J. reine angew. Math. **562** (2003), 201—235

Journal für die reine und angewandte Mathematik

Nonlinear Markov semigroups, nonlinear Dirichlet forms and applications to minimal surfaces

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Abstract. We introduce the notions of nonlinear Markov semigroups and nonlinear Dirichlet forms on a Hilbert space $L^2(X,m)$. Dirichlet forms are meant to be convex lower semicontinuous functionals on $L^2(X,m)$, enjoying contraction properties w.r.t. projections onto suitable closed and convex sets. We prove the one-to-one correspondence between these two classes of objects, by establishing Beurling-Deny-like criteria which characterize separately the non-expansion in $L^\infty(X,m)$ and the order preserving properties of the semigroup. Γ-limits of functionals enjoying the suitable contraction properties are nonlinear Dirichlet forms, and in particular this holds for relaxed functionals. Examples include elliptic, subelliptic and subriemannian p-Laplacians on Riemannian manifolds, possibly with measurable, non necessarily uniformly elliptic coefficients, nonlinear operators constructed from derivations in Hilbert C^*-modules, and convex functionals of the gradient including the area and the perimeter functionals. We apply the theory to construct Markov evolutions which approach minimal surfaces with given boundary contour, as well as Markov evolutions which converge to the solution of a Dirichlet problem with given boundary data.

1. Introduction

In a pair of celebrated papers [BD1], [BD2], A. Beurling and J. Deny introduced the notion of *Dirichlet spaces* and *Dirichlet forms* as function spaces continuously embedded in a L^1_{loc} space on which every normal contraction operates. These quadratic forms abstract the characteristic properties of the classical Dirichlet integral over Euclidean domains and allow to investigate from a functional analytic point of view the potential theory and the spectral synthesis of a wide class of second order differential or finite difference operators in divergence form, with measurable, possibly degenerate or singular coefficients.

On the other hand, the efforts of M. Fukushima and M. L. Silverstein established the deep connection between the theory of Dirichlet forms, potential theory and the theory of Hunt processes, allowing the construction and the detailed analysis of such processes under very minimal conditions on the coefficients of their generators. This connection was carried over the infinite dimensional setting, with applications to mathematical physics, by the work of S. Albeverio, R. Höegh-Krohn, Z. M. Ma, M. Röckner and coworkers: see e.g. [A], [AH], [AR] and references quoted. More recently the work of K.-T. Sturm (see [St]) showed the connection between the theory of Dirichlet forms with analysis and probability in metric spaces. The books [FOT], [MR], [BH] are excellent general references and can also be used to track the wide bibliography on the subject.

One of the main achievements of the Beurling-Deny work was the discovery of the one-to-one correspondence between the class of (quadratic) Dirichlet forms and the class of (linear) *Markovian semigroups*.

The aim of the present paper is to introduce a notion of *nonlinear Dirichlet form* as the class of convex, lower semicontinuous functionals $\mathcal{E}$ on a Hilbert space $L^2(X,m)$ satisfying certain contraction properties. We will then show that there is a one-to-one correspondence between such class of functionals and the class of *nonlinear Markovian semigroups* $\{T_t : t \geqq 0\}$ on $L^2(X,m)$. They are defined to be those (nonlinear) nonexpansive semigroups in $L^2(X,m)$ such that, for any $u, v \in L^2(X,m)$ and $\alpha > 0$

$$(1.1) \qquad 0 \leqq u - v \leqq \alpha \Rightarrow 0 \leqq T_t u - T_t v \leqq \alpha \quad \forall t > 0,$$

or equivalently such that T_t is order preserving and can be extended to a nonexpansive semigroup on $L^\infty(X,m)$. It is also required that their generator is *cyclically monotone* (see [B2]); this latter condition simply corresponds to the fact that the generator of $\{T_t : t \geqq 0\}$ is the subdifferential $\partial\mathcal{E}$ of a convex, lower semicontinuous functional $\mathcal{E}$ and it is equivalent, in the linear case, to self-adjointness of the generator.

We then arrive to a natural extension of the classical Beurling-Deny theory of quadratic Dirichlet forms by *characterizing separately* the order preserving and the nonexpansion in L^∞ properties. In this respect we would like to comment that our theory bears some relationship, but essential differences as well, with [BP] and with the theory of *semigroups of complete contractions* developed in [BC]. A more detailed comparison is given in the last section. We comment also that the terminology "nonlinear Dirichlet forms" also appears in a different context in [J].

Among the main examples which shall be discussed we mention the following *energy functionals*:

- the p-energy functional ($p > 1$) on Euclidean domains Ω

$$\mathcal{E}_p(u) = \int_\Omega |\nabla u|^p \, dx,$$

associated to the p-Laplacian operator $\triangle_p u := \nabla \cdot (|\nabla u|^{p-2}\nabla u)$, as well as functionals associated to second order nonlinear differential operators in divergence form with *measurable* coefficients which are, in a suitable sense, locally strictly elliptic (and hence possibly singular or degenerate) w.r.t. the p-Laplacian, and their natural counterparts on manifolds;

- functionals constructed from closed derivations with values in Hilbert C^*-modules, which in particular allow to discuss analogues of the p-Laplacian in subriemannian geometry. One of the most relevant examples is the subelliptic p-Laplacian on a riemannian manifold (M, g) associated to a collection of Hörmander vector fields $\{X_i : i = 1, \dots, m\}$, whose associated functional is:

$$\mathscr{E}(u) = \int_M \left(\sum_{i=1}^m |X_i u|^2 \right)^{p/2} m_g(\mathrm{d}x);$$

- the "area functional"

$$\mathscr{E}(u) = \int_\Omega \sqrt{1 + |Du|^2};$$

- the "perimeter functional"

$$\int_\Omega |Du|$$

for $u \in BV(\Omega) \cap \mathrm{L}^2(\Omega)$, where Du is the vector valued Radon measure representing the distributional derivative of u. This are examples from a wide class of convex functionals of the gradient.

The process of constructing Markov evolutions is *equivalent*, by the present theory, to the construction of convex, lower semicontinuous functionals which in addition satisfy appropriate contraction properties. Since, as in the quadratic case, one often starts with functionals first defined on a dense subset $\mathscr{F}$ of $\mathrm{L}^2(X, m)$ (which need not be l.s.c. when extended to $+\infty$ outside $\mathscr{F}$), the first problem to face is to prove existence of convex lower semicontinuous extensions of the initial functional $\mathscr{E}$, and possibly to characterize them, although this latter task will not be considered here. We notice that the problem of the existence of convex, lower semicontinuous extensions is the analogue, in the present situation, of the problem of *closability* of quadratic forms in Hilbert spaces, and in this connection the concept of Γ-*convergence* and of *relaxation* furnish the appropriate setting. For example, it is well-known that if $\mathscr{E}$ is l.s.c. on its initial domain, then its relaxed functional $\mathrm{sc}^-\mathscr{E}$ is an extension of $\mathscr{E}$. We will say in this situation, by analogy with the quadratic case, that $\mathscr{E}$ is *closable*.

The point here is the validity of the appropriate contraction properties for Γ-limits. In fact, we prove that if a family of convex, non necessarily l.s.c. functionals, enjoy such contraction properties, their Γ-limits are Dirichlet forms. This in particular holds for the relaxed functionals. Thus, convex functionals defined initially on a domain $\mathscr{F}$, l.s.c. on $\mathscr{F}$ and enjoying suitable contraction properties on $\mathscr{F}$, have an extension which is a Dirichlet form. Such result is basic in several applications given here.

The theory of quadratic Dirichlet forms has found natural applications in the study of the approach to equilibrium for particle systems in statistical mechanics. In the final part of this paper we shall show that the concept of nonlinear Dirichlet form can be used to study similarly the approach to equilibrium in a geometric setting (approach to minimal surfaces with fixed boundary contour) and in potential theory (approach to the solution of a Dirichlet problem).

To start with, consider the area functional

$$\mathscr{E}(u) = \int_{\Omega}\sqrt{1+|Du|^2} + \int_{\partial\Omega}|\mathrm{tr}_{\Omega}\, u - \varphi|\,\mathrm{d}\mathscr{H}^{n-1} \tag{1.2}$$

for $u \in BV(\Omega)$, and $+\infty$ otherwise: here Ω is a bounded and sufficiently regular euclidean domain, tr_{Ω} denotes the trace operator and $\mathscr{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, and $\varphi \in \mathrm{L}^1(\partial\Omega)$ is a function whose graph Γ represents the boundary contour. The graphs $\Sigma(u)$ of minimizers u of $\mathscr{E}$ are well-known to be the so-called *minimal surfaces*. We show that $\mathscr{E}$ is a Dirichlet form, so that it gives rise to a Markov evolution, which deform surfaces and makes them approach, in suitable senses, to minimal surfaces. For example, if φ is continuous, then for any initial surface $\Sigma(u)$ represented by a function u, the time evolved surface $\Sigma(T_t u)$ converges, as time tends to $+\infty$, to the unique minimal surface $\Sigma(v)$ in the sense that $T_t u \to v$ in strong L^1 sense. The Markov property makes the geometric properties of the evolution very clear: evolving surfaces which do not intersect initially do not cross at any time (a *barrier-like* property), and the L^∞ distance between the functions representing the time-evolved surface and the minimal surface does not increase.

A similar discussion is then given for the problem of constructing a Markov evolution approaching a solution of the *Dirichlet problem*

$$\begin{cases} \triangle u_\varphi = 0 & \text{if } x \in \Omega, \\ u_\varphi = \varphi & \text{if } x \in \partial\Omega. \end{cases}$$

The functional under consideration is defined as

$$\mathscr{E}_\varphi(u) = \int_{\Omega}|\nabla u|^2\,\mathrm{d}x$$

on those functions of the Sobolev space $H^1(\Omega)$ such that $\mathrm{tr}_{\Omega}\, u = \varphi$, and $+\infty$ otherwise in $\mathrm{L}^2(\Omega)$. $\mathscr{E}$ is shown again to be a Dirichlet form, and the time evolved function $T_t v$ approaches in suitable senses the solution u_φ as $t \to +\infty$, for any initial datum v.

The paper is organized as follows: in Section 2 we introduce the notion of nonlinear Markovian semigroup on a Hilbert space $\mathrm{L}^2(X,m)$ and investigate its more immediate properties like the L^p-interpolation property for $p \geqq 2$.

In Section 3, we introduce the class of nonlinear Dirichlet forms on a Hilbert space $\mathrm{L}^2(X,m)$. Then we show our main achievement, namely the above mentioned one-to-one correspondence between nonlinear Markovian semigroups whose generator is cyclically monotone, and nonlinear Dirichlet forms. This is achieved by proving a general result, of independent interest: given a closed and convex set $C \subset \mathscr{H}$ with $\mathscr{H}$ a Hilbert space, P_C the Hilbert projection onto C, and a convex l.s.c. functional $\mathscr{E}$ on $\mathscr{H}$, the property $\mathscr{E}(P_C x) \leqq \mathscr{E}(x)$ for all $x \in \mathscr{H}$ is *equivalent* to the fact that the (nonlinear) semigroup associated to $\mathscr{E}$ leaves C globally invariant. This fact will be crucial also in Section 6.

Section 4 is devoted to the construction of classes of examples. The first one is the semigroup generated by the p-Laplacian operator, possibly with measurable coefficients, on a Riemannian manifold M. Then we consider p-energy functionals constructed from closed derivations on Hilbert C^*-modules, particular cases of which give rise to the semigroups associated to subriemannian and subelliptic p-Laplacians. We then prove a basic result, namely the Dirichlet properties for Γ-limits of convex functionals enjoining the relevant contraction properties. This latter result allows to treat a wide class of convex functionals of the gradient, including the area and the perimeter functionals.

Section 5 is devoted to the construction and to the study of the ergodic properties of the nonlinear Markov evolution associated to the geometric functional (1.2). A similar construction is provided for a nonlinear Markov evolution which approaches harmonic functions with prescribed boundary data.

Our goal in Section 6 is to prove a domination principle between the semigroup associated to certain quasilinear operators and the unperturbed (nonlinear) semigroup. To this end we use a characterization of the comparison property $|T_t u| \leqq S_t|u|$ for two (nonlinear) semigroups the "larger" of which is order preserving, proved by methods similar to the previous one in terms of the associated energy functionals, a problem studied first in [Ba]. Finally the last section contains a comparison with the results of [BC] and [BP].

2. Nonlinear Markovian semigroups

We recall a basic definition. See for example [S], [B2] and references quoted.

