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Uniform Bounds for Solutions to Quasilinear Parabolic Equations

Fabio Cipriani

Dipartimento di Matematica, Politecnico di Milano,
Piazza Leonardo da Vinci 32, 20133 Milano, Italy
E-mail: fabcip@mate.polimi.it

and

Gabriele Grillo

Dipartimento di Matematica, Politecnico di Torino,
Corso Duca degli Abruzzi 24, 10129 Turin, Italy
E-mail: grillo@calvino.polito.it

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We consider a class of quasilinear parabolic equations whose model is the heat equation corresponding to the p-Laplacian operator, $\dot{u} = \Delta_p u := \sum_{i=1}^{d} \partial_i (|\nabla u|^{p-2} \partial_i u)$ with $p \in [2, d)$, on a domain $D \subset \mathbb{R}^d$ of finite measure. We prove that $|u(t, x)| \leqslant c\, |D|^{\alpha}\, t^{-\beta}\, \|u_0\|_r^{\gamma}$ for all $t > 0$, $x \in D$ and for all initial data $u_0 \in L^r(D)$, provided r is not smaller than a suitable r_0, where α, β, γ are positive constants explicitly computed in terms of d, p, r. The nonlinear cases associated with the case $p = 2$ display exactly the same contractivity properties which hold for the linear heat equation. We also show that the nonlinear evolution considered is contractive on any L^q space for any $q \in [2, +\infty]$.

Key Words: logarithmic Sobolev inequalities; quasilinear parabolic equations.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The purpose of the present paper is to prove global quantitative estimates for weak solutions to quasilinear parabolic equations of the form

$$\dot{u}(t, x) = \operatorname{div} \mathbf{a}(t, x, u(t, x), \nabla u(t, x)), \qquad t > 0 \tag{1.1}$$

on a domain $D \subset \mathbb{R}^d$ having finite measure. We consider initial data u_0 belonging to some $\mathrm{L}^q(D)$ space and solutions to (1.1) corresponding to Dirichlet boundary conditions.

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Concerning the structure function $\mathbf{a}\colon (0,+\infty)\times D\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$, $d\geqslant 3$, we assume that it satisfies a Caratheodory condition, that is that $\mathbf{a}(t,x,s,\xi)$ is measurable in (t,x) and continuous in (s,ξ), and that suitable structure conditions, of uniform elliptic type, hold true (see (1.9) below). The first model case is

$$\mathbf{a}^{(1)}(t,x,s,\xi)=|\xi|^{p-2}\,\xi$$

which corresponds to the nonlinear heat equation associated with the p-Laplacian operator

$$\dot{u}=\Delta_p u:=\operatorname{div}\,(|\nabla u|^{p-2}\,\nabla u) \tag{1.2}$$

with $2\leqslant p<d$. The second model case corresponds to the choice:

$$\mathbf{a}^{(2)}(t,x,s,\xi)=(|\xi_1|^{p-2}\,\xi_1,\,\ldots,\,|\xi_d|^{p-2}\,\xi_d). \tag{1.3}$$

The conditions required will be sufficiently general to consider, as a further example, the case

$$\mathbf{a}^{(3)}(t,x,s,\xi)=b(t,x,s,\xi)\,|\xi|^{p-2}\,\xi \tag{1.4}$$

with b Caratheodory and satisfying the pointwise conditions $c^{-1}\leqslant b(t,x,s,\xi)\leqslant c$ for a suitable positive constant c or similar modifications of $\mathbf{a}^{(2)}$.

By weak solution to Eq. (1.1) corresponding to the initial datum $u_0\in \mathrm{L}^2(D)$ we mean that $u\in \mathrm{L}^p((0,T);W^{1,p}_0(D))\cap C([0,T];\mathrm{L}^2(D))$ for any $T>0$ and that, for any positive and bounded test function

$$\varphi\in W^{1,2}(0,T;\mathrm{L}^2(D))\cap \mathrm{L}^p((0,T);W^{1,p}_0(D)),\qquad \varphi(T)=0,$$

one has:

(1.5)

$$\int_D u_0(x)\,\varphi(0,x)\,\mathrm{d}x=-\int_0^T\int_D u(t,x)\,\varphi'(t,x)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\int_D \mathbf{a}(t,x,u(t,x),\nabla u(t,x))\cdot\nabla\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

Let u be a weak solution to (1.1) corresponding to an initial datum $u_0=u(0)$. We aim at proving explicit global bounds for $\|u(t)\|_\infty$ depending on time t, on the Lebesgue measure of the domain D, on suitable L^r-norms of u_0, on d, on p and on the ellipticity constants involved.

The resulting bounds will be of the form

$$\|u(t)\|_\infty \leqslant C\frac{|D|^\alpha}{t^\beta}\|u(0)\|_r^\gamma \qquad \forall t>0 \tag{1.6}$$

and will be referred to, following the terminology of [DS], as *ultracontractive bounds*. Notice that similar *smoothing* properties are well-known for the linear heat equation. We find it remarkable that the ultracontractive bounds valid for the nonlinear cases associated with the choice $p=2$ (but with $\mathbf{a}(\cdot,\cdot,\cdot,\cdot)$ depending on *all* its variables) are *identical* to the bounds valid for the linear heat equations corresponding to uniformly elliptic second order operators in divergence form, that is to

$$\|u(t)\|_\infty \leqslant \frac{C}{t^{d/(2r)}}\|u(0)\|_r .$$

To the best of our knowledge no such bound seems to be present in the literature, although it is well-known (and will be used in the proofs) that, for the class of equations considered, the solutions corresponding to L^∞ data belong to L^∞ as well (see [CP]), no quantitatively precise bounds on the L^∞-norms of the solutions being known. Some *local* space-time bounds of a somewhat similar nature are given in [Di].

The basic idea is to show that, for any C^1 function $r(\cdot)$: $[0,t)\to[2,+\infty)$, the function $y(s):=\log(\|u(s,\cdot)\|_{r(s)})$ is differentiable and satisfies a suitable first order differential inequality whose coefficients depend only on $r(s)$, on the Lebesgue measure of the domain D, on d, on p and on the ellipticity constants. It will then be possible to integrate such a differential inequality so that, by an appropriate choice of $r(s)$, the above mentioned bounds will follow.

To arrive at such a differential inequality, the fundamental step (Proposition 2.1) will consist in showing that the usual Sobolev inequality

$$\|f\|_{pd/(d-p)} \leqslant c\,\|\nabla f\|_p ,$$

valid for any $f\in W_0^{1,p}(D)$, implies the validity of a new family of *energy-entropy* inequalities similar to the well-known Gross' *logarithmic Sobolev inequalities* (see [Gr] and the book of E. B. Davies [Da]) but involving the p-energy functional

$$Q_p(u) := \int_D |\nabla u|^p\,\mathrm{d}x \qquad u\in W^{1,p}(D) \tag{1.7}$$

naturally associated with the p-Laplacian operator: $Q_p(u) = -\int_D u\,\Delta_p u\,\mathrm{d}x$. We notice that an application of logarithmic Sobolev inequalities to the smoothing properties of solutions of nonlinear parabolic equations of Burgers's and Navier–Stokes type has been given by [CL].

We shall then prove that $y(s) := \log(\|u(s, \cdot)\|_{r(s)})$ is differentiable for $L^\infty(D)$ initial data and compute explicitly the derivative using the differential equation satisfied by u (Lemma 3.3); the derivative involves the p-energy and a (convex) entropy functional. By combining the ellipticity and growth assumptions, the logarithmic Sobolev inequalities and convexity arguments, we arrive at the above mentioned differential inequality (Lemma 3.7). These steps will use the fact (cf. [Di], [CP] and references quoted therein) that the solutions corresponding to L^∞ initial data are bounded as functions of space and time. The boundedness assumption on the initial datum is then removed by using the known space-time Hölder continuity for locally bounded solutions to Eq. (1.1) (cf. [Di] and references quoted therein) and standard results on the weak* topology of L^∞.

