lambda) B = (D_\lambda B) B = D_\lambda B^2.$$

Then (1.64) holds true for $k = 2$. An iteration of this argument shows (1.64) for $k \geq 2$. □

From this lemma, the Claim 1 easily follows. Indeed, for every $z = (t, x)$, $z' = (t', x') \in \mathbb{R}^{1+N}$ we have

$$\begin{aligned}(\delta_\lambda z) \circ (\delta_\lambda z') &= (\lambda t, D_\lambda x) \circ (\lambda t', D_\lambda x') = (\lambda t + \lambda t', D_\lambda x' + E(\lambda t') D_\lambda x) \\ \text{(by Lemma 1.5.1)} &= (\lambda(t+t'), D_\lambda x' + D_\lambda E(t')x) = \delta_\lambda(t+t', x' + E(t')x) \\ &= \delta_\lambda(z \circ z').\end{aligned}$$

Thus $\mathbb{B} = (\mathbb{R}^{1+N}, \circ, \delta_\lambda)$ is a homogeneous group whose first layer is $\mathbb{R} \times \mathbb{R}^{p_0} = \{(t, x^{(0)}) \,|\, t \in \mathbb{R}, x^{(0)} \in \mathbb{R}^{p_0}\}$. Moreover, the vector fields in the Jacobian basis related to this first layer are given by

$$Y = \partial_t + \langle Bx, \nabla_x \rangle, \quad \partial_{x_1}, \dots, \partial_{x_{p_0}}. \tag{1.65}$$

Claim 2 *We have* $\operatorname{rank}\big(\operatorname{Lie}\{Y, \partial_{x_1}, \dots, \partial_{x_{p_0}}\}(0,0)\big) = 1 + N$.

Once this claim is proved, it will follow that $(\mathbb{R}^{1+N}, \circ, \delta_\lambda)$ is a Carnot group of step $r+1$ and generators the $1 + p_0$ vector fields in (1.65). Thus the related canonical sub-Laplacian is given by

$$\Delta_{\mathbb{B}} = Y^2 + \Delta_{\mathbb{R}^{p_0}}, \qquad \Delta_{\mathbb{R}^{p_0}} = \textstyle\sum_{j=1}^{p_0} \partial_{x_j}^2. \tag{1.66}$$

This sub-Laplacian will be said of *Kolmogorov type*. To prove Claim 2, the following lemma will be useful.

Lemma 1.5.2. *In $\mathbb{R}^p \times \mathbb{R}^q$ let us consider the vector field*

$$Z = Ay \cdot (\nabla_z)^T, \text{ where } A \text{ is a } q \times p \text{ matrix, } y \in \mathbb{R}^p \text{ and } z \in \mathbb{R}^q.$$

Suppose $\operatorname{rank}(A) = q \le p$. *Then*

$$\operatorname{span}\big\{[\partial_{y_i}, Z] \,\big|\, i = 1, \dots, p\big\} = \operatorname{span}\{\partial_{z_1}, \dots, \partial_{z_q}\}. \tag{1.67}$$

Proof. Let $A = (a_{i,j})_{i \le q,\, j \le p}$. Then

$$[\partial_{y_i}, Z] = \textstyle\sum_{j=1}^{q} a_{j,i}\, \partial_{z_j}, \quad i = 1, \dots, p$$

so that, since $\operatorname{rank}(A) = q$,

$$\dim\big(\operatorname{span}\big\{[\partial_{y_i}, Z] \,\big|\, i = 1, \dots, p\big\}\big) = q.$$

This implies (1.67). □

We now prove Claim 2. Since B has the form (1.61), we can write

$$Y = \partial_t + \textstyle\sum_{i=1}^{r} B_i x^{(i-1)} \cdot (\nabla_{x^{(i)}})^T.$$

