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BV SPACES AND RECTIFIABILITY FOR CARNOT-CARATHÉODORY METRICS: AN INTRODUCTION

BRUNO FRANCHI

1. Introduction.

The aim of these lectures is to illustrate some recent results concerning rectifiable sets in Carnot groups, and to provide a short introduction to the subject, and, more generally, to some aspects of Geometric Measure Theory in Carnot–Carathéodory spaces.

I must thank the organizers of the Spring School NAFSA 7, and professors Bohumir Opic and Lubos Pick in particular, for this opportunity, for their warm hospitality, and for the friendly atmosphere of the School.

It is also a great pleasure to acknowledge the help and the support of several friends that made possible this work: first of all, all the results concerning BV functions and Geometric Measure Theory in Carnot–Carathéodory spaces presented here have been obtained jointly with Raul Serapioni and Francesco Serra Cassano. Our long collaboration has been always an invaluable source of scientific and human enrichment. Without their collaboration and their friendship, I would never have been able to attack this hard subject. I have to thank them also for permitting the large quotation of our joint papers.

Special thanks go also to Ermanno Lanconelli and Richard L. Wheeden. With them not only I shared mathematical interests and a fruitul scientific collaboration that goes far behind the number of joint papers we have written, but also the great pleasure of a long friendship. It is a pleasure to aknowledge that I owe to Ermanno Lanconelli the idea of approaching degenerate elliptic equations by means of the control metric associated with a family of vector fields (that is currently called Carnot–Carathéodory metric). This approach in the early 80's was the beginning of my interest toward the study of Carnot–Carathéodory spaces, and the origins of the present paper can be tracked to those pioneering works. I learned from Dick Wheeden plenty of mathematics and of new ideas. He introduced me to the magic of integral inequalities, and the section below concerning Poincaré inequality relies on several of our joint papers with Sylvain Gallot, Cristian Gutiérrez, Guozhen Lu, and Carlos Pérez.

I am very grateful to Valentino Magnani and Roberto Monti, who made their beautiful PhD theses [89] and [98] available to me. In fact, I followed [98] at several points.

I have to thank also several friends with whom I shared hours of fruitful discussions and whose work appears here, more or less explicitly: Luigi Ambrosio, Zoltan Balogh, Giovanna Citti, Thierry Coulhon, Piotr Hajłasz, Martin H. Reimann, Fulvio Ricci.

These notes are not meant to be a complete – and not even a partial – survey of the field of Carnot–Carathéodory metrics, since they are based on the content of few lectures given in Prague during the NAFSA 7. The reader interested to an exhaustive overview of the subject, with a full bibliography, sharp statements, and detailed proofs, may refer to P. Hajłasz [67], P. Hajłasz & P. Koskela [68], and to the PhD theses of V. Magnani and R. Monti [98], whereas, for more specific facets we restrict ourselves to recommend the reader to the general monographes [30], [68], [70], [65], [64], [116], [119], [96], to the papers [3], [4], [5], [20], [27], [29], [51], [54], [61], [63], [73], [105], [104], [106], [107], [108], [120] and to the references therein.

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Since these lectures are focused on Geometric Measure Theory and rectifiability theorems in particular, there are two wide fields of research that are not mentioned at all here, the fields of degenerate elliptic equations associated with a family of vector fields, or subelliptic equations, as they are currently called by several authors, and control theory. A not utterly unsatisfactory picture of these fields goes indeed behind the aim (and the size) of these lectures.

2. Sobolev spaces and Poincaré inequality.

2.1. **Vector fields.** Consider a family X of vector fields $X = (X_1, ..., X_m) \in \mathrm{Lip}\,(\mathbb{R}^n; \mathbb{R}^n)^m$. Since we are dealing with local properties, for sake of simplicity, we assume $X_1, ..., X_m$ are bounded in $\mathbb{R}^n$. This assumption gives a simpler form to some statements below. Later on, when the vector fields will be associated with a Carnot group structure, we shall drop the boundedness assumption. This will not yield contradiction or lack of coherence, since the local estimates we are dealing with are easily extended in groups to the whole space by translations and dilations.

As usual we shall identify vector fields and differential operators. If

$$X_j(x) = \sum_{i=1}^{n} c_i^j(x)\partial_i, \quad j = 1, ..., m,$$

we define the $m \times n$ matrix

$$C(x) = [c_i^j(x)]_{\substack{i=1,\dots,n \\ j=1,\dots,m}}.$$

We shall denote by X_j^* the operator formally adjoint to X_j in $L^2(\mathbb{R}^n)$, that is the operator which for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} \varphi(x) X_j \psi(x)\, dx = \int_{\mathbb{R}^n} \psi(x) X_j^* \varphi(x)\, dx.$$

Moreover, if $f \in L^1_{\mathrm{loc}}$ is a scalar function and $\varphi \in (L^1_{\mathrm{loc}})^m$ is a m-vector valued function, we define the X-*gradient* and X-*divergence* as the following distributions:

$$Xf := (X_1 f, ..., X_m f), \quad \mathrm{div}_X\,(\varphi) := -\sum_{j=1}^{m} X_j^* \varphi_j.$$

Let Ω be an open subset of $\mathbb{R}^n$. One can define the Sobolev space $W_X^{1,p}(\Omega)$, $1 \le p \le \infty$, associated with the family X as the space of all the functions with finite norm $\|u\|_{W_X^{1,p}} = \|u\|_p + \|Xu\|_p$, where $|Xu|^2 = \sum |X_j u|^2$ and the derivatives $X_j u$ are understood in the sense of distributions. The L^p-norms should be meant with respect to Lebesgue measure.

Throughout this paper, if $E \subset \mathbb{R}^n$, both $|E|$ and $\mathcal{L}^n(E)$ denote its Lebesgue measure. Analogously, if μ is a measure in a set X, we write $\mu(E)$ or $|E|_\mu$ for the μ-measure of the set $E \subset X$.

2.2. Sobolev spaces associated with vector fields.

Proposition 2.1. *Endowed with its natural norm, $W^{1,p}(\Omega)$, $1 \le p \le \infty$, is a Banach space, reflexive if $1 < p < \infty$. Moreover, $W^{1,2}(\Omega)$ is a Hilbert space.*

Another way to define the space for $1 \le p < \infty$ is to take the closure of C^∞ functions in the above norm. As in the Euclidean case, the two approaches are equivalent. This was obtained independently in [52] and [61]. The method goes, however, back to Friedrichs [60]. The result can be stated as follows. The statement for smooth manifolds in due to [34] and [35].

Theorem 2.2. *Let X be a family of Lipschitz continuous vector fields. Then, if $1 \le p < \infty$, we have*

$$\mathbf{C}^\infty(\Omega) \cap W_X^{1,p}(\Omega) \text{ is dense in } W_X^{1,p}(\Omega).$$

If in addition $\partial\Omega$ is a smooth manifold, then

$$\mathbf{C}^\infty(\overline{\Omega}) \text{ is dense in } W^{1,p}_X(\Omega).$$

The following definition is natural keeping in mind Theorem 2.2.

Definition 2.3. Let X be a family of Lipschitz continuous vector fields. Then, if $1 \le p < \infty$, we put

$$\overset{\circ}{W}{}^{1,p}_X(\Omega) := \overline{\mathcal{D}(\Omega)}^{W^{1,p}_X(\Omega)}.$$

Theorem 2.2 provides also a further characterization of the spaces $W^{1,p}_X(\Omega)$ when $1 < p < \infty$ through a relaxation argument. To this end, let $p \ge 1$ and let $f : \Omega \times \mathbb{R}^m \to [0,\infty)$ be a Carathéodory function such that

$$f(x,\cdot) \text{ is a convex function on } \mathbb{R}^m \text{ for every } x \in \Omega\,; \tag{1}$$

there exist two positive constants λ_0 and Λ_0 for which

$$\lambda_0|\eta|^p \le f(x,\eta) \le \Lambda_0(1+|\eta|^p) \text{ for every } (x,\eta) \in \Omega \times \mathbb{R}^m. \tag{2}$$

Let us define the functional $F_p : L^p(\Omega) \to [0,\infty]$

$$F_p(u) := \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in C^1_0(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

and its *relaxed* functional (see [114] and [17])

$$\bar{F}_p(u) := \inf\left\{ \liminf_{h\to\infty} F_p(u_h) : (u_h)_h \subset L^p(\Omega), u_h \to u \right\} \tag{3}$$

It is well known (see, for instance, [17]) that $\bar{F}_p$ is the greatest $L^p(\Omega)$-lower semicontinuous functional smaller or equal to F_p and that it coincides with F_p on $C^1_0(\Omega) \cap L^p(\Omega)$. Then the following characterization of the spaces $W^{1,p}_X(\Omega)$ holds when $1 < p < \infty$ (see [51]).

Theorem 2.4. *Let $p > 1$ and let Ω be an open subset of $\mathbb{R}^n$; let $f : \Omega \times \mathbb{R}^m \to [0,\infty)$ be a Carathéodory function for which (1) and (2) hold. Then*

(i) $\operatorname{dom} \bar{F}_p := \{u \in L^p(\Omega) : \bar{F}_p(u) < \infty\} = W^{1,p}_X(\Omega)$ *;*
(ii) $\bar{F}_p(u) = \int_\Omega f(x, Xu(x))dx$ *for every* $u \in W^{1,p}_X(\Omega)$ *.*

Remark 2.5. We have discussed here spaces of order 1. Fractional order spaces are discussed by D. Morbidelli in [102]. For higher order spaces, see for instance [40], [80], [7], [24], [26], [25], [82], [79] [22], [83].

2.3. **Carnot–Carathéodory distance.** Let us recall now the following standard definition of Carnot–Carathéodory metric associated with X (see, e.g., [38], [46], [103]).

Definition 2.6. We say that an absolutely continuous curve $\gamma : [0,T] \longrightarrow \mathbb{R}^n$ is a sub-unit curve with respect to X if for any $\xi \in \mathbb{R}^n$

$$\langle \dot\gamma(t), \xi\rangle^2 \le \sum_{j=1}^m \langle X_j(\gamma(t)), \xi\rangle^2$$

for a.e. $t \in [0,T]$. If $x_1, x_2 \in \mathbb{R}^n$, we define

$$\begin{aligned} d(x_1,x_2) = \inf \{T > 0 :\ & \text{there exists a sub-unit curve } \gamma \\ & \gamma : [0,T] \longrightarrow \mathbb{R}^n, \quad \gamma(0) = x_1,\ \gamma(T) = x_2\}. \end{aligned}$$

If the above set of curves is empty, we put $d(x_1,x_2) = \infty$.

Throughout this paper we shall assume the following hypothesis (H1) holds:

(H1) $d(x,y) < \infty$ for any $x, y \in \mathbb{R}^n$, so that d is a distance in $\mathbb{R}^n$. Moreover, the distance d is continuous with respect to the usual topology of $\mathbb{R}^n$.

If $x \in \mathbb{R}^n$ and $r > 0$ we will denote by $U_d(x,r) = \{y \in \mathbb{R}^n : d(x,y) < r\}$ the *metric balls* with respect to d. The boundedness of $X_1, \dots, X_m$ yields the existence of $C > 0$ such that

$$d(x,y) \geq C|x-y| \quad \text{for all } x, y \in \mathbb{R}^n.$$

In particular, metric balls are bounded with respect to the Euclidean distance.

We stress explicitly that in general Carnot–Carathéodory distances are not Euclidean at any scale, and hence not Riemannian. A beautiful proof can be found in [115] (for a more general statement see also [87]).

If X satisfies (H1), then the *total variation* of a curve $\gamma : [0,1] \to \mathbb{R}^n$ is by definition

$$\mathrm{Var}_X(\gamma) = \sup_{0 \leq t_1 < \ldots < t_k \leq 1} \sum_{i=1}^{k-1} d(\gamma(t_{i+1}), \gamma(t_i)).$$

The supremum is taken over all finite partition of $[0,1]$. If $\mathrm{Var}_X(\gamma) < +\infty$ the curve γ is said *rectifiable.*

A continuous rectifiable curve $\gamma : [0,1] \to \mathbb{R}^n$ is said to be a *geodesic,* or a *segment,* if $\mathrm{Var}_X(\gamma) = d(\gamma(0), \gamma(1))$. By an arclength reparametrization, a geodesic γ can always be reparametrized on the interval $[0, \mathrm{Var}_X(\gamma)]$ in such a way that $d(\gamma(t), \gamma(s)) = |t-s|$ for all $s, t \in [0, \mathrm{Var}_X(\gamma)]$ (see [16]).