We want to stress that the proof of similar results in the linear case, for parabolic equations associated to uniformly elliptic linear operators in divergence form, relies heavily on the Spectral Theorem, on complex interpolation and on the theory of Markovian semigroups. None of these tools is presently available in the nonlinear setting, causing several technical problems and somewhat involved calculations. It should also be mentioned that, while the assumption $2 \leqslant d < p$ makes the discussion of the present paper close in some sense to the linear case, it will be shown elsewhere that results of a completely similar nature also hold when $p \geqslant d$.

We also stress that our bounds have an *a-priori* nature and in particular do not rely either upon existence results and monotonicity assumptions on the generator of the evolution considered or on the theory of nonlinear semigroups ([B2], [Sh], [BMP]).

Our second main result will concern, under the same assumptions of Theorem 1.1, some other contractivity properties for the evolution equations considered. In fact we prove that, for any $q \in [2, +\infty]$ and $t > 0$, the L^q-norm of the solution $u(t)$ is not greater than the corresponding L^q-norm of the initial datum $u(0)$. It is remarkable that the result for $q = +\infty$ follows straight from Theorem 1.1, thus reversing the usual method of proof valid in the linear case.

We shall discuss in a companion paper, in the framework of nonlinear semigroups associated with convex, lower semicontinuous functionals on Hilbert spaces ([B2], [Sh]), a definition of Markovianity and ultracontractivity for nonlinear semigroups and shall show there that such properties are related to contractivity properties of the generating functional.

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We now state our main results.

THEOREM 1.1. *Let $D \subset \mathbb{R}^d$, $d \geqslant 3$, be a domain of finite measure, $2 \leqslant p < d$, and $r_0 := 2 \vee [p(p-2)/(d-p)]$. Let also u be a weak solution to the equation*

$$\dot{u}(t, x) = \operatorname{div} \mathbf{a}(t, x, u(t, x), \nabla u(t, x)), \tag{1.8}$$

belonging to the space

$$\mathrm{L}^p((0, T); W_0^{1,p}(D)) \cap C(0, T; \mathrm{L}^2(D))$$

for all $T > 0$ and corresponding to the initial datum $u(0) \in L^{q_0}(D)$, with $q_0 \geqslant r_0$, where the Caratheodory function $\mathbf{a}$: $(0, +\infty) \times D \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is assumed to satisfy the following ellipticity and growth conditions

$$\begin{aligned} &\mathbf{a}(t, x, u, \xi) \cdot \xi \geqslant c_1 \, |\xi|^p \\ &|\mathbf{a}(t, x, u, \xi)| \leqslant c_2 \, |\xi u|^{p-1} \end{aligned} \tag{1.9}$$

almost everywhere, for some positive costants c_1, c_2.

Then u belongs to the space $L^\infty((\varepsilon, +\infty); L^\infty(D))$ for all $\varepsilon > 0$, and the following ultracontractive bound holds true,

$$\|u(t)\|_\infty \leqslant C \frac{|D|^\alpha}{t^\beta} \|u(0)\|_{q_0}^\gamma, \tag{1.10}$$

for all $t > 0$ and for a suitable constant $C = C(d, p, q_0)$, where, if $p \neq 2$,

$$\begin{aligned} \alpha &= \frac{d-p}{d}\left[1 - \left(\frac{q_0}{q_0 + p - 2}\right)^{d/p}\right] \\ \beta &= \frac{1}{p-2}\left[1 - \left(\frac{q_0}{q_0 + p - 2}\right)^{d/p}\right] \\ \gamma &= \left(\frac{q_0}{q_0 + p - 2}\right)^{d/p} \end{aligned} \tag{1.11}$$

and, if $p = 2$:

$$\|u(t)\|_\infty \leqslant \frac{C}{t^{d/(2q_0)}} \|u(0)\|. \tag{1.12}$$

THEOREM 1.2. *Under the assumptions of Theorem 1.1, the nonlinear evolution under discussion is L^q contractive in the sense that*

$$\|u(t, \cdot)\|_q \leqslant \|u(0, \cdot)\|_q \qquad \textit{for any} \quad t > 0, \quad q \in [2, +\infty]. \tag{1.13}$$

2. SOBOLEV AND LOGARITHMIC SOBOLEV INEQUALITIES

In this section we first collect some known results concerning the connection between Sobolev inequalities, logarithmic Sobolev inequalities and L^∞ bounds for solutions to *linear* parabolic equations. Then we prove that the classical Sobolev inequality

$$\|f\|_{pd/(d-p)} \leqslant c\, \|\nabla f\|_p \qquad (2 \leqslant p < d), \tag{2.1}$$

valid for $f \in W_0^{1,p}(D)$, $D \subset \mathbb{R}^d$ being an open domain, also implies the validity of a family of logarithmic Sobolev inequalities which will be crucial in the proof of our main result

It has been shown by E. B. Davies [Da] that a family of Gross logarithmic Sobolev inequalities of the form [Gr]

$$\begin{aligned} \int_D |f|^2 \log|f|\,\mathrm{d}x - \left(\int_D |f|^2\,\mathrm{d}x\right) \log\left(\int_D |f|^2\,\mathrm{d}x\right)^{1/2} \\ \leqslant \varepsilon \int_D |\nabla f|^2\,\mathrm{d}x + \beta(\varepsilon) \int_D |f|^2\,\mathrm{d}x, \end{aligned} \tag{2.2}$$

valid for all $\varepsilon > 0$ and for all f belonging to the Sobolev space $W_0^{1,2}(D)$, with

$$\beta(\varepsilon) := \frac{d}{4}\log\varepsilon + \frac{d}{4}\log\left(\frac{cd}{4}\right) \qquad (c > 0) \tag{2.3}$$

is equivalent to the ordinary Sobolev inequality

$$\|f\|_{2d/(d-2)} \leqslant c\, \|\nabla f\|_2 \tag{2.4}$$

for all such functions f, with the *same* value of the constant $c > 0$. E. B. Davies also showed that the above inequalities also imply the validity of a different family of logarithmic Sobolev inequalities, namely,

$$\begin{aligned} \int_D |f|^q \log|f|\,\mathrm{d}x - \left(\int_D |f|^q\,\mathrm{d}x\right) \log\left(\int_D |f|^q\,\mathrm{d}x\right)^{1/q} \\ \leqslant \varepsilon \int_D (H_q f)\, f_q\,\mathrm{d}x + 2\beta(\varepsilon)\, q^{-1} \int_D |f|^q\,\mathrm{d}x, \end{aligned} \tag{2.5}$$

where $f_q := \operatorname{sgn} f\, |f|^{q-1}$. The above inequality holds for any f belonging to the domain of the operator H_q obtained by closure, in the Banach space $L^q(D)$, of the operator $-\Delta f = -\sum_{i=1}^d \partial^2 f / \partial x_i^2$, initially defined on smooth compactly supported functions in D.

In turn, any of the above inequalities has been shown again by E. B. Davies and B. Simon [DS] to be equivalent to the ultracontractive bound for the heat semigroup generated by the *Dirichlet Laplacian* Δ_D.

$$\|e^{+t\Delta_D} f\|_\infty \leqslant \frac{\text{const.}}{t^{d/(2r_0)}} \|f\|_{r_0} \tag{2.6}$$

for all $f \in \mathrm{L}^{r_0}(D)$, $t > 0$, $r_0 \geqslant 1$. A final result which is of special importance is the fact that *each* of the above inequalities is equivalent to a Gaussian, off-diagonal bounds for the Dirichlet heat kernel K_D associated to the heat semigroup on D, in the form

$$K_D(t, x, y) \leqslant \frac{1}{(4\pi t)^{d/2}} \exp\left[-\frac{|x-y|^2}{4t}\right] \tag{2.7}$$

for all (t, x, y) in $\mathbb{R}^+ \times D \times D$. It is remarkable that similar estimates also hold for the solutions to parabolic equations associated to uniformly elliptic, second order, differential operators with measurable coefficients. In particular, bounds similar to (2.7) (with different constants) still hold.

