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Stratified Groups and sub-Laplacians

1.1 Vector fields in $\mathbb{R}^N$. Exponential maps. Lie algebras of vector fields

Given an N-tuple of scalar functions $a_1, \dots, a_N$,

$$a_j : \mathbb{R}^N \to \mathbb{R}, \quad j \in \{1, \dots, N\},$$

the linear first order differential operator

$$X = \sum_{j=1}^N a_j \, \partial_j, \quad \partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}, \tag{1.1}$$

will be called a *vector field* in $\mathbb{R}^N$ with components $a_1, \dots, a_N$. We shall always deal with *smooth* vector fields, i.e., with vector fields whose components $a_1, \dots, a_N$ are functions of class C^∞. We shall denote by $T(\mathbb{R}^N)$ the set of all smooth vector fields in $\mathbb{R}^N$. Equipped with the natural operations, $T(\mathbb{R}^N)$ is a vector space over $\mathbb{R}$. We shall adopt the following notation: I will denote the identity map on $\mathbb{R}^N$ and, if X is the vector field in (1.1), then

$$XI := (a_1, \dots, a_N)^T \tag{1.2}$$

will be the column vector of the components of X. By consistency of notation, we may write

$$X = \nabla \cdot XI,$$

where $\nabla = (\partial_1, \dots, \partial_N)$ is the gradient operator in $\mathbb{R}^N$.

A path $\gamma : \mathcal{D} \to \mathbb{R}^N$, $\mathcal{D}$ = interval of $\mathbb{R}$, will be said an *integral curve* of X if $\dot\gamma(t) = XI(\gamma(t))$ for every $t \in \mathcal{D}$. If X is a smooth vector field, then, for every $x \in \mathbb{R}^N$, the Cauchy problem

$$\begin{cases} \dot\gamma = XI(\gamma), \\ \gamma(0) = x \end{cases} \tag{1.3}$$

has a unique solution $\gamma(\cdot,x):\mathcal{D}(X,x)\to\mathbb{R}^N$. We agree to denote by $\mathcal{D}(X,x)$ the greatest open interval of $\mathbb{R}$ on which $\gamma(\cdot,x)$ exists.

Since X is smooth, $t\mapsto\gamma(t,x)$ is a C^∞ function whose n-th Taylor expansion in a neighborhood of $t=0$ is given by

$$\begin{aligned}\gamma(t,x) = x + t\,X^{(1)}I(x) + \tfrac{t^2}{2!}\,X^{(2)}I(x) + \cdots + \tfrac{t^n}{n!}\,X^{(n)}I(x)\\ + \frac{1}{n!}\int_0^t (t-s)^n\,X^{(n+1)}I(\gamma(s,x))\,\mathrm{d}s.\end{aligned} \tag{1.4}$$

Hereafter we denote by $X^{(k)}$ the vector field

$$X^{(k)} = \textstyle\sum_{j=1}^N (X^{k-1}a_j)\,\partial_{x_j},$$

being $X^0=X$ and X^h, $h\ge 1$, the h-th order iterated of X, i.e.,

$$X^h := \underbrace{X\circ\cdots\circ X}_{h}.$$

We remark that X^h is a differential operator of order at most h, whereas $X^{(h)}$ is a differential operator of order at most 1. To check (1.4) we use (1.3). Writing $\gamma(t)$ instead of $\gamma(t,x)$, (1.3) gives: $\gamma(0)=x$, $(\mathrm{d}/\mathrm{d}t)|_{t=0}\gamma(t) = XI(x)$ and

$$\begin{aligned}\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\big|_{t=0}\gamma(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(XI)(\gamma(t)) = \mathcal{J}_{XI}(\gamma(0))\cdot\dot\gamma(0) = \mathcal{J}_{XI}(x)\cdot XI(x)\\ &= \begin{pmatrix}\nabla a_1(x)\cdot XI(x)\\ \vdots\\ \nabla a_N(x)\cdot XI(x)\end{pmatrix} = \begin{pmatrix}Xa_1(x)\\ \vdots\\ Xa_N(x)\end{pmatrix} = X^{(2)}I(x).\end{aligned}$$

By iterating this argument, we obtain

$$\gamma^{(k)}(0) := \tfrac{\mathrm{d}^k}{\mathrm{d}t^k}\big|_{t=0}\gamma(t) = X^{(k)}I(x), \quad k\ge 2.$$

Replacing this identity in the Taylor formula

$$\gamma(t) = x + \textstyle\sum_{k=1}^n \tfrac{t^k}{k!}\,\gamma^{(k)}(0) + \tfrac{1}{n!}\int_0^t (t-s)^n\,\gamma^{(n+1)}(s)\,\mathrm{d}s,$$

we obtain (1.4). We observe that, since the identity map I is linear and since the first order part of X^h coincides with $X^{(h)}$, then $X^{(h)}I\equiv X^hI$. Thus formula (1.4) can be rewritten as

$$\begin{aligned}\gamma(t,x) = x + t\,XI(x) + \tfrac{t^2}{2!}\,X^2I(x) + \cdots + \tfrac{t^n}{n!}\,X^nI(x)\\ + \frac{1}{n!}\int_0^t (t-s)^n\,X^{n+1}I(\gamma(s,x))\,\mathrm{d}s.\end{aligned} \tag{1.5}$$

This last expansion suggests to put

$$\exp(tX)(x) := \gamma(t,x). \tag{1.6}$$

Then, for every $n \in \mathbb{N}$,

$$\exp(tX)(x) = \sum_{k=0}^{n} \frac{t^k}{k!} X^k I(x) + \frac{1}{n!} \int_0^t (t-s)^n X^{n+1} I\big(\exp(sX)(x)\big)\,\mathrm{d}s. \tag{1.7}$$

In particular, for $n = 1$,

$$\exp(tX)(x) = x + t\,XI(x) + \int_0^t (t-s)\,X^2 I\big(\exp(sX)(x)\big)\,\mathrm{d}s. \tag{1.8}$$

If we define

$$\mathcal{U} := \{(t,x) \in \mathbb{R} \times \mathbb{R}^N \mid x \in \mathbb{R}^N,\ t \in \mathcal{D}(X,x)\},$$

from the basic theory of ordinary differential equations we know that $\mathcal{U}$ is open and the map

$$\mathcal{U} \ni (t,x) \mapsto \exp(tX)(x) \in \mathbb{R}^N$$

is smooth. Moreover, from the unique solvability of the Cauchy problem related to smooth vector fields we get: $t \in \mathcal{D}(-X,x)$ iff $-t \in \mathcal{D}(X,x)$ and

$$\exp(-tX)(x) = \exp(t(-X))(x), \tag{1.9}$$

$$\exp(-tX)\big(\exp(tX)(x)\big) = x, \tag{1.10}$$

$$\exp((t+\tau)X)(x) = \exp(tX)\big(\exp(tX)(x)\big), \tag{1.11}$$

when all the terms are defined. If $\mathcal{D}(X,x) = \mathbb{R}$, identities (1.9)-(1.11) hold for every t, $\tau \in \mathbb{R}$.

Remark 1.1.1. For our aims the vector fields of the following type

$$X = \textstyle\sum_{j=1}^{N} a_j(x_1,\dots,x_{j-1})\,\partial_{x_j} \tag{1.12}$$

will play a crucial rôle. In (1.12) the function a_j only depends on the variables $x_1,\dots,x_{j-1}$ and we agree to let $a_j(x_1,\dots,x_{j-1}) = \text{constant}$ when $j = 1$.

For any smooth vector field X of the form (1.12), *the map*

$$(x,t) \mapsto \exp(tX)(x)$$

is well defined for every $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

Indeed, if $\gamma = (\gamma_1,\dots,\gamma_N)$ is the solution to the Cauchy problem

$$\begin{cases} \dot\gamma = XI(\gamma) \\ \gamma(0) = x, \qquad x = (x_1,\dots,x_N), \end{cases}$$

then $\dot\gamma_1 = a_1$ and $\dot\gamma_j = a_j(\gamma_1,\dots,\gamma_{j-1})$ for $j = 2,\dots,N$. As a consequence

$$\gamma_1(x,t) = x_1 + ta_1, \quad \gamma_j(x,t) = x_j + \int_0^t a_j(\gamma_1(x,s),\dots,\gamma_{j-1}(x,s))\,\mathrm{d}s$$

and $\gamma_j(x,t)$ is defined for every $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Moreover, $\gamma_1(\cdot,t)$ only depends on x_1, whereas for $j = 2,\dots,N$, $\gamma_j(\cdot,t)$ only depends on $x_1,\dots,x_j$. Let us put $A_1(x,t) = A_1(x_1,t) = x_1 + ta_1$ and, for $j = 2,\dots,N$,

$$A_j(x,t) = A_j(x_1,\dots,x_{j-1},t) := \int_0^t a_j(\gamma_1(x,s),\dots,\gamma_{j-1}(x,s))\,\mathrm{d}s.$$

Then, for every $x \in \mathbb{R}^N$, $t \in \mathbb{R}$,

$$\exp(tX)(x) = \Big(x_1 + ta_1, x_2 + A_2(x_1,t),\dots,x_N + A_N(x_1,\dots,x_{N-1},t)\Big) \tag{1.13}$$

and the map $x \mapsto \exp(tX)(x)$ is a *global diffeomorphism of* $\mathbb{R}^N$ *onto* $\mathbb{R}^N$, for every fixed $t \in \mathbb{R}$. Its inverse map $y \mapsto L(y,t)$ is given by

$$y \mapsto L(y,t) = \exp(-tX)(y). \tag{1.14}$$

This last statement follows from identity (1.10).