Then, by applying Lemma 1.5.2, we get

$$\begin{aligned}\operatorname{span}\big\{[\partial_{x_i}, Y] \,\big|\, i = 1, \dots, p_0\big\} &= \operatorname{span}\big\{[\partial_{x_i}, B_1 x^{(0)} \cdot (\nabla_{x^{(1)}})^T] \,\big|\, i = 1, \dots, p_0\big\} \\ &= \operatorname{span}\big\{\partial_{x_i^{(1)}} \,\big|\, i = 1, \dots, p_1\big\}.\end{aligned}$$

Another application of Lemma 1.5.2 gives

$$\mathrm{span}\big\{[\partial_{x_i^{(1)}}, Y] \,|\, i = 1, \dots, p_1\big\} = \mathrm{span}\big\{\partial_{x_i^{(2)}} \,|\, i = 1, \dots, p_2\big\}.$$

Iterating this argument, we get

$$\mathrm{Lie}\{Y, \partial_{x_1}, \dots, \partial_{x_{p_0}}\} = \mathrm{Lie}\{Y, \partial_{x_1}, \dots, \partial_{x_N}\}.$$

This obviously shows the claim.

The groups of Kolmogorov type were introduced in [32] in studying a class of hypoelliptic ultraparabolic operators including the classical prototype operators of Kolmogorov-Fokker-Planck. The composition law in [32] was suggested by the structure of the fundamental solution of the operator in $\mathbb{R}^3$ $\partial_{x_1}^2 + x_1\,\partial_{x_2} - \partial_{x_3}$ given by Kolmogorov in [31].

Remark 1.5.3. Assume that the matrix B is as in (1.61). If we define

$$d_\lambda : \mathbb{R}^{1+N} \to \mathbb{R}^{1+N},$$
$$d_\lambda(t, x^{(0)}, \dots, x^{(r)}) = (\lambda^2 t, \lambda x^{(0)}, \lambda^3 x^{(1)}, \dots, \lambda^{2r+1} x^{(r)}),$$

then $\{d_\lambda\}_{\lambda>0}$ is a group of automorphisms of $\mathbb{B}$. For a proof of this statement, we directly refer to [32]. This remark shows that $(\mathbb{R}^{1+N}, \circ, d_\lambda)$ is a homogeneous Lie group. It can be also easily be proved that the *ultraparabolic operator*

$$L = \Delta_{p_0} + Y \tag{1.68}$$

is left-translation invariant and homogeneous of degree two with respect to $\{d_\lambda\}_{\lambda>0}$. Operator (1.68) generalizes the prototypes of the ones introduced by Kolmogorov in [31].

1.5.5 Homogeneous groups of step two

Let $\mathbb{R}^N$ be split as $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ and denote its point by $z = (x,t)$ with $x \in \mathbb{R}^m$ and $t \in \mathbb{R}^n$. Given an n-tuple $B^{(1)}, \dots, B^{(n)}$ of $m \times m$ matrices with real entries, let

$$(x,t) \circ (\xi, \tau) = (x + \xi, t + \tau + \tfrac{1}{2}\langle Bx, \xi\rangle). \tag{1.69}$$

Here $\langle Bx, \xi\rangle$ denotes the n-tuple $(\langle B^{(1)}x, \xi\rangle, \dots \langle B^{(n)}x, \xi\rangle)$ and $\langle , \rangle$ stands for the inner product in $\mathbb{R}^m$. One can easily verify that $(\mathbb{R}^N, \circ)$ is a Lie group whose identity is the origin and where the inverse is given by $(x,t)^{-1} = (-x, -t + \langle Bx, x\rangle)$. It is also quite easy to recognize that the dilation

$$\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_\lambda(x,t) = (\lambda x, \lambda^2 t) \tag{1.70}$$

is an automorphism of $(\mathbb{R}^N, \circ)$, for any $\lambda > 0$. Then $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ is a homogeneous group.