Our goal in the present section will be to show that Sobolev inequalities in $W_0^{1,p}(D)$ imply a new family of logarithmic Sobolev inequalities involving the $W^{1,p}$ norm. In fact we have

PROPOSITION 2.1. *The logarithmic Sobolev inequality*

$$\begin{aligned} \int_D |f|^p \log |f|^p \, \mathrm{d}x - \left(\int_D |f|^p \, \mathrm{d}x\right) \log\left(\int_D |f|^p \, \mathrm{d}x\right) \\ \leqslant \frac{d}{p}\left[-\|f\|_p^p \log \varepsilon + \varepsilon c \int_D |\nabla f|^p \, \mathrm{d}x\right] \end{aligned} \tag{2.8}$$

holds true for any $\varepsilon > 0$ and for all $f \in W_0^{1,p}$, where $2 \leqslant p < d$, and c is the constant appearing in the Sobolev inequality (2.1).

Proof. By homogeneity and because $\|\nabla f\|_p = \|(\nabla |f|)\|_p$ for all $f \in W_0^{1,p}(D)$, it suffices to prove the claim for nonnegative functions $f \in W_0^{1,p}(D)$ such that $\|f\|_p = 1$. Define the probability measure $\mu(x) = f(x)^p \, \mathrm{d}x$. Then

$$\begin{aligned} \int_D f^p \log f \, \mathrm{d}x &\leqslant \frac{d}{p^2} \log \|f\|_{dp/(d-p)}^p \\ &\leqslant \frac{d}{p^2} (-\log \varepsilon + \varepsilon \|f\|_{dp/(d-p)}^p) \\ &\leqslant \frac{d}{p^2} (-\log \varepsilon + \varepsilon c \|\nabla f\|_p^p), \end{aligned}$$

where the first inequality follows from Jensen's inequality (since log is concave), the second one from the numerical inequality $\log t < t$ and the last one from the Sobolev inequality. ∎

We remark that in the following it will be crucial that the constant c appearing in the above lemma does not depend upon the domain.

3. PROOF OF THE MAIN RESULTS

Our strategy will be to consider first *essentially bounded* initial data u_0. The cut-off will be removed in the last steps. We comment that the following results will be first proved in the case $p > 2$: the simpler case $p = 2$ will be discussed at the end of the section.

LEMMA 3.1. *Let u be a weak solution of* (1.1), *corresponding to an essentially bounded initial datum. For any $r \geqslant 2$ consider the function $f_r : (0, \infty) \to (0, \infty)$ defined by*

$$f_r(t) = \int_D |u(t, x)|^r \, dx.$$

Then f_r is differentiable and

$$\dot{f}_r(t) = -r(r-1) \int_D |u(t, x)|^{r-2} \, \mathbf{a}(t, x, u(t, x), \nabla u(t, x)) \cdot \nabla u(t) \, dx. \tag{3.1}$$

Proof. We first notice that f_r is well-defined because the solution u is bounded in $D \times (0, +\infty)$ ([CP], [Di]), and because D has finite measure. Let us then recall the definition of the Lebesgue–Steklov average u_h of the solution u, for $h > 0$:

$$u_h(t, x) := \frac{1}{h} \int_t^{t+h} u(s, x) \, ds.$$

This function is well-defined by definition of weak solution and takes values in the Sobolev space $W^{1,p}(D)$. Moreover, it is differentiable in time for all $h > 0$, and its derivative equals $[u(t+h) - u(t)]/h$. Define now

$$g_h(t) = \int_D |u_h(t, x)|^r \, dx,$$

which exists by the above mentioned boundedness properties of the solution and since $|D| < +\infty$. Then, for all positive s, ε,

(3.2)

$$g_h(s+\varepsilon)-g_h(s)=\int_s^{s+\varepsilon} \dot{g}_h(t)\,\mathrm{d}t=\int_s^{s+\varepsilon} \mathrm{d}t\,\frac{d}{\mathrm{d}t}\int_D |u_h(t,x)|^r\,\mathrm{d}x$$

$$=\int_s^{s+\varepsilon} \mathrm{d}t\int_D \frac{d}{\mathrm{d}t}|u_h(t,x)|^r\,\mathrm{d}x$$

$$=r\int_s^{s+\varepsilon} \mathrm{d}t\int_D |u_h(t,x)|^{r-1}\,(\dot{u}_h(t,x))\,\mathrm{sgn}[u_h(t,x)]\,\mathrm{d}x$$

$$=-r\int_s^{s+\varepsilon} \mathrm{d}t\int_D (\mathbf{a}(t,x,u(t,x)\,\nabla u(t,x)))_h\,\nabla\,|u_h(t,x)|^{r-1})$$

$$\times\,\mathrm{sgn}\,u_h(t,x)\,\mathrm{d}x,$$

where we have used the fact [Di] that u_h satisfies the equation

$$\int_{D\times\{t\}} \{\dot{u}_h(t,x)\,\varphi(t,x)+[\mathbf{a}(t,x,u(t,x),\nabla u(t,x))]_h\cdot\nabla\varphi(t,x)\}\,\mathrm{d}x=0$$

for all t and for all non-negative $\varphi\in W_0^{1,p}(D)\cap \mathrm{L}^\infty_{\mathrm{loc}}(D))$. We have therefore chosen $\varphi(t)=|u_h(t)|^{r-1}\,\mathrm{sgn}\,u_h(t)$, which satisfies the above requirements because, as a function of the spatial variable, u is essentially bounded, it belongs to $W^{1,p}(D)$, it vanishes in the sense of traces on ∂D by definition and so, by [B1], it also belongs to $W_0^{1,p}(D)$. We have also used the fact that

$$\frac{d}{\mathrm{d}t}\int_D |u_h(t,x)|^r\,\mathrm{d}x=\int_D \frac{d}{\mathrm{d}t}|u_h(t,x)|^r\,\mathrm{d}x$$

because

$$\dot{u}_h(t,x)\,|u_h(t,x)|^{r-1}\,\mathrm{sgn}[u_h(t,x)]$$

$$=\frac{1}{h}[u(t+h,x)-u(t,x)]\,|u_h(t,x)|^{r-1}\,\mathrm{sgn}[u_h(t,x)]$$

for almost all t, x, and the absolute value of the r.h.s. is bounded by an integrable function of x, locally uniformly in t, by the above mentioned boundedness properties of u and because $|D|<\infty$. Finally, the last term in the r.h.s. of (3.2) makes sense because

$$\int_D (\mathbf{a}(t,x,u(t,x),\nabla u(t,x)))_h\cdot\nabla\,|u_h(t,x)|^{r-1})\,\mathrm{sgn}[u_h(t,x)]\,\mathrm{d}x$$

is locally integrable in time. In fact, by using the convexity of the norm function, Jensen's and Hölder inequalities, we obtain

$$\left|\int_D (\mathbf{a}(t,x,u(t,x),\nabla u(t,x)))_h \cdot \nabla[|u_h(t,x)|^{r-1}] \operatorname{sgn}[u_h(t,x)]\,dx\right|$$

$$= (r-1)\left|\int_D (\mathbf{a}(t,x,u(t,x),\nabla u(t,x)))_h \cdot \nabla(u_h(t,x))\,|u_h(t,x)|^{r-2}\,dx\right|$$

$$= (r-1)\left|\int_D (\mathbf{a}(t,x,u(t,x),\nabla u(t,x)))_h \cdot (\nabla u(t,x))_h\,(x)\,|u_h(t,x)|^{r-2}\,dx\right|$$

$$\leqslant (r-1)\int_D |(\mathbf{a}(t,x,u(t,x),\nabla u(t,x)))_h|\,|(\nabla u(t,x))_h|\,|u_h(t,x)|^{r-2}\,dx$$

$$\leqslant (r-1)\int_D |(\mathbf{a}(t,x,u(t,x),\nabla u_h(t,x)))|\,|\nabla u(t,x)|_h\,u_h(t,x)^{r-2}\,dx$$

$$\leqslant (r-1)\,\|u\|^{r-2}_{\mathrm{L}^\infty(D\times(\varepsilon,T))}\int_D |(\mathbf{a}(t,x,u(t,x),\nabla u(t,x)))|_h\,|(\nabla u(t,x)|_h\,dx$$

$$\leqslant C\,\|(|\nabla u(t)|^{p-1})_h\|_{p/(p-1)}\,\|(|\nabla u(t)|)_h\|_p$$

$$\leqslant \|\,|\nabla u(t)|^{p-1}\|_{p/(p-1)}\,\|\,|\nabla u(t)|\,\|_p$$

$$= \|\,|\nabla u(t)|\,\|_p^p,$$

where C depends on r, on the constant appearing in the growth condition $|\mathbf{a}(t,x,u,\xi)| \leqslant c_2\,|\xi|^{p-1}$ and on the $\mathrm{L}^\infty(D\times(0,T))$-norm of u. The latter function of t is locally integrable in t by the very definition of weak solution of the equation at hand.