Let us now consider a smooth function $u : \mathbb{R}^N \to \mathbb{R}$ and the vector field in (1.1). Then

$$Xu(x) = \lim_{t\to 0} \frac{u(\exp(tX)(x)) - u(x)}{t}, \quad \forall\, x \in \mathbb{R}^N. \tag{1.15}$$

Indeed, since $\exp(tX)(x) = x + tXI(x) + \mathcal{O}(t^2)$, the limit on the right-hand side of (1.15) is equal to the following one:

$$\lim_{t\to 0} \frac{u(x + tXI(x)) - u(x)}{t} = \nabla u(x) \cdot XI(x) = Xu(x).$$

Given two smooth vector fields X and Y, we define the Lie-bracket $[X,Y]$ as follows

$$[X,Y] := XY - YX.$$

Then, if $X = \sum_{j=1}^N a_j \partial_j$ and $Y = \sum_{j=1}^N b_j \partial_j$, the Lie bracket $[X,Y]$ is the vector field

$$[X,Y] = \textstyle\sum_{j=1}^N (Xb_j - Ya_j)\partial_j.$$

As a consequence

$$[X,Y]I = (Xb_1,\dots,Xb_N)^T - (Ya_1,\dots,Ya_N)^T = \mathcal{J}_{YI} \cdot XI - \mathcal{J}_{XI} \cdot YI.$$

It is quite trivial to check that $(X,Y) \mapsto [X,Y]$ is a bilinear map on the vector space $T(\mathbb{R}^N)$ satisfying the *Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for every $X, Y, Z \in T(\mathbb{R}^N)$. We shall refer to $T(\mathbb{R}^N)$ (equipped with the above Lie-bracket) as the *Lie algebra* of the vector fields on $\mathbb{R}^N$. Any sub-algebra $\mathfrak{a}$ of $T(\mathbb{R}^N)$ will be called a *Lie algebra of vector fields*. More explicitly, $\mathfrak{a}$ is a Lie algebra of vector fields if $\mathfrak{a}$ is a *vector* subspace of $T(\mathbb{R}^N)$, closed with respect to $[\,,\,]$, i.e., $[X, Y] \in \mathfrak{a}$ for every $X, Y \in \mathfrak{a}$.

We now fix some other notation on algebras of vector fields. Given a set of vector fields $Z_1, \dots, Z_m \in T(\mathbb{R}^N)$, and given a multi-index

$$J = (j_1, \dots, j_k) \in \{1, \dots, m\}^k$$

we set

$$Z_J := [Z_{j_1}, \dots [Z_{j_{k-1}}, Z_{j_k}] \dots].$$

We say that Z_J is a commutator of length k of $Z_1, \dots, Z_m$. If $J = j_1$, we also say that $Z_J := Z_{j_1}$ is a commutator of length 1 of $Z_1, \dots, Z_m$.

If U is any subset of $T(\mathbb{R}^N)$, we denote by $\mathrm{Lie}\{U\}$ the least sub-algebra of $T(\mathbb{R}^N)$ containing U, i.e.,

$$\mathrm{Lie}\{U\} := \bigcap \mathfrak{h} \quad \text{where } \mathfrak{h} \text{ is a sub-algebra of } T(\mathbb{R}^N) \text{ with } U \subseteq \mathfrak{h}.$$

We define

$$\mathrm{rank}\big(\mathrm{Lie}\{U\}(x)\big) = \dim\{ZI(x) \mid Z \in \mathrm{Lie}\{U\}\}.$$

The following result holds.

Proposition 1.1.2. *Let $U \subseteq T(\mathbb{R}^N)$. We set*

$$U_1 := \mathrm{span}\{U\}, \qquad U_n := \mathrm{span}\{[u, v] \mid u \in U,\ v \in U_{n-1}\}, \quad n \geq 2.$$

Then, we have

$$\mathrm{Lie}\{U\} = \mathrm{span}\{U_n \mid n \in \mathbb{N}\}.$$

We explicitly remark that the very vector fields in U_n are linear combination of "nested" brackets, i.e., brackets of the following type

$$[u_1[u_2[u_3[\cdots[u_{n-1}, u_n]\cdots]]]],$$

with $u_1, \dots, u_n \in U$. The above proposition then states that *whatever* element of $\mathrm{Lie}\{U\}$ is a linear combination of nested brackets. To show the idea of the proof, let us take $u_1, u_2,\ v_1, v_2 \in U$ and prove that $[[u_1, u_2], [v_1, v_2]]$ is a linear combination of nested brackets. By the Jacobi identity, $[X, [Y, Z]] = -[Y, [Z, X]] - [Z, [X, Y]]$ one has

$$\begin{aligned}
[\underbrace{[u_1, u_2]}_{X}, [\underbrace{v_1}_{Y}, \underbrace{v_2}_{Z}]] &= -[v_1, [v_2, [u_1, u_2]]] - [v_2, [[u_1, u_2], v_1]] \\
&= -[v_1, [v_2, [u_1, u_2]]] + [v_2, [v_1, [u_1, u_2]]] \in U_4.
\end{aligned}$$

Proof. (of Proposition 1.1.2.) We set $U^* := \mathrm{span}\{U_n \,|\, n \in \mathbb{N}\}$. Obviously, U^* contains U and is contained in any algebra of vector fields which contains U. Hence, we are left to prove that U^* is closed under the bracket operation. Obviously, it is enough to show that, for any $i, j \in \mathbb{N}$ and for any $u_1, \dots, u_i$, $v_1, \dots, v_j \in U$ we have

$$\Big[[u_1[u_2[\cdots[u_{i-1}, u_i]\cdots]]]; [v_1[v_2[\cdots[v_{j-1}, v_j]\cdots]]]\Big] \in U_{i+j}.$$

We argue by induction on $k := i + j \geq 2$. For $k = 2$ and 3 the assertion is obvious. Let us now suppose the thesis holds for every $i + j \leq k$, with $k \geq 4$, and prove it also holds when $i + j = k + 1$. We can suppose, by skew-symmetry, $j \geq 3$. Exploiting repeatedly the induction hypothesis and the Jacobi identity, we have

$$\begin{aligned}
&\Big[u; [v_1[v_2[\cdots[v_{j-1}, v_j]\cdots]]]\Big] \\
&\quad = -[v_1, \underbrace{[[v_2, [v_3, \cdots]], u]}_{\text{length } k}] - [[v_2, [v_3, \cdots]], [u, v_1]] \\
&\quad = \{\text{element of } U_{k+1}\} - [[v_1, u], [v_2, [v_3, \cdots]]] \\
&\quad = \{\text{element of } U_{k+1}\} + [v_2, \underbrace{[[v_3, \cdots], [v_1, u]]}_{\text{length } k}] + [[v_3, \cdots], [[v_1, u]v_2]] \\
&\quad = \{\text{element of } U_{k+1}\} + [[v_2, [v_1, u]], [v_3, \cdots]] \\
&\text{(after finitely many steps)} \\
&\quad = \{\text{element of } U_{k+1}\} + (-1)^{j-1}[[v_{j-i}, [v_{j-2}, \cdots [v_1, u]]], v_j] \\
&\quad = \{\text{element of } U_{k+1}\} + (-1)^{j}[v_j, [v_{j-i}, [v_{j-2}, \cdots [v_1, u]]]] \\
&\quad \in U_{k+1}.
\end{aligned}$$

This ends the proof. □The following notation will be used when dealing with "stratified" Lie algebras. If V_1, V_2 are subsets of $T(\mathbb{R}^N)$, we denote

$$[V_1, V_2] := \mathrm{span}\big\{[v_1, v_2] \,|\, v_i \in V_i,\ i = 1, 2\big\}.$$

From Proposition 1.1.2 it follows that, if $Z_1, \dots, Z_m \in T(\mathbb{R}^N)$, then a system of generators spanning $\mathrm{Lie}\{Z_1, \dots, Z_m\}$ is given by the Z_J's with $J = (j_1, \dots, j_k) \in \{1, \dots, m\}^k$, $k \in \mathbb{N}$. This (non-trivial) fact will be used throughout the next sections.