Definition 2.1. A nonlinear, strongly continuous, non-expansive semigroup $\{T_t : t \geqq 0\}$ on a Hilbert space $\mathscr{H}$, is a family of maps from $\mathscr{H}$ to $\mathscr{H}$ satisfying the following properties:

$$T_0 = \mathbf{1}_{\mathscr{H}}; \tag{2.1}$$

$$T_{t+s} = T_t \circ T_s \quad \forall s, t \geqq 0; \tag{2.2}$$

$$\lim_{t\to 0} \|T_t x - x\|_{\mathscr{H}} = 0 \quad \forall x \in \mathscr{H}; \tag{2.3}$$

$$\|T_t x - T_t y\|_{\mathscr{H}} \leqq \|x - y\|_{\mathscr{H}} \quad \forall x, y \in \mathscr{H}, \forall t \geqq 0. \tag{2.4}$$

We shall only consider hereafter the case $\mathscr{H} = \mathrm{L}^2(X, m)$, where X is countably generated Borel space and m is σ-finite Borel measure. In the sequel $\|\cdot\|_p$ will denote the norm in $\mathrm{L}^p(X, m)$, $p \in [1, +\infty]$.

We now introduce one of basic definitions of the present work.

Definition 2.2. A nonlinear, strongly continuous, nonexpansive semigroup $\{T_t : t \geqq 0\}$ on $\mathscr{H} = \mathrm{L}^2(X, m)$ is said to be:

- *order preserving* if

$$T_t u \leqq T_t v \quad \forall t \geqq 0 \text{ whenever } u, v \in \mathrm{L}^2(X, m), u \leqq v; \tag{2.5}$$

- *Markovian* if it is order preserving, and if it satisfies:

$$\|T_t u - T_t v\|_\infty \leqq \|u - v\|_\infty \quad \forall t \geqq 0, \forall u, v \in \mathscr{H}. \tag{2.6}$$

Remark 2.3. Notice that condition (2.6) implies that if there exists $u_0 \in \mathrm{L}^\infty(X; m) \cap \mathrm{L}^2(X, m)$ whose orbit $\mathcal{O}(u_0) := \{T_t u_0 : t \geqq 0\}$ is bounded in $\mathrm{L}^\infty(X, m)$, then *all* orbits of essential bounded functions are, likewise, bounded in L^∞.

Theorem 2.4. *Any Markovian semigroup such that there exists an element $u_0 \in \mathrm{L}^2(X, M) \cap \mathrm{L}^\infty(X, m)$ whose orbit is bounded in $\mathrm{L}^\infty(X, m)$ can be extended to a strongly continuous, non-expansive semigroup on $\mathrm{L}^p(X, m)$ for any $p \in [2, \infty)$ and to a non-expansive semigroup on $\mathrm{L}^\infty(X, m)$. Each of such semigroups is order preserving.*

Proof. For any $t > 0$, the map T_t, initially defined on $\mathrm{L}^2(X, m) \cap \mathrm{L}^\infty(X, m)$, extends, by the Markov condition (see Definition 2.2), to a non-expansive map on $\mathrm{L}^\infty(X, m)$. By the nonlinear interpolation Theorem given in [Br] (see in particular Theorem 1 and the subsequent Corollary) this implies that it can be extended to a non-expansive map on any $\mathrm{L}^p(X, m)$ for any $p \geqq 2$. In particular, all maps T_t are continuous on L^p for all $p \in [2, +\infty)$. The semigroup property for the family $\{T_t : t \geqq 0\}$ of maps on $\mathrm{L}^p(X, m)$ readily follows by this continuity property and from the semigroup property, true on the dense set $\mathrm{L}^\infty(X, m) \cap \mathrm{L}^2(X, m)$. Finally, the strong continuity of the semigroups acting on $\mathrm{L}^p(X, m)$ is shown as follows. First notice that strong continuity holds if $u \in \mathrm{L}^\infty(X, m) \cap \mathrm{L}^2(X, m)$ by the bound

$$\begin{aligned}\|T_t u - u\|_p &\leqq \|T_t u - u\|_\infty^{(p-2)/p} \cdot \|T_t u - u\|_2^{2/p} \\ &\to 0 \quad \text{as } t \to 0,\end{aligned}$$

by the strong continuity of T_t on L^2 and by the fact that the orbit of u is bounded. To prove strong continuity for general u, fix $\varepsilon > 0$ and choose $v \in \mathrm{L}^\infty(X, m) \cap \mathrm{L}^2(X, m)$ such that $\|u - v\|_p \leqq \varepsilon$. Then

$$\begin{aligned}\|T_t u - u\|_p &\leqq \|T_t u - T_t v\|_p + \|T_t v - v\|_p + \|u - v\|_p \\ &\leqq 2\varepsilon + \|T_t v - v\|_p \leqq 3\varepsilon\end{aligned}$$

if t is sufficiently small. □

Clearly the requirement on the existence of a bounded orbit is satisfied if there exists a fixed point $u_0 \in \mathrm{L}^2 \cap \mathrm{L}^\infty$ for the semigroup T_t. We shall show in the following section (see Corollary 3.5) how this condition can be verified in terms of what we shall call *nonlinear Dirichlet forms*. If an invariant element exists, the following immediate properties hold.

Proposition 2.5. *Let $v \in \mathrm{L}^2(X, m) \cap \mathrm{L}^\infty(X, m)$ be an invariant element for a Markovian semigroup T_t, in the sense that for all $t \geqq 0$ one has $T_t v = v$. Then the following properties hold true*:

- *the balls in $\mathrm{L}^2(X, m)$ centered in v are left invariant by T_t*:

$$\|T_t u - v\|_2 \leqq \|u - v\|_2 \quad \textit{for all } u \in \mathrm{L}^2(X, m) \textit{ and } t \geqq 0;$$

- *the balls in* $L^\infty(X,m)$ *centered in* v *are left invariant by* T_t:

$$\|T_t u - v\|_\infty \leqq \|u - v\|_\infty \quad \text{for all } u \in L^\infty(X,m) \text{ and } t \geqq 0;$$

- *the set* $\{u \in L^2(X,m), u \geqq v\}$ *is left invariant by* T_t *for all* $t \geqq 0$.

The next lemma will be basic for all what follows. In fact, we show that the order preserving property and the Markov property are equivalent to invariance properties of suitable convex sets for suitable semigroups.

Lemma 2.6. *A nonlinear, strongly continuous, nonexpansive semigroup* $\{T_t : t \geqq 0\}$ *is order preserving if and only if the semigroup* $T_t^{(2)}$ *defined on* $L^2(X,m) \oplus L^2(X,m)$ *by*

$$T_t^{(2)}(u,v) = (T_t u, T_t v) \tag{2.7}$$

leaves invariant the following closed and convex set:

$$C_1 := \{(u,v) \in L^2(X,m) \oplus L^2(X,m) : u \leqq v\}. \tag{2.8}$$

It is Markovian if and only if $T_t^{(2)}$ *leaves invariant all the following closed and convex sets*:

$$C_2(\alpha) := \{(u,v) \in L^2(X,m) \oplus L^2(X,m) : \|u - v\|_\infty \leqq \alpha\}. \tag{2.9}$$

Proof. C_1 is closed and convex in $L^2(X,m) \oplus L^2(X,m)$ because $L^2_+(X,m)$ is closed and convex in $L^2(X,m)$. The stated equivalence is immediate. As concerns $C_2(\alpha)$, these sets are closed and convex because intersection with $L^2(X,m)$ of the closed balls of any radius α of L^∞ are convex and closed sets in $L^2(X,m)$. Again the stated equivalence is immediate. □

3. Nonlinear Dirichlet forms

We consider here lower semicontinuous, convex functionals $\mathscr{E} : L^2(X,m) \to [0,+\infty]$. It is well-known (see [B2]) that the subdifferentials $\partial\mathscr{E}$ of such functionals are exactly the maximally monotone operators A such that A^0 are cyclically monotone operators, where A^0 is the principal section of A defined as follows: $A^0 u$ is the element of $Au \in \mathscr{P}(L^2(X,m))$ (the set of parts of $L^2(X,m)$) with minimal L^2-norm.

We shall be interested in this paper in the classes of functionals discussed below; notice that in the following definition the notation Proj_C will denote the Hilbert projection onto the closed convex set C.

Definition 3.1. A lower semicontinuous, convex functional

$$\mathscr{E} : L^2(X,m) \to (-\infty,+\infty],$$

finite on a dense subset of $L^2(X,m)$, is said to be a *semi-Dirichlet form* if the functional $\mathscr{E}^{(2)} : L^2(X,m) \oplus L^2(X,m) \to [0,+\infty]$

$$\mathscr{E}^{(2)}(u,v) := \mathscr{E}(u) + \mathscr{E}(v) \tag{3.1}$$

for all $u, v \in \mathrm{L}^2(X,m)$ satisfies the following condition:

$$\mathscr{E}^{(2)}(\mathrm{Proj}_{C_1}(w)) \leqq \mathscr{E}^{(2)}(w) \tag{3.2}$$

for any $w \in \mathrm{L}^2(X,m) \oplus \mathrm{L}^2(X,m)$. It is said to be a *Dirichlet form* if, moreover, for all $\alpha > 0$:

$$\mathscr{E}^{(2)}(\mathrm{Proj}_{C_2(\alpha)}\, w) \leqq \mathscr{E}^{(2)}(w), \tag{3.3}$$

where C_1 and $C_2(\alpha)$ are defined in Lemma 2.6.

Hereafter we shall *always* deal without further comment with functionals which are finite on a dense set.

Remark 3.2. We stress that, although in the above definition the convexity of $\mathscr{E}$ is explicitely required, it follows that any lower semicontinuous functional satisfying (3.3) is automatically convex. Indeed, using the explicit expression for $\mathrm{Proj}_{C_2(\alpha)}$ given in the next lemma it follows that, when $\alpha \to 0$

$$\mathrm{Proj}_{C_2(\alpha)}(u,v) \to \left(\frac{u+v}{2}, \frac{u+v}{2}\right)$$

strongly in $\mathrm{L}^2(X,m)$, so that the lower semicontinuity of $\mathscr{E}$ implies that the inequality

$$2\mathscr{E}\left(\frac{u+v}{2}\right) \leqq \mathscr{E}(u) + \mathscr{E}(v).$$

This implies convexity again by lower semicontinuity.

The following lemma gives the explicit expression of the projections introduced above. Such explicit form shows that the projections at hand can be approximated by smoother function. This can be useful in the explicit verification of the Markov property, for example in combination with the concept of relaxed Dirichlet form as in the following section.

Lemma 3.3. • *The Hilbert projection P_1 onto the closed and convex set C_1 is given by the formula*:

$$P_1(u,v) = \begin{cases} (u,v) & \text{if } u \leqq v, \\ \left(\frac{1}{2}(u+v), \frac{1}{2}(u+v)\right) & \text{if } u > v \end{cases} \tag{3.4}$$

for all $(u,v) \in \mathrm{L}^2(X,m) \oplus \mathrm{L}^2(X,m)$. Equivalently one can write

$$P_1(u,v) = \left(\frac{u + u \wedge v}{2}, \frac{v + u \vee v}{2}\right)$$

or

$$P_1(u,v) = \left(u - \frac{1}{2}(u-v)_+, v + \frac{1}{2}(u-v)_+\right).$$

• *The Hilbert projection $P_{2,\alpha}$ onto the closed and convex set $C_2(\alpha)$ is given by the formula*:

$$(3.5) \qquad P_{2,\alpha}(u,v) = \begin{cases} (u,v) & \text{if } |u-v| \leqq \alpha, \\ \left(\frac{1}{2}(u+v-\alpha), \frac{1}{2}(u+v+\alpha)\right) & \text{if } u-v < -\alpha, \\ \left(\frac{1}{2}(u+v+\alpha), \frac{1}{2}(u+v-\alpha)\right) & \text{if } u-v > \alpha \end{cases}$$

for all $(u,v) \in \mathrm{L}^2(X,m) \oplus \mathrm{L}^2(X,m)$. Equivalently one can write

$$P_{2,\alpha}(u,v) = \Big(v + \frac{1}{2}[(u-v+\alpha)_+ - (u-v-\alpha)_-],$$
$$u - \frac{1}{2}[(u-v+\alpha)_+ - (u-v-\alpha)_-]\Big).$$

Proof. We shall use the fact that the projection Pf of f onto a closed and convex set C in the Hilbert space $\mathscr{H}$ is characterized by the fact that $Pf \in C$ and that

$$\langle f - Pf, g - Pf\rangle \leqq 0 \quad \forall g \in C.$$

Let $f = (u,v) \in \mathrm{L}^2(X,m) \oplus \mathrm{L}^2(X,m)$, $g = (a,b) \in C_1$. Then we have:

$$\begin{aligned} \langle f - P_1 f, g - P_1 f\rangle &= \int_{u \geqq v} \left(u - \frac{u+v}{2}\right)\left(a - \frac{u+v}{2}\right) \\ &\quad + \int_{u \geqq v} \left(v - \frac{u+v}{2}\right)\left(b - \frac{u+v}{2}\right) \\ &= \int_{u \geqq v} \left(\frac{u-v}{2}\right)\left(a - \frac{u+v}{2}\right) \\ &\quad + \int_{u \geqq v} \left(\frac{v-u}{2}\right)\left(b - \frac{u+v}{2}\right) \\ &= \int_{u \geqq v} \left(\frac{u-v}{2}\right)(a-b) \\ &\leqq 0 \end{aligned}$$

where we have used the fact that $a - b \leqq 0$.