Theorem 2.7. *Let X be a family of bounded Lipschitz continuous vector fields satisfying (H1). Then for all $x, y \in \mathbb{R}^n$ there exists a geodesic connecting them.*

Carnot–Carathódory metrics can be viewed as "limits" of Riemannian metrics (see [41], [65] and [98]).

Indeed, for sake of simplicity, assume $X = (X_1, ..., X_m)$ is a system of smooth vector fields. Then for any $k \in \mathbb{N}$ let $d^{(k)}$ be the C-C metric induced on $\mathbb{R}^n$ by the vector fields

$$X^{(k)} = (X_1, ..., X_m, \frac{1}{k}\partial_1, ..., \frac{1}{k}\partial_n).$$

The distance $d^{(k)}$ is in fact a Riemannian distance (see again [98]), basically since $X^{(k)}$ contains n linearly independent vector fields. Every $X^{(k)}-$subunit curve is $X^{(h)}-$subunit for all $h > k$ and also $X-$subunit. Then

$$d^{(k)}(x,y) \leq d^{(k+1)}(x,y) \leq d(x,y) \quad \text{for all } k \in \mathbb{N} \text{ and } x, y \in \mathbb{R}^n. \tag{4}$$

In addition, since C-C balls in the metric $d^{(1)}$ are bounded in the Euclidean metric, then, by an Ascoli-Arzelà argument, we can obtain that

$$\lim_{k \to \infty} d^{(k)}(x,y) = d(x,y) \tag{5}$$

for all $x, y \in \mathbb{R}^n$ and finally, by (H.1), the convergence is uniform on compact sets.

The following property is known as *doubling property* of d. It is not always satisfied by Carnot–Carathéodory distances associated with vector fields satisfying (H.1), but it holds in several important cases and most of the subsequent results rely on it.

(H2) For any compact $K \subset \mathbb{R}^n$ there exists a positive constant C_K such that

$$|U_d(x, 2r)| \leq C_K \, |U_d(x,r)|$$

for any $x \in K$ and $r < r_K$.

From now on we will call *geometric constant* any constant depending only on the dimension n, on the Lipschitz norm of the coefficients, and on the constants appearing in (H1) and (H2).

Moreover, for the sake of simplicity, we will omit the index d in U_d when there is no way of misunderstanding, and we will denote by the same letter C different geometric constants.

Remark 2.8. Assumptions (H1) and (H2) are satisfied by several important families of vector fields. For instance:

(i) If the vector fields are smooth and the rank of the Lie algebra generated by $X_1 \dots , X_m$ equals n at any point of $\mathbb{R}^n$ (Hörmander condition), then (H1) and (H2) hold ([103]).

(ii) If the vector fields are as in [46] and [42], then (H1) and (H2) hold. These assumptions still hold if the vector fields are as in [44].

On the other hand, keeping into account Proposition 2.9 (i) and Corollary 6.2 below, it is easy to see that the Carnot–Carathéodory distance associated with $X = (\partial_{x_1}, \exp(-1/x_1^2)\partial_{x_2})$ in $\mathbb{R}^2$ satisfies (H.1) but not (H.2).

The following properties of the metric balls follow straightforwardly from (H2).

Proposition 2.9. *Let (H1) and (H2) hold. If $K \subset\subset \mathbb{R}^n$, then there exist geometric constants $Q \geq n,\ r_K > 0,\ c_1 > 0,\ c_2 > 0,\ c_3 > 0,\ c_4 > 0$ such that*

(i) $|U(x,s)| \geq c_1 \left(\frac{s}{r}\right)^Q |U(x,r)| \quad \forall x \in K,\ \forall r, s \quad 0 < s < r \leq r_K$;

(ii) $|U(x,s)| \leq c_2\, s^n \quad \forall x \in K,\ \forall s \quad 0 < s \leq r_K$;

(iii) $c_3 |U(x, d(x,y))| \leq |U(y, d(x,y))| \leq c_4 |U(x, d(x,y))|$ *for any* $x, y \in K,\ d(x,y) \leq r_K$.

We refer to Q as to the (local) homogeneous dimension of $(\mathbb{R}^n, d, \mathcal{L}^n)$ (with some ambiguity, since Q is clearly not uniquely defined).

Lipschitz functions in general C-C spaces always have weak derivatives along the vector fields that are essentially bounded functions. When the function is the distance function this result was first proved in [52], and then in [62] for a generic Lipschitz function. A more precise result in the following one, taken from [45] (see also [19]).

Theorem 2.10. *Let $(\mathbb{R}^n, d)$ be a C-C space associated with a family of locally Lipschitz vector fields $X = (X_1, ..., X_m)$. Assume (H.1) holds. If $f : \mathbb{R}^n \to \mathbb{R}$ is a function such that for some $L \geq 0$*

$$|f(x) - f(y)| \leq L d(x,y) \qquad \text{for all } x, y \in \mathbb{R}^n, \tag{6}$$

then the derivatives $X_j f$, $j = 1, ..., m$ exist in distributional sense, are measurable functions and $|Xf(x)| \leq L$ for a.e. $x \in \mathbb{R}^n$.

Another relevant property of Carnot–Carathéodory distance is that it satisfies (at least in several important cases) an eikonal equation, like the Euclidean distance. This beautiful result has been proved by R. Monti & F. Serra Cassano.

Theorem 2.11. *Let X be a family of Lipschitz continuous vector fields in $\mathbb{R}^n$ and assume the associated Carnot–Carathéodory distance d satisfies (H1) and (H2). Suppose that the vector fields satisfy one of the cases A, B or C below:*

Case A. *$X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $m < n$, satisfy the Hörmander's rank condition, and they are of the form*

$$X_j = \partial_j + \sum_{i=m+1}^{n} a_{ij}(x)\partial_i, \quad j = 1, ..., m, \tag{7}$$

where $a_{ij} \in C^\infty(\mathbb{R}^n)$.

Case B. *$X_1, ..., X_n \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ are of the form*

$$X_1 = \partial_1, \quad X_2 = p_2(x_1)\partial_2, \quad ... \quad X_n = p_n(x_1, ..., x_{n-1})\partial_n, \tag{8}$$

where $p_j \in C^\infty(\mathbb{R}^{j-1})$, $j = 2, ..., n$, are functions vanishing on a set of null $(j-1)-$dimensional Lebesgue measure.

Case C. *$X_1, ..., X_m \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and* $\mathrm{span}\,\{X_1(x), ..., X_m(x)\} = \mathbb{R}^n$ *for every $x \in \mathbb{R}^n$.*

Let $K \subset \mathbb{R}^n$ be a closed set and let d_K be the distance from K. Then

$$|Xd_K(x)| = 1 \tag{9}$$

for a.e. $x \in \mathbb{R}^n \setminus K$.

Remark 2.12. Vector fields in Case A may be called "of Carnot type". This expression is motivated by the analogy with the canonical generating vector fields of a Carnot group (see below). Analogously, vector fields in Case B may be called "of Grushin type", since the model is provided by the so-called Grushin type vector fields studied in [46], [42], [44] (see below). Finally, vector fields in Case C may be called "of Riemann type", since in this case the distance d is the Riemannian distance associated with the matrix $C\,C^T$.

2.4. Poincaré inequality.

Definition 2.13. Let $1 \le p \le q < \infty$. We say that the system X satisfies a (p,q)-Poincaré inequality (in a compact set K) if for any $x \in K$, for any $r \in (0, r_K)$, and for any Lipschitz continuous function f the following Poincaré inequality holds: let $U = U(x, r(U))$ be a Carnot–Carathéodory ball, and denote by f_U the average of f in U. Then

$$\left(\frac{1}{|U|}\int_U |f(x) - f_U|^q\,dx\right)^{1/q} \le c\,r(U)\,\left(\frac{1}{|U|}\int_U |Xf(x)|^p\,dx\right)^{1/p}. \tag{10}$$

Examples of systems of vector fields satisfying a (p,q)-Poincaré inequality are provided by systems of smooth vector fields of Hörmander's type, as we see below. Further classes of non-smooth vector fields yielding a (p,q)-Poincaré inequality are introduced in [46], [42] (see also Appendix 6), [74], and [94].

Sometimes in the literature, when $p < q$ we refer to (10) as to a Sobolev-Poincaré inequality, the term "Poincaré inequality" being reserved to the case $q = p$. On the other hand, the expression "(p,q)-Sobolev inequality" indicates the weaker property

$$\left(\frac{1}{|U|}\int_U |f(x)|^q\,dx\right)^{1/q} \le c\,r(U)\,\left(\frac{1}{|U|}\int_U |Xf(x)|^p\,dx\right)^{1/p} \tag{11}$$

for all Lipschitz continuous functions f *supported in U.*

For systems of smooth vector fields of Hörmander's type, a (p,p)-Poincaré inequality was proved first by D. Jerison in [71]. This result was improved in case $p > 1$ in [77] by showing that the estimate holds for $1 < p < Q$ and $q = pQ/(Q-p)$. In fact, (10) holds for $1 \le p < q < \infty$ if p and q are related by a natural balance condition which involves the local doubling order of Lebesgue measure (for metric balls). The limit case $p = 1$ is very important, since it is equivalent, as we see later, to an intrinsic isoperimetric inequality. This inequality was proved independently by [21], [47], [68], and [90] (see also [12]). Here we give a simple formulation.

Theorem 2.14. *Let X be a system of smooth vector fields satisfying Hörmander's rank condition. Let $1 \le p < q < \infty$ be such that the following balance condition holds:*

$$\frac{r(\tilde U)}{r(U)}\left(\frac{|\tilde U|}{|U|}\right)^{1/q} \le C\,\left(\frac{|\tilde U|}{|U|}\right)^{1/p} \tag{12}$$

for all balls $\tilde U, U$ such that $\tilde U \subset U$. Then, denoting by f_U the average of f on U,

$$\left(\frac{1}{|U|}\int_U |f-f_U|^q\,dx\right)^{1/q} \le C\,r(U)\,\left(\frac{1}{|U|}\int_U |Xf|^p\,dx\right)^{1/p} \tag{13}$$

with C independent of f.

The proof of Theorem 2.14 can be carried out directly. However, the (p,q)-Poincaré inequality can be derived from the $(1,1)$-Poincaré inequality of [71]. This is a more elegant (and deeper) proof relying on the so-called self-improving property of Poincaré inequality. In fact, starting with work of Saloff-Coste [111], it is known that – thanks to the doubling property of the Carnot–Carathéodory metric with respect to Lebesgue measure – Poincaré inequalities have a self-improving nature, in the sense that it is possible to derive estimates for general p,q from particular special cases such as

$$\frac{1}{|U|}\int_U |f(x)-f_U|\,dx \le c\,r(U)\,\left(\frac{1}{|U|}\int_U |Xf|^{p_0}\,dx\right)^{1/p_0} \tag{14}$$

for some p_0, provided p and q satisfy a suitable balance condition involving the volume of the metric balls.

We refer to [112] for an introduction to this property of Poincaré inequalities.

Saloff-Coste's result has been successively extended to more general situations in [49] and [50]. In fact, Theorem 2.14 can be derived from the $(1,1)$-Poincaré inequality of [71] by means of the following result (Corollary 2.16 of [50]).

Theorem 2.15. *Let μ and ν be doubling Borel measures in $(\mathbb{R}^n, d)$, $p_0 > 0$ and T be a differential operator for which*

$$\frac{1}{|U|_\mu}\int_U |f-f_U|\,d\mu \le C\,r(U)\,\left(\frac{1}{|U|_\nu}\int_U |Tf|^{p_0}\,d\nu\right)^{1/p_0} \tag{15}$$

for all balls U and all Lipschitz functions f. Let $p_0 \le p < q < \infty$, and assume that ω is a doubling measure in $(\mathbb{R}^n, d)$, and the following balance condition holds:

$$\frac{r(\tilde U)}{r(U)}\left(\frac{|\tilde U|_\omega}{|U|_\omega}\right)^{1/q} \le C\,\left(\frac{|\tilde U|_\nu}{|U|_\nu}\right)^{1/p} \tag{16}$$

for all balls $\tilde U, U$ such that $\tilde U \subset U$. Then

$$\left(\frac{1}{|U|_\omega}\int_U |f-f_U|^q\,d\omega\right)^{1/q} \le C\,r(U)\,\left(\frac{1}{|U|_\nu}\int_U |Tf|^p\,d\nu\right)^{1/p} \tag{17}$$

with C independent of f and U.