1.2 Lie groups on $\mathbb{R}^N$

Let $\circ$ be a given group law on $\mathbb{R}^N$ and suppose that the map $(x, y) \mapsto y^{-1} \circ x$ is smooth. Then $\mathbb{G} = (\mathbb{R}^N, \circ)$ is called a *Lie group*. We shall assume that the origin 0 is the identity of $\mathbb{G}$.

We denote by $\tau_\alpha(x) = \alpha \circ x$ the left-translations on $\mathbb{G}$. A (smooth) vector field X on $\mathbb{R}^N$ is called *left-invariant* on $\mathbb{G}$ if

$$X(\varphi \circ \tau_\alpha) = (X\varphi) \circ \tau_\alpha,$$

for every $\alpha \in \mathbb{G}$ and for every smooth function φ. We denote by $\mathfrak{g}$ the set of left-invariant vector fields on $\mathbb{G}$. It is quite obvious to recognize that, for every $X, Y \in \mathfrak{g}$ and for every $\lambda, \mu \in \mathbb{R}$, $\lambda X + \mu Y \in \mathfrak{g}$ and $[X, Y] \in \mathfrak{g}$. Then, $\mathfrak{g}$ is a Lie algebra of vector fields , sub-algebra of $T(\mathbb{R}^N)$. It will be called the *Lie algebra* of $\mathbb{G}$.

From the Theorem of differentiation of composite functions, we easily get the following characterization of left-invariant vector fields on $\mathbb{G}$.

Proposition 1.2.1. *The vector field X belongs to $\mathfrak{g}$ if and only if*

$$(XI)(\alpha \circ x) = \mathcal{J}_{\tau_\alpha}(x) \cdot (XI)(x), \quad \forall\ \alpha,\, x \in \mathbb{G}. \tag{1.16}$$

Proof. For every smooth function φ on $\mathbb{R}^N$ we have

$$(X(\varphi \circ \tau_\alpha))(x) = \nabla(\varphi \circ \tau_\alpha)(x) \cdot XI(x) = \Big((\nabla\varphi)(\tau_\alpha(x)) \cdot \mathcal{J}_{\tau_\alpha}(x)\Big) \cdot XI(x)$$

and

$$(X\varphi)(\tau_\alpha(x)) = (\nabla\varphi)(\tau_\alpha(x)) \cdot XI(\tau_\alpha(x)).$$

Then, $X \in \mathfrak{g}$ if and only if

$$(\nabla\varphi)(\tau_\alpha(x)) \cdot \Big(\mathcal{J}_{\tau_\alpha}(x) \cdot XI(x)\Big) = (\nabla\varphi)(\tau_\alpha(x)) \cdot XI(\tau_\alpha(x)), \tag{1.17}$$

for every $\alpha, x \in \mathbb{R}^N$ and for every $\varphi \in C^\infty(C^\infty, \mathbb{R})$. By choosing $\varphi(x) = \sum_{j=1}^N h_j x_j$, with $h_j \in \mathbb{R}$ for $1 \le j \le N$, (1.17) gives $h^T \cdot \mathcal{J}_{\tau_\alpha}(x) \cdot XI(x) = h^T \cdot XI(\tau_\alpha(x))$ for every $h \in \mathbb{R}^N$, which obviously implies (1.16). □Swapping α with x in (1.16), we obtain $(XI)(x \circ \alpha) = \mathcal{J}_{\tau_x}(\alpha) \cdot (XI)(\alpha)$ for all $\alpha,\, x \in \mathbb{G}$, so that, when $\alpha = 0$,

$$(XI)(x) = \mathcal{J}_{\tau_x}(0)\,(XI)(0), \quad \forall\, x \in \mathbb{G}. \tag{1.18}$$

This identity says that a left-invariant vector field on $\mathbb{G}$ is determined by its value at the origin and by the Jacobian matrix at the origin of the left-translation. The following result shows that (1.18) characterizes the vector fields in $\mathfrak{g}$.

Proposition 1.2.2. *Let η be a fixed vector of $\mathbb{R}^N$ and define the vector field X as follows*

$$XI(x) = \mathcal{J}_{\tau_x}(0) \cdot \eta, \quad x \in \mathbb{R}^N. \tag{1.19}$$

Then $X \in \mathfrak{g}$.

Proof. Definition (1.19) gives

$$XI(\alpha \circ x) = \mathcal{J}_{\tau_{\alpha \circ x}}(0) \cdot \eta, \quad \alpha,\, x \in \mathbb{R}^N. \tag{1.20}$$

On the other hand, since the composition law on $\mathbb{G}$ is associative, we have $\tau_{\alpha\circ x} = \tau_\alpha \circ \tau_x$, so that $\mathcal{J}_{\tau_{\alpha\circ x}}(0) = \mathcal{J}_{\tau_\alpha}(x) \cdot \mathcal{J}_{\tau_x}(0)$. Replacing this identity in (1.20) we get $XI(\alpha \circ x) = \mathcal{J}_{\tau_\alpha}(x) \cdot \mathcal{J}_{\tau_x}(0) \cdot \eta$ which implies, by (1.19), $XI(\alpha \circ x) = \mathcal{J}_{\tau_\alpha}(x) \cdot XI(x)$. Then, by Proposition 1.2.1, $X \in \mathfrak{g}$. □From Proposition 1.2.1 and identity (1.18) it follows that $\mathfrak{g}$ is a vector space of dimension N. Indeed, the following proposition holds.

Proposition 1.2.3. *The map*

$$J : \mathbb{R}^N \to \mathfrak{g}, \quad \eta \mapsto J(\eta)$$

with $J(\eta)$ *defined by*

$$J(\eta)I(x) = \mathcal{J}_{\tau_x}(0) \cdot \eta,$$

is an isomorphism of vector spaces. In particular,

$$\dim \mathfrak{g} = N.$$

Proof. We first observe that J is well defined since, by Proposition 1.2.2, $J(\eta) \in \mathfrak{g}$ for every $\eta \in \mathbb{R}^N$. Moreover, by identity (1.18), $J(\mathbb{R}^N) = \mathfrak{g}$. The linearity of J is obvious. Then. it remains to prove that J is injective. Suppose $J(\eta) = 0$. This means that $\mathcal{J}_{\tau_x}(0) \cdot \eta = 0$ for every $x \in \mathbb{R}^N$. In particular $\mathcal{J}_{\tau_0}(0) \cdot \eta = 0$. On the other hand, since the left-translation τ_0 is the identity map, $\mathcal{J}_{\tau_0}(0) \cdot \eta = \eta$. Then $\eta = 0$ and J is one-to-one. □For what follows, the next remarks will be useful.

Remark 1.2.4. Let $X \in \mathfrak{g}$ and denote by η the value of XI at $t = 0$, i.e., $\eta = XI(0)$. Then, by the identity (1.18), $XI(x) = \mathcal{J}_{\tau_x}(0) \cdot \eta$. As a consequence, for every smooth function φ on $\mathbb{R}^N$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \varphi(x \circ t\eta) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \varphi(\tau_x(t\eta)) \\ &= \nabla\varphi(x) \cdot \mathcal{J}_{\tau_x}(0) \cdot \eta = \nabla\varphi(x) \cdot XI(x). \end{aligned}$$

Then

$$(X\varphi)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \varphi(x \circ t\eta), \quad \eta = XI(0). \tag{1.21}$$

Identity (1.21) characterizes the left-invariant vector fields on $\mathbb{G}$. This follows from the next remark.

Remark 1.2.5. Let X be a vector field on $\mathbb{R}^N$. Assume that, for every $\varphi \in C^\infty(\mathbb{R}^N, \mathbb{R})$,

$$(X\varphi)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \varphi(x \circ t\eta), \quad \forall\, x \in \mathbb{R}^N, \tag{1.22}$$

where $\eta = XI(0)$. Then $X \in \mathfrak{g}$.

Indeed, (1.22) and the associativity of $\circ$ imply

$$\begin{aligned} (X\varphi)(\alpha \circ x) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \varphi((\alpha \circ x) \circ t\eta) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\varphi \circ \tau_\alpha)(x \circ t\eta) \\ &= X(\varphi \circ \tau_\alpha)(x), \end{aligned}$$

for every $\alpha, x \in \mathbb{G}$. Then X is left-invariant on $\mathbb{G}$.