Let $g = (a, b) \in C_2(\alpha)$. We compute:

$$
\begin{aligned}
&\langle f - P_2(\alpha)f, g - P_2(\alpha)f \rangle \\
&= \int_{u-v<-\alpha} \left(u - \frac{u+v-\alpha}{2}\right)\left(a - \frac{u+v-\alpha}{2}\right) \mathrm{d}m \\
&\quad + \int_{u-v<-\alpha} \left(v - \frac{u+v+\alpha}{2}\right)\left(b - \frac{u+v+\alpha}{2}\right) \mathrm{d}m \\
&\quad + \int_{u-v>\alpha} \left(u - \frac{u+v+\alpha}{2}\right)\left(a - \frac{u+v+\alpha}{2}\right) \mathrm{d}m \\
&\quad + \int_{u-v>\alpha} \left(v - \frac{u+v-\alpha}{2}\right)\left(b - \frac{u+v-\alpha}{2}\right) \mathrm{d}m \\
&= \int_{u-v<-\alpha} \left(\frac{u-v+\alpha}{2}\right)\left(a - \frac{u+v-\alpha}{2}\right) \mathrm{d}m \\
&\quad + \int_{u-v<-\alpha} \left(\frac{v-u-\alpha}{2}\right)\left(b - \frac{u+v+\alpha}{2}\right) \mathrm{d}m \\
&\quad + \int_{u-v>\alpha} \left(\frac{u-v-\alpha}{2}\right)\left(a - \frac{u+v+\alpha}{2}\right) \mathrm{d}m \\
&\quad + \int_{u-v>\alpha} \left(\frac{v-u+\alpha}{2}\right)\left(b - \frac{u+v-\alpha}{2}\right) \mathrm{d}m \\
&= \int_{u-v<-\alpha} \left(\frac{u-v+\alpha}{2}\right)\left(a - \frac{u+v-\alpha}{2} - b - \frac{u+v+\alpha}{2}\right) \mathrm{d}m \\
&\quad + \int_{u-v>\alpha} \left(\frac{u-v-\alpha}{2}\right)\left(a - \frac{u+v+\alpha}{2} - b + \frac{u+v-\alpha}{2}\right) \mathrm{d}m \\
&= \int_{u-v<-\alpha} \left(\frac{u-v+\alpha}{2}\right)(a - b + \alpha)\, \mathrm{d}m \\
&\quad + \int_{u-v>\alpha} \left(\frac{u-v-\alpha}{2}\right)(a - b - \alpha)\, \mathrm{d}m \\
&\leqq 0
\end{aligned}
$$

where we have used in the last step the fact that $|a - b| \leqq \alpha$. □

Along the way to the main goal of the present section, Theorem 3.6 below, we first provide an elementary proof of a general result, of independent interest (see [BrP], [Ba]).

Theorem 3.4. *Let $\mathscr{E}$ be a lower semicontinuous, convex functional on a Hilbert space $\mathscr{H}$ with values in $(-\infty, +\infty]$, and T_t be the corresponding strongly continuous, nonexpansive semigroup on $\mathscr{H}$, generated by the subdifferential $A = \partial\mathscr{E}$. Then T_t leaves invariant a closed and convex set $C \subset \mathscr{H}$ if and only if $\mathscr{E}(\mathrm{Proj}_C(x)) \leqq \mathscr{E}(x)$ for all $x \in \mathscr{H}$.*

Proof. We denote by $J_\beta := (I + \beta A)^{-1}$ the resolvent of A, given $\beta > 0$. We shall use in the sequel the known fact that a closed and convex set $C \subset \mathscr{H}$ is left invariant by T_t for all t if and only if it is left invariant by J_β for all β (see [B2], Proposition 4.5). By Moreau's Theorem ([S] and references quoted) the functional

$$\mathscr{E}_\beta(w) := \frac{1}{2\beta}\|w - z\|^2_{\mathscr{H}} + \mathscr{E}(w)$$

for fixed $z \in \mathrm{L}^2(X, m)$, attains its minimum value at $w = J_\beta(z)$. Suppose that $\mathscr{E}(\mathrm{Proj}_C(w)) \leqq \mathscr{E}(w)$. If $z \in C$, the contraction property of $\mathscr{E}$ and the contraction property of any Hilbert projection Proj_C onto a closed and convex set C (see [B2], p. 80), we have, denoting by z'_β the Hilbert projection on C of $J_\beta z$:

$$\begin{aligned}
\mathscr{E}_\beta(z'_\beta) &= \frac{1}{2\beta}\|z'_\beta - z\|^2_{\mathscr{H}} + \mathscr{E}(z'_\beta) \\
&\leqq \frac{1}{2\beta}\|z'_\beta - z\|^2_{\mathscr{H}} + \mathscr{E}(J_\beta z) \\
&\leqq \frac{1}{2\beta}\|J_\beta z - z\|^2_{\mathscr{H}} + \mathscr{E}(J_\beta z).
\end{aligned}$$

By the uniqueness of the minimizer of the functional $\mathscr{E}_\beta$, this implies that $z'_\beta = J_\beta z$, so that $J_\beta z$ belongs to C. Since

$$T_t w = s - \lim_{n \to +\infty}\left(I + \frac{t}{n}A\right)^{-n}$$

(cf. [B2], Corollary 4.4), this easily implies the stated assertion.

Conversely, let

$$\tilde{\mathscr{E}}_\beta(x) := \min_w\left(\frac{1}{2\beta}\|w - x\|^2_{\mathscr{H}} + \mathscr{E}(w)\right).$$

It is known (see e.g. [S], Proposition 1.8) that the Frechet derivative of $\tilde{\mathscr{E}}_\beta$ exists and coincides with $A_\beta := \beta^{-1}(I - J_\beta)$. Thus:

$$\tilde{\mathscr{E}}_\beta(x) - \tilde{\mathscr{E}}_\beta(y) \geqq \langle A_\beta y, x - y\rangle_{\mathscr{H}}.$$

Therefore

$$\begin{aligned}
\tilde{\mathscr{E}}_\beta(x) - \tilde{\mathscr{E}}_\beta(\mathrm{Proj}_C x) &\geqq \langle A_\beta(\mathrm{Proj}_C x), x - \mathrm{Proj}_C x\rangle_{\mathscr{H}} \\
&= \frac{1}{\beta}\langle \mathrm{Proj}_C x - J_\beta \mathrm{Proj}_C x, x - \mathrm{Proj}_C x\rangle_{\mathscr{H}} \\
&\geqq 0
\end{aligned}$$

by the well-known characterization of a Hilbert projection onto a closed and convex set, since J_β leaves, by assumption, C invariant. □

One should comment that we use, for our characterization of Dirichlet forms, a characterization of the invariance, under the action of a semigroup, of closed and convex sets in a Hilbert space in terms of the *functional* associated to the semigroup; in [O] (which uses techniques originating from the former work of H. Brezis) a similar characterization is given in terms of the *generator* of the semigroup at hand (for the linear and nonlinear cases respectively). Our approach seems more natural in view of the aim of studying a nonlinear extension of the concept of Dirichlet forms.

The following corollary, besides of its independent interest, is relevant in connection with the previous Proposition 2.5.

Corollary 3.5. *With the notations of Theorem* 3.4, *a function* $u \in \mathrm{L}^2(X,m)$ *is an absolute minimum of the convex, lower semicontinuous functional* $\mathscr{E}$, *if and only if* $T_t u = u$ *for all* $t > 0$.

Our analogue of the classical linear Beurling-Deny criterion is then an application of the above result.

Theorem 3.6. *Let* $\{T_t : t \geqq 0\}$ *be a* (*nonlinear*) *strongly continuous nonexpansive semigroup on the Hilbert space* $\mathscr{H} = \mathrm{L}^2(X,m)$, *whose generator is the subdifferential of a convex, lower semicontinuous functional* $\mathscr{E} : \mathscr{H} \to [0,+\infty]$. *Then*:

- T_t *is order preserving if and only if* $\mathscr{E}$ *is a semi-Dirichlet form*;
- T_t *is Markovian if and only if* $\mathscr{E}$ *is a Dirichlet form.*

Moreover, if T_t *is Markovian then it is nonexpansive on all* L^p *spaces for* $p \in [1,+\infty]$.

Proof. It suffices to combine Lemma 2.6 with Theorem 3.4. For the last statement one combines the above mentioned Browder interpolation Theorem with the duality argument given in [BP], p. 21. □

We conclude this section by noticing that in some special cases the verification of the Dirichlet properties can be somewhat simplified.

Corollary 3.7. *Let* $\mathscr{E}$ *be a convex, lower semicontinuous functional* $\mathscr{E} : \mathscr{H} \to [0,+\infty]$, *which is homogeneous of degree* $k > 0$. *Then the Markovian property* (3.3) *is satisfied for any* $\alpha > 0$ *provided it is satisfied for* $\alpha = 1$.

Proof. We compute, for any $w \in \mathrm{L}^2(X,m) \oplus \mathrm{L}^2(X,m)$ and all positive α:

$$\begin{aligned}
\mathscr{E}\big(P_{2,\alpha}(w)\big) &= \mathscr{E}\big(P_{2,\alpha}\big(\alpha(w/\alpha)\big)\big) \\
&= \mathscr{E}\big(\alpha P_{2,\alpha=1}(w/\alpha)\big) = \alpha^k \mathscr{E}\big(P_{2,\alpha=1}(w/\alpha)\big) \\
&\leqq \alpha^k \mathscr{E}(w/\alpha) = \mathscr{E}(w)
\end{aligned}$$

where we have used the property $P_{2,\alpha}(\alpha v) = \alpha P_{2,\alpha=1}(v)$, valid for all real α and all $v \in \mathrm{L}^2(X,m) \oplus \mathrm{L}^2(X,m)$. □

An alternative characterization of the semi-Dirichlet property has been given in [Ba], and reads as follows:

Theorem 3.8. *A nonnegative convex and lower semicontinuous functional $\mathscr{E}$ is a semi-Dirichlet form if and only if*

$$\mathscr{E}[u \wedge v] + \mathscr{E}[u \vee v] \leqq \mathscr{E}[u] + \mathscr{E}[v] \quad \forall u, v \in \mathrm{L}^2(X, m). \tag{3.6}$$

Proof. We give an elementary proof of the sufficiency part, computing:

$$\begin{aligned}
\mathscr{E}^{(2)}\big(P_1(u,v)\big) &= \mathscr{E}^{(2)}\left(\frac{1}{2}(u + u \wedge v), \frac{1}{2}(v + u \vee v)\right) \\
&= \mathscr{E}\left(\frac{1}{2}(u + u \wedge v)\right) + \mathscr{E}\left(\frac{1}{2}(v + u \vee v)\right) \\
&\leqq \frac{1}{2}\big(\mathscr{E}(u) + \mathscr{E}(u \wedge v) + \mathscr{E}(v) + \mathscr{E}(u \vee v)\big) \\
&\leqq \mathscr{E}(u) + \mathscr{E}(v) = \mathscr{E}^{(2)}(u, v).
\end{aligned}$$

For the converse, see [Ba]. □

We notice finally that (3.2) and (3.3) give an alternative characterization of the usual class of quadratic (symmetric) Dirichlet forms.

Corollary 3.9. *A nonnegative, convex, lower semicontinuous quadratic form $\mathscr{E}$ is a semi-Dirichlet form in the sense of* [MR] *if and only if* (3.2) *holds. It is a Dirichlet form if and only if* (3.2) *and* (3.3) *hold for some, hence for all* $\alpha > 0$.

4. Constructions and examples

We shall collect in the present section a number of examples of nonlinear Dirichlet forms. We also refer to [CG1] for an example of infinite dimensional nonlinear Dirichlet form on abstract Wiener spaces.