Remark 2.16. We stress that the self-improving property of Theorem 2.15 does not rely on any smoothness of the vector fields. In fact, the smoothness of the vector fields – together with Hörmander's rank hypothesis – is required only in order to obtain the doubling property of the d-balls and the $(1,1)$-Poincaré inequality providing the starting point in order to apply Theorem 2.15. Thus, Theorem 2.15 applies whenever the doubling property of the d-balls and the $(1,1)$-Poincaré inequality hold.

There is another proof of Theorem 2.14 starting from the $(1,1)$-Poincaré inequality, that relies on a representation formula of a function f with zero average on a metric ball in terms of the norm of its X-gradient $|Xf|$. In fact, it is possible to prove that the $(1,1)$-Poincaré inequality associated with X is equivalent to such a formula. This result was proved first under supplementary hypotheses in [48], and then in the present sharp form in [59] and [81].

Theorem 2.17. *Let $(\mathcal{S}, \varrho, m)$ be a complete metric measure space, where ϱ is a distance in $\mathcal{S}$, and m is a doubling Borel measure in $\mathcal{S}$. Suppose that $(\mathcal{S}, \varrho)$ has the segment property, i.e. suppose for each pair of points $x, y \in \mathcal{S}$ there exists a continuous curve γ connecting x and y such that $\varrho(\gamma(t), \gamma(s)) = |t-s|$. Let μ, ν be locally doubling measures on $(\mathcal{S}, \varrho, m)$ with doubling constants A_μ and A_ν, respectively. Let $U_0 = U(x_0, r_0)$ be a ball, and let $f, g \in L^1(U_0)$ be given functions. Assume there exists $P > 0$ such that, for all balls $U \subseteq U_0$,*

$$\frac{1}{\nu(U)} \int_U |f - f_{U,\nu}|\, d\nu \le P \frac{r(U)}{\mu(U)} \int_U |g|\, d\mu,$$

where $f_{U,\nu} = \frac{1}{\nu(U)} \int_U f d\nu = \fint_U f d\nu$. If there is a constant $\vartheta > 0$ such that for all balls U, $\tilde{U}$ with $\tilde{U} \subseteq U \subseteq U_0$,

$$\frac{\mu(U)}{\mu(\tilde{U})} \ge \vartheta \frac{r(U)}{r(\tilde{U})},$$

then for $(d\nu)$-a.e. $x \in U_0$,

$$|f(x) - f_{U_0,\nu}| \le C \int_{U_0} |g(y)| \frac{\varrho(x,y)}{\mu(U(x, \varrho(x,y)))}\, d\mu(y),$$

were C is a geometric constant depending on P, A_μ, A_ν.

As it is proved in [59], $\mathcal{S} = \mathbb{R}^n$, $\varrho = d$, $m = \mu = \nu = \mathcal{L}^n$, and $g = |Xf|$ satisfy the assumptions of Theorem 2.17 and then the following representation formula holds:

$$|f(x) - f_{U_0}| \le C \int_{U_0} |Xf(y)| \frac{d(x,y)}{|U(x, d(x,y)|}\, dy \qquad \text{for a.e. } x \in U_0. \tag{18}$$

Once (18) is proved, then Theorem 2.14 can be derived by means of $L^p - L^q$ continuity theorems for singular integral operators of potential type, as in [44].

A typical example of this kind of (weak type) continuity results is provided by Theorem 4.1 in [44] that reads as follows.

Theorem 2.18. *Let $(X, \tilde{\varrho}, d\nu)$ be a space of homogeneous type in the sense of [23], i.e. a metric space $(X, \tilde{\varrho})$ endowed with a doubling Radon measure ν, and let κ be the quasi-metric constant of $\tilde{\varrho}$. Let $\tilde{K}$ be a nonnegative kernel and put*

$$\tilde{T}f(x) = \int_{U_0} \tilde{K}(x,y) f(y) d\nu(y) \tag{19}$$

where $f \ge 0$ and $U_0 = U(x_0, r_0)$ is a fixed ball. Assume for simplicity that $\nu(\{x\}) = 0$ for $x \in U_0$ and that $\nu(U(x,r))$ is a continuous function of r for $x \in U_0$. If $1 \le p < q < \infty$ and $\tilde{u}, \tilde{v}$ are weights (i.e. nonnegative locally summable functions), then

$$\int_{U_0 \cap \{Tf > t\}} \tilde{u} d\nu \le c \left(\frac{\tilde{L} ||f||_{L^p_{\tilde{v}d\nu}(U_0)}}{t} \right)^q, \; t > 0, \tag{20}$$

with

$$\tilde{L} = \begin{cases} \sup \left(\int_{c_1 U(x,r)} \tilde{u} d\nu \right)^{\frac{1}{q}} \left(\int_{U_0 \setminus U(x,r)} \tilde{K}(x,y)^{p'} \tilde{v}(y)^{-\frac{1}{p-1}} d\nu(y) \right)^{\frac{1}{p'}}, & \text{if } p > 1 \\ \sup \left(\int_{c_1 U(x,r)} \tilde{u} d\nu \right)^{\frac{1}{q}} \left(\operatorname{ess\,sup}_{y \in U_0 \setminus U(x,r)} \tilde{K}(x,y) \frac{1}{\tilde{v}(y)} \right), & \text{if } p = 1, \end{cases}$$

where the sup is taken over all x, r such that $U(x,r) \subset c_2 U_0$ and $x \in U_0$, and the ess sup is taken with respect to the measure $\tilde{v} d\nu$. The constants c_1 and c_2 can be written explicitly and depend only on the constant κ.

In fact, Theorem 2.18 provides only a weak type continuity estimate, but here we can pass from the weak type estimate to the strong type one, thanks to the fact that the right hand side of Poincaré inequality contains a first order differential operator. Indeed, the main property we need to pass from weak type estimates to strong type estimates is a certain "stability" property under truncations. This idea was originally introduced in [76] and exploited in [113], [43], [47] and [8]. We refer to [49] and [50] for a detailed presentation of this technique.

The proof of the $(1,1)$-Poincaré inequality relies on the lifting technique for vector fields introduced by Rotschild & Stein [110], but it becomes particularly simple and elegant in the setting of groups, when X is a complete system of left invariant vector fields in a Carnot group identified with $\mathbb{R}^n$ through the exponential map. The notion of Carnot group, together with all related definitions and properties, will be the subject of Section 4. The following proof is due to Varopoulos ([118]); in the present form it is taken from [98].

Proof of $(1,1)$*-Poincaré inequality for Carnot groups.* Let a structure of Carnot group induced by $X = (X_1, ..., X_m)$ be given on $\mathbb{R}^n$. The group product of $x, y \in \mathbb{R}^n$ will be denoted by $x \cdot y$. We shall see below that $|U(x,r)| = kr^Q$ for all $x \in \mathbb{R}^n$ and $r \geq 0$ with $k = |U(0,1)|$.

Fix $U = U(x_0, r)$ with $x_0 \in \mathbb{R}^n$ and $r > 0$ and let $u \in C_0^1(\mathbb{R}^n)$. Notice that

$$\int_U |u(x) - u_U|\,dx = \int_U \Big| \fint_U (u(x) - u(y))dy \Big|\,dx \leq \fint_U \int_U |u(x) - u(y)|\,dxdy.$$

We perform in the inner integral the change of variable $z = y^{-1} \cdot x$, which has Jacobian identically 1, getting

$$\int_U |u(x) - u_U|\,dx \leq \fint_U \int_{y^{-1}\cdot U} |u(y \cdot z) - u(y)|\,dzdy \leq \fint_U \int_{U(0,2r)} |u(y \cdot z) - u(y)|\,dzdy.$$

Indeed, if $y \in U$ then $y^{-1} \cdot U \subset U(0, 2r)$.

Let now $z \in U(0, 2r)$ be fixed, let $\delta = d(0, z)$ and take a geodesic $\gamma : [0, \delta] \to \mathbb{R}^n$ such that $\gamma(0) = 0$ and $\gamma(\delta) = z$ with $\delta \leq 2r$. For some $h \in L^\infty(0, \delta)^m$

$$\dot\gamma(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t)) \quad \text{and} \quad |h(t)| \leq 1 \quad \text{for a.e. } t \in [0, \delta].$$

Then

$$\begin{aligned} u(y \cdot z) - u(y) &= \int_0^\delta \frac{d}{dt} u(y \cdot \gamma(t))\,dt = \int_0^\delta \langle Du(y \cdot \gamma(t)), \frac{d}{dt}(y \cdot \gamma(t)) \rangle\,dt \\ &= \int_0^\delta \langle Du(y \cdot \gamma(t)), \sum_{j=1}^m h_j(t) X_j(y \cdot \gamma(t)) \rangle\,dt \\ &= \int_0^\delta \langle Xu(y \cdot \gamma(t)), h(t) \rangle\,dt. \end{aligned}$$

We used the left invariance of $X_1, ..., X_m$. As $h_\infty \leq 1$ we obtain

$$\begin{aligned} \int_U |u(x) - u_U|\,dx &\leq \fint_U \int_{U(0,2r)} \int_0^\delta |Xu(y \cdot \gamma(t))|\,dtdzdy \\ &\leq \int_0^\delta \int_{U(0,2r)} \fint_U |Xu(y \cdot \gamma(t))|\,dydzdt. \end{aligned}$$

The curve γ depends on z. Since $\gamma(t) \in U(0, 2r)$ for all $t \in [0, \delta]$, if $y \in U$ then $y \cdot \gamma(t) \in 3U = U(x_0, 3r)$. Indeed

$$d(y \cdot \gamma(t), x_0) \leq d(y \cdot \gamma(t), y) + d(y, x_0) = d(\gamma(t), 0) + d(y, x_0) \leq 3r.$$

Thus we finally get

$$\begin{aligned}\int_U |u(x)-u_U|\,dx &\leq \frac{1}{|U(0,r)|}\int_0^\delta \int_{U(0,2r)}\int_{3U} |Xu(y)|\,dydzdt\\ &\leq 2r\frac{|U(0,2r)|}{|U(0,r)|}\int_{3U}|Xu(y)|\,dy = r2^{Q+1}\int_{3U}|Xu(y)|\,dy.\end{aligned}$$

Finally, we can get rid of the constant 3 in the last integral $\int_{3U}|Xu(y)|\,dy$ by means of an argument that goes back to Boman, and that was generalized to the setting of doubling metric spaces in [44]. It relies on the fact that – as proved in [44] – metric balls are Boman domains, as they will be defined below.

From Poincaré inequality Theorem 2.14 we can derive the following Rellich-type theorem.

Theorem 2.19. *Suppose the assumptions of Theorem 2.14 hold, and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then the seminorm*

$$|u|_{\overset{\circ}{W}{}^{1,p}_X(\Omega)} := \left(\int_\Omega |Xu|^p dx\right)^{1/p}$$

is a norm in $\overset{\circ}{W}{}^{1,p}_X(\Omega)$. Moreover $\overset{\circ}{W}{}^{1,p}_X(\Omega)$ is compactly embedded in $L^q(\Omega)$.

Another interesting consequence of the Poincaré inequality for Hörmander's vector fields is that the associated Sobolev spaces fit in the general setting of Sobolev spaces on metric spaces, as defined by Hajłasz [66]. We refer the reader to [45].

2.5. **Geometry of domains.** The present section is largely taken from [98]. We refer also to the exhaustive bibliography of [98] for a detailed account of the different contributions to this field.

We consider a metric space (M,d). A *domain* $\Omega \subset M$ is a connected open set. The metric space (M,d) will be said *with geodesics* if every couple of point $x,y \in M$ can be connected by a continuous rectifiable (i.e. of finite length) curve with length $d(x,y)$. By Theorem 2.7, Carnot–Carathéodory distances yield a metric space with geodesics.