Remark 1.2.6. For every $x \in \mathbb{R}^N$ and $X \in \mathfrak{g}$ the following expansion holds

$$\exp(tX)(x) = x \circ t\eta + o(t) \text{ as } t \to 0, \quad \eta = XI(0). \tag{1.23}$$

Indeed, since $XI(x) = \mathcal{J}_{\tau_x}(0) \cdot \eta$,

$$x \circ t\eta = \tau_x(t\eta) = \tau_x(0) + t\mathcal{J}_{\tau_x}(0) \cdot \eta + o(t) = x + tXI(x) + o(t).$$

Then (1.23) follows from (1.8).

Remark 1.2.7. From Proposition 1.2.3 it follows that any basis of $\mathfrak{g}$ is the image via J of a basis of $\mathbb{R}^N$.

If $\{e_1, \dots, e_N\}$, is the canonical basis of $\mathbb{R}^N$, we call

$$\{Z_1, \dots, Z_N\}, \quad Z_j = J(e_j)$$

the *Jacobian basis* of $\mathfrak{g}$. From the very definition of J, we obtain

$$Z_jI(x) = \mathcal{J}_{\tau_x}(0) \cdot e_j = j\text{-th column of } \mathcal{J}_{\tau_x}(0), \quad \forall\, x \in \mathbb{R}^N, \tag{1.24}$$

so that, since $\mathcal{J}_{\tau_x}(0) = \mathbb{I}_N$,

$$Z_jI(0) = e_j.$$

Form Remark 1.2.5 we also have

$$(Z_j\varphi)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \varphi(x \circ te_j) = \frac{\partial}{\partial y_j}\Big|_{y=0} \varphi(x \circ y),$$

for every $\varphi \in C^\infty(\mathbb{R}^N)$ and every $x \in \mathbb{G}$.

Then, collecting these results: the Jacobian basis $\{Z_1, \dots, Z_N\}$ of $\mathfrak{g}$ is given by the N column of the Jacobian matrix $\mathcal{J}_{\tau_x}(0)$ (whence the name). Moreover $Z_j(0) = \partial/\partial x_j$ and

$$(Z_j\varphi)(x) = (\partial/\partial y_j)|_{y=0}\varphi(x \circ y), \quad \forall\, \varphi \in C^\infty(\mathbb{R}^N),\ x \in \mathbb{G}.$$

In the sequel, to endow $\mathfrak{g}$ with a differentiable structure, we shall fix a system of coordinates on $\mathfrak{g}$ by choosing the Jacobian basis, then identifying $\mathfrak{g}$ with $\mathbb{R}^N$.

We remark that two vector fields can be linearly independent in $T(\mathbb{R}^N)$ without being linearly independent at every point. Take, for example, ∂_{x_1} and $x_1\,\partial_{x_2}$ in $\mathbb{R}^2$. Moreover, two vector fields can be linearly dependent at every point without being linearly dependent in $T(\mathbb{R}^N)$. Take, for example, ∂_{x_1} and $x_1\,\partial_{x_1}$ in $\mathbb{R}^2$. The following result shows that neither of the previous situations can occur for left-invariant vector fields on a Lie group. Indeed, given a family of vector fields $X_1, \dots, X_m \in \mathfrak{g}$, the rank of the subset of $\mathbb{R}^N$ spanned by $\{X_1I(x), \dots, X_mI(x)\}$ is independent of x. More precisely we have:

Proposition 1.2.8. *Let $X_1, \dots, X_m \in \mathfrak{g}$. Then the following statements are equivalent:*

(i) $X_1, \dots, X_m$ *are linearly independent (in* $\mathfrak{g}$*);*
(ii) $X_1 I(0), \dots, X_m I(0)$ *are linearly independent (in* $\mathbb{R}^N$ *);*
(iii) there exists $x_0 \in \mathbb{R}^N$ *such that* $X_1 I(x_0), \dots, X_m I(x_0)$ *are linearly independent (in* $\mathbb{R}^N$ *);*
(iv) $X_1 I(x), \dots, X_m I(x)$ *are linearly independent (in* $\mathbb{R}^N$ *), for every* $x \in \mathbb{R}^N$*.*

Proof. We first recall that, by identity (1.18),

$$X_j I(x) = \mathcal{J}_{\tau_x}(0) \cdot \eta_j, \quad \text{with } \eta_j = X_j I(0),$$

for every $x \in \mathbb{R}^N$. On the other hand, since $\tau_{x^{-1}} \circ \tau_x = I$, $\mathcal{J}_{\tau_{x^{-1}}}(x) \cdot \mathcal{J}_{\tau_x}(0) = \mathbb{I}_N$. Hence $\mathcal{J}_{\tau_x}(0)$ is non singular for every $x \in \mathbb{R}^N$. Then (ii), (iii) and (iv) are equivalent. The equivalence between (i) and (ii) follows from Proposition 1.2.3. Indeed, with the notation of that proposition, for every $j \in \{1, \dots, m\}$, $X_j = J(\eta_j)$ with $\eta_j = X_j I(0)$ and J is an isomorphism of $\mathbb{R}^N$ onto $\mathfrak{g}$. $\square$

The next Lemma will be useful to define the notion of *exponential map* of $\mathfrak{g}$ in $\mathbb{G}$, one of the most important tools in Lie group theory.

Lemma 1.2.9. *Let* $X \in \mathfrak{g}$ *and let* $\gamma : [t_0, t_0 + T] \to \mathbb{R}^N$ *be an integral curve of* X*. Then*

(i) $\alpha \circ \gamma$ *is an integral curve of* X*, for every* $\alpha \in \mathbb{G}$*.*
(ii) γ *can be continued to an integral curve of* X *on the interval* $[t_0 - T, t_0 + 2T]$*.*

Proof. (i): For every $t \in [t_0, t_0 + T]$ we have (by (1.16))

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\alpha \circ \gamma(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}(\tau_\alpha(\gamma(t))) = \mathcal{J}_{\tau_\alpha}(\gamma(t)) \cdot \dot{\gamma}(t) \\ &= \mathcal{J}_{\tau_\alpha}(\gamma(t)) \cdot XI(\gamma(t)) = X(\alpha \circ \gamma(t)). \end{aligned}$$

(ii): Define $\Gamma : [t_0 - T, t_0 + 2T] \to \mathbb{R}^N$ as follows:

$$\Gamma(t) := \begin{cases} \gamma(t_0) \circ (\gamma(t_0 + T))^{-1} \circ \gamma(t + T), & \text{if } t_0 - T \le t \le t_0, \\ \gamma(t), & \text{if } t_0 \le t \le t_0 + T, \\ \gamma(t_0 + T) \circ (\gamma(t_0))^{-1} \circ \gamma(t - T), & \text{if } t_0 + T \le t \le t_0 + 2T. \end{cases}$$

Then, by (i), Γ is an integral curve of X and, obviously, $\Gamma|_{[t_0, t_0+T]} \equiv \gamma$. $\square$From assertion (ii) of this Lemma, we immediately obtain the following statement: *for every* $X \in \mathfrak{g}$*, the map*

$$(x, t) \mapsto \exp(tX)(x)$$

is well-defined for every $x \in \mathbb{R}^N$ *and every* $t \in \mathbb{R}$*.*

From the assertion (i) of Lemma 1.2.9, the next important corollary easily follows.

Corollary 1.2.10. *Let $X \in \mathfrak{g}$ and $x, y \in \mathbb{G}$. Then*

$$x \circ \exp(tX)(y) = \exp(tX)(x \circ y), \tag{1.25}$$

for every $t \in \mathbb{R}$. In particular, for $y = 0$,

$$\exp(tX)(x) = x \circ \exp(tX)(0).$$

Proof. By Lemma 1.2.9-(i), $t \mapsto x \circ \exp(tX)(y)$ is an integral curve of X. Moreover

$$(x \circ \exp(tX)(y))|_{t=0} = x \circ y.$$

Then (1.25) follows. $\square$

The *exponential map* of $\mathfrak{g}$ in $\mathbb{G}$ is defined as

$$\mathrm{Exp}\,: \mathfrak{g} \to \mathbb{G}, \quad \mathrm{Exp}\,(X) = \exp(X)(0).$$

From Corollary 1.2.10 and identity (1.10) (with $\tau = -t$), we get

$$\mathrm{Exp}\,(-X) \circ \mathrm{Exp}\,(X) = 0.$$

Indeed,

$$\begin{aligned}\mathrm{Exp}\,(-X) \circ \mathrm{Exp}\,(X) &= \mathrm{Exp}\,(-X) \circ \exp(X)(0) = \exp(X)(\mathrm{Exp}\,(-X)) \\ &= \exp(X)(\exp(-X)(0)) = 0.\end{aligned}$$