4.1. The *p*-Laplacian. The first example is the functional naturally associated with the p-Laplacian. It could also be seen as a particular case of the more general situation considered in Section 4.7, but we consider it separatly because of its special relevance and because the methods used here admit simple generalizations to much more general contexts. Consider a smooth and connected Riemannian manifold (M, g) without boundary, where g is the Riemannian metric, the associated Riemannian gradient ∇ and the Riemannian measure m_g. Define, for $p > 1$ the functional

$$\mathscr{E}_p(u) := \int_M |\nabla u(x)|_x^p m_g(\mathrm{d}x) \tag{4.1}$$

(with values in $[0, +\infty]$), where $|\cdot|_x$ denotes the length of vectors belonging to the tangent space T_xM, defined on the whole $\mathrm{L}^2(M, m_g)$, where we use the convention that the functional equals $+\infty$ if ∇u (in distributional sense) does not belong to $\mathrm{L}^p(TM)$, where

$L^p(TM)$ denotes the space of L^p sections of the tangent bundle. We notice that, *formally* $\mathcal{E}_p(u) = -(u, \triangle_p u)$, where

$$\triangle_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

is the so called p-Laplacian operator.

Theorem 4.1. *The functional $\mathcal{E}_p$ is a Dirichlet form for all $p > 1$.*

Proof. The convexity of the above functional is clear. As for the lower semicontinuity, take u_n converging in L^2 to a function u and consider the sequence $a_n = \mathcal{E}_p(u_n)$. Suppose that $a := \liminf_{n\to+\infty} a_n$ is finite, otherwise there is nothing to prove. Take any subsequence, still denoted by u_n, such that $\mathcal{E}_p(u_n) \to a$. Then the set $\{\nabla u_n\}_{n\in\mathbb{N}}$ is bounded in $L^p(TM)$ so that, since this latter space is reflexive, it is relatively weakly compact. We can then extract a subsequence, still indicated by u_n, such that ∇u_n is weakly convergent to an element $X \in L^p(TM)$. We shall prove that $X = \nabla u$ in de Rham distributional sense. In fact, denoting by $\langle\cdot\,,\cdot\rangle_x$ the scalar product on T_xM:

$$\begin{aligned}
&\lim_{n\to+\infty} \int_M \langle \nabla u_n(x), \chi(x)\rangle_x m_g(\mathrm{d}x) \\
&\quad = -\lim_{n\to+\infty} \int_M u_n(x) \operatorname{div}\chi(x) m_g(\mathrm{d}x) \\
&\quad = \int_M u(x) \operatorname{div}\chi(x) m_g(\mathrm{d}x)
\end{aligned}$$

for any $\chi \in C_c^\infty(TM)$, so that $X = \nabla u \in L^p(TM)$. By the weak lower semicontinuity of the L^p norm we thus have:

$$\mathcal{E}_p(u) \leqq \liminf_{n\to+\infty} \mathcal{E}_p(u_n)$$

with the present choice of u_n. This holds for all such $L^p(TM)$ weakly convergent subsequences of ∇u_n, which thus converge weakly to the same limit ∇u. Lower semicontinuity then holds.

To prove that $\mathcal{E}_p$ is a Dirichlet form, take $u, v \in L^2(M)$ and notice that

$$\begin{aligned}
\mathcal{E}_p^{(2)}[P_1(u,v)] &= \int_{u\leqq v} |\nabla u(x)|_x^p m_g(\mathrm{d}x) + \int_{u\leqq v} |\nabla v(x)|_x^p m_g(\mathrm{d}x) \\
&\quad + 2\int_{u>v} \left|\nabla\left(\frac{u+v}{2}\right)\right|_x^p m_g(\mathrm{d}x) \\
&\leqq \int_{u\leqq v} |\nabla u(x)|_x^p m_g(\mathrm{d}x) + \int_{u\leqq v} |\nabla v(x)|_x^p m_g(\mathrm{d}x) \\
&\quad + \int_{u>v} |\nabla u(x)|_x^p m_g(\mathrm{d}x) + \int_{u>v} |\nabla v(x)|_x^p m_g(\mathrm{d}x) \\
&= \int_M |\nabla u|_x^p m_g(\mathrm{d}x) + \int_M |\nabla v|_x^p m_g(\mathrm{d}x) \\
&= \mathcal{E}_p^{(2)}[u,v],
\end{aligned}$$

where we have used the convexity of the functional

$$\mathscr{E}_A[u] := \int_A |\nabla u(x)|_x^p m_g(\mathrm{d}x)$$

for any measurable set $A \subset M$. The condition

$$\mathscr{E}_p^{(2)}[P_{2,\alpha}(u,v)] \leqq \mathscr{E}_p^{(2)}[u,v]$$

is shown likewise, using in addition the fact that $\nabla(u+c) = \nabla u$ for all real constants c. □

4.2. The *p*-Laplacian with measurable coefficients. The setting of this subsection will be the same of the previous one, but we will consider the functionals associated to the operator formally given, in local coordinates, by

$$Hu := \sum_{i,j=1}^{d} \partial_i\big(a_{i,j}(\cdot)|\nabla u|^{p-2}\partial_j u\big),$$

where d is the dimension of M and $\{a_{i,j}\}$ is a positive symmetric matrix with locally integrable entries, satisfying suitable conditions to be made precise below, but not being required to be uniformly elliptic. More precisely, we shall consider the functional

$$\mathscr{E}_p^{(a)}(u) := \int_M a_x\big(\nabla u(x), \nabla u(x)\big)|\nabla u(x)|_x^{p-2} m_g(\mathrm{d}x) \tag{4.2}$$

where a is a measurable metric on TM.

Theorem 4.2. *Let a be a measurable metric on TM satisfying the following local strict ellipticity condition*

$$a_x\big(\xi(x), \xi(x)\big) \geqq \lambda_K g_x\big(\xi(x), \xi(x)\big) \tag{4.3}$$

for a.a. $x \in K$, *for all compact sets* $K \in M$, *for all smooth vector fields* ξ *on* M, *and for suitable* $\lambda_K > 0$. *Then the functional* $\mathscr{E}_p^{(a)}$ *is a Dirichlet form.*

Proof. The convexity is again clear, and the contraction properties can be proved exactly as in the above theorem. We now show the lower semicontinuity of the functional. As above, take u_n converging in L^2 to a function u and consider the sequence $c_n = \mathscr{E}_p^{(a)}(u_n)$. Suppose that $c := \liminf_{n\to+\infty} c_n$ is finite and consider any minimizing sequence, still indicated by u_n. By the local strict ellipticity, the set $\{\nabla u_n\}_{n\in\mathbb{N}}$ is locally bounded in $\mathrm{L}^p(TM)$ so that there exists a locally weakly convergent subsequence. Consider any such subsequence, still indicated by u_n and denote by X the weak limit (a priori depending on the subsequence) of ∇u_n. As above we can prove that $\nabla u_n \to \nabla u$ in de Rham distributional sense so that $X = \nabla u$. Again by the weak lower semicontinuity of the L^p-norm in any measure space we obtain

$$\int_K |\nabla u(x)|_x^p m_g(\mathrm{d}x) \leqq \liminf_{n\to+\infty} \int_K |\nabla u_n(x)|_x^p m_g(\mathrm{d}x)$$
$$\leqq \liminf_{n\to+\infty} \int_M |\nabla u_n(x)|_x^p m_g(\mathrm{d}x)$$

for all compact subsets K of M, whence the assertion follows. □

4.3. Derivations and Hilbert C*-modules. The setting of the present section is motivated by the following basic example. Let (M, g) be a Riemannian manifold as in the previous sections, and consider the space $E = C_0(TM)$ of continuous sections of TM vanishing at infinity. Then E is a module under pointwise multiplication in the fibers and is endowed with a natural $C_0(M)$-valued scalar product defined by

$$\langle \xi, \eta \rangle(x) = g_x\big(\xi(x), \eta(x)\big), \quad x \in M.$$

To generalize the above setting, we shall introduce the concept of Hilbert C*-monomodule, referring to [C] for a complete discussion. Let X be a locally compact Hausdorff space. We say that E is an inner product monomodule over $C_0(X)$ if there is an action of $C_0(X)$ over E (written equivalently on the right and on the left), and if E is endowed with a sesquilinear symmetric map $\langle \cdot, \cdot \rangle$ from $E \times E$ to $C_0(X)$ with the following properties:

- $\langle \xi, u\eta \rangle = u \langle \xi, \eta \rangle$ for all $u \in C_0(X)$, for all $\xi, \eta \in E$;
- $\langle \xi, \xi \rangle \geqq 0$ for all $\xi \in E$ and it equals zero if and only if $\xi = 0$.

A Hilbert C*-monomodule is an inner product monomodule which is complete under the seminorm

$$\|\xi\|_E := \| \, |\xi| \, \|_\infty, \quad \xi \in E$$

where we have defined $|\xi| := \langle \xi, \xi \rangle^{1/2}$.

To illustrate the next concept, let us come back to the motivating example, and recall that the Riemannian gradient ∇ is a closed linear operator from $\mathrm{L}^2(X, m_g)$ to $\mathrm{L}^2(TX, m_g)$, and that it is a *derivation* in the sense that it satisfies the *Leibniz rule*.

To generalize such example in the present setting, we first define the "L^p-spaces over E", $\mathrm{L}^p(E, \mu)$, μ being a finite Radon measure over X, as the completion of E under the norm

$$\|\xi\|_p := \| \, |\xi| \, \|_{\mathrm{L}^p(X,\mu)}.$$

The next central object will be an *E-valued derivation* $\partial : D(\partial) \to E$ defined on a dense subalgebra $D(\partial)$ of $C_0(X)$. It is required that ∂ is closable from $\mathrm{L}^2(X, \mu)$ to $\mathrm{L}^2(E, \mu)$, and that for all $u, v \in D(\partial)$ the Leibniz rule

$$\partial(uv) = u\partial v + v\partial u$$

holds true. Moreover, we define the "*Sobolev space*" $W^{1,p}(X, \partial)$ associated to the derivation at hand as the completion of the domain $D(\partial)$ under the norm

$$\|u\|_{W^{1,p}(X,\partial)} := \|u\|_{\mathrm{L}^2(X,m)} + \|\partial u\|_{\mathrm{L}^p(E,\mu)}.$$

It is required (if $p > 2$) that $W^{1,p}(X,\partial)$ is dense in $\mathrm{L}^2(X,m)$.

The energy functional we shall consider is:

$$\mathscr{E}_{p,\partial}(u) := \|\partial u\|^p_{\mathrm{L}^p(E,\mu)}$$

on $W^{1,p}(X,\partial)$, and $+\infty$ elsewhere in $\mathrm{L}^2(X,\mu)$.

We need a preliminary result.

Lemma 4.3. *The space $\mathrm{L}^p(E,\mu)$ is reflexive for all $p > 1$.*

Proof. We first notice that, by [C], p. 152, any Hilbert C^*-module can be canonically represented as the Hilbert C^* module of continuous sections of a continuous field of Hilbert spaces. If E is represented in such a way, then $\mathrm{L}^2(E,\mu)$ represents as the direct integral $\int_X^{\oplus} E_x \,\mathrm{d}\mu$ and therefore, if $a, b \in \mathrm{L}^2(X,\mu)$ are represented as $a = \big(a(x)\big)_{x\in X}$, $b = \big(b(x)\big)_{x\in X}$, one has:

$$(a,b)_{\mathrm{L}^2(E,\mu)} = \int_X \langle a(x), b(x)\rangle_x \,\mathrm{d}\mu(x).$$

The $\mathrm{L}^p(E,\mu)$ norms enjoy similar representations. We first consider the case $p \geqq 2$. We then notice that a Clarkson-type inequality holds true for the modulus function $|\cdot|$ on E by proceeding as in [B1], p. 59, also using the fact that $\langle\,\cdot\,\rangle$ is a sesquilinear symmetric map. To deal with the case $1 < p < 2$ one proceeds as in [B1], p. 60 using the above representation of $\mathrm{L}^p(E,\mu)$. □

Theorem 4.4. *The functional $\mathscr{E}_{p,\partial}$ is a Dirichlet form for all $p > 1$.*

Proof. The convexity being clear, we proceed to prove lower somicontinuity first. Let now $u_n \in W^{1,p}(X,\partial)$ be a sequence converging in $\mathrm{L}^2(X,m)$ to a function u, consider as usual the sequence $a_n := \mathscr{E}_{p,\partial}(u_n)$. If $\liminf_{n\to+\infty} a_n = +\infty$ there is nothing to prove. Otherwise suppose that $\liminf_{n\to+\infty} a_n = \alpha < +\infty$ and take any subsequence u_n such that $\mathscr{E}_{p,\partial}(u_n) \to \alpha$. Then $\{\partial u_n\}$ is a bounded sequence in $\mathrm{L}^p(E,\mu)$. It is easy to show that this latter space is a normed space, this making use of a Cauchy-Schwarz-like inequality. Moreover such space is reflexive as proved above.