We want now to discuss Poincaré inequality in open sets different from balls. Clearly, not any open set admits a Poincaré inequality (as already happens in the Euclidean setting), and the main issue consists of providing a reasonable class of sets. Let us start with few general definitions.

Definition 2.20. Let (M,d) be a metric space. A bounded open set $\Omega \subset M$ is a *John domain* if there exist $x_0 \in \Omega$ and $C>0$ such that for every $x \in \Omega$ there exists a continuous rectifiable curve parametrized by arclength $\gamma : [0,T] \to \Omega$, $T \geq 0$, such that $\gamma(0)=x$, $\gamma(T)=x_0$ and

$$\operatorname{dist}(\gamma(t);\partial\Omega) \geq Ct. \tag{21}$$

Definition 2.21. Let (M,d) be a metric space. A bounded open set $\Omega \subset M$ is a *weak John domain* if there exist $x_0 \in \Omega$ and $0 < C \leq 1$ such that for every $x \in \Omega$ there exists a continuous curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0)=x$, $\gamma(1)=x_0$ and

$$\operatorname{dist}(\gamma(t);\partial\Omega) \geq Cd(\gamma(t),x). \tag{22}$$

The following result is basically proved in [44] and provides a key tool in the setting of Poincaré inequalities for Carnot–Carathéodory spaces.

Remark 2.22. If (M,d) is a metric space with geodesics, then every ball $U(x_0,r)$, $x_0 \in M$ and $r>0$, is a John domain with constant $C=1$ in (21).

Definition 2.23. Let (M,d) be a metric space. A set $E \subset M$ satisfies the *interior* (*exterior*) *corkscrew condition* if there exist $r_0 > 0$ and $k \geq 1$ such that for all $0 < r \leq r_0$ and $x \in \partial E$ there exist $y \in E$ ($y \in M \setminus E$) such that

$$\frac{r}{k} \leq \mathrm{dist}(y;\partial E) \quad \text{and} \quad d(x,y) \leq r.$$

A set E satisfies the *corkscrew condition* if it satisfies both the interior and the exterior corkscrew condition. The constant k will be called the *corkscrew constant* of E .

Clearly, if Ω is a John domain then it satisfies the interior corkscrew condition.

Proposition 2.24. *Let (M,d,μ) be a doubling metric space with arcwise connected balls. If $E \subset M$ satisfies the interior corkscrew condition then there exist $r_0 > 0$ and $C > 0$ such that for all $x \in \partial E$ and $0 \leq r \leq r_0$*

$$\mu(E \cap U(x,r)) \geq C\mu(U(x,r)).$$

Theorem 2.25. *Let (M,d,μ) be a doubling metric space with geodesics. Then $\Omega \subset M$ is a weak John domain if and only if it is a John domain.*

Corollary 2.26. *Suppose X is a system of bounded Lipschitz continuous vector fields in $\mathbb{R}^n$ satisfying (H.1) and (H.2). Then $\Omega \subset \mathbb{R}^n$ is a weak John domain for the Carnot–Carathéodory distance d if and only if it is a John domain for d.*

The proof of Theorem 2.25 can be found in [68, Proposition 9.6] and for the Euclidean case in [91, Lemma 2.7].

Definition 2.27. An open set $\Omega \subset M$ is a *Boman domain* if there exists a covering $\mathcal{F}$ of Ω with balls and there exist $N \geq 1$, $\lambda > 1$ and $\nu \geq 1$ such that

(i) $\lambda U \subset \Omega$ for all $U \in \mathcal{F}$;
(ii) $\sum_{U\in\mathcal{F}} \mathbf{1}_{\lambda U}(x) \leq N$ for all $x \in \Omega$;
(iii) there exists $U_0 \in \mathcal{F}$ such that for any $U \in \mathcal{F}$ there exist $U_1, ..., U_k$ such that $U_k = U$, $\mu(U_i \cap U_{i+1}) \geq 1/N \max\{\mu(U_i), \mu(U_{i+1})\}$ and $U \subset \nu U_i$ for all $i = 0, 1, ..., k$.

Under additional hypotheses on the metric space the definition of John domain is equivalent to that of Boman domain (see [15] and [61, section 6]).

Theorem 2.28. *Let (M,d,μ) be a doubling metric space. If $\Omega \subset M$, $\Omega \neq M$, is a weak John domain then it is a Boman domain.*

Theorem 2.29. *Let (M,d,μ) be a doubling metric space with geodesics. If $\Omega \subset M$ is a Boman domain then it is a John domain.*

Corollary 2.30. *Suppose X is a system of bounded Lipschitz continuous vector fields in $\mathbb{R}^n$ satisfying (H.1) and (H.2). Then $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, is a John domain for the Carnot–Carathéodory distance d if and only if it is a Boman domain for d. In particular, metric balls are Boman domains.*

We can state now a Poincaré inequality for Boman (= John) domains (see [47]).

Theorem 2.31. *Let X be a family of vector fields satisfying Hörmander's rank condition, and let Ω be a Boman (= John) domain. Suppose p and q as in Theorem 2.14 can be chosen uniformly in Ω. Then*

$$\left(\int_\Omega |f - f_\Omega|^q \, dx\right)^{1/q} \leq C_\Omega \left(\int_\Omega |Xf|^p \, dx\right)^{1/p} \tag{23}$$

with C_Ω independent of f.

If $1 \leq p < Q$ we can always choose $q = p^* := pQ/(Q-p)$, if Q is the homogeneous dimension of a compact neighborhood of Ω.

Sharp characterization of John domains with respect to families of vector fields are given in [99] and [100].

Theorem 2.31 yields the following Rellich type theorem.

Theorem 2.32. *Suppose the assumption of Theorem 2.31 hold. Then:*

(i) *if $1 \le p < Q$ and $1 \le q < p^*$ the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact;*
(ii) *if $p \ge Q$ and $q \ge 1$ the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact.*

3. BV space

Let us remind now the notion of functions of bounded X-variation and recall some of their properties (see [51] and [61]). Let $\Omega \subset \mathbb{R}^n$ be an open set and set

$$F(\Omega;\mathbb{R}^m) := \{\varphi \in C_0^1(\Omega;\mathbb{R}^m) : |\varphi(x)| \le 1 \ \ \forall x \in \Omega\}. \tag{24}$$

The space $BV_X(\Omega)$ is the set of functions $f \in L^1(\Omega)$ such that

$$\|Xf\|(\Omega) := \sup_{\varphi \in F(\Omega;\mathbb{R}^m)} \int_\Omega f(x)\, \mathrm{div}_X\,(\varphi)(x)\, dx < \infty. \tag{25}$$

The space $BV_{X,\mathrm{loc}}(\Omega)$ is the set of functions belonging to $BV_X(U)$ for each open set $U \subset\subset \Omega$.

Observe that if $f \in W^{1,1}_{X;\mathrm{loc}}(\Omega)$ then

$$\int_\Omega d\|Xf\| = \int_\Omega |Xf|\, dx.$$

A measurable set $E \subset \mathbb{R}^n$ is of locally finite X-perimeter in Ω (or is a X-Caccioppoli set) if the indicatrix function $\mathbf{1}_E \in BV_{X,\mathrm{loc}}(\Omega)$, namely if

$$|\partial E|_X(U) := \|X\mathbf{1}_E\|(U) < \infty \tag{26}$$

for every open set $U \subset\subset \Omega$.

For each $f \in BV_X(\Omega)$ the functional Xf can be extended to all of $\mathbf{C}_0^0(\Omega;\mathbb{R}^m)$. We keep calling Xu such an extension. By means of Riesz representation theorem one can prove that if $f \in BV_{X,\mathrm{loc}}(\Omega)$ then $\|Xf\|$ is a Radon measures on Ω (see [37], 2.2.5). Moreover, the following results hold (see [51] and [20], respectively).

Proposition 3.1 (Lower semicontinuity)**.** *Let $f, f_k \in L^1(\Omega)$, $k \in \mathbb{N}$, be such that $f_k \to f$ in $L^1(\Omega)$; then*

$$\liminf_{k\to\infty} \|Xf_k\|(\Omega) \ge \|Xf\|(\Omega).$$

Proposition 3.2. *If E is a X-Caccioppoli set with C^1 boundary, then the X-perimeter has the following representation*

$$|\partial E|_X(\Omega) = \int_{\partial E \cap \Omega} \big(\sum_j \langle X_j, n\rangle^2\big)^{1/2} d\mathcal{H}^{n-1},$$

here $n(x)$ is the Euclidean unit outward normal to E and $\mathcal{H}^s$ is the Euclidean s-dimensional Hausdorff measure.

Theorem 3.3 (Structure of BV_X functions)**.** *Let $f \in BV_X(\Omega)$ and write $\mu = \|Xf\|$. There exists a μ-measurable function $\sigma_f : \Omega \to \mathbb{R}^m$ such that $|\sigma_f| = 1$ μ-almost everywhere and*

$$\int_\Omega f(x)\mathrm{div}_X\,(\varphi)(x)\, dx = \int_\Omega \langle \varphi(x), \sigma_f(x)\rangle d\mu,$$

for all $\varphi \in F(\Omega;\mathbb{R}^m)$.

When $f = \mathbf{1}_E$ in Theorem 3.3, then we call X-*generalized inner normal* of E in Ω the vector

$$\nu_E(x) := -\sigma_{\mathbf{1}_E}(x). \tag{27}$$

As for the Sobolev spaces $W^{1,p}_X$, $1 < p < \infty$, the space BV_X can be defined as the domain of a relaxed functional. In particular, this shows that our space BV_X fits into the setting of BV-spaces in metric spaces introduced by M. Miranda jr. [93] and L. Ambrosio [1].

To this end, let us state preliminarily an approximation theorem in BV_X that is the exact counterpart of the corresponding result for usual BV functions proved by Anzellotti & Giaquinta [6]. The following result is proved in [51], Theorem 2.2.2.

Theorem 3.4. *Let $u \in BV_X(\Omega)$. Then there exists a sequence $(u_h)_h \subset \mathbf{C}^\infty_0(\Omega)$ such that*

$$\lim_{h\to+\infty} \|u_h - u\|_{L^1(\Omega)} = 0$$
$$\lim_{h\to+\infty} \int_\Omega d\|Xu_h\| = \int_\Omega d\|Xu\|.$$

Moreover ([51], Corollary 2.2.3) we have:

Corollary 3.5. *For $u \in L^1(\Omega)$ we define*

$$\int_\Omega \sqrt{1+|Xu|^2}$$
$$= \sup\left\{\int_\Omega (\varphi + u\,\mathrm{div}_X\psi)\,dx : (\varphi,\psi) \in C^\infty_0(\Omega, \mathbb{R}\times\mathbb{R}^m), |\varphi(x)|^2 + |\psi(x)|^2 \le 1\right\}.$$

Then the following facts hold:

(i)

$$\int_\Omega d|Xu| \le \int_\Omega \sqrt{1+|Xu|^2} \le |\Omega| + \int_\Omega d|Xu| \quad \textit{for every } u \in L^1(\Omega),$$
$$\int_\Omega \sqrt{1+|Xu|^2} = \int_\Omega \sqrt{1+|Xu(x)|^2}dx \ \ \textit{for every } u \in W^{1,1}_{X;\mathrm{loc}}(\Omega).$$

(ii) *Let $(u_h)_h$, $u \in L^1(\Omega)$ be such that $u_h \to u$ in $L^1(\Omega)$. Then*

$$\int_\Omega \sqrt{1+|Xu|^2} \le \liminf_{h\to\infty} \int_\Omega \sqrt{1+|Xu_h|^2}.$$

(iii) *Let $u \in BV(\Omega)$; then there exists a sequence $(u_h)_h$ in $C^1(\Omega)\cap BV_X(\Omega)$ such that*

$$u_h \to u \ \textit{in } L^1(\Omega), \ \textit{and} \int_\Omega \sqrt{1+|Xu_h|^2}dx \to \int_\Omega \sqrt{1+|Xu|^2}.$$

Thanks to the above approximation theorem (Theorem 3.4), we can pass to the limit in the Poincaré inequality of Theorem 2.14 and we obtain an intrinsic isoperimetric inequality. This result is proved in [61], but appears also in a slightly less general form in [47] (see also [43]). However, in the setting of the Heisenberg group (see below), a (different but *a posteriori* equivalent, by Theorem 5.7) isoperimetric inequality was proved by P.Pansu in [106] (see also [105]).