Next we notice that $u_h(s)\to u(s)$ as $h\to 0$ in $\mathrm{L}^p(D)$ for all $p\geqslant 1$, s–a.e.. In fact, u is also locally integrable in time with values in $\mathrm{L}^p(D)$ and, by Lebesgue theorem, the statement follows.

We want to prove that f_r is differentiable and that its derivative has the form given by (3.1). To this end we have proved that g_h is differentiable and

(3.3)
$$\dot g_h(t) = -r\int_D (\mathbf{a}(t,x,u(t,x),\nabla u(t,x)))_h\,\nabla(|u_h(t,x)|^{r-1})\operatorname{sgn}[u_h(t,x)]\,dx$$

$$= -r(r-1)\int_D (\mathbf{a}(t,x,u(t,x),\nabla u(t,x)))_h\,\nabla u_h(t,x)\,|u_h(t,x)|^{r-2}\,dx.$$

Next we shall prove that, as $h\to 0$, $g_h\to f$ so that $\dot g_h\to\dot f$ in the sense of distributions. Moreover, since the convergence in the sense of distributions restricted to locally integrable functions coincides with the usual convergence in L^1_{loc}, the convergence in L^1_{loc} of both sides of (3.3) to the corresponding quantities in (3.1) implies the thesis.

We prove that $g_h \to f_r$ in the sense of distributions as $h \to 0$. In fact we have proved above that $u_h(t) \to u(t)$ in $L^p(D)$ a.e. in t and, by Jensen's inequality

$$\begin{aligned} |u_h(t,x)|^r &= \left| \frac{1}{h} \int_t^{t+h} u(s,x)\, ds \right|^r \\ &\leqslant \frac{1}{h} \int_t^{t+h} |u(s,x)|^r\, ds \\ &\leqslant \|u\|^r_{L^\infty(D\times(t,t+1))} \end{aligned}$$

for h sufficiently small. Dominated convergence can therefore be used to obtain that $g_h(t) \to f_r(t)$ for almost all t. Moreover, using again Jensen's inequality in the third step:

$$\begin{aligned} g_h(t) &= \int_D |u_h(t,x)|^r\, dx \\ &= \int_D dx \left| \frac{1}{h} \int_t^{t+h} u(s,x)\, ds \right|^r \\ &\leqslant \int_D dx \frac{1}{h} \int_t^{t+h} |u(s,x)|^r\, ds \\ &= \frac{1}{h} \int_t^{t+h} \|u(s)\|_r^r\, ds \\ &\leqslant |D|^r \|u\|_{L^\infty(D\times(t,t+1))} \end{aligned}$$

so that $g_h(t)$ is locally uniformly bounded as a function of t for h sufficiently small. Dominated convergence can therefore be used again to prove that, as $h \to 0$,

$$\int_{\mathbb{R}^+} g_h(s)\, \varphi(s)\, ds \to \int_{\mathbb{R}^+} f(s)\, \varphi(s)\, ds$$

for every test function belonging to $\mathscr{D}(\mathbb{R}^+)$. This means that $g_h \to f_r$ in the sense of distributions, as claimed. Then $\dot{g}_h \to \dot{f}_r$ in the sense of distributions as well.

Next we shall identify the limit, as $h \to 0$ of the r.h.s. of (3.3), as a function of t, for a.e. t. We first observe that $\nabla u_h = (\nabla u)_h$ for almost all t, x by dominated convergence, because ∇u is locally essentially bounded in (t, x) by [Di].

We want to prove that the r.h.s. of (3.3) converges as $h \to 0$, for almost all t, to

$$-r(r-1)\int_D \mathbf{a}(t, x, u(t, x), \nabla u(t, x)) \cdot \nabla u(t, x))\, |u(t, x)|^{r-2}\, \mathrm{d}x. \tag{3.4}$$

First notice that the last integral exists since $u(s) \in L^\infty(D)$ for almost all s and, by definition of weak solution, $u(s) \in W^{1,p}(D)$ for almost all s. Moreover, since $\nabla u_h = (\nabla u)_h$, we have, for almost all s:

$$\begin{aligned}\nabla(|u_h(s, x)|^{r-1}) &= (r-1)\, |u_h(s, x)|^{r-2}\, \nabla\, |u_h(s, x)|\\ &= (r-1)\, |u_h(s, x)|^{r-2}\, (\nabla u(s, x))_h\, \mathrm{sgn}[u_h(s, x)].\end{aligned}$$

Thus, we have to consider the following quantity (as a function of time):

$$k_h(s) := \int_D (\mathbf{a}(s, x, u(s, x), \nabla u(s, x)))_h\, (\nabla u(s, x))_h\, |u(s, x)|_h^{r-2}\, \mathrm{d}x.$$

By the growth assumption $|\mathbf{a}(x, u, \xi)| \leqslant c_2\, |\xi|^{p-1}$ and by the fact that $u \in W^{1,p}$, we can observe that the factor $\mathbf{a}$ belongs to $\mathrm{L}^{p/(p-1)}(D; \mathbb{R}^d)$; in addition $\nabla u(s)$ belongs to $\mathrm{L}^p(D; \mathbb{R}^d)$) and $u(s)$ is essentially bounded. Notice that $q = p/(p-1)$ is the conjugate exponent of p. Since the Steklov average of an L^s function g converges in L^s to g, an application of Hölder inequality implies that, for almost all s, one has, as $h \to 0$:

$$k_h(s) \to \int_D \mathbf{a}(s, x, u(s, x), \nabla u(s, x)) \cdot \nabla u(s, x)\, |u(s, x)|^{r-2}\, \mathrm{d}x.$$

Thus $\dot{f}_r(s)$ is a locally integrable function of time and equals, a.e.:

$$-r(r-1)\int_D \mathbf{a}(s, x, u(s, x), \nabla u(s, x)) \cdot \nabla u(s, x)\, |u(s, x)|^{r-2}\, \mathrm{d}x. \quad \blacksquare$$

LEMMA 3.2. *Let u be a weak solution to $\dot{u} = \mathrm{div}\, \mathbf{a}(t, x, u, \nabla u)$, under the running assumptions on $\mathbf{a}$, corresponding to an essentially bounded initial datum. Let also $r\colon [0, +\infty) \to [2, +\infty)$ be a nondecreasing C^1 function. Then*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}((\|u(s)\|_{r(s)})^{r(s)}) &= \dot{r}(s)\int_D |u(s, x)|^{r(s)} \log |u(s, x)|\, \mathrm{d}x\\ &\quad - r(s)(r(s)-1)\int_D \mathbf{a}(s, x, u(s, x), \nabla u(s, x))\\ &\quad \cdot \nabla u(s, x)\, |u(s, x)|^{r(s)-2}\, \mathrm{d}x.\end{aligned} \tag{3.5}$$

Proof. For any fixed $r \geqslant 2$ define $g(r, s) := \|u(s)\|_r^r$ for all $s > 0$. By the previous lemma we have:

$$\frac{\partial}{\partial s} g(r, s) = -r(r-1) \int_D \mathbf{a}(s, x, u(s, x), \nabla u(s, x)) \cdot \nabla u(s, x)\, |u(s, x)|^{r-2}\, \mathrm{d}x.$$