Then, possibly by passing to a subsequence, we can assume that $\{\partial u_n\}$ converges weakly in $\mathrm{L}^p(E,\mu)$ to an element ξ of such space. Since the natural injection of $\mathrm{L}^p(E,\mu)$ in $\mathrm{L}^2(E,\mu)$ is continuous (because μ is finite), hence weakly continuous, it follows that $\{\partial u_n\}$ converges to ξ weakly in $\mathrm{L}^2(E,\mu)$ as well. Since ∂ is a closed operator, ξ equals ∂u and, by the weak lower semicontinuity of the norm function in any Banach space, we have $\mathscr{E}_{p,\partial}(u) \leqq \liminf_{n\to+\infty} \mathscr{E}_{p,\partial}(u_n)$. Hence lower semicontinuity follows.

As concerns the contraction property of $\mathscr{E}_{p,\partial} \oplus \mathscr{E}_{p,\partial}$ w.r.t. P_1 we first show that $\mathscr{E}_{p,\partial} \oplus \mathscr{E}_{p,\partial}$ is finite on $P_1(u,v)$ for all u, v belonging to the domain $W^{1,p}(X,\partial)$. In fact we shall use the following *chain rule*: if $w \in W^{1,p}(X,\partial) \cap C_0(X)$ and $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function vanishing at the origin and with bounded derivative, then

$$\partial f(w) = f'(w)\partial w.$$

This follows from Lemma 7.2 in [CS], where a more general version for derivations on bimodules is given. When applied to the special case of derivations on monomodules, it gives the above equality. Then

$$\mathscr{E}_{p,\partial}\big(f(w)\big) \leqq \|f'\|_{C_0(\mathbb{R})}\mathscr{E}_{p,\partial}(w).$$

By the assumptions and the lower semicontinuity of $\mathscr{E}_{p,\partial}$ the same property and estimate hold for $w \in W^{1,p}(X,\partial)$. Let now $g(x) = x \vee 0$, and choose f_n to be a sequence of C^1 functions vanishing at the origin, with $|f'| \leqq 1$ and such that $f_n(w) \to g(w)$ in $\mathrm{L}^2(X,m)$ given $w \in W^{1,p}(X,\partial)$ (see [FOT], p. 8). Then

$$\mathscr{E}_{p,\partial}\big(f_n(w)\big) \leqq \mathscr{E}_{p,\partial}(w)$$

and the lower semicontinuity of $\mathscr{E}_{p,\partial}$ then implies that

$$\mathscr{E}_{p,\partial}\big(g(w)\big) \leqq \mathscr{E}_{p,\partial}(w) < +\infty.$$

Recall that we can write

$$P_1(u,v) = \left(u - \frac{u \wedge v}{2}, v + \frac{u \vee v}{2}\right) = \left(u - \frac{1}{2}g(u-v), v + \frac{1}{2}g(u-v)\right)$$

for all $u, v \in \mathrm{L}^2(X,m)$. It then follows that $\mathscr{E}_{p,\partial}\big(P_1(u,v)\big)$ is finite.

We can assume that the functions f_n above also satisfy, besides the preceding assumptions, the following ones: $f_n(s) = s$ if $s > 0$, $f_n(s) = -1/n$ if $s \leqq -2/n$, $|f_n(s)| \leqq -1/n$ if $-2/n \leqq s < 0$. Define

$$P_1^n(u,v) = \left(u - \frac{f_n(u-v)}{2}, v + \frac{f_n(u-v)}{2}\right)$$

and the regions

$$A := \{u \geqq v\},$$

$$B_n := \left\{-\frac{2}{n} < u - v < 0\right\},$$

$$C_n := \left\{u - v \leqq -\frac{2}{n}\right\}.$$

Then $X = A \cup B_n \cup C_n$ for all n and, by using the convexity of the functional at hand in the penultimate step:

$$
\begin{aligned}
\mathscr{E}_p \oplus \mathscr{E}_{p,\partial}\big(P_1^n(u,v)\big) &= \mathscr{E}_{p,\partial}\left(u - \frac{f_n(u-v)}{2}\right) + \mathscr{E}_{p,\partial}\left(v + \frac{f_n(u-v)}{2}\right) \\
&= \int_X \left|\partial u - \frac{1}{2} f_n'(u-v)\partial(u-v)\right|^p \mathrm{d}\mu \\
&\quad + \int_X \left|\partial v + \frac{1}{2} f_n'(u-v)\partial(u-v)\right|^p \mathrm{d}\mu \\
&= \int_A \left|\partial\left(\frac{u+v}{2}\right)\right|^p \mathrm{d}\mu + \int_{C_n} |\partial u|^p \,\mathrm{d}\mu \\
&\quad + \int_{B_n} \left|\partial u - \frac{1}{2} f_n'(u-v)\partial(u-v)\right|^p \mathrm{d}\mu \\
&\quad + \int_A \left|\partial\left(\frac{u+v}{2}\right)\right|^p \mathrm{d}\mu + \int_{C_n} |\partial v|^p \,\mathrm{d}\mu \\
&\quad + \int_{B_n} \left|\partial v + \frac{1}{2} f_n'(u-v)\partial(u-v)\right|^p \mathrm{d}\mu \\
&\leqq \int_{A \cup C_n} (|\partial u|^p + |\partial v|^p)\,\mathrm{d}\mu + K \int_{B_n} (|\partial u|^p + |\partial v|^p)\,\mathrm{d}\mu \\
&\to \mathscr{E}_{p,\partial} \oplus \mathscr{E}_{p,\partial}(u,v)
\end{aligned}
$$

as $n \to +\infty$, because the measure of B_n tends to zero as n tends to infinity. The contraction property then follows again by the lower semicontinuity of $\mathscr{E}_p$ because $P_1^n(u,v)$ converges to $P_1(u,v)$ in $\mathrm{L}^2(X,m) \oplus \mathrm{L}^2(X,m)$.

The contraction property relative to $P_{2,\alpha}$ is proved likewise, by only using the fact that in addition $\partial 1 = 0$. $\square$

We notice that in [CG2] it has been shown that a class of quasilinear evolution equations driven by operators whose model is the *Euclidean* p-Laplacian is not only contractive on any L^p space, but even *ultracontractive* in the sense that it brings (continuously) L^q data into solutions which belong to L^∞ at all times. This does not make use of the Markov property, which is instead proved directly in that paper. In [CG3] it is shown how to extend such result to the evolution equation driven by the subdifferential of $\mathscr{E}_{p,\partial}$, using crucially the Markov property proved here.

4.4. Subriemannian structures and the subelliptic p-Laplacian. We specialize the above setting to discuss two particularly relevant examples: the *subriemannian p-Laplacian* and the special case of the *subelliptic p-Laplacian*.

We consider here a smooth, connected, orientable manifold M without boundary, and a distribution on M, that is a smooth subbundle of TM, say $\mathscr{D}$, such that the Lie algebra generated by $\mathscr{D}$ at any point $m \in M$ coincides with the tangent space T_mM. A Riemannian metric on $\mathscr{D}$ is a C^∞ real function on $\mathscr{D}$ such that each restriction of g on the fibers $\mathscr{D}(m)$ is a positive definite quadratic form. A *subriemannian structure* (in the sense of

R. S. Strichartz) on M is a couple $(\mathscr{D}, g)$ where $\mathscr{D}$ is a distribution on M and g a Riemannian metric on $\mathscr{D}$ (see [Gr] and references quoted).

Finally, let $d_{\mathscr{D}} f : \mathscr{D} \to \mathbb{R}$ be the restriction to $\mathscr{D}$ of the operator $df : TM \to \mathbb{R}$. Let us define g^* as the metric naturally associated to g on the cotangent space T^*M, and $\langle d_{\mathscr{D}} f(m) \rangle_{g^*}$ as the length, in the metric g^*, of $d_{\mathscr{D}} f(m)$. Let us choose a volume form v on M, and consider in the sequel all integrals w.r.t. to such form.

We shall consider, for any $p > 1$, the functional given by

$$\mathscr{E}_{p,\mathscr{D}}(u) = \int_M \langle d_{\mathscr{D}} u \rangle_{g^*}^p \, dv \tag{4.4}$$

on the Sobolev space $W_{\mathscr{D}}^{1,p}(M)$ and whose value is $+\infty$ otherwise in $L^2(M)$.

The semigroup associated to the above functional satisfies *formally* the following evolution equation:

$$\dot{u} = -d_{\mathscr{D}}^*(\langle d_{\mathscr{D}} f \rangle_{g^*}^{p-2} \, d_{\mathscr{D}} u) := \triangle_{p,\mathscr{D}} u. \tag{4.5}$$

The operator $\triangle_{p,\mathscr{D}}$ will be called *subriemannian p-Laplacian.*

The present setting falls within the previous discussion by choosing as C^*-module the space $C_0(\mathscr{D})$ of bounded continuous section of $\mathscr{D}$ vanishing at infinity, and as derivation ∂ the operator $d_{\mathscr{D}S}$. One then has:

Theorem 4.5. *The functional $\mathscr{E}_{p,\mathscr{D}}$ is a Dirichlet form for all $p > 1$.*

Example 4.6 (The subelliptic p-Laplacian)**.** A particularly relevant case is the following. Let $\{X_i\}_{i=1}^m$ be a collection of smooth vector fields on M satisfying the Hörmander condition, that is such that their brackets generate the tangent space at each point, and let $(\mathscr{D}, g)$ be the subriemannian structure canonically associated to it (see [Gr]). The *subelliptic p-Laplacian* is the operator formally given by

$$L_X u = \sum_{i=1}^m X_i(|Xu|^{p-2} X_i u),$$

where $|Xu| := \left(\sum_{i=1}^m |X_i u|^2 \right)^{1/2}$. As before, to give sense to it one defines the functional

$$\mathscr{E}_{p,X}(u) := \int_M |Xu|^p \, dv.$$

4.5. Γ-convergence and closability. We recall some basic definition on Γ-convergence: see [DM] for a complete reference.

Definition 4.7. Let X be a topological space and, for any $x \in X$, let us denote by $\mathscr{U}(x)$ the set of all open neighbourhoods of x. Let $\mathscr{E}_n$ be a sequence of mappings from X to $(-\infty, +\infty]$. Then the Γ-lower (resp. upper) limit of $\mathscr{E}_n$ is defined

$$\left(\Gamma\text{-}\liminf_{n\to+\infty} \mathscr{E}_n\right)(x) := \sup_{U\in\mathscr{N}(x)} \liminf_{n\to+\infty} \inf_{y\in U} \mathscr{E}_n(y)$$

(resp.

$$\left(\Gamma\text{-}\limsup_{n\to+\infty} \mathscr{E}_n\right)(x) := \sup_{U\in\mathscr{N}(x)} \limsup_{n\to+\infty} \inf_{y\in U} \mathscr{E}_n(y)).$$

If these two quantities coincide and equal, say, $\mathscr{E}(x)$, we say that $\mathscr{E}_n$ Γ-converges to $\mathscr{E}$.