Theorem 3.6 (Isoperimetric inequality). *Let X be a system of smooth vector fields satisfying Hörmander's rank condition. Let $1 \le q < \infty$ be such that the following balance condition holds:*

$$\frac{r(\tilde U)}{r(U)}\left(\frac{|\tilde U|}{|U|}\right)^{1/q} \le C\,\frac{|\tilde U|}{|U|} \tag{28}$$

for all balls $\tilde U, U$ *such that* $\tilde U \subset U$. *Then*

$$\min\{|E\cap\Omega|, |(\mathbb{R}^n\setminus E)\cap\Omega|\}^{\frac{q-1}{q}} \le C\frac{r(U)}{|U|^{1/q}}|\partial E|_X(\Omega). \tag{29}$$

with C *independent of* E.

A similar result with balls replaced by John (= Boman)-domains can be analogously derived from Theorem 2.31.

A coarea formula for vector fields has been proved in [62], [51], [85], [88] [86] and [101]. A similar coarea formula in the setting of metric spaces has been proved also in [3] and [93]. In the coarea formula a solid integral is expressed as a superposition of surface integrals and the integration measure is the perimeter of the boundary of the level sets of a Lipschitz function. The following statement follows that of [98].

Theorem 3.7. *Let* $X_1, ..., X_m \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ *and let* $\Omega\subset\mathbb{R}^n$ *be an open set. If* $f\in BV_X(\Omega)$ *then*

$$\|Xf\|(\Omega) = \int_{-\infty}^{+\infty} |\partial E_t|_X(\Omega)\,dt, \tag{30}$$

where $E_t = \{x\in\Omega : f(x) > t\}$.

Moreover, if (H.1) holds, $f\in \mathrm{Lip}(\Omega, d)$ *and* $u\in L^1(\Omega)$, *then*

$$\int_\Omega u\,|Xf|\,dx = \int_{-\infty}^{+\infty}\Big(\int_{\{x\in\Omega: f(x)=t\}} u\,d|\partial E_t|_X\Big)\,dt. \tag{31}$$

Finally we recall that from the approximation result and the coarea formula we get the following approximation result for bounded subsets of $\mathbb{R}^n$ of finite X-perimeter.

Corollary 3.8. *Let* E *be a bounded subset of* $\mathbb{R}^n$ *of finite* X*-perimeter. Then* E *can be approximated by a sequence of* C^∞ *sets* E_h *such that*

$$\int_{\mathbb{R}^n}|\mathbf{1}_{E_h}-\mathbf{1}_E|dx\to 0, \qquad \int_{\mathbb{R}^n} d\|X\mathbf{1}_{E_h}\|\to\int_{\mathbb{R}^n} d\|X\mathbf{1}_E\|.$$

Let now $f:\Omega\times\mathbb{R}^m\to[0,\infty)$ be a Borel function verifying (1). We denote by $f^\infty:\Omega\times\mathbb{R}^m\to[0,\infty)$ the *recession* function of f, that is

$$f^\infty(x,\eta) := \lim_{t\to 0^+} f(x,\frac{\eta}{t})t \quad \text{for every } x\in\Omega,\ \eta\in\mathbb{R}^m$$

and by $\bar f:\Omega\times\mathbb{R}^m\times[0,\infty)\to[0,\infty)$ the function

$$\bar f(x,\eta,t) := \begin{cases} f(x,\frac{\eta}{t})t & t>0\\ f^\infty(x,\eta) & t=0\end{cases}. \tag{32}$$

Moreover, if μ is a m-vector-valued Radon measure, let us set

$$\int_\Omega f(x,\mu) := \int_\Omega f(x,[\mu]_a(x))dx + \int_\Omega f^\infty(x,\frac{d[\mu]_s}{d|[\mu]_s|}(x))d|[\mu]_s| \tag{33}$$

where $\mu = [\mu]_a dx + [\mu]_s$ is the Lebesgue decomposition of μ in its absolutely continuous and singular parts with respect to Lebesgue measure. As usual $\frac{d[\mu]_s}{d|[\mu]_s|}(x)$ and $[\mu]_a(x)$ are respectively the density of $[\mu]_s$ with respect to $|[\mu]_s|$ and the density of $[\mu]_a dx$ with the respect to Lebesgue measure.

The following semicontinuity and continuity properties of functional (33) on the set of m-vector-valued Radon measures are extensions of well known results proved by Reschetnyak in [109] (for a proof of these versions see the appendix of [84] or Theorems 4.4 and 4.6 in [33]).

Theorem 3.9. *Let $f : \Omega \times \mathbb{R}^m \to [0, \infty)$ be a Borel function verifying (1), and assume that the function $\bar{f}$ defined in (32) is lower semicontinuous. Then, for every $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$ and $(\mu_h)_h \subset \mathcal{M}(\Omega; \mathbb{R}^m)$ with $\mu_h \to \mu$ weakly in $\mathcal{M}(\Omega; \mathbb{R}^m)$*

$$\int_\Omega f(x, \mu) \leq \liminf_{h\to\infty} \int_\Omega f(x, \mu_h)\,.$$

Theorem 3.10. *Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $f : \Omega \times \mathbb{R}^m \to [0, \infty)$ be a Borel function verifying (1) and (2) with $p = 1$. Let us suppose that the function $\bar{f}$ defined in (32) is continuous. Then, for every $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$ and $(\mu_h)_h \subset \mathcal{M}(\Omega; \mathbb{R}^m)$ with*

$$\mu_h \to \mu \text{ weakly in } \mathcal{M}(\Omega; \mathbb{R}^m) \quad \text{and} \quad \int_\Omega \sqrt{1 + |\mu_h|^2} \to \int_\Omega \sqrt{1 + |\mu|^2}\,,$$

it follows

$$\lim_{h\to\infty} \int_\Omega f(x, \mu_h) = \int_\Omega f(x, \mu)\,.$$

We are now in position to state the characterization result for the relaxed functional $\bar{F}_1$, which extends well known results for the classical Euclidean case that is when $X = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$.

Theorem 3.11. *Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $f : \Omega \times \mathbb{R}^m \to [0, \infty)$ be a Borel function verifying (1) and (2) with $p = 1$. Let us suppose that the function $\bar{f}$ defined in (32) is continuous. Then*

(i) $\operatorname{dom} \bar{F}_1 := \{u \in L^1(\Omega) : \bar{F}_1(u) < \infty\} = BV_X(\Omega)$*;*
(ii) $\bar{F}_1(u) = \int_\Omega f(x, Xu)$ *for every* $u \in BV_X(\Omega)$*.*

Remark 3.12. If f verifies (1), the continuity of the function $\bar{f}$ defined in (32) is equivalent to the existence, for every x and $x_0 \in \Omega$ and for every $\varepsilon > 0$, of a $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$|x - x_0| < \delta \Longrightarrow |f(x, \eta) - f(x_0, \eta)| \leq \varepsilon\,(1 + |\eta|) \quad \text{for every } \eta \in \mathbb{R}^m\,.$$

By Theorem 3.11 and Remark 3.12, we get the following characterization of the relaxed area functional.

Corollary 3.13. *Let Ω be a bounded open subset of $\mathbb{R}^n$, then for every $u \in BV_X(\Omega)$*

$$\int_\Omega \sqrt{1 + |Xu|^2} = \int_\Omega \sqrt{1 + |[Xu]_a(x)|^2}dx + \int_\Omega d|[Xu]_s|.$$

The original definition of perimeter given by De Giorgi [31], [32] involves the approximation by means of polyhedral hypersurfaces. It may be surprising to see that the same result holds for the X-perimeter, even if there are no intrinsic polyhedral hypersurfaces. This result has been proved by F. Montefalcone [95].

Definition 3.14. Let $A(n, n-1)$ denote the set of $(n-1)$-dimensional affine manifolds (i.e. the hyperplanes) in $\mathbb{R}^n$. We say that Σ is an Euclidean polyhedral domain if there exist $\kappa \in \mathbb{N}$ and $\mathcal{J} := \{\mathcal{J}_i\}_{i=1}^{\kappa} \subseteq A(n, n-1)$ such that

$$\mathrm{Fr}(\Sigma) \subseteq \bigcup_{i=1}^{\kappa} \mathcal{J}_i \quad .$$

By $\mathcal{P}^n$ we denote the set of all Euclidean polyhedral domains of $\mathbb{R}^n$.

Then the following approximation result holds.

Theorem 3.15. *Let X be a family of Lipschitz continuous vector fields. Let $E \subseteq \mathbb{R}^n$ with $|E| < \infty$. Then there exists a family Σ of polyhedral domains, $\Sigma := \{\Sigma_i\}_{i\in\mathbb{N}} \subseteq \mathcal{P}^n$, such that*

1. $\lim_i \|\mathbf{1}_{\Sigma_i} - \mathbf{1}_{\mathrm{E}}\|_{\mathcal{L}^1(\Omega)} = 0$

2. $\lim_i \|\partial\Sigma_i\|_X(\Omega) = \|\partial \mathrm{E}\|_X(\Omega)$

for any open set $\Omega \subset \mathbb{R}^n$.

When a family of Lipschitz continuous vector fields $X = (X_1, \dots, X_m)$ is given, we can define the j-th partial perimeter $\|\partial\mathrm{E}\|_{X_j}$ of a set $E \subseteq \mathbb{R}^n$ as the perimeter associated with the family (X_j) given by the vector field X_j alone. Then the following characterization of X-Caccioppoli sets is proved in [95].

Theorem 3.16. *Let X, E, and Ω be as in Theorem 3.15. If for each $j = 1, \dots, m$ there exist $\{\Sigma_i^j\}_{i\in\mathbb{N}} \subseteq \mathcal{P}^n$ and $\mathcal{A}_j < \infty$ such that*

(i) $\lim_i \|\mathbf{1}_{\Sigma_i^j} - \mathbf{1}_{\mathrm{E}}\|_{\mathcal{L}^1(\Omega)} = 0$,

(ii) $\sup_{i\in\mathbb{N}} \|\partial\Sigma_i^j\|_{X_j}(\Omega) \le \mathcal{A}_j$,

then E *has finite X-perimeter in Ω and there exists $\{\Sigma_i\}_{i\in\mathbb{N}} \subseteq \mathcal{P}^n$ such that*

(iii) $\lim_i \|\mathbf{1}_{\Sigma_i} - \mathbf{1}_{\mathrm{E}}\|_{\mathcal{L}^1(\Omega)} = 0$,

(iv) $\lim_i \|\partial\Sigma_i\|_X(\Omega) = \|\partial\mathrm{E}\|_X(\Omega)$.

The perimeter appears in the Euclidean setting also in connection with the notion of *Minkowski content*, that is, roughly speaking, the derivative with respect to ε of the volume of a ε-neighborhood of the boundary. It is well known that in the Euclidean setting the two notion coincide for sufficiently regular sets. A similar result for the X-perimeter has been proved by R. Monti & F. Serra Cassano in [101].

Let $E \subset \mathbb{R}^n$ be a bounded open set, and let $X = (X_1, ..., X_m)$ be a family of *smooth* vector fields. Suppose (H.1) and (H.2) hold, and let d be the Carnot–Carathéodory distance associated with $X_1, ..., X_m$. Set $d_{\partial E}(x) = \inf_{y\in\partial E} d(x, y)$, and for $r > 0$ define the tubular neighborhood $I_{r,X}(\partial E) = \{x \in \mathbb{R}^n : d_{\partial E}(x) < r\}$. The *upper* and *lower Minkowski content* of ∂E in an open set $\Omega \subset \mathbb{R}^n$ are respectively defined by

$$M_X^+(\partial E)(\Omega) := \limsup_{r\to 0^+} \frac{|I_{r,X}(\partial E) \cap \Omega|}{2r},$$
$$M_X^-(\partial E)(\Omega) := \liminf_{r\to 0^+} \frac{|I_{r,X}(\partial E) \cap \Omega|}{2r}.$$

The following theorem states that if E is regular and Ω has regular boundary, then

$$M_X^+(\partial E)(\Omega) = M_X^-(\partial E)(\Omega),$$

and this common value, which we shall call *X-Minkowski content* of ∂E in Ω and we denote by $M_X(\partial E)(\Omega)$, coincides with the $X-$perimeter of E in Ω as defined in (26). The proof is based on a Riemannian approximation of the C-C space $(\mathbb{R}^n, d)$. Here $\mathcal{H}^{n-1}$ stands for the $(n-1)-$dimensional Euclidean Hausdorff measure.