Moreover

$$\frac{\partial}{\partial r} g(r, s) = \frac{\partial}{\partial r} \int_D |u(s, x)|^r\, \mathrm{d}x = \int_D \frac{\partial}{\partial r} e^{r \log |u(s, x)|}\, \mathrm{d}x$$

$$= \int_D |u(s, x)|^r \log |u(s, x)|\, \mathrm{d}x.$$

Since $\|u(s)\|_{r(s)}^{r(s)} = g(r(s), s)$ we have:

$$\frac{\mathrm{d}}{\mathrm{d}s} ((\|u(s)\|_{r(s)})^{r(s)})$$

$$= \dot{r}(s) \frac{\partial}{\partial r} g(r(s), s)) + \frac{\partial}{\partial s} g(r(s), s)$$

$$= \dot{r}(s) \int_D |u(s, x)|^{r(s)} \log |u(s, x)|\, \mathrm{d}x - r(s)(r(s)-1)$$

$$\times \int_D \mathbf{a}(s, x, u(s, x), \nabla u(s, x)) \cdot \nabla u(s, x)\, |u(s, x)|^{r(s)-2}\, \mathrm{d}x. \quad \blacksquare$$

LEMMA 3.3. *Under the same assumptions of the previous lemma:*

(3.6) $$\frac{\mathrm{d}}{\mathrm{d}s} \log \|u(s)\|_{r(s)}$$

$$= \frac{\dot{r}(s)}{r(s)} \int_D \frac{|u(s, x)|^{r(s)}}{\|u(s)\|_{r(s)}^{r(s)}} \log \frac{|u(s, x)|}{\|u(s)\|_{r(s)}}\, \mathrm{d}x - \frac{(r(s)-1)}{\|u(s)\|_{r(s)}^{r(s)}}$$

$$\times \int_D \mathbf{a}(s, x, u(s, x), \nabla u(s, x)) \cdot \nabla u(s, x)\, |u(s, x)|^{r(s)-2}\, \mathrm{d}x.$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}s} \log \|u(s)\|_{r(s)}$$

$$= \frac{\mathrm{d}}{\mathrm{d}s} r(s)^{-1} \log \|u(s)\|_{r(s)}^{r(s)}$$

$$= -\frac{\dot{r}(s)}{r(s)^2} \log \|u(s)\|_{r(s)}^{r(s)} + r(s)^{-1} \frac{\mathrm{d}}{\mathrm{d}s} \log \|u(s)\|_{r(s)}^{r(s)}$$

$$
\begin{aligned}
&= -\frac{\dot r(s)}{r(s)} \log \|u(s)\|_{r(s)} + \frac{r(s)^{-1}}{\|u(s)\|_{r(s)}^{r(s)}} \frac{\mathrm{d}}{\mathrm{d}s} \|u(s)\|_{r(s)}^{r(s)} \\
&= -\frac{\dot r(s)}{r(s)} \log \|u(s)\|_{r(s)} + \frac{r(s)^{-1}}{\|u(s)\|_{r(s)}^{r(s)}} \\
&\quad \times \Big(\dot r(s) \int_D |u(s,x)|^{r(s)} \log |u(s,x)| \,\mathrm{d}x - r(s)(r(s)-1) \\
&\quad \times \int_D \mathbf{a}(s,x,u(s,x),\nabla u(s,x)) \cdot \nabla u(s,x)\, |u(s,x)|^{r(s)-2} \,\mathrm{d}x \Big) \\
&= \frac{\dot r(s)}{r(s)} \int_D \frac{|u(s,x)|^{r(s)}}{\|u(s)\|_{r(s)}^{r(s)}} \log \frac{|u(s,x)|}{\|u(s)\|_{r(s)}} \,\mathrm{d}x - \frac{(r(s)-1)}{\|u(s)\|_{r(s)}^{r(s)}} \\
&\quad \times \int_D \mathbf{a}(s,x,u(s,x),\nabla u(s,x)) \cdot \nabla u(s,x)\, |u(s,x)|^{r(s)-2} \,\mathrm{d}x. \quad \blacksquare
\end{aligned}
$$

LEMMA 3.4. *Let us define, for any $p \geqslant 1$, $v \in W^{1,p}(D)$,*

$$
Q_p(v) := \int_D |\nabla v|^p \,\mathrm{d}x. \tag{3.7}
$$

Then, for any function $u(s,x)$ which, for almost all s belongs to $\mathrm{L}^\infty(D) \cap W^{1,p}(D)$, the following inequality holds true, for all $r \geqslant 2$,

$$
\begin{aligned}
&c_1 \left(\frac{p}{r+p-2} \right)^p Q_p(|u(s)|^{(r+p-2)/p}) \\
&\quad \leqslant \int_D \mathbf{a}(s,x,u(s,x),\nabla u(s,x)) \cdot \nabla u(s,x)\, |u(s,x)|^{r-2} \,\mathrm{d}x,
\end{aligned} \tag{3.8}
$$

where c_1 is the constant of strict ellipticity appearing in the Assumption.

Proof. We compute

$$
\begin{aligned}
&\int_D \mathbf{a}(s,x,u(s,x),\nabla u(s,x)) \cdot \nabla u(s,x)\, |u(s,x)|^{r-2} \,\mathrm{d}x \\
&\qquad \geqslant c_1 \int_D |\nabla u(s,x)|^p \, |u(s,x)|^{r-2} \,\mathrm{d}x \\
&\qquad = c_1 \left(\frac{p}{r+p-2} \right)^p \int_D |\nabla u(s,x)^{(r+p-2)/p}|^p \,\mathrm{d}x
\end{aligned}
$$

which is the above statement. $\blacksquare$

LEMMA 3.5. *Under the assumption of Lemma 3.2 the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} \leqslant \frac{\dot r(s)}{r(s)}\int_D \frac{|u(s,x)|^{r(s)}}{\|u(s)\|_{r(s)}^{r(s)}}\log\frac{|u(s,x)|}{\|u(s)\|_{r(s)}}\,\mathrm{d}x - c_1\left(\frac{p}{r+p-2}\right)^p\frac{(r(s)-1)}{\|u(s)\|_{r(s)}^{r(s)}}Q_p(|u(s)|^{(r(s)+p-2)/p}) \tag{3.9}$$

holds true.

Proof. It suffices to combine Lemmata 3.2 and 3.4. ∎

Let us define $X := \bigcap_{p\geqslant 1}\mathrm{L}^p$. Then, let us consider the Young functional $J\colon [1,+\infty)\times X\to[0,+\infty]$ as follows:

$$J(q,u) := \int_D \frac{|u|^q}{\|u\|_q^q}\log\left(\frac{|u|}{\|u\|_q}\right).$$

From the previous lemma and from the logarithmic Sobolev inequalities proved above starting from ordinary Sobolev inequalities we obtain the following result.

LEMMA 3.6. *Under the assumptions of Lemma 3.2 the following inequality holds true, for any $\varepsilon>0$:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} &\\ \leqslant \frac{r'(s)}{r(s)}J(r(s),u(s)) &\\ -c_1(r(s)-1)\left(\frac{p}{r(s)+p-2}\right)^{p-1}&\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}\\ \times\left(\frac{p^2}{cd\varepsilon}J(r(s)+p-2,u(s))+\frac{p}{r(s)+p-2}\frac{\log\varepsilon}{c\varepsilon}\right)&.\end{aligned} \tag{3.10}$$

Proof. The lemma is proved by combining Lemma 3.5 with Proposition 2.1, which is used choosing $f=|u(s)|^{r(s)+p-2}$; this is possible because p and r are not smaller then 2 and because $u(s)$ is both bounded and belonging to $W_0^{1,p}$. ∎

LEMMA 3.7. *Under the assumptions of Lemma 3.2, define* $r_0 := 2 \vee [p(p-2)/(d-p)]$. *Suppose also that* r *is a* C^1 *function from* $[0, +\infty)$ *to* $[r_0, +\infty)$. *Then, for any* $s > 0$, *the inequality*

(3.11)
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} \\ &\leqslant -\frac{d}{p}\frac{\dot r(s)}{r(s)}\frac{p-2}{r(s)+p-2}\log(\|u(s)\|_{r(s)}) \\ &+\frac{\dot r(s)}{r(s)}\frac{p-2}{r(s)+p-2}\left(\frac{d}{p}-1\right)\log|D| \\ &-\frac{d}{p}\frac{\dot r(s)}{r(s)}\frac{1}{r(s)+p-2}\log\left[\frac{c_1 p^2}{cd}\frac{r(s)(r(s)-1)}{\dot r(s)}\left(\frac{p}{r(s)+p-2}\right)^{p-1}\right]\end{aligned}$$

holds true.