We explicitly remark that the composition law of any Lie group in $\mathbb{R}^m \times \mathbb{R}^n$, homogeneous w.r.t. the dilations $\{\delta_\lambda\}_\lambda$ as in (1.70), takes the form (1.69), see Theorem 1.3.11.

The Jacobian matrix at $(0,0)$ of the left translation $\tau_{(x,t)}$ takes the following block form

$$\mathcal{J}_{\tau_{(x,t)}}(0,0) = \begin{pmatrix} \mathbb{I}_m & 0 \\ \frac{1}{2}\,Bx & \mathbb{I}_n \end{pmatrix},$$

where if $B^{(k)} = (b_{i,j}^{(k)})_{i,j\le m}$, for $k = 1,\dots,n$, Bx denotes the matrix

$$\big(\textstyle\sum_{j=1}^m b_{i,j}^{(k)}\,x_j\big)_{k\le m,\, j\le n}.$$

Then, the Jacobian basis of $\mathfrak{g}$, the Lie algebra of $\mathbb{G}$, is given by

$$\begin{aligned} &X_i = (\partial/\partial x_i) + \tfrac{1}{2}\textstyle\sum_{k=1}^n \big(\sum_{l=1}^m b_{i,l}^{(k)} x_l\big)\,(\partial/\partial t_k), \qquad (i = 1,\dots,m),\\ &T_k = \partial/\partial t_k, \qquad (k = 1,\dots,n). \end{aligned} \tag{1.71}$$

An easy computation shows that

$$[X_j, X_i] = \textstyle\sum_{k=1}^n \frac{1}{2}\big(b_{i,j}^{(k)} - b_{j,i}^{(k)}\big)\,\partial_{t_k} =: \sum_{k=1}^n c_{i,j}^{(k)}\,\partial_{t_k}.$$

We have denoted by $C^{(k)} = (c_{i,j}^{(k)})_{i,j\le m}$ the skew-symmetric part of $B^{(k)}$. Let us now assume that $C^{(1)},\dots,C^{(n)}$ are linearly independent. This implies that the $m^2 \times n$ matrix

$$\begin{pmatrix} C_{1,1}^{(1)} & \cdots & C_{1,1}^{(n)} \\ \vdots & \cdots & \vdots \\ C_{m,m}^{(1)} & \cdots & C_{m,m}^{(n)} \end{pmatrix}$$

has rank equal to n. As a consequence

$$\operatorname{span}\{[X_j, X_i]\,|\,i,j = 1,\dots,m\} = \operatorname{span}\{\partial_{t_1},\dots,\partial_{t_n}\}.$$

Therefore,

$$\begin{aligned} &\operatorname{rank}\big(\operatorname{Lie}\{X_1,\dots,X_m\}(0,0)\big) \\ &= \dim\big(\operatorname{span}\{\partial_{x_1},\dots,\partial_{x_m},\partial_{t_1},\dots,\partial_{t_n}\}\big) = m+n. \end{aligned}$$

This shows that $\mathbb{G}$ is a Carnot group of step two and generators $X_1,\dots,X_m$.

We explicitly remark that the linear independence of the matrices

$$C^{(1)},\dots,C^{(n)}$$

is also *necessary* for $\mathbb{G}$ to be a Carnot group. Then, *every Carnot group of step two and m generators is of the type described here.* Moreover, the above arguments show that there exist Carnot groups of any dimension $m \in \mathbb{N}$ of the first layer and any dimension $n \le m(m-1)/2$ of the second layer: it suffices to choose n linear independent matrices $B^{(1)},\dots,B^{(n)}$ in the vector space of the skew-symmetric $m\times m$ matrices (which has dimension $m(m-1)/2$) and define the composition law as in (1.69). Finally, by means of the general results on stratified groups in Section 1.8, we have the following remarkable result.

Theorem 1.5.4. *The N-dimensional stratified groups of step two and m generators are characterized by being (canonically isomorphic to) $(\mathbb{R}^N,\circ)$ with the Lie group law as in* (1.69) *where the $B^{(k)}$'s are $m\times m$ linearly independent skew-symmetric matrices. The group of dilations is given by* (1.70).