Then we have

$$(\mathrm{Exp}\,(X))^{-1} = \mathrm{Exp}\,(-X). \tag{1.26}$$

Let $\{X_1, \ldots, X_N\}$ be a basis of $\mathfrak{g}$. Then, for every $X \in \mathfrak{g}$,

$$X = \textstyle\sum_{j=1}^N \xi_j X_j, \quad \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N,$$

so that

$$\mathrm{Exp}\,(X) = \exp(\textstyle\sum_{j=1}^N \xi_j X_j)(0).$$

From the classical theory of ODE's, we know that the map

$$(\xi_1, \ldots, \xi_N) \mapsto \exp(\textstyle\sum_{j=1}^N \xi_j X_j)(0)$$

is smooth. Then, we can say that $X \mapsto \mathrm{Exp}\,(X)$ is smooth. From the Taylor expansion (1.8), we get

$$\mathrm{Exp}\,(X) = \textstyle\sum_{j=1}^N \xi_j \,\eta_j + \mathcal{O}(|\xi|^2), \quad \text{as } |\xi| \to 0,$$

where $\eta_j = X_j I(0)$. It follows that, denoting by E the matrix whose column vectors are $\eta_1, \ldots, \eta_N$,

$$\mathcal{J}_{\mathrm{Exp}}\,(0) = E.$$

In particular, if $\{X_1, \ldots, X_N\} = \{Z_1, \ldots, Z_N\}$ is the Jacobian basis of $\mathfrak{g}$, then

$$\mathcal{J}_{\mathrm{Exp}}(0) = \mathbb{I}_N. \tag{1.27}$$

As a consequence, Exp is a diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ onto a neighborhood of $0 \in \mathbb{G}$. Where defined, we denote by Log the inverse map of Exp. The next proposition is an easy consequence of Corollary 1.2.10 and shows an important link between the composition law in $\mathbb{G}$ and the exponential map.

Proposition 1.2.11. *Let $x, y \in \mathbb{G}$. Assume* $\mathrm{Log}\,(y)$ *is defined. Then*

$$x \circ y = \exp(\mathrm{Log}\,(y))(x). \tag{1.28}$$

Proof. Let $X = \mathrm{Log}\,(y)$. This means that $y = \mathrm{Exp}\,(X) = \exp(X)(0)$. Then, by Corollary 1.2.10, $x \circ y = x \circ \exp(X)(0) = \exp(X)(x)$. This is (1.28). □

We end this section with the following important remark.

Remark 1.2.12. Let $\mathbb{G} = (\mathbb{R}^N, \circ)$ be a Lie group on $\mathbb{R}^N$ and let $Z_1, \dots, Z_N$ be the Jacobian basis of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$. For any differentiable function u defined on an open set $\Omega \subseteq \mathbb{R}^N$, we consider a sort of *intrinsic* gradient of u given by $(Z_1 u, \dots, Z_N u)$. Then, from (1.24) it follows that

$$(Z_1 u(x), \dots, Z_N u(x)) = \nabla u(x) \cdot \mathcal{J}_{\tau_x}(0) \qquad \forall\, x \in \Omega. \tag{1.29}$$

On the other hand, since $\mathcal{J}_{\tau_x}(0)$ is non-singular and its inverse is given by $\mathcal{J}_{\tau_{x^{-1}}}(0)$, we can write the Euclidean gradient of u in terms of its intrinsic gradient in the following way

$$\nabla u(x) = (Z_1 u(x), \dots, Z_N u(x)) \cdot \mathcal{J}_{\tau_{x^{-1}}}(0) \qquad \forall\, x \in \Omega. \tag{1.30}$$

From (1.30), we immediately obtain the following result. We shall follow the notation of Remark 1.2.12.

Proposition 1.2.13. *Let $\Omega \subseteq \mathbb{R}^N$ be an open connected set. A function $u \in C^1(\Omega, \mathbb{R})$ is constant in Ω if and only if its intrinsic gradient $(Z_1 u, \dots, Z_N u)$ vanishes identically on Ω.*

Proof. From (1.29) and (1.30), it follows that the intrinsic gradient of u vanishes at $x \in \Omega$ if and only if $\nabla u(x) = 0$. □

1.3 Homogeneous Lie groups on $\mathbb{R}^N$

A Lie group $\mathbb{G} = (\mathbb{R}^N, \circ)$ is a *homogeneous group* if the following property holds:

(H.1) *There exists an N-tuple of real numbers $\sigma = (\sigma_1, \dots, \sigma_N)$, with $1 \le \sigma_1 \le \ldots \le \sigma_N$, such that the* dilation

$$\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_\lambda(x_1, \dots, x_N) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_N} x_N)$$

is an automorphism of the group $\mathbb{G}$, for every $\lambda > 0$.

The family of dilations $\{\delta_\lambda\}_{\lambda>0}$ forms a group whose identity is $\delta_1 = I$, the identity of $\mathbb{R}^N$. Moreover, $(\delta_\lambda)^{-1} = \delta_{\lambda^{-1}}$. In the sequel, $\{\delta_\lambda\}_{\lambda>0}$ will be referred to as the dilation group of $\mathbb{G}$.

We shall denote by $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ a homogeneous Lie group with composition law $\circ$ and dilation group $\{\delta_\lambda\}_{\lambda>0}$. From (H.1) it follows that

$$\delta_\lambda(x \circ y) = (\delta_\lambda\, x) \circ (\delta_\lambda\, y), \qquad \forall\, x,\, y \in \mathbb{G} \tag{1.31}$$

and, if e denotes the identity of $\mathbb{G}$, $\delta_\lambda(e) = e$ for every $\lambda > 0$. This obviously implies that $e = 0$. This is consistent with our previous assumption that the origin is the identity of $\mathbb{G}$.

Before we continue the analysis of homogeneous Lie groups, we show some basic properties of homogeneous functions and homogeneous differential operators.

A real function a defined on $\mathbb{R}^N$ is called δ_λ-*homogeneous* of degree $m \in \mathbb{R}$ if, for every $x \in \mathbb{R}^N$ and $\lambda > 0$, it holds

$$a(\delta_\lambda\,(x)) = \lambda^m a(x).$$

A linear differential operator X is called δ_λ-*homogeneous* of degree $m \in \mathbb{R}$ if, for every $\varphi \in C^\infty(\mathbb{R}^N)$, $x \in \mathbb{R}^N$ and $\lambda > 0$, it holds

$$X\big(\varphi(\delta_\lambda(x))\big) = \lambda^m \big(X\varphi\big)(\delta_\lambda(x)).$$

Let a be a smooth δ_λ-homogeneous function of degree m and X be a differential operator δ_λ-homogeneous of degree n. Then Xa is a δ_λ-homogeneous function of degree $m - n$. Indeed, for every $x \in \mathbb{R}^N$ and $\lambda > 0$, we have

$$\lambda^n (Xa)(\delta_\lambda(x)) = X(a(\delta_\lambda(x))) = X(\lambda^m a(x)) = \lambda^m (Xa)(x).$$

Given a multi-index $\alpha \in \big(\mathbb{N} \cup \{0\}\big)^N$, $\alpha = (\alpha_1, \ldots, \alpha_N)$, we define the $\mathbb{G}$-*length* of α as

$$|\alpha|_{\mathbb{G}} = \langle \alpha, \sigma \rangle = \textstyle\sum_{i=1}^N \alpha_i\, \sigma_i.$$

Then, since $x \mapsto x_j$ and $\partial/\partial x_j$, $j \in \{1, \ldots, N\}$, are δ_λ-homogeneous of degree σ_j, the function $x \mapsto x^\alpha$ and the differential operator D^α are both δ_λ-homogeneous of degree $|\alpha|_{\mathbb{G}}$.