Theorem 3.17. *Let $\Omega \subset \mathbb{R}^n$ be an open set with C^∞ boundary or $\Omega = \mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be a bounded open set with C^∞ boundary and suppose that $\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0$. Then $M_X^+(\partial E)(\Omega) = M_X^-(\partial E)(\Omega)$ and in addition*

$$M_X(\partial E)(\Omega) = \|\partial E\|_X(\Omega).$$

There is another important characterization of the X-perimeter of a set $E \subset \mathbb{R}^n$ in terms of variational convergence (De Giorgi's Γ-convergence) of "solid" integrals. In the Euclidean setting, this result is known in the literature as Modica-Mortola convergence result.

This variational characterization has been extended to the X-perimeter by R. Monti & F. Serra Cassano in [101].

We recall first the definition of $\Gamma-$convergence (for a comprehensive introduction see [28]).

Definition 3.18. Let (M,d) be a metric space, and let $F, F_h : M \to [-\infty,+\infty]$, $h \in \mathbb{N}$. F is said to be the $\Gamma-$limit of the sequence $(F_h)_{h\in\mathbb{N}}$, and we shall write $F = \Gamma(M) - \lim_{h\to\infty} F_h$, if the following conditions hold

$$\text{if } x \in M \text{ and } x_h \to x \quad \text{then} \quad F(x) \le \liminf_{h\to\infty} F_h(x_h), \tag{34}$$

$$\forall x \in M\ \exists (x_h)_{h\in\mathbb{N}} \quad \text{such that } x_h \to x \quad \text{and } F(x) \ge \limsup_{h\to\infty} F_h(x_h). \tag{35}$$

First, in [101] the authors prove that the X-perimeter is the Γ-limit of a family of Riemannian perimeters, as the Carnot–Carathéodory distance is the limit of Riemannian distances.

For $\varepsilon > 0$ define the new family $X_\varepsilon = (X_1, ..., X_m, \varepsilon\partial_1, ..., \varepsilon\partial_n)$. Let $\Omega \subset \mathbb{R}^n$ be an open set and define the functionals $P, P_\varepsilon : L^1(\Omega) \to [0,+\infty]$

$$P(u) = \begin{cases} \|\partial E\|_X(\Omega) & \text{if } u = \chi_E \in BV_X(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$P_\varepsilon(u) = \begin{cases} \|\partial E\|_{X_\varepsilon}(\Omega) & \text{if } u = \chi_E \in BV_{X_\varepsilon}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\varepsilon_h \to 0$ and write $P_h = P_{\varepsilon_h}$. In the following theorem we prove that the "elliptic-Riemannian" regularization of the perimeter $\Gamma-$converges to the perimeter.

Theorem 3.19. *If $\Omega \subset \mathbb{R}^n$ is a bounded open set with C^∞ boundary then*

$$P = \Gamma(L^1(\Omega)) - \lim_{h\to\infty} P_h.$$

Finally, fix a bounded open set $\Omega \subset \mathbb{R}^n$. For $\varepsilon > 0$ define the functionals $F, F_\varepsilon : L^1(\Omega) \to [0,+\infty]$

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega (\varepsilon|Xu|^2 + \frac{1}{\varepsilon} W(u))dx & \text{if } u \in W^{1,2}_X(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{36}$$

where $W(u) = u^2(1-u)^2$, and

$$F(u) = \begin{cases} 2\alpha\|\partial E\|_X(\Omega) & \text{if } u = \chi_E \in BV_X(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{37}$$

where $\alpha = \displaystyle\int_0^1 \sqrt{W(s)}ds$. Let $\varepsilon_h \to 0$ and write $F_h := F_{\varepsilon_h}$.

Theorem 3.20. *Suppose that $X_1, ..., X_m \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ satisfy hypotheses (H1) and (H2). If $\Omega \subset \mathbb{R}^n$ is a bounded open set with C^∞ boundary then*

$$F = \Gamma(L^1(\Omega)) - \lim_{h\to\infty} F_h. \tag{38}$$

4. Carnot groups.

4.1. **Definition and first properties.** The present subsection is largely taken from [57] and [54] (see also [56]). A *Carnot group* $\mathbb{G}$ *of step* k (see [39], [71], [101], [69], [104], [118], and [119]) is a connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a *step* k *stratification*, i.e. there exist linear subspaces $V_1, ..., V_k$ such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad V_k \neq \{0\}, \quad V_i = \{0\} \text{ if } i > k, \tag{39}$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. Let $m_i = \dim(V_i)$, for $i = 1, \ldots, k$ and $h_i = m_1 + \cdots + m_i$ with $h_0 = 0$ and, clearly, $h_k = n$. Choose a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$ adapted to the stratification, that is such that

$$e_{h_{j-1}+1}, \ldots, e_{h_j} \text{ is a base of } V_j \ \text{ for each } j = 1, \ldots, k.$$

Let $X = X_1, \ldots, X_n$ be the family of left invariant vector fields such that $X_i(0) = e_i$. Given (39), the subset $X_1, \ldots, X_{m_1}$ generates by commutations all the other vector fields; we will refer to $X_1, \ldots, X_{m_1}$ as *generating vector fields* of the group. The exponential map is a one to one map from $\mathfrak{g}$ onto $\mathbb{G}$, i.e. any $p \in \mathbb{G}$ can be written in a unique way as $p = \exp(p_1X_1 + \cdots + p_nX_n)$. Using these *exponential coordinates*, we identify p with the n-tuple $(p_1, \ldots, p_n) \in \mathbb{R}^n$ and we identify $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$ where the explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula (see [39]) and some of its features are described in the following Proposition 4.2. If $p \in \mathbb{G}$ and $i = 1, \ldots, k$, we put $p^i = (p_{h_{i-1}+1}, \ldots, p_{h_i}) \in \mathbb{R}^{m_i}$, so that we can also identify p with $[p^1, \ldots, p^k] \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k} = \mathbb{R}^n$.

The subbundle of the tangent bundle $T\mathbb{G}$ that is spanned by the vector fields $X_1, \ldots, X_{m_1}$ plays a particularly important role in the theory, it is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H\mathbb{G}_x = \text{span } \{X_1(x), \ldots, X_{m_1}(x)\}, \qquad x \in \mathbb{G}.$$

A subriemannian structure is defined on $\mathbb{G}$, endowing each fiber of $H\mathbb{G}$ with a scalar product $\langle \cdot, \cdot \rangle_x$ and with a norm $|\cdot|_x$ that make the basis $X_1(x), \ldots, X_{m_1}(x)$ an orthonormal basis. That is if $v = \sum_{i=1}^{m_1} v_i X_i(x) = (v_1, \ldots, v_{m_1})$ and $w = \sum_{i=1}^{m_1} w_i X_i(x) = (w_1, \ldots, w_{m_1})$ are in $H\mathbb{G}_x$, then $\langle v, w \rangle_x := \sum_{j=1}^{m_1} v_j w_j$ and $|v|_x^2 := \langle v, v \rangle_x$.
The sections of $H\mathbb{G}$ are called *horizontal sections*, a vector of $H\mathbb{G}_x$ is an *horizontal vector* while any vector in $T\mathbb{G}_x$ that is not horizontal is a vertical vector. Each horizontal section is identified by its canonical coordinates with respect to this moving frame $X_1(x), \ldots, X_{m_1}(x)$. This way, an horizontal section φ is identified with a function $\varphi = (\varphi_1, \ldots, \varphi_{m_1}) : \mathbb{R}^n \to \mathbb{R}^{m_1}$. When dealing with two such sections φ and ψ whose argument is not explicitly written, we drop the index x in the scalar product writing $\langle \psi, \varphi \rangle$ for $\langle \psi(x), \varphi(x) \rangle_x$. The same convention is adopted for the norm.

Two important families of automorphism of $\mathbb{G}$ are the so called intrinsic translations and the intrinsic dilations of $\mathbb{G}$. For any $x \in \mathbb{G}$, the *(left) translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined as

$$z \mapsto \tau_x z := x \cdot z.$$

For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

$$\delta_\lambda(x_1, ..., x_n) = (\lambda^{\alpha_1} x_1, ..., \lambda^{\alpha_n} x_n), \tag{40}$$

where $\alpha_i \in \mathbb{N}$ is called *homogeneity of the variable* x_i in $\mathbb{G}$ (see [40] Chapter 1) and is defined as

$$\alpha_j = i \quad \text{whenever } h_{i-1} + 1 \leq j \leq h_i, \tag{41}$$

hence $1 = \alpha_1 = ... = \alpha_{m_1} < \alpha_{m_1+1} = 2 \leq ... \leq \alpha_n = k$.

The simplest example of Carnot group is provided by the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$. We denote the points of $\mathbb{H}^n$ by $P = [z, t] = [x + iy, t]$, $z \in \mathbb{C}^n$, $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$. If $P = [z, t]$,

$Q = [\zeta, \tau] \in \mathbb{H}^n$ and $r > 0$, following the notations of [116], where the reader can find an exhaustive introduction to the Heisenberg group, we define the group operation

$$P \cdot Q := [z + \zeta, t + \tau + 2\Im m(z\bar{\zeta})]$$

and the family of non isotropic dilations

$$\delta_r(P) := [rz, r^2 t].$$

The Lie algebra of left invariant vector fields in $\mathbb{H}^n$ is given by

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad j = 1, \ldots, n \\ Y_j &= \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad j = 1, \ldots, n \\ T &= \frac{\partial}{\partial t}, \end{aligned}$$

the only non-trivial commutator relations being

$$[X_j, Y_j] = -4T, \qquad j = 1, \ldots, n.$$

Thus the vector fields $X_1, \ldots, X_n, Y_1 \ldots, Y_n$ satisfy Hörmander's rank condition, and $\mathbb{H}^n$ is a step 2 Carnot group, the stratification of the Lie algebra of left invariant vector fields being given by

$$V_1 = \text{span}\,\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\} \quad \text{and} \quad V_1 = \text{span}\,\{T\}.$$

An alternative approach to Carnot groups is given by A. Bonfiglioli & F. Uguzzoni and A. Bonfiglioli in [14] and [13]. Let us sketch it. Basically, it is an alternative presentation that corresponds to the standard definition when the last one is seen in a particular coordinate system (the exponential coordinates).

Theorem 4.1. *If $x, y \in \mathbb{R}^n$, let $(x, y) \to x \circ y$ be a multiplication in $\mathbb{R}^n$. Assume the origin is the identity element and $\mathbb{G} = (\mathbb{R}^n, \circ)$ is a Lie group, i.e. the multiplication and the inverse $x \to x^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ operations are smooth maps.*

Assume also $\mathbb{G}$ is a homogenous group (see [116]*, 13.5), in the following sense: we write $n = m_1 + m_2 + \cdots + m_k$, and, given $x \in \mathbb{R}^n$, we put $x = [x^1, x^2, \cdots, x^k]$ with $x^j \in \mathbb{R}^{m_j}$ for $j = 1, \cdots, k$. Then assume that the family of dilations*

$$\delta_\lambda x = [\lambda x^1, \lambda^2 x^2 \cdots, \lambda^k x^k], \qquad \lambda > 0, \tag{42}$$

forms a group of automorphisms of $\mathbb{G}$, i.e. $\delta_\lambda(x \circ y) = \delta_\lambda x \circ \delta_\lambda y$.

Let $\mathfrak{g}$ denote the Lie algebra of $\mathbb{G}$, i.e. the class of left invariant vector fields on $\mathbb{G}$, and take a basis $X_1, \cdots, X_N$ of $\mathfrak{g}$ such that $X_j(0) = D_j$, $j = 1, \cdots, n$ (left invariant vector fields are fully determined by their value at the origin).

Assume that the Lie algebra generated by $X_1, \ldots, X_{m_1}$ coincides with $\mathfrak{g}$. Then $\mathbb{G} = (\mathbb{R}^n, \circ)$ is a Carnot group of step k with m_1 generators.

We collect in the following proposition some more or less elementary properties of the group operation and of the canonical vector fields .