1.5.6 Groups of Heisenberg type

Let $\mathbb{H}=(\mathbb{R}^{m+n},\circ,\delta_\lambda)$ be the homogeneous Lie group with composition law (1.69) and the group of dilations as in (1.70). Let us also assume that the matrices $B^{(1)},\dots,B^{(n)}$ have the following properties:

1. $B^{(j)}$ is a $m\times m$ skew-symmetric and orthogonal matrix, for every $j\le n$;
2. $B^{(i)}\,B^{(j)}=-B^{(j)}\,B^{(i)}$, for every $i,\,j\in\{1,\dots,n\}$ with $i\ne j$.

If all these conditions are satisfied, $\mathbb{G}$ is called a group of Heisenberg-type, in short, a H-type group.

An H-type group is a Carnot group, since conditions (1) and (2) imply the linear independence of $B^{(1)},\dots,B^{(n)}$. Indeed, if $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^N\setminus\{0\}$ then $\frac{1}{|\alpha|}\sum_{s=1}^n\alpha_s\,B^{(s)}$ is orthogonal (hence non-vanishing) as the following computation shows:

$$\Big(\tfrac{1}{|\alpha|}\textstyle\sum_{s=1}^n\alpha_sB^{(s)}\Big)\cdot\Big(\tfrac{1}{|\alpha|}\sum_{s=1}^n\alpha_sB^{(s)}\Big)^T=-\tfrac{1}{|\alpha|^2}\sum_{r,s\le n}\alpha_r\alpha_s\,B^{(r)}B^{(s)}$$
$$=-\frac{1}{|\alpha|^2}\sum_{r\le n}\alpha_r^2\,(B^{(r)})^2-\frac{1}{|\alpha|^2}\sum_{r,s\le n,\,r\ne s}\alpha_r\alpha_s\,B^{(r)}B^{(s)}=\mathbb{I}_m.$$

Here we used the following facts: $(B^{(r)})^2=-\mathbb{I}_m$ since $B^{(r)}$ is skew-symmetric and orthogonal; $B^{(r)}B^{(s)}=-B^{(s)}B^{(r)}$ from condition (2) above.

The generators of $\mathbb{G}$ are the vector fields (see (1.71))

$$X_i=\partial x_i+\tfrac12\textstyle\sum_{k=1}^n\big(\sum_{l=1}^m b_{i,l}^{(k)}x_l\big)\,\partial t_k,\qquad i=1,\dots,m.$$

A direct computation shows that the canonical sub-Laplacian $\varDelta_{\mathbb{G}}=\sum_{i=1}^m X_i^2$ can be written as follows

$$\varDelta_{\mathbb{G}}=\varDelta_x+\tfrac14\textstyle\sum_{h,k=1}^n\langle B^{(h)}x,B^{(k)}x\rangle\,\partial_{t_ht_k}$$
$$+\textstyle\sum_{k=1}^m\langle B^{(k)}x,\nabla_x\rangle\,\partial_{t_k}+\sum_{k=1}^n\mathrm{trace}(B^{(k)})\,\partial_{t_k}.$$

On the other hand, by conditions (1) and (2), $\langle B^{(h)}x,B^{(h)}x\rangle=|x|^2$ while, for $h\ne k$, $\langle B^{(h)}x,B^{(k)}x\rangle=0$ since $\langle B^{(h)}x,B^{(k)}x\rangle=-\langle B^{(k)}\,B^{(h)}x,x\rangle=\langle B^{(h)}\,B^{(k)}x,x\rangle=-\langle B^{(k)}x,B^{(h)}x\rangle$. We also have $\mathrm{trace}(B^{(k)})=0$ since $B^{(k)}$ is skew-symmetric. Then $\varDelta_{\mathbb{G}}$ takes the form

$$\varDelta_{\mathbb{G}}=\varDelta_x+\tfrac14\,|x|^2\,\varDelta_t+\textstyle\sum_{k=1}^n\langle B^{(k)}x,\nabla_x\rangle\,\partial_{t_k}. \tag{1.72}$$

Remark 1.5.5. The first layer of a group of Heisenberg type has even dimension m. Indeed, if B is a $m \times m$ skew-symmetric orthogonal matrix, we have $\mathbb{I}_m = B\,B^T = -B^2$, whence $1 = (-1)^m (\det B)^2$.