If a is a continuous function δ_λ-homogeneous of degree m and $a(x_0) \neq 0$ for some $x_0 \in \mathbb{R}^N$, then $m \geq 0$. Indeed, from $a(\delta_\lambda(x_0)) = \lambda^m a(x_0)$ we get

$$\lim_{\lambda \to 0} \lambda^m = \lim_{\lambda \to 0} \frac{a(\delta_\lambda(x_0))}{a(x_0)} = \frac{a(0)}{a(x_0)}.$$

Let us now consider a smooth function a δ_λ-homogeneous of degree m and a multi-index α, and assume $D^\alpha a$ is not identically zero. Then, since $D^\alpha a$ is smooth and δ_λ-homogeneous of degree $m - |\alpha|_{\mathbb{G}}$, it has to be $m - |\alpha|_{\mathbb{G}} \geq 0$, i.e., $|\alpha|_{\mathbb{G}} \leq m$. This result can be restated as follows:

$$D^\alpha a \equiv 0 \quad \forall\ \alpha:\ |\alpha|_{\mathbb{G}} > m.$$

Thus a is a *polynomial function*. Let $a(x) = \sum_{\alpha\in\mathcal{A}} a_\alpha\, x^\alpha$, where $\mathcal{A}$ is a finite set of multi-indices and $a_\alpha \in \mathbb{R}$ for every $\alpha \in \mathcal{A}$. Since a is δ_λ-homogeneous of degree m, we have

$$\textstyle\sum_{\alpha\in\mathcal{A}} \lambda^m\, a_\alpha\, x^\alpha = \lambda^m\, a(x) = a(\delta_\lambda(x)) = \sum_{\alpha\in\mathcal{A}} a_\alpha\, \lambda^{|\alpha|_{\mathbb{G}}}\, x^\alpha.$$

Hence $\lambda^m\, a_\alpha = \lambda^{|\alpha|_{\mathbb{G}}}\, a_\alpha$ for every $\lambda > 0$, so that $|\alpha|_{\mathbb{G}} = m$ if $a_\alpha \neq 0$. Then

$$a(x) = \sum_{|\alpha|_{\mathbb{G}}=m} a_\alpha\, x^\alpha. \tag{1.32}$$

It is quite obvious that every polynomial function of the form (1.32) is δ_λ-homogeneous of degree m. Thus, we have proved the following proposition.

Proposition 1.3.1. *Let $a \in C^\infty(\mathbb{R}^N, \mathbb{R})$. Then a is δ_λ-homogeneous of degree m iff a takes the form* (1.32).

From the proposition above one easily obtains the following characterization of the smooth δ_λ-homogeneous vector fields.

Proposition 1.3.2. *Let X be a smooth vector field on $\mathbb{R}^N$:*

$$X = \textstyle\sum_{j=1}^N a_j(x)\, \partial_{x_j}.$$

Then X is δ_λ-homogeneous of degree n iff a_j is a polynomial function δ_λ-homogeneous of degree $\sigma_j - n$.

Proof. A direct computation shows the "if" part of the proposition. Viceversa, if $X\big(\varphi\circ\delta_\lambda\big) = \lambda^n\,\big(X\,\varphi\big)\circ\delta_\lambda$, the choice $\varphi(x) = x_j$ yields $\lambda^{\sigma_j}\, a_j(x) = \lambda^n\, a_j(\delta_\lambda(x))$, whence a_j is a (smooth) δ_λ-homogeneous function of degree $\sigma_j - n$. By Proposition 1.3.1, a_j is a polynomial function. □

Corollary 1.3.3. *Let X be a smooth vector field. Then X is δ_λ-homogeneous of degree n iff*

$$\delta_\lambda\big(XI(x)\big) = \lambda^n\, XI(\delta_\lambda(x)).$$

Proof. Let $X = \sum_{j=1}^N a_j\, \partial_{x_j}$. By Proposition 1.3.2, X is δ_λ-homogeneous of degree n iff $a_j(\delta_\lambda(x)) = \lambda^{\sigma_j - n}\, a_j(x)$ for any $j \in \{1, \dots, N\}$. This is equivalent to say that

$$\begin{aligned}\delta_\lambda(XI(x)) &= \delta_\lambda\big(a_1(x), \dots, a_N(x)\big)^T = \big(\lambda^{\sigma_1} a_1(x), \dots, \lambda^{\sigma_N} a_N(x)\big)^T \\ &= \lambda^n\big(a_1(\delta_\lambda(x)), \dots, a_N(\delta_\lambda(x))\big)^T = \lambda^n\, XI(\delta_\lambda(x)).\end{aligned}$$

This ends the proof. □

As a straightforward consequence we have the following simple fact.

Remark 1.3.4. With the notation of the previous proposition, if a_j is not identically zero, then $n \le \sigma_j$. As a consequence, if $X \neq 0$, it has to be $n \le \sigma_N$ and

$$X = \sum_{j \le N,\, \sigma_j \ge n} a_j(x)\, \partial/\partial x_j.$$

Since a_j is a polynomial function of degree $\sigma_j - n$, if $n > 0$ then a_j *does not depend on* $x_j, \dots, x_N$:

$$a_j(x) = a_j(x_1, \dots, x_{j-1})$$

(we agree to let $a_j(x_1, \dots, x_{j-1}) = \text{constant}$ when $j = 1$).

From this remark the next proposition straightforwardly follows.

Proposition 1.3.5. *Let* $X = \sum_{j=1}^N a_j(x)\, \partial_{x_j}$ *be a smooth vector field* δ_λ*-homogeneous of degree* $n > 0$*. Then its adjoint* $X^* = -X$ *and*

$$X^2 = \operatorname{div}(A \cdot \nabla^T), \tag{1.33}$$

where A *is the square matrix* $(a_i\, a_j)_{i,j \le N}$.

Proof. By the previous remark, the coefficient a_j does not depend on x_j. Then, for every smooth function φ,

$$X^*\varphi = -\textstyle\sum_{j=1}^N \partial_j(a_j\, \varphi) = -\sum_{j=1}^N a_j\, \partial_j \varphi = -X\varphi.$$

Moreover

$$X^2 = \textstyle\sum_{i,j=1}^N a_i \partial_i (a_j\, \partial_j) = \sum_{i=1}^N \partial_i \big(\sum_{j=1}^N a_i\, a_j\, \partial_j \big) = \operatorname{div}(A \cdot \nabla^T),$$

where A is in the assertion. □

Vector fields with different degree of homogeneity are linearly independent, if they do not vanish at the origin. Indeed, the following proposition holds.

Proposition 1.3.6. *Let* $X_1, \dots, X_k \in T(\mathbb{R}^N)$ *be* δ_λ*-homogeneous vector fields of degree* $n_1, \dots, n_k$*, respectively. If* $n_i \neq n_j$ *for* $i \neq j$ *and if* $X_j I(0) \neq 0$ *for every* $j \in \{1, \dots, k\}$*, then* $X_1, \dots, X_k$ *are linearly independent.*

Proof. Let $c_1, \dots, c_k \in \mathbb{R}$ be such that $\sum_{j=1}^k c_j\, X_j = 0$. Then, for every smooth function φ

$$0 = \textstyle\sum_{j=1}^k c_j\, X_j(\varphi(\delta_\lambda x)) = \sum_{j=1}^k c_j\, \lambda^{n_j}\, (X_j \varphi)(\delta_\lambda x), \qquad \forall\, x \in \mathbb{R}^N.$$

If we take $\varphi(x) = \langle h, x \rangle = \sum_{j=1}^N h_j\, x_j$, this identity at $x = 0$ gives

$$0 = \textstyle\sum_{j=1}^k c_j\, \lambda^{n_j}\, \langle \eta_j, h \rangle, \qquad \forall\, h \in \mathbb{R}^N, \quad \forall\, \lambda > 0,$$

where $\eta_j = X_j I(0)$. Then $\sum_{j=1}^k c_j\, \lambda^{n_j}\, \eta_j = 0$ for all $\lambda > 0$, so that, since $n_i \neq n_j$ if $i \neq j$, $c_j\, \eta_j = 0$ for any $j \in \{1, \dots, k\}$. This implies $c_j = 0$ since $\eta_j \neq 0$ (for $j = 1, \dots, k$) by hypothesis. □

Corollary 1.3.7. *Let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$ and let $X_1, \ldots, X_k \in \mathfrak{g}$ be not-identically vanishing and δ_λ-homogeneous of degree $n_1, \ldots, n_k$, respectively. If $n_i \neq n_j$ for $i \neq j$, then $X_1, \ldots, X_k$ are linearly independent.*

Proof. Since $X_j I(x) = \mathcal{J}_{\tau_x} X_j I(0)$ for every $x \in \mathbb{R}^N$, and X_j is not-identically vanishing, then $X_j I(0) \neq 0$ for any $j \in \{1, \ldots, k\}$. Hence the assertion follows from the previous proposition. □

The following simple proposition will be useful in the sequel.

Proposition 1.3.8. *Let X_1, $X_2 \in \mathfrak{g}$ be δ_λ-homogeneous vector fields of degree n_1, n_2, respectively. Then $[X_1, X_2]$ is δ_λ-homogeneous of degree $n_1 + n_2$.*

Proof. It suffices to note that, for every smooth function φ on $\mathbb{R}^N$, one has

$$(X_1 X_2)(\varphi(\delta_\lambda(x))) = \lambda^{n_2} X_1((X_2\varphi)(\delta_\lambda(x))) = \lambda^{n_2+n_1} (X_1 X_2)(\varphi(\delta_\lambda(x))).$$

This ends the proof. □

By using the elementary properties of the homogeneous functions showed above, we shall obtain a structure theorem for the composition law in $(\mathbb{R}^N, \circ, \delta_\lambda)$. We first prove two lemmas.