Theorem 4.8. *Let $\mathscr{E}_n$ be a sequence of positive convex functionals on $\mathscr{H} := \mathrm{L}^2(X,m)$, satisfying the contraction properties* (3.2), (3.3) *and Γ-converging to a functional $\mathscr{E}$ on $\mathrm{L}^2(X,m)$. Then $\mathscr{E}$ is a Dirichlet form.*

Proof. The lower semicontinuity of $\mathscr{E}$ is standard, see [DM], Prop. 6.8. As for the convexity, it is known that it holds for the Γ-upper limit (see [DM], Theorem 11.1) and hence for the Γ-limit, here assumed to exist. Therefore we are left with verifying the contraction properties for the Γ-limit. Since the Borel space X has been assumed to be countably generated, the Hilbert space $\mathrm{L}^2(X,m)$ is separable as well as $\mathrm{L}^2(X,m)\oplus\mathrm{L}^2(X,m)$, and hence the well-known sequential characterization of Γ-limits holds (see [DM], Ch. 4). In fact, let us define for all sequences of functionals $\mathscr{E}_n$

$$\mathscr{E}' = \Gamma\text{-}\liminf_{n\to+\infty} \mathscr{E}_n, \quad \mathscr{E}'' = \Gamma\text{-}\limsup \mathscr{E}_n.$$

Then, for all sequences x_n strongly converging to $x \in \mathscr{H}$,

$$\mathscr{E}'(x) \leqq \liminf_{n\to+\infty} \mathscr{E}_n'(x_n)$$

and there is at least one such sequence such that

$$\mathscr{E}'(x) = \liminf_{n\to+\infty} \mathscr{E}_n'(x_n).$$

Similarly, for all such sequences

$$\mathscr{E}''(x) \leqq \limsup_{n\to+\infty} \mathscr{E}_n''(x_n)$$

and there is at least one such sequence such that

$$\mathscr{E}''(x) = \limsup_{n\to+\infty} \mathscr{E}_n''(x_n). \tag{4.6}$$

Let $\mathscr{E}_n$ be a sequence of functionals satisfying the running assumptions. Let $\{(u_n, v_n)\}$ be a sequence strongly converging to (u,v) and such that (4.6) holds. Let Q be any of the projections involved in the definition of Dirichlet forms. We denote by Q_1 and Q_2 its components. We notice that, by the continuity of Q, the sequences $Q_1(u_n,v_n)$ (resp. $Q_2(u_n,v_n)$) converge to $Q_1(u,v)$ (resp. $Q_2(u,v)$). Then

$$\begin{aligned}
\mathscr{E}^{(2)}\big(P(u,v)\big) &= \mathscr{E}\big(Q_1(u,v)\big) + \mathscr{E}\big(Q_2(u,v)\big) \\
&\leqq \liminf_{n\to+\infty} \mathscr{E}_n\big(Q_1(u_n,v_n)\big) + \liminf_{n\to+\infty} \mathscr{E}_n\big(Q_2(u_n,v_n)\big) \\
&\leqq \liminf_{n\to+\infty} \big(\mathscr{E}_n\big(Q_1(u_n,v_n)\big) + \mathscr{E}_n\big(Q_2(u_n,v_n)\big)\big) \\
&= \liminf_{n\to+\infty} \mathscr{E}_n^{(2)}\big(Q(u_n,v_n)\big) \\
&\leqq \liminf_{n\to+\infty} \mathscr{E}_n^{(2)}(u_n,v_n) \\
&\leqq \limsup_{n\to+\infty} \mathscr{E}_n^{(2)}(u_n,v_n) \\
&= \limsup_{n\to+\infty} \big(\mathscr{E}_n(u_n) + \mathscr{E}_n(v_n)\big) \\
&\leqq \limsup_{n\to+\infty} \mathscr{E}_n(u_n) + \limsup_{n\to+\infty} \mathscr{E}_n(v_n) \\
&\leqq \mathscr{E}''(u) + \mathscr{E}''(v) \\
&= \mathscr{E}(u) + \mathscr{E}(v) = \mathscr{E}^{(2)}(u,v). \quad \square
\end{aligned}$$

To state an immediate corollary to the above theorem, we recall that, given a functional $\mathscr{E}$, its relaxed functional $\mathrm{sc}^-\mathscr{E}$ is defined as follows:

$$\mathrm{sc}^-\mathscr{E} := \sup_G G$$

where the supremum is taken over all lower semicontinuous functionals such that $G \leqq \mathscr{E}$.

Corollary 4.9. *The relaxed functional* $\mathrm{sc}^-\mathscr{E}$ *of a positive convex functional* (*not necessarily lower-semicontinuous*) $\mathscr{E}$, *satisfying the contraction properties* (3.2), (3.3), *is a Dirichlet form.*

Proof. It suffices to notice that $\mathrm{sc}^-\mathscr{E}$ is the Γ-limit of the constant sequence of functionals $\mathscr{E}_n = \mathscr{E}$ (see again [DM]). $\square$

We conclude this section by giving a simple criterion for a functional to admit an extension which is a Dirichlet form. First we say that a convex functional $\mathscr{E} : \mathrm{L}^2(X,m) \to [0,+\infty]$ is *closed* if it is lower semicontinuous, and that it is *closable* if it admits a lower semicontinuous extension, i.e. if there exists a lower semicontinuous $\tilde{\mathscr{E}} : \mathrm{L}^2(X,m) \to [0,+\infty]$ such that

$$\tilde{\mathscr{E}}(u) = \mathscr{E}(u)$$

whenever $\mathscr{E}(u)$ is finite.

Corollary 4.10. *Assume that* $\mathscr{E} : D(\mathscr{E}) \to [0,+\infty]$ *is a functional defined only on a convex and dense set* $D(\mathscr{E})$, *and assume that it is convex and lower semicontinuous. Denote again by* $\mathscr{E}$ *the functional on* $\mathrm{L}^2(X,m)$ *obtained by extending* $\mathscr{E}$ *to be* $+\infty$ *in* $\mathrm{L}^2(X,m)\backslash D(\mathscr{E})$. *Then the relaxed functional* $\mathrm{sc}^-\mathscr{E}$ *is a convex and lower semicontinuous extension of* $\mathscr{E}$, *so* *that* $\mathscr{E}$ *is closable. Moreover if* $\mathscr{E}$ *enjoys the contraction properties* (3.2) *and* (3.3) *on* $D(\mathscr{E})$ *then* $\mathrm{sc}^{-}\mathscr{E}$ *is a Dirichlet form.*

Proof. One needs only to observe that the very definition of relaxed functional and the lower semicontinuity assumption of $\mathscr{E}$ on $D(\mathscr{E})$ imply that $\mathrm{sc}^{-}\mathscr{E}$ is an extension of $\mathscr{E}$, the other statements being known from the above corollary. □

4.6. Convex functionals of ∇u. We shall notice here that the following theorem is an immediate consequence of the form of the projections $P_1, P_{2,\alpha}$ and of well-known results (see [DM] and references quoted). Hereafter, Ω is a Euclidean domain.

Theorem 4.11. *Let* $\mathscr{E}$ *be a functional on* $\mathrm{L}^2(\Omega)$ *defined by*

$$\mathscr{E}(u) = \int_\Omega \varphi(\nabla u(x)) m_g(\mathrm{d}x) \tag{4.7}$$

whenever u *is a* C_c^1 *function on* Ω, *and* $+\infty$ *otherwise, where* φ *is a positive convex function. Then* $\mathscr{E}$ *is closable and its closure is a Dirichlet form.*

Proof. The convexity is obvious. As for the lower semicontinuity in C^1 with respect to the L^2-topology, this is a special case of the results of [DBD]. The closability then follows by Corollary 4.10, and the contraction properties follow from that corollary as well by using the strategy outlined in Section 4.1, using the approximation procedure outlined in Section 4.3. □

Example 4.12 (The area functional)**.** As a special example, we notice that the theorem covers the case in which $\mathscr{E}$ is the *area functional* obtained by relaxing the functional defined initially on C_c^1 by

$$\mathscr{E}(u) = \int_M \sqrt{1 + |\nabla u|_x^2}\, m_g(\mathrm{d}x).$$

4.7. The perimeter functional. Let Ω be an open subset of $\mathbb{R}^n$. Define the functional

$$\mathscr{E}(u) = \int_\Omega |Du|, \quad u \in BV(\Omega) \cap \mathrm{L}^2(\Omega) \tag{4.8}$$

and $+\infty$ otherwise in $\mathrm{L}^2(\Omega)$, where Du is the vector-valued Radon measure representing the distributional derivative of $u \in BV(\Omega)$.

If Ω is bounded and has a Lipschitz boundary, define the functional

$$\mathscr{E}_0(u) = \int_\Omega |Du| + \int_{\partial\Omega} |\mathrm{tr}_\Omega\, u| \,\mathrm{d}\mathscr{H}^{n-1}, \quad u \in BV(\Omega) \cap \mathrm{L}^2(\Omega) \tag{4.9}$$

and $+\infty$ otherwise in $\mathrm{L}^2(\Omega)$, where $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure and tr_Ω is the usual trace operator. We recall (see [Gi]) that the BV function has a well-defined trace which belongs to $\mathrm{L}^1(\partial\Omega, \mathscr{H}^{n-1})$ if Ω is bounded and has a Lipschitz boundary.

It is well-known that if u is the indicator function of a Borel set $E \subset \Omega$ then $\mathscr{E}(u)$ is the definition of the *perimeter* of E in Ω. If it is finite for all bounded open sets Ω then E is called a Caccioppoli set (see [Gi]).

We are going to see that, like its close relative, the area functional, the perimeter functional generates a "good" evolution in all L^p spaces with $p \in [1, \infty]$.

Theorem 4.13. *The functionals $\mathscr{E}$ and $\mathscr{E}_0$ are Dirichlet forms.*

Proof. The functionals under consideration are the restrictions to L^2 of the functionals defined by the same formula on L^1_{loc}. Such functionals are obtained by relaxation in L^1_{loc} of the functionals

$$\mathscr{E}'(u) = \int_\Omega |\nabla u| \, dx, \quad u \in C^1(\Omega),$$

$$\mathscr{E}_0'(u) = \int_\Omega |\nabla u| \, dx, \quad u \in C^1_c(\Omega),$$

respectively, see Example 3.14 of [DM] and [Gi]. Then this implies that $\mathscr{E}$ and $\mathscr{E}_0$ are lower semicontinuous in L^2 since they enjoy this property w.r.t. the L^1_{loc} topology. Moreover, they also enjoy the relevant contraction properties: it suffices to proceed as in the proof of Theorem 4.8 noticing in addition that, if $(u_n, v_n) \to (u, v)$ in $L^2 \oplus L^2$ then $(Pu_n, Pv_n) \to (Pu, Pv)$ in $L^2 \oplus L^2$ and then also in $L^1_{\mathrm{loc}} \oplus L^1_{\mathrm{loc}}$. $\square$

5. Markovian approach to minimal surfaces and to solutions of Dirichlet problems

5.1. Minimal surfaces. One of the main problems in the theory of minimal surfaces (see e.g. [Gi]) is the following: given a bounded open set $\Omega \subset \mathbb{R}^n$ and a function φ on the (sufficiently regular) boundary $\partial\Omega$, find a function u (of bounded variation) on Ω whose boundary values coincide with φ (in the sense of traces) and whose graph has minimal area.

This problem has solutions under very minimal conditions on the data and, moreover, the minimizer is unique under stronger conditions.

Our aim is to contruct evolutions of Markov type (in particular order preserving) which deform surfaces in the course of time and eventually approach the minimal surfaces. The construction will then show that in the present setting the Markov conditions have a precise geometric meaning.

Let then Ω be a bounded Euclidean domain with Lipschitz boundary and $\varphi \in L^1(\partial\Omega, \mathscr{H}^{n-1})$. The functional we are dealing with is the following: $\mathscr{E} : L^2(\Omega) \to [0, +\infty]$ is given by

$$\mathscr{E}(u) = \int_\Omega \sqrt{1 + |Du|^2} + \int_{\partial\Omega} |\mathrm{tr}_\Omega\, u - \varphi| \, d\mathscr{H}^{n-1} \tag{5.1}$$

for $u \in BV(\Omega)$, and $+\infty$ otherwise. We recall that, by definition, for $u \in BV(\Omega)$ one defines

$$\int_\Omega \sqrt{1+|Du|^2} := \sup\left\{ \int_\Omega g_{n+1}\,\mathrm{d}x - \sum_{i=1}^{n} \int_\Omega u\partial_i g\,\mathrm{d}x : \right.$$
$$\left. g = (g_1,\dots,g_{n+1}) \in C_c^1(\Omega,\mathbb{R}^{n+1}), |g| \leqq 1 \right\}.$$

For regular functions the latter object indeed coincides with the integral

$$\int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x,$$

so that the above functional is a natural extension of the one considered in Example 4.12.

We also recall that the quantity $\int_\Omega \sqrt{1+|Du|^2}$ is interpreted, for any $u \in BV(\Omega)$ as the *area* of the graph $\Sigma(u)$ of u: moreover the second term in the definition of $\mathscr{E}$ represents the area of the portion $S(u,\varphi)$ of the cylinder with basis $\partial\Omega$ between the graphs of $\mathrm{tr}_\Omega\, u$ and φ, so that the whole value $\mathscr{E}(u)$ represents the area of $\Sigma(u)$ plus the area of $S(u,\varphi)$.

Under the running assumptions, the functional $\mathscr{E}$ admits a minimum. The minizers are then called *minimal surfaces* with fixed contour φ (see [Gi]).

Theorem 5.1. *The functional $\mathscr{E}$ is a Dirichlet form.*

Proof. Convexity and lower semicontinuity are well-known, so that we will only deal with the verification of the contraction properties. The functional

$$\mathscr{E}_1(u) = \int_\Omega \sqrt{1+|Du|^2}$$

on BV and $+\infty$ otherwise, is the relaxed functional of the functional defined likewise on $C^1(\Omega)$ functions, and $+\infty$ otherwise. For this latter functional the contraction property hold, by Theorem 4.11, so that $\mathscr{E}_1$ is a Dirichlet form because of Corollary 4.9.

