1. Some properties of the linear heat equation

The linear heat equation

$$\dot{u} = \triangle u, \quad u(0) = u_0$$

(e.g. on $\mathbb{R}^n$) defines a linear semigroup $\{T_t\}_{t \geq 0}$ on each L^p space, $p \in [1, +\infty]$. Moreover

- T_t is positivity preserving: $u_0 \geq 0 \Rightarrow T_t u_0 \geq 0 \ \forall t > 0$;

- T_t is a contraction on each L^p: $\|T_t u_0\|_p \leq \|u_0\|_p \ \forall u_0 \in \mathsf{L}^p$, $p \in [1, +\infty], t > 0$.

This corresponds, (via Beurling–Deny theory) to properties of the *energy functional* associated to the operator $-\triangle$:

$$\mathcal{E}(u) = \int |\nabla u|^2 \mathrm{d}x$$

which decreases under *normal contractions*: e.g.

$$\mathcal{E}((u \wedge 1) \vee 0) \le \mathcal{E}(u) \quad \forall u :$$

$\mathcal{E}$ is a *Dirichlet form*.

Finer L^p–L^q regularizing properties hold: if $q \ge p \in [1, +\infty]$ then

$$\|u(t)\|_q \le C_{p,q} \frac{\|u_0\|_p}{t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}}.$$

This is called *supercontractivity* if $q < +\infty$, *ultracontractivity* if $q = +\infty$.

These facts are related to stronger properties of the energy functional. e.g.:

• Sobolev or Nash inequalities;

• Logarithmic Sobolev inequalities (in principle weaker, valid also in some infinite dimensional setting).

Problems. 1) To prove, in a suitable nonlinear setting an analogue of the implication

$$\begin{array}{l}\text{Markov property}\\ \Updownarrow \\ \text{Beurling}--\text{Deny conditions.}\end{array}$$

2) To relate L^p–L^q regularization to Sobolev–type inequalities.

2. Nonlinear Dirichlet forms

(see F. Cipriani, G.G., J. reine angew. Math. (2003)).

Definition A (nonlinear) *Markov semigroup* $(T_t)_{t\geq 0}$ on $\mathcal{H} = \mathsf{L}^2(X, m)$ is a strongly continuous, nonexpansive semigroup on $\mathcal{H}$ which preserves order (i.e. $u \leq v \Rightarrow T_t u \leq T_t v$) and is non–expansive in the L^∞ norm (i.e. $\|T_t u - T_t v\|_\infty \leq \|u - v\|_\infty$).

Consequence: the semigroup is non expansive in each L^p.

In view of the Beurling–Deny theory, we shall consider those operators which "come from a functional". Let then

$$\mathcal{E} : \mathcal{H} \to (-\infty, +\infty]$$

be convex and lower semicontinuous in the strong topology of $\mathcal{H}$. Consider the semigroup associated to the subgradient ∂E of such a functional:

$$\mathcal{E}(u) - \mathcal{E}(v) \leq (\partial E(u), u - v) \quad \forall v \in \mathcal{H}.$$

It is well–known (Brezis) that such semigroups are strongly continuous and non–expansive.

A *sufficient condition* for Markovianity has been given by Benilan and Crandall.

Lemma 1. *The order preserving property for* (T_t) *is equivalent to the fact that*

$$T_t^{(2)}(u,v) := (T_t u, T_t v)$$

leaves invariant the closed and convex set:

$$C_1 := \{(u,v) \in \mathcal{H} \oplus \mathcal{H} : u \leq v\}.$$

The Markov property is equivalent to the fact that $T_t^{(2)}$ *leaves invariant the closed and convex sets*

$$C_2(\alpha) := \{(u,v) \in \mathcal{H} \oplus \mathcal{H} : \|u - v\|_\infty \leq \alpha\}.$$

Fundamental Lemma *A strongly continuous non–expansive semigroup associated to the convex l.s.c.* $\mathcal{E}$ *leaves invariant a closed and convex set* C *if and only if*

$$\mathcal{E}(\mathrm{Proj}_C(x)) \leq \mathcal{E}(x) \quad \forall x.$$

Theorem *A A strongly continuous non–expansive semigroup associated to the convex l.s.c.* $\mathcal{E}$ *is order preserving iff the functional* $\mathcal{E}^{(2)}(u,v) := \mathcal{E}(u) + \mathcal{E}(v)$ *satisfies*

$$\mathcal{E}^{(2)}(\mathrm{Proj}_{C_1}(u,v)) \leq \mathcal{E}^{(2)}(u,v).$$

It is non–expansive in L^∞ *iff,* $\forall \alpha > 0$*:*

$$\mathcal{E}^{(2)}(\mathrm{Proj}_{C_{2(\alpha)}}(u,v)) \leq \mathcal{E}^{(2)}(u,v).$$

We then call a functional as above a *nonlinear Dirichlet form*.