Proof. In inequality (3.10), let us choose

$$\varepsilon(s) = \frac{c_1 p^2}{cd}\frac{r(s)(r(s)-1)}{\dot r(s)}\left(\frac{p}{r(s)+p-2}\right)^{p-1}\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}.$$

The above mentioned inequality then becomes

(3.12)
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} &\leqslant \frac{\dot r(s)}{r(s)}[J_{r(s)}(u(s)) - J_{r(s)+p-2}(u(s))] \\ &\quad -c_1(r(s)-1)\left(\frac{p}{r(s)+p-2}\right)^p\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}\frac{\log\varepsilon(s)}{c\varepsilon(s)}.\end{aligned}$$

Consider the funcion $N\colon [1, +\infty) \times X \to \mathbb{R}$ by

$$N(q, u) := \log\|u\|_q^q.$$

For every fixed $u \in X$ this is a convex function of q so that its derivative exists a.e. and

$$\frac{\mathrm{d}}{\mathrm{d}q}N(q, u) = \int_D \frac{|u|^q}{\|u\|_q^q}\log|u|\,\mathrm{d}x = J_q(u) + \log\|u\|_q$$

a.e. By convexity, the above derivative is a monotonically nondecreasing function. Thus, for any $q_1 \leqslant q_2$:

$$J(q_1, u) - J(q_2, u) = \frac{\mathrm{d}}{\mathrm{d}q} N(q, u)\big|_{q_1} - \log \|u\|_{q_1} - \left(\frac{\mathrm{d}}{\mathrm{d}q} N(q, u)\big|_{q_2} - \log \|u\|_{q_2}\right)$$
$$\leqslant \log \frac{\|u\|_{q_2}}{\|u\|_{q_1}}.$$

Using this inequality in (3.12) yields, by recalling that $p \geqslant 2$ so that $r(s)+p-2$ is not smaller than $r(s)$ for all s:

$$\frac{\mathrm{d}}{\mathrm{d}s} \log \|u(s)\|_{r(s)} \leqslant \frac{\dot{r}(s)}{r(s)} \log \frac{\|u\|_{r(s)+p-2}}{\|u\|_{r(s)}} - c_1(r(s)-1) \times \left(\frac{p}{r(s)+p-2}\right)^p \frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}} \frac{\log \varepsilon(s)}{c\varepsilon(s)}. \tag{3.13}$$

The last term in (3.13) becomes, with the present choice of $\varepsilon(s)$:

$$\begin{aligned} c_1(r(s)-1) &\left(\frac{p}{r(s)+p-2}\right)^p \frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}} \frac{\log \varepsilon(s)}{c\varepsilon(s)} \\ &= c_1(r(s)-1) \left(\frac{p}{r(s)+p-2}\right)^p \frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}} \frac{cd\dot{r}(s)}{cc_1 p^2 r(s)(r(s)-1)} \\ &\quad\times \left(\frac{r(s)+p-2}{p}\right)^{p-1} \frac{\|u(s)\|_{r(s)}^{r(s)}}{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}} \log \varepsilon(s) \\ &= \frac{d}{p^2} \frac{\dot{r}(s)}{r(s)} \left(\frac{p}{r(s)+p-2}\right) \\ &\quad\times \log \left[\frac{c_1 p^2}{cd} \frac{r(s)(r(s)-1)}{\dot{r}(s)} \left(\frac{p}{r(s)+p-2}\right)^{p-1} \frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}\right]. \end{aligned}$$

From the latter formulas we get:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} \log \|u(s)\|_{r(s)} \leqslant{}& \frac{\dot{r}(s)}{r(s)} \log \frac{\|u\|_{r(s)+p-2}}{\|u\|_{r(s)}} \\ &- \frac{d}{p} \frac{\dot{r}(s)}{r(s)} \frac{1}{r(s)+p-2} \log \frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}} \\ &- \frac{d}{p} \frac{\dot{r}(s)}{r(s)} \frac{1}{r(s)+p-2} \\ &\times \log \left[\frac{c_1 p^2}{cd} \frac{r(s)(r(s)-1)}{\dot{r}(s)} \left(\frac{p}{r(s)+p-2}\right)^{p-1}\right]. \end{aligned} \tag{3.14}$$

Now we proceed estimating the logarithm of the ratio of L^q norms appearing above. In fact, since

$$\|u(s)\|_{r(s)+p-2} \geqslant |D|^{-(p-2)/[r(s)(r(s)+p-2)]} \, \|u(s)\|_{r(s)} \tag{3.15}$$

the following calculations hold:

$$\begin{aligned}
\log \frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}} &= \log \left[\frac{\|u(s)\|_{r(s)+p-2}^{r(s)}}{\|u(s)\|_{r(s)}^{r(s)}} \, \|u(s)\|_{r(s)+p-2}^{p-2} \right] \\
&= r(s) \log \frac{\|u(s)\|_{r(s)+p-2}}{\|u(s)\|_{r(s)}} + (p-2) \log \|u(s)\|_{r(s)+p-2} \, .
\end{aligned} \tag{3.16}$$

From (3.14), (3.15) and (3.16) we get, recalling also that r is nondecreasing and that $r(0) \geqslant r_0$ with r_0 as in the statement

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s} \log \|u(s)\|_{r(s)} \\
&\quad \leqslant \frac{\dot r(s)}{r(s)} \left[1 - \frac{d}{p} \, \frac{r(s)}{r(s)+p-2} \right] \log \frac{\|u(s)\|_{r(s)+p-2}}{\|u(s)\|_{r(s)}} \\
&\qquad - \frac{d}{p} \, \frac{\dot r(s)}{r(s)} \, \frac{p-2}{r(s)+p-2} \log \left(\|u(s)\|_{r(s)+p-2} \right) \\
&\qquad - \frac{d}{p} \, \frac{\dot r(s)}{r(s)} \, \frac{1}{r(s)+p-2} \log \left[\frac{c_1 p^2}{cd} \, \frac{r(s)(r(s)-1)}{\dot r(s)} \left(\frac{p}{r(s)+p-2} \right)^{p-1} \right] \\
&\quad \leqslant \frac{\dot r(s)}{r(s)} \left[1 - \frac{d}{p} \, \frac{r(s)}{r(s)+p-2} \right] \log\left(|D|^{-(p-2)/(r(s)+p-2)} \right) \\
&\qquad - \frac{d}{p} \, \frac{\dot r(s)}{r(s)} \, \frac{p-2}{r(s)+p-2} \log\left(|D|^{-(p-2)/(r(s)+p-2)} \, \|u(s)\|_{r(s)} \right) \\
&\qquad - \frac{d}{p} \, \frac{\dot r(s)}{r(s)} \, \frac{1}{r(s)+p-2} \log \left[\frac{c_1 p^2}{cd} \, \frac{r(s)(r(s)-1)}{\dot r(s)} \left(\frac{p}{r(s)+p-2} \right)^{p-1} \right]
\end{aligned} \tag{3.17}$$

$$\leqslant -\frac{d}{p}\frac{\dot r(s)}{r(s)}\frac{p-2}{r(s)+p-2}\log(\|u(s)\|_{r(s)})$$

$$+\frac{\dot r(s)}{r(s)}\frac{p-2}{r(s)+p-2}\left(\frac{d}{p}-1\right)\log|D|$$

$$-\frac{d}{p}\frac{\dot r(s)}{r(s)}\frac{1}{r(s)+p-2}\log\left[\frac{c_1 p^2}{cd}\frac{r(s)(r(s)-1)}{\dot r(s)}\left(\frac{p}{r(s)+p-2}\right)^{p-1}\right]$$

as stated. ∎

From the previous lemma and from elementary calculus considerations it is easy to obtain our next result.