Proposition 4.2. *The group product has the form*

$$x \cdot y = x + y + \mathcal{Q}(x, y), \qquad \forall x, y \in \mathbb{R}^n \tag{43}$$

where $\mathcal{Q} = (\mathcal{Q}_1, \ldots, \mathcal{Q}_n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and each $\mathcal{Q}_i$ is a homogeneous polynomial of degree α_i with respect to the intrinsic dilations of $\mathbb{G}$ defined in (40), that is

$$\mathcal{Q}_i(\delta_\lambda x, \delta_\lambda y) = \lambda^{\alpha_i} \mathcal{Q}_i(x, y), \quad \forall x, y \in \mathbb{G}.$$

Moreover, again $\forall x, y \in \mathbb{G}$

$$\mathcal{Q}_1(x,y) = ... = \mathcal{Q}_{m_1}(x,y) = 0,$$

$$\mathcal{Q}_j(x,0) = \mathcal{Q}_j(0,y) = 0 \quad \text{and} \quad \mathcal{Q}_j(x,x) = Q_j(x,-x) = 0, \quad \text{for } m_1 < j \le n, \tag{44}$$

$$\mathcal{Q}_j(x,y) = \mathcal{Q}_j(x_1,\dots,x_{h_{i-1}},y_1,\dots,y_{h_{i-1}}), \quad \text{if} \quad 1 < i \le k \quad \text{and} \quad j \le h_i. \tag{45}$$

Proof. For the first part see [116], Chapter 12, Section 5. The last statement follows the homogeneity of Q_j. □

Note that from Proposition 4.2 it follows that

$$\delta_\lambda x \cdot \delta_\lambda y = \delta_\lambda(x \cdot y)$$

and that the inverse x^{-1} of an element $x = (x_1, \dots, x_n) \in (\mathbb{R}^n, \cdot)$ has the form

$$x^{-1} = (-x_1, \dots, -x_n),$$

(see [40], Proposition 2.1 and also [71]).

Proposition 4.3. *The vector fields X_j have polynomial coefficients and if $h_{\ell-1} < j \le h_\ell$, $1 \le \ell \le k$, then*

$$X_j(x) = \partial_j + \sum_{i > h_l}^{n} q_{i,j}(x)\partial_i, \tag{46}$$

where $q_{i,j}(x) = \frac{\partial Q_i}{\partial y_j}(x,y)|_{y=0}$ so that if $h_{\ell-1} < j \le h_\ell$, then $q_{i,j}(x) = q_{i,j}(x_1,...,x_{h_{l-1}})$ and $q_{i,j}(0) = 0$.

By (39), the rank of the Lie algebra generated by $X_1, \dots, X_{m_1}$ is n; hence $X = (X_1, \dots, X_{m_1})$ is a system of smooth vector fields satisfying Hörmander's condition.

Several distances equivalent to d have been used in the literature. Later on, we shall use the following one, that can also be computed explicitly

$$d_\infty(x,y) = d_\infty(y^{-1} \cdot x, 0),$$

where, if $p = [p^1, \dots, \tilde{p}^k] \in \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_k} = \mathbb{R}^n$, then

$$d_\infty(p,0) = \max\{\varepsilon_j ||p^j||_{\mathbb{R}^{m_j}}^{1/j}, j = 1, \dots, k\}. \tag{47}$$

Here $\varepsilon_1 = 1$, and $\varepsilon_2, \dots \varepsilon_k \in (0,1)$ are suitable positive constants depending on the group structure. As above, we shall denote $U_\infty(p,r)$ and $B_\infty(p,r)$ respectively the open and closed balls associated with d_∞.
Both the Carnot–Carathéodory metric d and the metric d_∞ are well behaved with respect to left translations and dilations, that is

$$\begin{aligned} d(z \cdot x, z \cdot y) = d(x,y) \quad &, \quad d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x,y) \\ d_\infty(z \cdot x, z \cdot y) = d_\infty(x,y) \quad &, \quad d_\infty(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_\infty(x,y) \end{aligned} \tag{48}$$

for $x, y, z \in \mathbb{G}$ and $\lambda > 0$.

Related with these distances, different Hausdorff measures, obtained by Carathédory's construction as in [37] Section 2.10.2, are used in this paper: we denote by $\mathcal{H}^m$ the m-dimensional Hausdorff measure obtained from the Euclidean distance in $\mathbb{R}^n \simeq \mathbb{G}$, by $\mathcal{H}_c^m$ the m-dimensional Hausdorff measure obtained from the distance d in $\mathbb{G}$, and by $\mathcal{H}_\infty^m$ the m-dimensional Hausdorff measure obtained from the distance d_∞ in $\mathbb{G}$. Analogously, $\mathcal{S}^m$, $\mathcal{S}_c^m$, and $\mathcal{S}_\infty^m$ denote the corresponding spherical Hausdorff measures.

The integer

$$Q = \sum_{j=1}^{n} \alpha_j = \sum_{i=1}^{k} i \dim V_i \tag{49}$$

is the *homogeneous dimension* of $\mathbb{G}$. It is also the Hausdorff dimension of $\mathbb{R}^n$ with respect to the Carnot–Carathéodory distance d. For this statement, see [97]. However, in the setting of Carnot groups, this property follows easily from (50) below. Indeed, (50) implies that Lebesgue measure is Q-Ahlfors-David regular, and hence that it is equivalent to $\mathcal{H}_c^Q$ (for instance by [37], 2.10-17 and 2.10-18).

The n-dimensional Lebesgue measure $\mathcal{L}^n$, is the Haar measure of the group $\mathbb{G}$. Hence if $E \subset \mathbb{R}^n$ is measurable, then $\mathcal{L}^n(x \cdot E) = \mathcal{L}^n(E)$ for all $x \in \mathbb{G}$. Moreover, if $\lambda > 0$ then $\mathcal{L}^n(\delta_\lambda(E)) = \lambda^Q \mathcal{L}^n(E)$. We explicitly observe that

$$\mathcal{L}^n(U(p,r)) = r^Q \mathcal{L}^n(U(p,1)) = r^Q \mathcal{L}^n(U(0,1)). \tag{50}$$

4.2. **Calculus in Carnot groups.** This section is entirely taken from [57]. The following definitions and results about intrinsic differentiability in Carnot groups are basically due to P. Pansu ([104]), or are inspired by his ideas.

A map $L: \mathbb{G} \to \mathbb{R}$ is $\mathbb{G}$-*linear* if it is a homomorphism from $\mathbb{G} \equiv (\mathbb{R}^n, \cdot)$ to $(\mathbb{R}, +)$ and if it is positively homogeneous of degree 1 with respect to the dilations of $\mathbb{G}$, that is $L(\delta_\lambda x) = \lambda L x$ for $\lambda > 0$ and $x \in \mathbb{G}$. The $\mathbb{R}$-linear set of $\mathbb{G}$-*linear* functionals $\mathbb{G} \to \mathbb{R}$ is indicated as $\mathcal{L}_\mathbb{G}$ and it is endowed with the norm

$$\|L\|_{\mathcal{L}_\mathbb{G}} := \sup\{|L(p)| : \ d_c(p,0) \le 1\}.$$

Given a basis $X_1, \dots, X_n$, all $\mathbb{G}$-linear maps are represented as the follows.

Proposition 4.4. *A map $L : \mathbb{G} \to \mathbb{R}$ is $\mathbb{G}$-linear if and only if there is $a = (a_1, \dots, a_{m_1}) \in \mathbb{R}^{m_1}$ such that, if $x = (x_1, \dots, x_n) \in \mathbb{G}$, then $L(x) = \sum_{i=1}^{m_1} a_i x_i$.*

Definition 4.5. Let Ω be an open set in $\mathbb{G}$, then $f : \Omega \to \mathbb{R}$ is *Pansu-differentiable* (differentiable in the sense of Pansu: see [104] and [73]) at x_0 if there is a $\mathbb{G}$-linear map L such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - L(x_0^{-1} \cdot x)}{d(x, x_0)} = 0.$$

Remark 4.6. The above definition is equivalent to the following one: there exists a homomorphism L from $\mathbb{G}$ to $(\mathbb{R}, +)$ such that

$$\lim_{\lambda \to 0+} \frac{f(\tau_{x_0}(\delta_\lambda v)) - f(x_0)}{\lambda} = L(v)$$

uniformly with respect to v belonging to compact sets in $\mathbb{G}$. In particular, L is unique and we shall write $L = d_\mathbb{G} f(x_0)$. Notice that this definition of differential depends only on $\mathbb{G}$ and not on the particular choice of the canonical generating vector fields. Indeed any two Carnot–Carathéodory distances induced by different choices of (equivalent) scalar products in $H\mathbb{G}$ are equivalent as distances.

Definition 4.7. If Ω is an open set in $\mathbb{G}$, we denote by $\mathbf{C}^1_\mathbb{G}(\Omega)$ the set of continuous real functions in Ω such that $d_\mathbb{G} f : \Omega \to \mathcal{L}_\mathbb{G}$ is continuous in Ω. Moreover, we shall denote by $\mathbf{C}^1_\mathbb{G}(\Omega, H\mathbb{G})$ the set of all sections φ of $H\mathbb{G}$ whose canonical coordinates $\varphi_j \in \mathbf{C}^1_\mathbb{G}(\Omega)$ for $j = 1, \dots, m_1$.

Remark 4.8. We recall that $\mathbf{C}^1(\Omega) \subset \mathbf{C}^1_\mathbb{G}(\Omega)$ and that the inclusion may be strict, for an example see Remark 6 in [54].

We say that f is *differentiable along* X_j, $j = 1, \dots, m_1$, at x_0 if the map $\lambda \mapsto f(\tau_{x_0}(\delta_\lambda e_j))$ is differentiable at $\lambda = 0$, where e_j is the j-th vector of the canonical basis of $\mathbb{R}^n$.

Once a generating family of vector fields $X_1, \dots, X_{m_1}$ is fixed, we define, for any function $f : \mathbb{G} \to \mathbb{R}$ for which the partial derivatives $X_j f$ exist, the horizontal gradient of f, denoted by $\nabla_{\mathbb{G}} f$, as the horizontal section

$$\nabla_{\mathbb{G}} f := \sum_{i=1}^{m_1} (X_i f) X_i.$$

whose coordinates are $(X_1 f, ..., X_{m_1} f)$. Moreover, if $\varphi = (\varphi_1, \dots, \varphi_{m_1})$ is an horizontal section such that $X_j \varphi_j \in L^1_{\mathrm{loc}}(\mathbb{G})$ for $j = 1, \dots, m_1$, we define $\mathrm{div}_{\mathbb{G}}\, \varphi$ as the real valued function

$$\mathrm{div}_{\mathbb{G}}\, (\varphi) := -\sum_{j=1}^{m_1} X_j^* \varphi_j = \sum_{j=1}^{m_1} X_j \varphi_j$$

(see also Section 2.1).

Remark 4.9. The notation we have used for the gradient in a group is partially imprecise, indeed $\nabla_{\mathbb{G}} f$ really depends on the choice of the basis $X_1, ..., X_{m_1}$. If we choose a different base, say $Y_1, ..., Y_{m_1}$, then in general $\sum_i (X_i f) X_i \neq \sum_i (Y_i f) Y_i$. Only if the two bases are one orthonormal with respect to the scalar product induced by the other, we have that

$$\sum_i (X_i f) X_i = \sum_i (Y_i f) Y_i.$$

On the contrary, the notation $\mathrm{div}_{\mathbb{G}}$ used for the divergence is correct. Indeed $\mathrm{div}_{\mathbb{G}}$ is an intrinsic notion and it can be computed using the previous formula for any fixed generating family.

Finally, if $x = (x_1, \dots, x_n) \in \mathbb{R}^n \equiv \mathbb{G}$ and $x_0 \in \mathbb{G}$ are given, we set

$$\pi_{x_0}(x) = \sum_{j=1}^{m_1} x_j X_j(x_0).$$

The map $x_0 \to \pi_{x_0}(x)$ is a smooth section of $H\mathbb{G}$.

Proposition 4.10. *If f is Pansu-differentiable at x_0, then it is differentiable along X_j at x_0 for $j = 1, \dots, m_1$, and*

$$d_{\mathbb{G}} f(x_0)(v) = \langle \nabla_{\mathbb{G}} f, \pi_{x_0}(v) \rangle_{x_0}. \tag{51}$$

For a proof see [101] Remark 3.3. The following proposition can be proved via an approximation argument as in Proposition 5.8 of [54].