We now show that the trace operator commutes with Lipschitz functional calculus: namely, for any Lipschitz continuous function $p:\mathbb{R}\to\mathbb{R}$ and $u \in BV(\Omega)$, then $p(u) \in BV(\Omega)$ and

$$\mathrm{tr}_\Omega\big(p(u)\big) = p(\mathrm{tr}_\Omega\, u).$$

The stability of $BV(\Omega)$ under Lipschitz functional calculus is clear from the very definitions. As for the trace formula, we use the fact that $\mathrm{tr}_\Omega\, u$ is characterized by the condition:

$$\lim_{r\to 0} r^{-n} \int_{B(x,r)\cap\Omega} |u(z) - \mathrm{tr}_\Omega\, u(x)|\,\mathrm{d}x = 0$$

for $\mathscr{H}^{n-1}$ almost all $x \in \partial\Omega$. We then notice that, if k is the Lipschitz constant of p, then

$$|p(u)(z) - p(\mathrm{tr}_\Omega\, u)(x)| = \big|p\big(u(z)\big) - p\big(\mathrm{tr}_\Omega\, u(x)\big)\big|$$
$$\leqq k|u(z) - \mathrm{tr}_\Omega\, u(x)|.$$

We can now deal with the contraction properties for

$$\mathscr{E}_2(u) = \|\mathrm{tr}_\Omega\, u - \varphi\|_{\mathrm{L}^1(\partial\Omega,\, \mathscr{H}^{n-1})}.$$

Notice indeed that, given p as above, and defining $\|\cdot\|_1$ to be the norm in $\mathrm{L}^1(\partial\Omega, \mathscr{H}^{n-1})$:

$$\begin{aligned}
&\mathscr{E}_2\big(u - p(u-v)\big) + \mathscr{E}_2\big(v + p(u-v)\big)\\
&\quad = \big\|\mathrm{tr}_\Omega\big(u - p(u-v)\big) - \varphi\big\|_1 + \big\|\mathrm{tr}_\Omega\big(v + p(u-v)\big) - \varphi\big\|_1\\
&\quad = \|u' - p(u'-v') - \varphi\|_1 + \|v' + p(u'-v') - \varphi\|_1,
\end{aligned}$$

where we have used the linearity of the trace and have defined $u' = \mathrm{tr}_\Omega\, u$ and $v' = \mathrm{tr}_\Omega\, v$. By the explicit form of the projections we are dealing with, it then suffices to verify the relevant contraction property for the (convex l.s.c.) functional

$$\mathscr{E}'(w) = \|w - \varphi\|_1.$$

This is achieved by noting, using a strategy of [BC], that defining

$$\lambda = I_{u \neq v} \frac{p(u-v)}{u-v}$$

one has $\lambda \in [0,1]$ with the choice of p appearing in the contraction properties (3.2) and (3.3), so that the convexity of the integrand appearing in the definition of $\mathscr{E}'$ implies

$$\begin{aligned}
&\mathscr{E}'\big(u - p(u-v)\big) + \mathscr{E}'\big(v + p(u-v)\big)\\
&\quad = \int_{\partial\Omega} |\lambda v + (1-\lambda)u - \varphi|\,\mathrm{d}\mathscr{H}^{n-1} + \int_{\partial\Omega} |\lambda u + (1-\lambda)v - \varphi|\,\mathrm{d}\mathscr{H}^{n-1}\\
&\quad \leqq \int_{\partial\Omega} |u - \varphi|\,\mathrm{d}\mathscr{H}^{n-1} + \int_{\partial\Omega} |v - \varphi|\,\mathrm{d}\mathscr{H}^{n-1}\\
&\quad = \mathscr{E}'(u) + \mathscr{E}'(v). \quad \square
\end{aligned}$$

The Markov property for the present evolution has an intuitive geometric description, given in the following corollary. Given a $BV(\Omega)$ function u we shall denote by $\Sigma(t,u)$ the graph of the function $T_t u$, where $\{T_t : t \geqq 0\}$ is the Markov semigroup associated to $\mathscr{E}$. Also, the notation $\Sigma \leqq \Sigma'$ means that the surface Σ lies below Σ'.

Corollary 5.2. *The area of the time evolved surfaces* $\Sigma(t,u)$ *is finite for any* $t > 0$ *and* $u \in \mathrm{L}^2(\Omega)$ *(even when* $\Sigma(0,u)$ *has infinite area).*

Moreover, if

$$\Sigma(0,u) \leqq \Sigma(0,v)$$

then

$$\Sigma(t,u) \leqq \Sigma(t,v) \quad \forall t > 0.$$

In other words, if a surface Σ *lies below another surface* Σ', *the same property holds for the corresponding time evolved surfaces at any time.*

Moreover, let $v \in BV(\Omega)$ *be a minimal surface and the cylinder* $\Gamma(h,v)$ *be defined, for any* $h > 0$, *by*

$$\Gamma(h,v) = \{(x,s) \in \Omega \times \mathbb{R} : |s - v(x)| \leqq h\}.$$

If $h > 0$ *is such that*

$$\Sigma(0,u) \subset \Gamma(h,v)$$

then

$$\Sigma(t,u) \subset \Gamma(h,v) \quad \forall t \geqq 0.$$

In other words, if a surface lies in a cylinder of height $2h$ *constructed as above from a minimal surface* $\Sigma(0,v)$, *then so do the evolved surfaces at any time.*

Finally, let $S(t,u)$ *with* $t \geqq 0$, $u \in BV(\Omega)$ *be the portion of the cylinder with basis* $\partial\Omega$ *which lies between the graphs of* $\mathrm{tr}_\Omega(T_t u)$ *and* φ, *so that*

$$S(t,u) = \big\{(x,s) \in \partial\Omega \times \mathbb{R} : \min\big(\mathrm{tr}_\Omega(T_t u)(x), \varphi(x)\big) \leqq s \leqq \max\big(\mathrm{tr}_\Omega(T_t u)(x), \varphi(x)\big)\big\}.$$

Then the area of $\Sigma(t,u) \cup S(t,u)$ *decreases in time.*

Proof. Notice that $\mathscr{E}(T_t u) < +\infty$ for all $t > 0$ and $u \in \mathrm{L}^2(\Omega)$, so that $T_t u \in BV(\Omega)$ for all $t > 0$ as stated. The first mentioned property is just the order preserving property, while the second one coincides with non-expansion in L^∞ for the evolution considered. The last statement amounts to saying that the energy functional decreases along the evolution (see [S]). □

The non-expansion property in L^∞ can also be rewritten in analytical terms, when v is a minimal surface, as

$$\|T_t u - v\|_\infty \leqq \|u - v\|_\infty$$

for all $t > 0$ so that, applying such bound to $u = T_s w$ one readily concludes that

$$\|T_t u - v\|_\infty$$

is *decreasing*. One then wonders whether there is reasonable sense in which the time evolved surfaces *approach* a minimal surface. The answer is positive, and to this end we start with some general results on the asymptotic properties of the semigroup, of independent interest, which can be seen as a development upon a result of [Bru].

Proposition 5.3. *Let $\mathscr{E} : \mathscr{H} \to [0, +\infty]$ be a convex, lower semicontinuous functional on a Hilbert space $\mathscr{H}$. Suppose that there is a bounded orbit for $\{T_t : t \geqq 0\}$. Then $\mathscr{E}$ has a minimum. Moreover, the limit points for $t \to +\infty$, in the weak topology of $\mathscr{H}$, of any orbit $\{T_t u : t \geqq 0\}$, are minimizers of $\mathscr{E}$.*

Proof. Given $u \in L^2(X, m)$, the set $\{T_t u : t \geqq 0\}$ is by assumption bounded in $L^2(X, m)$, so that there exists a sequence $v(t_n)$ converging weakly in $L^2(X, m)$ to a limit v. It is known ([B2], Theorem 3.10) that

$$\partial\mathscr{E}^{\circ}\big(u(t)\big) \to 0 \quad \text{as } t \to +\infty$$

in strong L^2 sense, where A° denotes as usual the principal section of a monotone operator A (see [B2]). This shows that

$$\lim_{n \to +\infty} \big(u(t_n), \partial\mathscr{E}^{\circ}\big(u(t_n)\big)\big) = 0$$

so that, by Prop. 2.5 of [B2],

$$\partial\mathscr{E}^{\circ}(v) = 0.$$

This implies that v is a fixed point of the semigroup, and, by Theorem 3.4, it follows that v is a minimizer of $\mathscr{E}$. $\square$

Corollary 5.4. *A convex, lower semicontinuous functional $\mathscr{E} : \mathscr{H} \to [0, +\infty]$ has a minimum point if and only if the corresponding semigroup has a bounded orbit.*

The final result of this subsection will show that, in a suitable sense, the time evolved surfaces really approach the minimal surfaces. Hereafter we shall use the notation u to denote both a function and its graph.

Theorem 5.5. *Let $\mathscr{E}$ be the Dirichlet form defined in (5.1) and consider a generic orbit $\{T_t u : t \geqq 0\}$ of the associated semigroup, for $u \in L^2(\Omega)$. Then all the limit points as $t \to \infty$ of such an orbit, in the weak topology of $L^2(\Omega)$, are minimizers of $\mathscr{E}$ and are limit points in the strong topology of $L^1(\Omega)$ as well.*

Finally, if the boundary contour φ is continuous, then all the orbits $\Sigma(t, u)$ converge strongly in $L^1(\Omega)$ as $t \to +\infty$ to the unique minimal surface v. If the initial surface belongs to $L^\infty(\Omega)$ and $\partial\Omega$ has nonnegative mean curvature in the sense of [Gi], Def. 15.6, the convergence takes place in all $L^p(\Omega)$ spaces, $p \in [1, +\infty)$.

Proof. The first part of the theorem is an application of Proposition 5.3 as far as the weak topology of L^2 is involved. As for the limits in the strong L^1-topology, just notice that a sequence $T_{t_n} u$ converging weakly in $L^2(\Omega)$ to a minimizer v is a bounded set in $BV(\Omega)$ so that it is precompact in $L^1(\Omega)$ (see [Gi]).

If φ is continuous it is known that the minimal surface is unique (see [Gi]). This implies that all the above limit points coincide so that the limit in $\mathrm{L}^1(\Omega)$ exists and equals v. If moreover the boundary has nonnegative mean curvature then the minimal surface is continuous in $\bar{\Omega}$ so that there exists a bounded fixed point for $\{T_t : t \geqq 0\}$, so that all orbits corresponding to an initial surface in $\mathrm{L}^\infty(\Omega)$ are bounded, and hence convergence in $\mathrm{L}^1(\Omega)$ implies convergence in $\mathrm{L}^p(\Omega)$ for such data. □

Remark 5.6. The contraction properties of the functional considered in the present subsection have a geometrical meaning. Indeed, let $\Sigma(v)$ be a minimal surface with fixed contour, represented by the function v, and let $\Sigma(u)$ be another surface with the same boundary contour. Let also $p : \mathbb{R} \to \mathbb{R}$ be one of the Lipschitz functions appearing in the definition of Dirichlet form. Then:

$$\mathscr{E}\big(u - p(u - v)\big) \leqq \mathscr{E}(u),$$
$$\mathscr{E}\big(v + p(u - v)\big) \leqq \mathscr{E}(u).$$

In fact, for example

$$\begin{aligned}\mathscr{E}\big(u - p(u - v)\big) + \mathscr{E}(v) &\leqq \mathscr{E}\big(u - p(u - v)\big) + \mathscr{E}\big(v + p(u - v)\big)\\ &\leqq \mathscr{E}(u) + \mathscr{E}(v).\end{aligned}$$

This can be interpreted by saying that the map which associates to the surface $\Sigma(u)$ the new surface $\Sigma\big(u - p(u - v)\big)$ decreases areas.

5.2. The Dirichlet problem. Let us consider a bounded open domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary and let φ belong to the fractional Sobolev space $H^{1/2}(\partial\Omega)$. Let also u_φ be the unique solution to the Dirichlet problem

$$\begin{cases}\triangle u_\varphi = 0 & \text{if } x \in \Omega,\\ u_\varphi = \varphi & \text{if } x \in \partial\Omega.\end{cases}$$

We shall construct here a nonlinear Dirichlet form whose associated nonlinear Markov semigroup allows to approach, as the time variable tends to $+\infty$, the solution u_φ. The functional under consideration is the following:

$$\mathscr{E}_\varphi(u) = \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \infty \cdot \int_{\partial\Omega} |\mathrm{tr}_\Omega\, u - \varphi|^2 \,\mathrm{d}\mathscr{H}^{n-1}$$

on the Sobolev space $H^1(\Omega)$, and $+\infty$ otherwise in $\mathrm{L}^2(\Omega)$, where tr_Ω is the usual trace operator in $H^1(\Omega)$ and $\mathscr{H}^{n-1}$ is the Hausdorff measure on the boundary. This is just a symbol for defining $\mathscr{E}_\varphi$ to be the functional given by

$$\mathscr{E}_\varphi(u) = \int_\Omega |\nabla u|^2 \,\mathrm{d}x$$

on those H^1 function whose boundary trace is φ, and $+\infty$ elsewhere. Clearly $\mathscr{E}$ is convex and lower semicontinuous.