PROPOSITION 3.8. *Define the following functions of the time variable* $s>0$*:*

$$y(s)=\log\|u(s)\|_{r(s)};$$

$$p(s)=\frac{d}{p}\frac{\dot r(s)}{r(s)}\frac{p-2}{r(s)+p-2};$$

$$q(s)=-\frac{\dot r(s)}{r(s)}\frac{p-2}{r(s)+p-2}\left(\frac{d}{p}-1\right)\log|D|+\frac{d}{p}\frac{\dot r(s)}{r(s)}\frac{1}{r(s)+p-2}\times\log\left[\frac{c_1 p^2}{cd}\frac{r(s)(r(s)-1)}{\dot r(s)}\left(\frac{p}{r(s)+p-2}\right)^{p-1}\right]. \tag{3.18}$$

Then y *satisfies the following differential inequality:*

$$\dot y(s)+p(s)\,y(s)+q(s)\leqslant 0\qquad \forall s\geqslant 0.$$

Thus $y(s)\leqslant \bar y(s)$*, provided* $y(0)\leqslant \bar y(0)$*, where*

$$\bar y(s)=\exp\left[-\int_0^s p(u)\,du\right]\left(\bar y(0)-\int_0^s q(u)\exp\left[\int_0^u p(v)\,dv\right]\right) \tag{3.19}$$

is a solution of the ordinary differential equation

$$\dot z(s)+p(s)\,z(s)+q(s)=0\qquad \forall s\geqslant 0.$$

LEMMA 3.9. *Let us fix* $t>0$, $q_0\geqslant r_0$ *where* r_0 *is as in Lemma 3.7. Then the solution* $\bar y$ *to Eq. (3.19) with the choice* $r(s)=q_0 t/(t-s)$ *satisfies*

$$
\begin{aligned}
(3.20)\qquad w(t) &:= \lim_{s\to t^-} \bar{y}(s) \\
&= \left(\frac{q_0}{q_0+p-2}\right)^{d/p}\left[\bar{y}(0)-\frac{1}{p-2}\left(\left(\frac{q_0+p-2}{q_0}\right)^{d/p}-1\right)\log t\right. \\
&\quad \left.-\frac{d-p}{d}\left[\left(\frac{q_0+p-2}{q_0}\right)^{d/p}-1\right]\log|D|+K\right],
\end{aligned}
$$

where K depends only upon r_0, p and d.

Proof. We apply the above Proposition with the choice $r(s) = q_0 t/(t-s)$ for all $s \in [0, t)$, so that we also have $\dot{r}(s)/r(s) = 1/(t-s)$. By the explicit expression for $p(s)$ and $q(s)$ and by the present choice of the function $r(s)$, we have

$$
p(v) = \frac{d(p-2)}{p}\,\frac{1}{tq_0+(t-v)(p-2)}
$$

$$
q(v) = \sum_{i=1}^{5} q_i(v),
$$

where

$$
\begin{aligned}
q_1(v) &= \frac{d}{p}[tq_0+(t-v)(p-2)]^{-1}\log\left(\frac{c_1 p^{p+1}}{cd}\right) \\
q_2(v) &= \frac{d}{p}(p-1)[tq_0+(t-v)(p-2)]^{-1}\log(t-v) \\
q_3(v) &= \frac{d}{p}[tq_0+(t-v)(p-2)]^{-1}\log[tq_0-(t-v)] \\
q_4(v) &= -\frac{d}{p}(p-1)[tq_0+(t-v)(p-2)]^{-1} \\
&\quad \times\log[tq_0+(t-v)(p-2)] \\
q_5(v) &= \frac{p-d}{p}\,\frac{p-2}{tq_0+(t-v)(p-2)}\log|D|.
\end{aligned}
$$

Thus:

$$
\begin{aligned}
(3.21)\qquad \int_0^s p(v)\,\mathrm{d}v &= \frac{d(p-2)}{p}\int_0^s \frac{1}{tq_0+(t-v)(p-2)}\,\mathrm{d}v \\
&= \frac{d}{p}\log\frac{t((q_0+p-2)}{tq_0+(t-s)(p-2)}.
\end{aligned}
$$

We then have

$$
\begin{aligned}
&\int_0^t \mathrm{d}v\, q(v) \exp\left[\int_0^v p(w)\,\mathrm{d}w\right] \\
(3.22)\qquad &= \int_0^t \mathrm{d}v\, q(v) \left[\frac{t(q_0+p-2)}{tq_0+(t-v)(p-2)}\right]^{d/p} = \sum_{i=1}^{5} I_i,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \frac{d}{p}[t(q_0+p-2)]^{d/p} \log\left[\frac{c_1 p^{p+1}}{cd}\right] \int_0^t \mathrm{d}v[tq_0+(t-v)(p-2)]^{-1-(d/p)} \\
I_2 &= \frac{d}{p}[t(q_0+p-2)]^{d/p}(p-1) \int_0^t \mathrm{d}v[tq_0+(t-v)(p-2)]^{-1-(d/p)} \log(t-v) \\
I_3 &= \frac{d}{p}[t(q_0+p-2)]^{d/p} \int_0^t \mathrm{d}v[tq_0+(t-v)(p-2)]^{-1-(d/p)} \log[tq_0-(t-v)] \\
I_4 &= -\frac{d}{p}[t(q_0+p-2)]^{d/p}(p-1) \\
&\quad \times \int_0^t \mathrm{d}v[tq_0+(t-v)(p-2)]^{-1-(d/p)} \log[tq_0+(t-v)(p-2)] \\
I_5 &= -\left[(p-2)\left(\frac{d}{p}-1\right)\log|D|\right][t(q_0+p-2)]^{d/p} \\
&\quad \times \int_0^t \mathrm{d}v[tq_0+(t-v)(p-2)]^{-1-(d/p)}.
\end{aligned}
$$

By elementary calculation this yields the following results

$$
\begin{aligned}
I_1 &= \frac{1}{p-2}\left(\left(\frac{q_0+p-2}{q_0}\right)^{d/p}-1\right)\log\left(\frac{c_1 p^{p+1}}{cd}\right) \\
I_2 &= \left(\frac{p-1}{p-2}\right)\left(\left(\frac{q_0+p-2}{q_0}\right)^{d/p}-1\right)\log t + R_2 \\
I_3 &= \left(\frac{1}{p-2}\right)\left(\left(\frac{q_0+p-2}{q_0}\right)^{d/p}-1\right)\log t + R_3 \\
I_4 &= -\left(\frac{p-1}{p-2}\right)\left(\left(\frac{q_0+p-2}{q_0}\right)^{d/p}-1\right)\log t + R_4 \\
I_5 &= -\frac{d-p}{d}\left[\left(\frac{q_0+p-2}{q_0}\right)^{d/p}-1\right]\log|D|,
\end{aligned}
$$

where the terms R_2, R_3, R_4 depend only upon d, p and q_0 and are independent from t and $|D|$. In fact their explicit values are:

$$R_2 = \frac{d}{p}(p-1)(q_0+p-2)^{d/p} \int_0^1 dv[q_0+v(p-2)]^{-1-(d/p)} \log v$$

$$R_3 = \frac{d}{p}(q_0+p-2)^{d/p} \int_0^1 dv[q_0+v(p-2)]^{-1-(d/p)} \log(q_0-v)$$

$$R_4 = -\frac{d}{p}(p-1)(q_0+p-2)^{d/p} \int_0^1 dv[q_0+v(p-2)]^{-1-(d/p)} \times \log(q_0+v(p-2)).$$

One can therefore conclude that

$$w(t) = \left(\frac{q_0}{q_0+p-2}\right)^{d/p} \left\{ \bar{y}(0) - \frac{1}{p-2}\left(\left(\frac{r_0+p-2}{r_0}\right)^{d/p} - 1\right) \log t - \left(-\frac{d-p}{d}\left[\left(\frac{q_0+p-2}{q_0}\right)^{d/p} - 1\right] \log |D| + I_1 + R_2 + R_3 + R_4\right)\right\}. \quad \blacksquare$$

Proof of Theorem 1.1. If u is a solution corresponding to a bounded initial datum, we notice that, by Lemma 3.1, the following contractivity property holds true for all $0 \leqslant s \leqslant t$ and for all $r \geqslant 2$:

$$\|u(t)\|_r \leqslant \|u(s)\|_r .$$

Therefore, by Proposition 3.8 and Lemma 3.9, one has, for all such s and t,

$$\begin{aligned} \|u(t)\|_{r(s)} &\leqslant \|u(s)\|_{r(s)} \\ &= \exp[\log \|u(s)\|_{r(s)}] = e^{y(s)} \leqslant e^{\bar{y}(s)}, \end{aligned}$$

whence, letting $s \to t^-$ and recalling that $r(s) \to +\infty$ as $s \to t^-$, we deduce, by also using the explicit form of $\bar{y}$,

$$\begin{aligned} \|u(t)\|_\infty &= \lim_{s \to t^-} \|u(t)\|_{r(s)} \\ &\leqslant \lim_{s \to t^-} e^{\bar{y}(s)} = e^{w(t)} \\ &= C(d, p, q_0) \frac{|D|^\alpha}{t^\beta} \|u(0)\|_{q_0}^\gamma , \end{aligned}$$

where the values of α, β, γ are those appearing in the statement of the theorem. Therefore, the claimed $L^{r(0)}$–L^∞ contractivity property holds true for the solutions which correspond to bounded initial data.