The projections can be calculated explicitly, e.g.:

$$\mathrm{Proj}_{C_1}(u,v) = \begin{cases} (u,v) & \text{if } u \leq v \\ \left(\dfrac{u+v}{2}, \dfrac{u+v}{2}\right) & \text{otherwise.} \end{cases}$$

3. Examples

3.1 The p–Laplacian. It is associated to

$$\mathcal{E}_p(u) = \int |\nabla u|^p \mathrm{d}x.$$

The generator is given formally by

$$Au = \mathrm{div}\,(|\nabla u|^{p-2}\nabla u).$$

Immediate generalization: the p–Laplacian *with measurable coefficients* on a manifold M, given formally in local coordinates by

$$H_p u := \sum_{i,j=1}^{d} \partial_i(a_{i,j}(\cdot)|\nabla u|^{p-2}\partial_j u)$$

((a) measurable, nonnegative and symmetric).
It is associated to the functional

$$\mathcal{E}_p^{(a)}(u) := \int_M a_x(\nabla u(x), \nabla u(x))|\nabla u(x)|_x^{p-2} m_g(\mathrm{d}x).$$

Theorem. *Under the local strict ellipticity condition*

$$a_x(v,v) \geq \lambda_K g_x(v,v)$$

valid $\forall K \subset\subset M$, $\forall x \in K, v \in TM$ *and a suitable* λ_K, $\mathcal{E}_p^{(a)}$ *is a nonlinear Dirichlet form.*

(generalizable to convex l.s.c functionals of the gradient).

• The Riemannian gradient can be generalized by "any" *derivation* (closable operator satisfying the Leibniz rule), possibly vector valued. Technically difficult. E.g.:

• *The subelliptic* p*–laplacian*. Let $\{X_i\}_{i=1}^m$ be a collection of closable vector fields on an open connected set or on a manifold. Let

$$|Xu|^2 = \sum_{i=1}^{m} |X_i u|^2.$$

Let finally

$$\mathcal{E}_X(u) = \int |Xu|^p \mathrm{d}x,$$

associated formally to

$$Au = \sum X_i^*(|Xu|^{p-2} X_i u).$$

It is a Dirichlet form. It can be generalized to the subriemannian p–Laplacian associated to a subriemannian structure on a manifold.

3.2 Γ–limits. The Dirichlet properties *pass to the* Γ*–limit*. If $\mathcal{E}_n$ is a sequence of convex functionals, it Γ–converges to $\mathcal{E}$ iff for all $(x_n) \subset \mathcal{H}$ with $x_n \to x$ in $\mathcal{H}$

$$\mathcal{E}(x) \leq \liminf \mathcal{E}_n(x_n)$$

and there exists at least on of such sequences s.t.

$$\mathcal{E}(x) = \lim \mathcal{E}_n(x_n).$$

Then:

$\mathcal{E}_n$ proper, convex (not nec. l.s.c.)

$+$contraction properties

$\Rightarrow \ \Gamma - \lim \mathcal{E}_n$ is a Dirichlet form.

This is true in particular for the relaxed functional $\text{sc}^-\mathcal{E}$:

$$\text{sc}^-\mathcal{E} := \sup\{G \text{ l.s.c.}, G \leq \mathcal{E}\}$$

of a convex functional satisfying the requested contraction properties. $\text{sc}^-\mathcal{E}$ *is then a Dirichlet form*.

Technical application: *closability*. If a convex l.s.c. functional satisfies the requested contraction properties on a dense subset of $\mathcal{H}$, it has an extension which is a Dirichlet form.

3.3 The perimeter functional. Let $\Omega \subset \mathbb{R}^n$ be bounded with Lipschitz boundary. Let

$$\mathcal{E}(u) = \int_\Omega |Du| \quad \forall u \in \mathsf{BV}(\Omega) \cap \mathsf{L}^2(\Omega)$$

where Du is the vector–valued Radon measure representing the distributional derivative of u. It is a nonlinear Dirichlet form, giving rise to the so called *total variation flow*. It is obtained as the relaxed functional of

$$\mathcal{E}^0(u) = \int_\Omega |\nabla u| \mathrm{d}x \quad u \in C^1(\Omega)$$

3.4 The area functional. Let Ω be bounded with Lipschitz boundary. Let

$$\mathcal{E}(u) = \int_\Omega \sqrt{1 + |Du|^2} + \int_{\partial\Omega} |\mathrm{tr}_\Omega u - \varphi| \mathrm{d}\mathcal{H}^{n-1}$$

on $\mathsf{BV}(\Omega) \cap \mathsf{L}^2(\Omega)$, with φ *continuous*. Then:

$$\int_\Omega \sqrt{1 + |Du|^2} := \sup \left\{ \int_\Omega \left(g_{n+1} - \sum_{i=1}^n u \partial_i g \right) \mathrm{d}x, \right.$$
$$\left. g = (g_1, \dots g_{n+1}) \in C_c^1(\Omega; \mathbb{R}^{n+1}) : |g| \leq 1, \right\}.$$

This is called the *area functional* with boundary contour φ. It is a Dirichlet form. Indeed the first quantity is obtained by relaxation of

$$\mathcal{E}_0(u) = \int_\Omega \sqrt{1+|Du|^2}, \quad u \in C^1(\Omega).$$

As for the second, for all Lipschitz function $p : \mathbb{R} \to \mathbb{R}$ and all $u \in BV(\Omega)$, then $p(u) \in BV(\Omega)$ and

$$\mathrm{tr}_\Omega(p(u)) = p(\mathrm{tr}_\Omega u).$$

4. Ultracontractivity and asymptotic behaviour

In the linear case:

Markov property
+ Sobolev (or log-Sobolev, or Nash)
+spectral theorem...
$\Rightarrow$ ultracontractivity.