We can call the associated semigroup *the heat semigroup on* Ω *with nonlinear boundary conditions.*

Theorem 5.7. *The functional* $\mathcal{E}_\varphi$ *is a Dirichlet form and the associated Markov semigroup* $\{T_t : t \geqq 0\}$ *is ergodic in the sense that*

$$T_t v \xrightarrow[t \to +\infty]{} u_\varphi \quad \forall v \in \mathrm{L}^2(\Omega)$$

where the convergence takes place strongly in any $\mathrm{L}^p(\Omega)$ *with* $p \in [1, 2n/(n-2))$. *If the initial datum belongs to* $\mathrm{L}^\infty(\Omega)$, *the convergence takes place strongly in any* $\mathrm{L}^p(\Omega)$ *with* $p \in [1, +\infty)$.

Proof. The convexity is clear. Lower semicontinuity follows from the fact that the trace operator is bounded from $H^1(\Omega)$ to $\mathrm{L}^2(\partial\Omega)$. The relevant contraction properties are obvious as soon as one notices that the projections involved leave the domain of $\mathcal{E}$ invariant.

The limiting properties follow from the well-known Rellich-Kondrachov compact embedding of $H^1(\Omega)$ in $\mathrm{L}^p(\Omega)$ with $p \in [1, 2n/(n-2))$, proceeding as in the proof of Theorem 5.1. □

6. Comparison of semigroups

We shall prove in this section a domination principle between the nonlinear semigroup associated to an integral functional of the gradient, and the nonlinear semigroup obtained by a quasilinear perturbation of the original semigroup.

To this end let $\{S_t : t \geqq 0\}$ be a strongly continuous, contraction semigroup (in general nonlinear) in $\mathrm{L}^2(X, m)$, which is also *order preserving* in the sense of Section 2. Let $\{T_t : t \geqq 0\}$ be another strongly continuous, contraction semigroup in $\mathrm{L}^2(X, m)$. We look for conditions ensuring that a semigroup T_t is *dominated* by S_t, in the sense that $|T_t u| \leqq S_t|u|$ for all $u \in \mathrm{L}^2(X, m)$. In fact, such property is characterized as follows for semigroups associated to convex lower semicontinuous functionals: we refer to [Ba] for more details.

Theorem 6.1. *Let* S_t *be a strongly continuous, contraction and order preserving semigroup on* $\mathcal{H} := \mathrm{L}^2(X, m)$ *associated to a convex lower semicontinuous functional* $\mathcal{E}_S$, *and* T_t *be another strongly continuous contraction semigroup on* $\mathrm{L}^2(X, m)$ *associated to a convex lower semicontinuous functional* $\mathcal{E}_T$. *Then the following conditions are equivalent*:

•

$$|T_t u| \leqq S_t |u|$$

for all $u \in \mathrm{L}^2(X, m)$.

•

$$\mathscr{E}_T \oplus \mathscr{E}_S\big(Q(u,v)\big) \leqq \mathscr{E}_T \oplus \mathscr{E}_S(u,v)$$

for all $u, v \in \mathrm{L}^2(X,m)$, *where* Q *is the Hilbert projection in the Hilbert space* $\mathrm{L}^2(X,m) \oplus \mathrm{L}^2(X,m)$ *onto the closed and convex set*

$$\mathscr{K} := \{(u,v) : |u| \leqq v\}.$$

Proof. In view of Theorem 3.3 it suffices to prove that the first assertion is equivalent to the fact that $T_t \oplus S_t$ leaves $\mathscr{K}$ invariant. In fact, if this latter property holds, since $(u,|u|)$ belongs to $\mathscr{K}$ it follows that $|T_t u| \leqq S_t|u|$. For the converse, notice that if $(u,v) \in \mathscr{K}$

$$|T_t u| \leqq S_t|u| \leqq S_t v$$

because S_t is order preserving, so that

$$T_t \oplus S_t(u,v) = (T_t u, S_t v) \in \mathscr{K}. \quad \square$$

By proceeding exactly as in Section 3, one gets the explicit expression of the projection appearing above. In fact we have:

Proposition 6.2. *The projection onto the closed and convex set* $\mathscr{K}$ *is given by*

$$Q(u,v) = \begin{cases} (u,v) & \text{if } |u| \leqq v, \\ (0,0) & \text{if } |u| \leqq -v, \\ \big((u+v)/2, (u+v)/2\big) & \text{if } u > |v|, \\ \big((u-v)/2, (v-u)/2\big) & \text{if } u < -|v|. \end{cases} \tag{6.1}$$

We then have, for any Riemannian manifold M, the above mentioned comparison theorem between semigroups associated to integral functionals of the gradient and their quasilinear perturbations.

Theorem 6.3. *Consider the functional*

$$\mathscr{E}(u) := \int_M \varphi\big(\nabla u(x)\big) m_g(\mathrm{d}x)$$

where $\varphi : TM \to [0, +\infty)$ *is such that its restriction to every fibre is convex. Define also*

$$\mathscr{E}_\psi(u) := \mathscr{E}(u) + \int_M \psi(|u|) V(x) m_g(\mathrm{d}x),$$

V *being a positive measurable function and* ψ *being positive and monotonically increasing. Then the semigroup associated to* $\mathscr{E}_\psi$ *is dominated by the semigroup associated to* $\mathscr{E}$.

Proof. We compute:

$$\mathscr{E}_\psi \oplus \mathscr{E}\big(Q(u,v)\big) = \int_{|u| \leqq v} \big(\varphi(\nabla u) + \psi(|u|)V(x)\big)m_g(\mathrm{d}x)$$

$$+ \int_{|v|<u} \Big(\varphi\big(\nabla(u+v)/2\big) + \psi\big(|(u+v)/2|\big)V(x)\Big)m_g(\mathrm{d}x)$$

$$+ \int_{u<-|v|} \Big(\varphi\big(\nabla(u-v)/2\big) + \psi\big(|(u-v)/2|\big)V(x)\Big)m_g(\mathrm{d}x)$$

$$+ \int_{|u| \leqq v} \varphi(\nabla v)m_g(\mathrm{d}x) + \int_{|v|<u} \varphi\big(\nabla(u+v)/2\big)m_g(\mathrm{d}x)$$

$$+ \int_{u<-|v|} \varphi\big(\nabla(v-u)/2\big)m_g(\mathrm{d}x).$$

We have to prove that the previous quantities are not larger than

$$\int_M \big(\varphi(\nabla u) + \psi(|u|)V(x)\big)m_g(\mathrm{d}x) + \int_M \varphi(\nabla v)m_g(\mathrm{d}x).$$

To this end, we first investigate the terms not involving gradients. In fact, on the region $|v| \leqq u$ we have

$$|u+v| \leqq |u| + |v| \leqq |u| + u = 2|u|$$

since on such region u is nonnegative. Similarly, on the region $u < -|v|$:

$$|u-v| \leqq |u| + |v| \leqq |u| - u = 2|u|$$

since on such region u is negative. Therefore, by the monotonicity of ψ:

$$\int_{|u| \leqq v} \psi(|u|)V(x)m_g(\mathrm{d}x) + \int_{|v|<u} \psi\big(|(u+v)/2|\big)V(x)m_g(\mathrm{d}x)$$

$$+ \int_{u<-|v|} \psi\big(|(u-v)/2|\big)V(x)m_g(\mathrm{d}x)$$

$$\leqq \int_M \psi(|u|)V(x)m_g(\mathrm{d}x).$$

As for the terms involving gradients, they are not larger, by the convexity of φ, than

$$\int_{|u| \leqq v} \varphi(\nabla u)m_g(\mathrm{d}x) + \frac{1}{2}\int_{|v|<u} \big(\varphi(\nabla u) + \varphi(\nabla v)\big)m_g(\mathrm{d}x)$$

$$+ \frac{1}{2}\int_{u<-|v|} \big(\varphi(\nabla u) + \varphi(\nabla v)\big)m_g(\mathrm{d}x) + \int_{|u| \leqq v} \varphi(\nabla v)m_g(\mathrm{d}x)$$

$$+ \frac{1}{2}\int_{\{|v|<u\}\cup\{u<-|v|\}} \big(\varphi(\nabla u) + \varphi(\nabla v)\big)\,\mathrm{d}m_g(x)$$

$$\leqq \int_M \big(\varphi(\nabla u) + \varphi(\nabla v)\big)\,\mathrm{d}m_g(x)$$

as was to be proved. $\square$

Remark 6.4. If also T_t is order preserving, the domination property is equivalent to the fact that the closed and convex set $\hat{\mathscr{K}} := \{(u,v) : 0 \leqq u < v\}$ is left invariant by $T_t \oplus S_t$. The corresponding equivalent contraction property on the energy functionals is

$$\mathscr{E}_T \oplus \mathscr{E}_S\big(\hat{Q}(u,v)\big) \leqq \mathscr{E}_T \oplus \mathscr{E}_S(u,v),$$

$\hat{Q}$ being the projection on $\hat{\mathscr{K}}$, given explicitely by:

$$\hat{Q}(u,v) = \begin{cases} (u,v) & \text{if } u,v \geqq 0, u \leqq v, \\ (0,0) & \text{if } v < 0 \wedge (-u), \\ \big((u+v)/2, (u+v)/2\big) & \text{if } u > |v|, \\ (0,v) & \text{if } u < 0, v > 0. \end{cases} \tag{6.2}$$

7. Comparison with previous results

In this section we compare our results with those of Benilan, Crandall and Picard given in [BP], [BC].

In [BC] the authors introduce the concept of *semigroup of complete contractions* $\{T_t\}_{t \geqq 0}$. This means (in the setting of semigroups on $\mathrm{L}^2(X,m)$) that each map T_t is order preserving and is nonexpansive on each $\mathrm{L}^p(X,m)$ for any $p \in [1,+\infty]$. They pass throught the equivalent condition

$$\int (T_t u - T_t v - k)^+ \, \mathrm{d}m \leqq \int (u - v - k)^+$$

for any $u, v \in \mathrm{L}^2(X,m)$ and for all $k > 0$ (see [BC], eq. (0.8)). They prove the following:

If $\mathscr{E}$ is a lower semicontinuous functional on $\mathrm{L}^2(X,m)$ satisfying the condition

$$\mathscr{E}\big(u \wedge (v+k)\big) + \mathscr{E}\big((u-k) \vee v\big) \leqq \mathscr{E}(u) + \mathscr{E}(v) \tag{7.1}$$

for all positive k, then:

- $\mathscr{E}$ is convex;
- the resolvent operator

$$R_\lambda = (I + \lambda \partial \mathscr{E})^{-1}$$

is a complete contraction for all positive λ.

This *sufficient* condition clearly implies the validity of the contraction properties (3.2), (3.3): this can be seen either via the contraction properties of the semigroup, or indagating directly the functional by using Prop. 7.2 and Remark 7.8 of [BC]. Besides noticing that (7.1) is in principle a *sufficient* condition for the resolvent to be an order preserving contraction on all L^p spaces, we also comment that a *separate* characterization of the order preserving property on the one hand, and of the L^p contraction properties on the other hand, is not available in [BC].

We finally remark that the terminology "nonlinear Dirichlet form" has been already used in the paper [BP], where they generalize the classical linear Beurling-Deny theory to the nonlinear setting as far the *contraction* properties of the semigroup at hand are involved. In fact, they characterize those nonlinear semigroups associated to convex lower semi-continuous functionals, whose associated resolvent is *simultaneously* both *negativity preserving*, i.e. $u \leqq 0 \Rightarrow R_\lambda u \leqq 0$ for all positive λ, and *contractive* on each L^p, $p \in [1, +\infty]$, in the sense that $\|R_\lambda u\|_p \leqq \|u\|$ for all $u \in L^p(X,m) \cap L^2(X,m)$. Such properties and the ones investigated in the present paper coincide in the linear setting, but do not overlap in the nonlinear setting.

We finally notice that in both papers [BC], [BP] the role of projections is not investigated: in fact it is easy to show that the contraction property (7.1) does *not* involve a Hilbert projection.

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Eingegangen 21. Januar 2002, in revidierter Fassung 27. September 2002