Remark 1.5.6. With the previous notation, if $\mathbb{H} = (\mathbb{R}^{m+n}, \circ, \delta_\lambda)$ is a H-type group, then $\mathfrak{z} = \{(0,t) \,|\, t \in \mathbb{R}^n\}$ is the center of $\mathbb{H}$. Indeed, let $(y,t) \in \mathbb{H}$ be such that $(x,s) \circ (y,t) = (y,t) \circ (x,s)$ for every $(x,s) \in \mathbb{H}$. This holds iff $\langle B^{(k)} x, y \rangle = \langle B^{(k)} y, x \rangle$ for any $x \in \mathbb{R}^m$ and any $k \in \{1, \dots, n\}$. Then, since $(B^{(k)})^T = -B^{(k)}$,

$$\langle B^{(k)} y, x \rangle = 0, \qquad \forall\, x \in \mathbb{R}^m, \ \forall\, k \in \{1, \dots, n\},$$

so that $y = 0$ because $B^{(k)}$ is orthogonal, hence non-singular.

Remark 1.5.7. The classical Heisenberg group on $\mathbb{R}^{2k+1}$ is a H-type group. It corresponds to the case $m = 2k$, $n = 1$ and

$$B^{(1)} = \mathrm{diag}\Big\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Big\}, \quad \text{the block occurring } k \text{ times.}$$

The classical Heisenberg groups are the only (up to isomorphism) H-type groups with one-dimensional center.

Remark 1.5.8. Groups of Heisenberg type with center of dimension $n \geq 2$ do exist. For example, the following two matrices

$$B^{(1)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ B^{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

satisfy conditions (1)-(2) and hence they define in $\mathbb{R}^6 = \mathbb{R}^4 \times \mathbb{R}^2$ a H-type group whose center has dimension 2. The above matrices $B^{(1)}$ and $B^{(2)}$ together with

$$B^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

define in $\mathbb{R}^7 = \mathbb{R}^4 \times \mathbb{R}^3$ a H-type group whose center has dimension 3.

Remark 1.5.9. The following result holds (see [30, Corollary 1]). *Let m, n be two positive integers. Then there exists a H-type group of dimension $m + n$ whose center has dimension n if and only if it holds $n < \rho(m)$, where ρ is the so-called Hurwitz-Radon function, i.e.,*

$$\rho : \mathbb{N} \to \mathbb{N}, \qquad \rho(m) := 8p + q, \quad \textit{ove} \quad m = (\textit{odd}) \cdot 2^{4p+q}, \ 0 \leq q \leq 3.$$

We explicitly remark that if m is odd, then $\rho(m) = 0$, whence the first layer of any H-type group has even dimension (as we already proved in Remark 1.5.5).

Remark 1.5.10. The groups of Heisenberg-type were introduced by A. Kaplan in [30]. Kaplan's definition of H-type groups is more abstract than the one given here. We shall show that, up to an isomorphism, the two definitions are equivalent.