Essential tools are lacking in the nonlinear case. We dealt with the problem in two *model* cases:

1) the evolution equation driven by the p–sublaplacian

$$\dot{u} = \triangle_{p,X} u := \sum_{i=1}^{n} X_i^*(|Xu|^{p-2} X_i u), \quad p > 2.$$

2) similar generalizations of the porous media equation.

Existing results only in nondegenerate, euclidean case. (Kamin, Vazquez, del Pino, Dolbeault, Andreucci, Aronsson, Vazquez, Kamin, Benilan, Friedman, Alikakos, Toscani...)

Our aim will be to show that ultracontractive estimates follow from one single assumption on the fields at hand: *a Sobolev–type inequality*.

Sample result: the p–sublaplacian on a compact manifold.

Step 1. *A logarithmic Sobolev inequality.*:

$$\int_M |f|^p \log\left(\frac{|f|}{\|f\|_p}\right)^p \, \mathrm{d}x$$
$$\leq \varepsilon \| |Xf| \|_p^p + \|f\|_p^p \, (C\varepsilon - \log \varepsilon) \, .$$

proved starting from a Sobolev inequality (to be assumed!) for the nonlinear Dirichlet form.

$$\mathcal{E}_p(u) := \| |Xf| \|_p^p.$$

Step 2. *Derive the norm w.r.t. a parameter.* Let

$$\overline{u} = \frac{1}{\mathrm{Vol}(M)} \int_M u \, \mathrm{d}m_g(x).$$

The quantity

$$y(s) := \log \|T_s u - \overline{u}\|_{r(s)}^{r(s)}$$

assume u bounded to start with and $r \in C^1$ differentiable w.r.t. s.

The derivative is what is expected. *The Markov property is crucial*. The derivative contains the energy functional and suitable "entropic terms" like

$$\int_M |f|^p \log \left(\frac{|f|}{\|f\|_p} \right)^p .$$

Step 3. *A differential inequality for* $y(s)$. Entirely different from the linear case. Use the log-Sobolev inequalities and further entropic bounds.

Integrating the inequality and using again the Markov property one gets, for bounded data, choosing $r(s) \to +\infty$ if $s \to t$:

$$\|T_t u - \overline{u}\|_\infty \leq \frac{c}{t^\alpha} \|u - \overline{u}\|_q^\beta \qquad (*)$$

for small times. An approximation argument allows to prove the L^q–L$^\infty$ regularizing property.

Step 4. One proves a polynomial L^q–L^q time decay The semigroup property and previous step allow to prove that $(*)$ holds for large times as well, with a different power of t.

The Markov property again allows to remove the boundedness assumption on the initial data.

Generalizations. 1) L^q–L^∞ Hölder continuity: if u and v *have the same mean value* then

$$\|T_t u - T_t v\|_\infty \leq \frac{c}{t^\alpha}\|u-v\|_q^\beta.$$

2) Dirichlet b.c. on bounded domains: similar bounds, but $T_t u \to 0$ uniformly. No difference if the b.c. are of Neumann type.

Conversely: suppose that a *hypercontractive* bound holds:

$$\|T_t u\|_{r(t)} \le K(t)\|u\|_2^{\beta(t)} \quad \forall t > 0, \forall u \in \mathsf{L}^2$$

with $r(0) = 2$, r increasing, $\beta(0) = 1$. This is in principle *much weaker* than the previous conditions. There are linear semigroups which are hypercontractive but not super or ultracontractive (e.g. the Ornstein–Uhlenbeck semigroup).

Assume that the generator H of (T_t) is subhomogeneous in the sense that there exists $p > 2$ s.t.

$$(H(\lambda u), \lambda u) \le M\lambda^p (Hu, u) \quad \forall u \in D(H)$$

and that it is positive in the sense that

$$(Hu, u) > 0 \quad \forall u \in D(H)$$

Then Nash type inequalities hold for a suitable $\vartheta \in (0, 1)$:

$$\|u\|_p \le C(Hu, u)^{\vartheta/p}\|u\|_1^{1-\vartheta}.$$

This is a striking *nonlinear* feature: in the linear case

$$\text{Nash} \Longleftrightarrow \text{Sobolev.}$$

In fact, under natural further assumptions on H (true e.g. for the p–sublaplacian), results by Bakry, Coulhon, Ledoux and Saloff–Coste show that a whole bunch of Gagliardo–Nirenberg inequalities then hold, including in particular the appropriate Sobolev one.