Lemma 1.3.9. *Let $P : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a smooth function such that*

$$P(\delta_\lambda(x), \delta_\lambda(y)) = \lambda^{\sigma_j} P(x, y), \qquad \forall\, x, y \in \mathbb{R}^N,\ \forall\, \lambda > 0,$$

where $1 \le j \le N$. Assume also that

$$P(x, 0) = x_j, \quad P(0, y) = y_j \tag{1.34}$$

where $1 \le j \le N$. Then

$$P(x, y) = x_j + y_j + \widetilde{P}(x_1, \ldots, x_{j-1}, y_1, \ldots, y_{j-1}),$$

where $\widetilde{P}$ is a polynomial sum of mixed monomials in $x_1, \ldots, x_{j-1}, y_1, \ldots, y_{j-1}$. Moreover, $\widetilde{P}(\delta_\lambda(x), \delta_\lambda(y)) = \lambda^{\sigma_j} \widetilde{P}(x, y)$.

Proof. By Proposition 1.3.1, P is a polynomial function of the following type:

$$P(x, y) = \sum_{|\alpha|_{\mathbb{G}} + |\beta|_{\mathbb{G}} = \sigma_j} c_{\alpha,\beta}\, x^\alpha\, y^\beta, \qquad c_{\alpha,\beta} \in \mathbb{R}.$$

On the other hand, by (1.34),

$$x_j = P(x, 0) = \textstyle\sum_{|\alpha|_{\mathbb{G}} = \sigma_j} c_{\alpha,0}\, x^\alpha$$

and

$$y_j = P(0, y) = \textstyle\sum_{|\beta|_{\mathbb{G}} = \sigma_j} c_{0,\beta}\, y^\alpha.$$

Then
$$P(x,y) = x_j + y_j + \sum_{|\alpha|_{\mathbb{G}}+|\beta|_{\mathbb{G}}=\sigma_j,\ \alpha,\beta\neq 0} c_{\alpha,\beta}\, x^\alpha\, y^\beta.$$

We can complete the proof by noticing that the condition $|\alpha|_{\mathbb{G}} + |\beta|_{\mathbb{G}} = \sigma_j$, $\alpha, \beta \neq 0$ is empty when $j = 1$, whereas it implies $\alpha = (\alpha_1, \dots, \alpha_{j-1}, 0, \dots, 0)$, $\beta = (\beta_1, \dots, \beta_{j-1}, 0, \dots, 0)$ when $j \geq 2$. $\square$

Lemma 1.3.10. *Let $Q : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a smooth function such that*

$$Q(\delta_\lambda(x), \delta_\lambda(y)) = \lambda^m\, Q(x,y), \qquad \forall\ x, y \in \mathbb{R}^N,\ \forall\ \lambda > 0,$$

where $m \geq 0$. Then
$$x \mapsto \frac{\partial\, Q}{\partial\, y_j}(x, 0)$$
is δ_λ-homogeneous of degree $m - \sigma_j$.

Proof. By Proposition 1.3.9, Q is a polynomial of the following type

$$Q(x,y) = \sum_{|\alpha|_{\mathbb{G}}+|\beta|_{\mathbb{G}}=m} c_{\alpha,\beta}\, x^\alpha\, y^\beta.$$

Then, if we denote by e_j the j-th element of the canonical basis of $\mathbb{R}^N$, we have
$$\frac{\partial\, Q}{\partial\, y_j}(x,y) = \sum_{|\alpha|_{\mathbb{G}}+|\beta|_{\mathbb{G}}=m} c_{\alpha,\beta}\, \beta_j\, x^\alpha\, y^{\beta - e_j},$$
so that, since $|e_j|_{\mathbb{G}} = \sigma_j$,
$$\frac{\partial\, Q}{\partial\, y_j}(x,0) = \sum_{|\alpha|_{\mathbb{G}}=m-\sigma_j,\, \beta=e_j} c_{\alpha,\beta}\, x^\alpha.$$

This completes the proof. $\square$

Now, we are in the position to prove the previously mentioned structure theorem for the composition law.

Theorem 1.3.11. *Let $(\mathbb{R}^N, \circ, \delta_\lambda)$ be a homogeneous Lie group. Then $\circ$ has polynomial component functions. Furthermore we have*

$$(x \circ y)_1 = x_1 + y_1, \quad (x \circ y)_j = x_j + y_j + Q_j(x,y), \quad 2 \leq j \leq N$$

where

1. *Q_j only depends on $x_1, \dots, x_{j-1}$ and $y_1, \dots, y_{j-1}$;*
2. *Q_j is a sum of mixed monomials in x, y;*
3. *$Q_j(\delta_\lambda x, \delta_\lambda y) = \lambda^{\sigma_j}\, Q_j(x,y)$.*

Proof. Let $j \in \{1, \dots, N\}$ and define

$$P_j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}, \qquad P_j(x,y) = (x \circ y)_j.$$

Since δ_λ is an automorphism of $\mathbb{G}$, we have

$$P_j(\delta_\lambda(x), \delta_\lambda(y)) = (\delta_\lambda(x \circ y))_j = \lambda^{\sigma_j} (x \circ y)_j = \lambda^{\sigma_j}\, P_j(x,y).$$

Moreover, since $x \circ 0 = x$, $0 \circ y = y$, we have

$$P_j(x,0) = x_j, \quad P_j(0,y) = y_j.$$

Then, the proof follows from Lemma 1.3.9. $\square$

Corollary 1.3.12. *Let $(\mathbb{R}^N, \circ, \delta_\lambda)$ be a homogeneous Lie group. Then, we have*

$$\mathcal{J}_{\tau_x}(0) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_2^{(1)} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_N^{(1)} & \cdots & a_N^{(N-1)} & 1 \end{pmatrix} \tag{1.35}$$

where $a_i^{(j)}$ is a polynomial function δ_λ-homogeneous of degree $\sigma_i - \sigma_j$. As a consequence, if we let

$$Z_j = \partial_{x_j} + \textstyle\sum_{i=j+1}^N a_i^{(j)}\, \partial_{x_i} \quad \text{for } 1 \le j \le N-1 \text{ and } \quad Z_N = \partial_{x_N},$$

then Z_j is a left-invariant vector field δ_λ-homogeneous of degree σ_j. Moreover

$$\mathcal{J}_{\tau_x}(0) = \big(Z_1(x) \cdots Z_N(x)\big).$$

Proof. By Theorem 1.3.11, the Jacobian matrix $\mathcal{J}_{\tau_x}(0)$ takes the form (1.35) with

$$a_i^{(j)}(x) = \frac{\partial\, Q_i}{\partial\, y_j}(x,0).$$

Then, by Lemma 1.3.10, $a_i^{(j)}(x)$ is a polynomial function δ_λ-homogeneous of degree $\sigma_i - \sigma_j$. This proves the first part of the corollary. The second one follows from Proposition 1.3.2. $\square$

The structure theorem of the composition law in $(\mathbb{R}^N, \circ, \delta_\lambda)$ implies that the Lebesgue measure on $\mathbb{R}^N$ is invariant under left and right translations on $\mathbb{G}$. Indeed, by Theorem 1.3.11, the Jacobian matrices of the functions $x \mapsto \alpha \circ x$ and $x \mapsto x \circ \alpha$ have the following lower triangular form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \bigstar & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \bigstar & \cdots & \bigstar & 1 \end{pmatrix}.$$

Then, we can state the following proposition.

Proposition 1.3.13. *The Lebesgue measure on $\mathbb{R}^N$ is invariant with respect to the left and the right translations on $\mathbb{G}$.*

If we denote by $|E|$ the Lebesgue measure of a measurable set $E \subseteq \mathbb{R}^N$, we then have

$$|\alpha \circ E| = |E| = |E \circ \alpha| \quad \forall\, \alpha \in \mathbb{G}.$$

We also have that the Lebesgue measure is homogeneous with respect to the dilations $\{\delta_\lambda\}_{\lambda>0}$. More precisely, as a trivial computation shows,

$$|\delta_\lambda(E)| = \lambda^Q\, |E|,$$

where

$$Q = \textstyle\sum_{j=1}^N \sigma_j. \tag{1.36}$$

The positive number Q is called the *homogeneous dimension* of the group $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$.

Remark 1.3.14. From Corollary 1.3.12, we easily obtain the splitting of $\mathfrak{g}$ as direct sum of linear spaces spanned by vector fields of constant degree of homogeneity.