1.5.7 Sum of Carnot Groups

Suppose we are given two homogeneous stratified groups $\mathbb{G}^{(1)} = (\mathbb{R}^N, \circ^{(1)})$, $\mathbb{G}^{(2)} = (\mathbb{R}^M, \circ^{(2)})$ with dilations

$$\delta_\lambda^{(1)}(x) = (\lambda x^{(1)}, \dots, \lambda^r x^{(r)}), \quad x \in \mathbb{G}^{(1)};$$
$$\delta_\lambda^{(2)}(y) = (\lambda y^{(1)}, \dots, \lambda^s y^{(s)}), \quad y \in \mathbb{G}^{(2)}$$

where $x^{(i)} \in \mathbb{R}^{N_i}$, $i \le r$, $N_1 + \cdots + N_r = N$ and $y^{(i)} \in \mathbb{R}^{M_i}$, $i \le s$, $M_1 + \cdots + M_s = M$. Let $\Delta_{\mathbb{G}^{(1)}} = \sum_{j=1}^{N_1} X_j^2$ and $\Delta_{\mathbb{G}^{(2)}} = \sum_{j=1}^{M_1} Y_j^2$ be the canonical sub-Laplacians on $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$, respectively. We define a homogeneous stratified group $\mathbb{G}$ on $\mathbb{R}^{N+M}$ as follows. Suppose $r \le s$. If $(x, y) \in \mathbb{R}^N \times \mathbb{R}^M$, we consider the following permutation of the coordinates

$$R(x, y) = (x^{(1)}, y^{(1)}, \dots, x^{(r)}, y^{(r)}, y^{(r+1)}, \dots, y^{(s)}).$$

We then denote the point of $\mathbb{G} \equiv \mathbb{R}^{N+M}$ by $z = R(x, y)$. We finally define the group law $\circ$ and the dilations δ_λ on $\mathbb{G}$ as one can expect: for every $z = R(x, y)$, $\zeta = R(\xi, \eta) \in \mathbb{G}$, we set

$$z \circ \zeta = R(x \circ^{(1)} \xi, y \circ^{(2)} \eta), \quad \delta_\lambda z = R(\delta_\lambda^{(1)} x, \delta_\lambda^{(2)} y).$$

It is then easily checked that $(\mathbb{G}, \circ, \delta_\lambda)$ is a homogeneous stratified group of step s and $N_1 + M_1$ generators. Moreover, the canonical sub-Laplacian on $\mathbb{G}$ is the sum of the sub-Laplacians on $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$:

$$\Delta_{\mathbb{G}} = \Delta_{\mathbb{G}^{(1)}} + \Delta_{\mathbb{G}^{(2)}} = \textstyle\sum_{j=1}^{N_1} X_j^2 + \sum_{j=1}^{M_1} Y_j^2.$$

1.5.8 Carnot groups with homogeneous dimension $Q \le 3$

Let $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ be a Carnot group with homogeneous dimension $Q \le 3$. We recall that $Q = \sum_{j=1}^r j\, N_j$, where r and $N_1, \dots, N_r$ are, respectively, the step of $\mathbb{G}$ and the dimensions of the layers $\mathfrak{g}_1, \dots, \mathfrak{g}_r$ of $\mathbb{G}$. Obviously, the group is not the Euclidean group in $\mathbb{R}^N$ iff $\mathfrak{g}_2 \ne \{0\}$, i.e., $r \ge 2$. In this case, the first layer $\mathfrak{g}_1$ must be at least two-dimensional since $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2 \ne \{0\}$. This shows that any non-Euclidean Carnot group has homogeneous dimension $Q \ge 4$. Thus, if $Q \le 3$ then $\mathbb{G}$ is the Euclidean group in $\mathbb{R}^N$, i.e., $\circ = +$ and $\delta_\lambda(x) = \lambda x$. The sub-Laplacians on $\mathbb{G}$ are the second order elliptic operators with constant coefficients. The canonical sub-Laplacians are $\mathrm{d}^2/\mathrm{d}x_1^2$ in $\mathbb{R}$ ($Q = N = 1$), $\partial_{x_1}^2 + \partial_{x_2}^2$ in $\mathbb{R}^2$ ($Q = N = 2$), and $\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$ in $\mathbb{R}^3$ ($Q = N = 3$).