We conclude the proof of the present theorem by removing the boundedness assumption on the initial datum.

Consider an initial datum $u(0) \in L^{q_0}(D)$ with $q_0 \geqslant r_0$. Let $u_k(0)$ be a sequence of L^∞ functions on D, converging to $u(0)$ in L^{q_0} as $k \to +\infty$. Let also $u_k(t)$ be the solution to the equation at hand corresponding to the essentially bounded initial datum $u_k(0)$. By the previous calculations:

$$\|u_k(t)\|_\infty \leqslant C(d, p, q_0) \frac{|D|^\alpha}{t^\beta} \|u_k(0)\|_{q_0}^\gamma .$$

By letting k tend to infinity, we notice that the r.h.s. converges to

$$C(d, p, q_0) \frac{|D|^\alpha}{t^\beta} \|u(0)\|_{q_0}^\gamma .$$

Hence, u_k is a bounded sequence in $L^\infty((\varepsilon, T); L^\infty(D))$. Possibly by passing to a subsequence we can suppose that u_k converges to a suitable function v in the weak* topology of the above space. Hence, it also converges to v in the weak* topology of the space $L^\infty((\varepsilon, T); L^2(D))$, since D has finite measure and thus $L^2(D) \subset L^1(D)$. We claim that $v(t)$ coincides with the solution $u(t)$ corresponding to the initial datum $u(0)$, for almost all $t > 0$. In fact, by [Li, p. 159], one knows that u_k converges, in the weak* topology of $L^\infty((\varepsilon, T); L^2(D))$ to u. Thus, $u = v$ as functions of $L^\infty((\varepsilon, T);$ $L^\infty(D))$. We can conclude that, at least for almost all $t > 0$ (because ε and T are arbitrary):

$$\|u(t)\|_\infty \leqslant C(d, p, q_0) \frac{|D|^\alpha}{t^\beta} \|u(0)\|_{q_0}^\gamma \qquad \text{for almost all} \quad t > 0. \tag{3.23}$$

We now show that the latter estimate holds for all positive t. To this end, first notice that, since u is a solution to the equation at hand, it is locally bounded above as a function of x and t by [Di, Chap. 5]. Moreover, the function $v = -u$ satisfies the parabolic equation

$$\dot{v} = \operatorname{div}(\mathbf{b}(t, x, v, \nabla v))$$

with $\mathbf{b}(t, x, s, \xi) = -\mathbf{a}(t, x, -s, -\xi)$. The function $\mathbf{b}$ satisfies, by elementary calculations, the same ellipticity and growth bounds satisfied by the function $\mathbf{a}$, so that $v = -u$ is locally bounded above as a function of x and t as well. Thus u is locally bounded. We can therefore apply the results of [Di, Chap. 2] to prove the local Hölder continuity of u in space and time, so that in particular, for any compact set $K \subset D$, $u(\cdot)$ is continuous in time with values in $L^\infty(K)$. The validity of the bound (3.23) obviously implies the validity of a similar bound with $\|\cdot\|_{L^\infty(K)}$ replacing $\|\cdot\|_{L^\infty(D)}$ in the left-hand side. By using this fact, the above mentioned continuity property of $u(\cdot)$ and the lower semicontinuity of the norm $\|\cdot\|_{L^\infty(K)}$ with respect to the weak* topology, we arrive at proving that

$$\|u(t)\|_{L^\infty(K)} \leqslant C(d, p, q_0) \frac{|D|^\alpha}{t^\beta} \|u(0)\|_{q_0}^\gamma \qquad \text{for all} \quad t > 0$$

and for all compact sets $K \subset D$. The thesis is thus proven for the case $p > 2$. If $p = 2$ we return to the calculations which led to to Lemma 3.9. It can be shown by an identical procedure that the function $\bar{y}(t)$ appearing in that Lemma takes the form

$$w(t) := \lim_{s \to t^-} \bar{y}(s) = \bar{y}(0) - \frac{d}{2q_0} \log t + K,$$

where K depends only on d since $p = 2$ implies $r_0 = 2$. The thesis follows as above. ∎

Proof of Theorem 1.2. We start by proving the contractivity property on L^∞. In fact, consider the ultracontractive estimate proven in Theorem 1.1. The constants α, β, γ involved depend on q_0 and satisfy:

$$\lim_{q_0 \to +\infty} \alpha(q_0) = \lim_{q_0 \to +\infty} \beta(q_0) = 0; \qquad \lim_{q_0 \to +\infty} \gamma(q_0) = 1.$$

Moreover, the explicit expression of the constants I_1, R_2, R_3, R_4 appearing in the proof of Theorem 1.1 show that all of them converge to zero as $q_0 \to +\infty$. Therefore, the bound

$$\|u(t)\|_\infty \leqslant \exp[I_1 + R_2 + R_3 + R_4] \frac{|D|^\alpha}{t^\beta} \|u(0)\|_{q_0}^\gamma$$

appearing in the proof of Theorem 1.1 implies, when $u(0) \in L^\infty(D)$ and, thus, to all $L^q(D)$ spaces, that

$$\|u(t)\|_\infty \leqslant \|u(0)\|_\infty$$

for all $t > 0$, since $\|\cdot\|_q \to \|\cdot\|_\infty$ as $q \to \infty$. The contraction property on L^∞ is thus proven.

To prove the analogous statement on L^q with $2 \leqslant q < +\infty$, we use an approximation argument similar to the one used in the final part of the proof of Theorem 1.1. In fact, let $u_0 \in L^q(D)$ for $2 \leqslant q < +\infty$. Consider a sequence of essentially bounded functions $u_k(0)$ converging to $u(0)$ in $L^q(D)$. We denote by $u_k(t)$ the solutions to the equation at hand corresponding to the initial data $u_k(0)$. Then

$$\begin{aligned}\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|u_k(t)\|_q^q &= -(q-1)\int_D |u_k(t,x)|^{q-2}\,\mathbf{a}(t,x,u_k(t,x),\nabla u_k(t,x))\\ &\qquad \times \nabla u_k(t,x)\,\mathrm{d}x\\ &\leqslant -c_1(q-1)\int_D |u_k(t,x)|^{q-2}\,|\nabla u_k(t,x)|^p\,\mathrm{d}x\\ &\leqslant 0\end{aligned}$$

by the ellipticity conditions, since all the above integrals make sense. Then

$$\|u_k(t)\|_q \leqslant \|u_k(0)\|_q$$

for any $t > 0$, $k \in \mathbb{N}$. Let now $k \to +\infty$; since $u_k(0)$ converges to $u(0)$ in L^q, the sequence $\{u_k\}$ is bounded in $L^\infty((0,+\infty); L^q(D))$. Possibly by choosing a subsequence u_{k_h}, we can suppose that it is weakly* convergent in such a space to a function v. Since $L^2(D) \subset L^{q'}(D)$, where $q^{-1} + q'^{-1} = 1$, we can conclude that is also weakly* convergent to v in the space $L^\infty((0,+\infty); L^2(D))$. By the above mentioned Lions's result it follows that $u = v$ in such space and

$$\|u(t)\|_q \leqslant \|u(0)\|_q$$

at least for almost all positive t. The passage from this to the analogous statement for all positive t is accomplished exactly as in the final part of the proof of Theorem 1.1. ∎

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