Let us denote by $n_1, \dots, n_r$ and $N_1, \dots, N_r$ real and natural numbers, respectively, such that

$$n_1 < n_2 < \ldots < n_r, \qquad N_1 + N_2 + \cdots + N_r = N,$$

defined by

$$\begin{cases} n_1 = \sigma_j & \text{for } 1 \le j \le N_1, \\ n_2 = \sigma_j & \text{for } N_1 < j \le N_1 + N_2, \\ \quad\vdots & \\ n_r = \sigma_j & \text{for } N_1 + \cdots + N_{r-1} < j \le N_1 + \cdots + N_{r-1} + N_r. \end{cases}$$

Let $Z_1, \dots, Z_N$ be the Jacobian basis of $\mathfrak{g}$, and define

$$\begin{aligned} &\mathfrak{g}_1 = \operatorname{span}\{Z_j \,|\, 1 \le j \le N_1\}, \qquad \text{and, for } i = 2, \dots, r \\ &\mathfrak{g}_i = \operatorname{span}\{Z_j \,|\, N_1 + \cdots + N_{i-1} < j \le N_1 + \cdots + N_{i-1} + N_i\}. \end{aligned}$$

By Corollary 1.3.12, the generators of $\mathfrak{g}_i$ are δ_λ-homogeneous vector fields of degree n_i, $1 \le i \le r$. Moreover, we obviously have

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r.$$

We also explicitly notice that, by Proposition 1.3.6, a vector field $X \in \mathfrak{g}$ is δ_λ-homogeneous of degree n iff, for a suitable $i \in \{1, \dots, r\}$, $n = n_i$ and $X \in \mathfrak{g}_i$.

In the next section, we shall deal with homogeneous groups in which $n_i = i$ for $1 \le i \le r$, and the *layer* $\mathfrak{g}_i$, $i \in \{1, \dots, r\}$, is generated by commutators of length i of vector fields in $\mathfrak{g}_1$.

The Exp map on the Lie algebra $\mathfrak{g}$ has some remarkable properties, due to the homogeneous structure of $\mathbb{G}$. We prove them in what follows.

Let $Z_1, \ldots, Z_N$ be the Jacobian basis of $\mathfrak{g}$. By Corollary 1.3.12, Z_j is δ_λ-homogeneous of degree σ_j and takes the form

$$Z_j = \textstyle\sum_{k=j}^N a_k^{(j)}(x_1, \ldots, x_{k-1})\, \partial_{x_k}, \tag{1.37}$$

where $a_k^{(j)}$ is a polynomial function δ_λ-homogeneous of degree $\sigma_k - \sigma_j$ and $a_j^{(j)} \equiv 1$. We now introduce on $\mathfrak{g}$ a dilation group, again denoted by $\{\delta_\lambda\}_{\lambda>0}$, defining

$$\delta_\lambda : \mathfrak{g} \longrightarrow \mathfrak{g}$$

as follows:

$$\delta_\lambda\big(\textstyle\sum_{j=1}^N \xi_j\, Z_j\big) := \textstyle\sum_{j=1}^N \lambda^{\sigma_j}\, \xi_j\, Z_j. \tag{1.38}$$

Remark 1.3.15. The dilation (1.38) is *consistent* with the one in $\mathbb{R}^N$. More precisely, if $Z \in \mathfrak{g}$ then

$$\delta_\lambda(ZI(x)) = (\delta_\lambda Z)I(\delta_\lambda(x)), \qquad \forall\, x \in \mathbb{R}^N. \tag{1.39}$$

We first check this identity in the case $Z = Z_j$, $j = 1, \ldots, N$. Since Z_j is homogeneous of degree σ_j, by Corollary 1.3.3, we have $\delta_\lambda(Z_jI(x)) = \lambda^{\sigma_j}(Z_jI)(\delta_\lambda(x))$ so that $\delta_\lambda(Z_jI(x)) = (\delta_\lambda Z_j)I(\delta_\lambda(x))$. Then, given $Z = \sum_{j=1}^N \xi_j\, Z_j \in \mathfrak{g}$, we have

$$\begin{aligned}\delta_\lambda(ZI(x)) &= \textstyle\sum_{j=1}^N \xi_j\, \delta_\lambda(Z_jI(x)) = \sum_{j=1}^N \xi_j\, \big((\delta_\lambda Z_j)I(\delta_\lambda(x))\big)\\ &= \big(\textstyle\sum_{j=1}^N \xi_j\, (\delta_\lambda Z_j)\big)I(\delta_\lambda(x)) = (\delta_\lambda Z)I(\delta_\lambda(x)).\end{aligned}$$

From the previous remark, we easily obtain the following lemma.

Lemma 1.3.16. *Let $\gamma : [0,T] \to \mathbb{R}^N$ be an integral curve of Z, with $Z \in \mathfrak{g}$. Then $\Gamma := \delta_\lambda(\gamma)$ is an integral curve of $\delta_\lambda(Z)$.*

Proof. Identity (1.39) gives

$$\dot{\Gamma} = \delta_\lambda(\dot{\gamma}) = \delta_\lambda(ZI(\gamma)) = (\delta_\lambda Z)I(\delta_\lambda(\gamma)) = (\delta_\lambda Z)I(\Gamma).$$

This ends the proof. □

We are now in the position to prove the following important theorem.

Theorem 1.3.17. *Let $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ be a homogeneous Lie group. Then* Exp $: \mathfrak{g} \to \mathbb{G}$ *and* Log $: \mathbb{G} \to \mathfrak{g}$ *are globally defined diffeomorphisms with polynomial components. Moreover, for every $Z \in \mathfrak{g}$ and $x \in \mathbb{G}$*

$$\mathrm{Exp}\,\big(\delta_\lambda(Z)\big) = \delta_\lambda(\mathrm{Exp}\,(Z)) \quad \textit{and} \quad \mathrm{Log}\,(\delta_\lambda(x)) = \delta_\lambda(\mathrm{Log}\,(x)). \tag{1.40}$$

Proof. Let $Z \in \mathfrak{g}$, $Z = \sum_{j=1}^N \xi_j Z_j$. From (1.37) we obtain

$$Z = \sum_{k=1}^N \big(\sum_{j=1}^k \xi_j \, a_k^{(j)}(x_1, \dots, x_{k-1}) \big) \partial_{x_k}. \tag{1.41}$$

Then, the first part of the theorem follows from Remark 1.1.1. In order to prove the first identity in (1.40), we consider the solution γ to the Cauchy problem

$$\dot\gamma = ZI(\gamma), \quad \gamma(0) = 0.$$

By the very definition of $\operatorname{Exp}(Z)$, we have $\gamma(1) = \operatorname{Exp}(Z)$. Let us put $\varGamma = \delta_\lambda(\gamma)$. By Lemma (1.3.16), $\varGamma$ is an integral curve of $\delta_\lambda(Z)$. Moreover $\varGamma(0) = \delta_\lambda(\gamma(0)) = \delta_\lambda(0) = 0$. Then $\varGamma(1) = \operatorname{Exp}(\delta_\lambda(Z))$, so that

$$\operatorname{Exp}(\delta_\lambda(Z)) = \varGamma(1) = \delta_\lambda(\gamma(1)) = \delta_\lambda(\operatorname{Exp}(Z)).$$

This proves the first identity in (1.40). The second one is trivially equivalent to the first one. □

The first part of this theorem together with (1.26) and Proposition 1.2.11 give the following corollary.

Corollary 1.3.18. *For every $x, y \in \mathbb{G}$ we have*

$$x \circ y = \exp(\operatorname{Log}(y))(x) \quad \textit{and} \quad x^{-1} = \operatorname{Exp}\big(-\operatorname{Log}(x)\big).$$

Remark 1.3.19. If Z is the vector field (1.41), then

$$ZI(x) = \big(\xi_1, \xi_2 + \xi_1 \, a_2^{(1)}(x_1), \dots, \xi_N + \sum_{j=1}^{N-1} a_N^{(j)}(x_1, \dots, x_{N-1})\big).$$

This implies (see (1.13))

$$\operatorname{Exp}(Z) = \exp(Z)(0) = \big(\xi_1, \xi_1 + B_2(\xi_1), \dots, \xi_N + B_N(\xi_1, \dots, \xi_{N-1})\big),$$

where the B_j's are suitable polynomial functions. Then, the Jacobian matrix of the map

$$(\xi_1, \dots, \xi_N) \mapsto \operatorname{Exp}(\xi_1 Z_1 + \cdots + \xi_N Z_N)$$

takes the following form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \bigstar & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \bigstar & \cdots & \bigstar & 1 \end{pmatrix}.$$

Thus, with respect to the Jacobian basis of $\mathfrak{g}$ and the canonical basis of $\mathbb{G}$, Exp *preserves the Lebesgue measure. The same property holds for the map* Log *since* Log $= (\operatorname{Exp})^{-1}$.