Proposition 4.11. *A continuous function belongs to $\mathbf{C}^1_{\mathbb{G}}(\Omega)$ if and only if its distributional derivatives $X_j f$ are continuous in Ω for $j = 1, \dots, m_1$.*

Remark 4.12. As we observed both $\nabla_{\mathbb{G}}$ and the Carnot–Carathéodory distance d depend on the choice of the canonical generating family X_j. But the *eikonal equation* connecting the two notions

$$|\nabla_{\mathbb{G}} d(0, x)| = 1 \tag{52}$$

holds for $\mathcal{L}^n$-a.e. $x \in \mathbb{G}$ and for all generating family (see Theorem 3.1 of [101]).

An extension theorem of Whitney type holds.

Theorem 4.13. (Whitney Extention Theorem) *Let $F \subset \mathbb{G}$ be a closed set, and let $f : F \to \mathbb{R}$, $k : F \to H\mathbb{G}$ be, respectively, a continuous real function and a continuous horizontal section. We set*

$$R(x, y) := \frac{f(x) - f(y) - \langle k(y), \pi_y(y^{-1} \cdot x) \rangle_y}{d(y, x)},$$

and, if $K \subset F$ is a compact set,

$$\varrho_K(\delta) := \sup\{|R(x,y)| : \quad x, y \in K,\ 0 < d(x,y) < \delta\}.$$

Assume

$$\varrho_K(\delta) \to 0 \text{ as } \delta \to 0 \text{ for every compact set } K \subset F,$$

then there exist $\tilde f : \mathbb{G} \to \mathbb{R}$, $\tilde f \in \mathbf{C}^1_{\mathbb{G}}(\mathbb{G})$ such that

$$\tilde f_{|F} = f, \quad \nabla_{\mathbb{G}} \tilde f_{|F} = k.$$

4.3. **BV-functions and finite perimeter sets.** Since with any Carnot group we can associate a Hörmander's family of smooth vector fields, then all our previous definitions and results still hold in this setting. In particular, within a Carnot group, we can define BV spaces in a form equivalent to that of the previous section as follows.

If $\Omega \subseteq \mathbb{R}^n$ is open, the space of compactly supported smooth sections of $H\mathbb{G}$ is denoted by $\mathbf{C}^\infty_0(\Omega, H\mathbb{G})$. If $k \in \mathbb{N}$, $\mathbf{C}^k_0(\Omega, H\mathbb{G})$ is defined analogously.

The space $BV_{\mathbb{G}}(\Omega)$ is the set of functions $f \in L^1(\Omega)$ such that

$$\|\nabla_{\mathbb{G}} f\|(\Omega) := \sup\left\{\int_\Omega f(x)\mathrm{div}_G\varphi(x)\,dx : \varphi \in \mathbf{C}^1_0(\Omega, H\mathbb{G}),\ |\varphi(x)|_x \le 1\right\} < \infty. \tag{53}$$

The space $BV_{\mathbb{G},\mathrm{loc}}(\Omega)$ is the set of functions belonging to $BV_{\mathbb{G}}(\mathcal{U})$ for each open set $\mathcal{U} \subset\subset \Omega$. Notice the use of the intrinsic fiber norm inside the previous definition.

It is easy to see that $f \in BV_{\mathbb{G}}(\Omega)$ if and only if $f \in BV_X(\Omega)$, where X is a family of vector fields that generate the horizontal layer.

In the setting of Carnot groups, the structure theorem for BV-functions reads as follows.

Theorem 4.14. (Structure of $BV_{\mathbb{G}}$ functions) *If $f \in BV_{\mathbb{G},\mathrm{loc}}(\Omega)$ then $\|\nabla_{\mathbb{G}} f\|$ is a Radon measure on Ω. Moreover, there exists a $\|\nabla_{\mathbb{G}} f\|$-measurable horizontal section $\sigma_f : \Omega \to H\mathbb{G}$ such that $|\sigma_f(x)|_x = 1$ for $\|\nabla_{\mathbb{G}} f\|$-a.e. $x \in \Omega$, and*

$$\int_\Omega f(x)\, div_{\mathbb{G}}\, \varphi(x)\, dx = \int_\Omega \langle \varphi, \sigma_f\rangle\, d\|\nabla_{\mathbb{G}} f\|,$$

for all $\varphi \in \mathbf{C}^1_0(\Omega, H\mathbb{G})$. Finally the notion of gradient $\nabla_{\mathbb{G}}$ can be extended from regular functions to functions $f \in BV_{\mathbb{G}}$ defining $\nabla_{\mathbb{G}} f$ as the vector valued measure

$$\nabla_{\mathbb{G}} f := -\sigma_f \llcorner \|\nabla_{\mathbb{G}} f\| = \left(-(\sigma_f)_1 \llcorner \|\nabla_{\mathbb{G}} f\|, \dots, -(\sigma_f)_{m_1} \llcorner \|\nabla_{\mathbb{G}} f\|\right),$$

where $(\sigma_f)_j$ are the components of σ_f with respect to the moving base X_j.

It is well known that the usefulness of these definitions for the Calculus of Variations, relies mainly in the validity of the two following theorems. In the context of subriemannian geometries they are proved respectively in [61] and [51].

Theorem 4.15. (Compactness) *$BV_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$ is compactly embedded in $L^p_{\mathrm{loc}}(\mathbb{G})$ for $1 \le p < \frac{Q}{Q-1}$ where Q, defined in (49), is the homogeneous dimension of $\mathbb{G}$.*

Theorem 4.16. (Lower semicontinuity) *Let $f, f_k \in L^1(\Omega)$, $k \in \mathbb{N}$, be such that $f_k \to f$ in $L^1(\Omega)$; then*

$$\liminf_{k\to\infty} \|\nabla_{\mathbb{G}} f_k\|(\Omega) \ge \|\nabla_{\mathbb{G}} f\|(\Omega).$$

Definition 4.17. A measurable set $E \subset \mathbb{R}^n$ is of *locally finite $\mathbb{G}$-perimeter* in Ω (or is a *$\mathbb{G}$-Caccioppoli set*) if the characteristic function $\mathbf{1}_E \in BV_{\mathbb{G},\mathrm{loc}}(\Omega)$. In this case we call *perimeter of* E the measure

$$|\partial E|_{\mathbb{G}} := \|\nabla_{\mathbb{G}} \mathbf{1}_E\| \tag{54}$$

and we call *(generalized inward)* $\mathbb{G}$*-normal* to ∂E in Ω the vector

$$\nu_E(x) := -\sigma_{\mathbf{1}_E}(x). \tag{55}$$

Remark 4.18. This remark is analogous to remark 4.9. The symbol $|\partial E|_{\mathbb{G}}$ is somehow incorrect, indeed the value of the $\mathbb{G}$-perimeter depends on the choice of the generating vector fields $X_1, \dots, X_{m_1}$, precisely through the bound $|\varphi| \le 1$ in (53). The values of the perimeters induced by two different families of generating vector fields coincide only if the two families are mutually orthonormal; nevertheless the perimeters induced by different families are equivalent as measures and, as a consequence, the notion of being a $\mathbb{G}$-Caccioppoli set is an intrinsic one depending only on the group $\mathbb{G}$.

Remark 4.19. The $\mathbb{G}$-perimeter is invariant under group translations, that is

$$|\partial E|_{\mathbb{G}}(A) = |\partial(\tau_p E)|_{\mathbb{G}}(\tau_p A), \qquad \forall p \in \mathbb{G}, \quad \text{and for any Borel set } A \subset \mathbb{G};$$

indeed $\mathrm{div}_{\mathbb{G}}$ is invariant under group translations and the Jacobian determinant of $\tau_p : \mathbb{G} \to \mathbb{G}$ equals 1. Moreover the $\mathbb{G}$-perimeter is homogeneous of degree $Q-1$ with respect to the dilations of the group, that is

$$|\partial(\delta_\lambda E)|_{\mathbb{G}}(A) = \lambda^{1-Q} |\partial E|_{\mathbb{G}}(\delta_\lambda A), \quad \text{for any Borel set } A \subset \mathbb{G}; \tag{56}$$

also this fact is elementary and can be proved by changing variables in formula (53).

By (50), the isoperimetric inequality in a Carnot group takes the following form ([61]).

Proposition 4.20. (Isoperimetric inequality) *There is a positive constant $c_I > 0$ such that for any $\mathbb{G}$-Caccioppoli set E, for all $x \in \mathbb{G}$ and $r > 0$,*

$$\min\{\mathcal{L}^n(E \cap U(x,r)), \mathcal{L}^n(E^c \cap U(x,r))\}^{\frac{Q-1}{Q}} \le c_I |\partial E|_{\mathbb{G}}(U(x,r)) \tag{57}$$

and

$$\min\{\mathcal{L}^n(E), \mathcal{L}^n(E^c)\}^{\frac{Q-1}{Q}} \le c_I |\partial E|_{\mathbb{G}}(\mathbb{R}^n). \tag{58}$$

Isoperimetric sets have been recently studied in [75].

5. Regular hypersurfaces in Carnot groups and rectifiability.

5.1. **Regular hypersurfaces.** This section relies totally on [55]. We define $\mathbb{G}$-regular hypersurfaces in a Carnot group $\mathbb{G}$, mimicking Definition 6.1 in [54], as non critical level sets of functions in $\mathbf{C}^1_{\mathbb{G}}(\mathbb{R}^n, \mathbb{R})$.

Definition 5.1. ($\mathbb{G}$-regular hypersurfaces) Let $\mathbb{G}$ be a Carnot group. We shall say that $S \subset \mathbb{G}$ is a $\mathbb{G}$-*regular hypersurface* if for every $x \in S$ there exist a neighborhood $\mathcal{U}$ of x and a function $f \in \mathbf{C}^1_{\mathbb{G}}(\mathcal{U})$ such that

$$S \cap \mathcal{U} = \{y \in \mathcal{U} : f(y) = 0\}; \tag{i}$$

$$\nabla_{\mathbb{G}} f(y) \ne 0 \quad \text{for } y \in \mathcal{U}. \tag{ii}$$

$\mathbb{G}$-regular surfaces have a unique tangent plane at each point. This follows from a Taylor formula for functions in $\mathbf{C}^1_{\mathbb{G}}$ that is basically proved in [104].

Proposition 5.2. *If $f \in \mathbf{C}^1_{\mathbb{G}}(U(p,r))$, then*

$$f(x) = f(p) + \sum_{j=1}^{m} (X_j f)(p)(x_j - p_j) + o(d(x,p)), \qquad \text{as } x \to p. \tag{59}$$

If $S = \{x : f(x) = 0\} \subset \mathbb{G}$ is a $\mathbb{G}$-regular hypersurface the *tangent group* $T^g_{\mathbb{G}}S(x)$ to S at x is

$$T^g_{\mathbb{G}}\ S(x) := \{v = (v_1, \dots, v_n) \in \mathbb{G} : \sum_{j=1}^{m} X_j f(x) v_j = 0\}.$$

By (4.2), $T^g_{\mathbb{G}}S(x)$ is a proper subgroup of $\mathbb{G}$. We can define the *tangent plane* to S at x as

$$T_{\mathbb{G}}S(x) := x \cdot T^g_{\mathbb{G}}S(x).$$

We stress that this is a good definition. Indeed the tangent plane does not depend on the particular function f defining the surface S because of point *(iii)* of Implicit Function Theorem below that yields

$$T^g_{\mathbb{G}}S(x) = \{v \in \mathbb{G} :\ \langle \nu_E(x), \pi_x v\rangle_x = 0\}$$

where ν_E is the generalized inward unit normal defined in (55) and $\pi_x(v) = \sum_{j=1}^{m} v_j X_j(x)$. Notice that the map $v \mapsto \pi_x(v)$, for $x \in \mathbb{G}$ fixed,

$$\pi_x(v) = \sum_{j=1}^{m} v_j X_j(x). \tag{60}$$

is a smooth section of $H\mathbb{G}$.

Notice also that, once more from *(iii)* of Theorem 5.5, it follows that ν_E is a continuous function.

If $v^0 = \sum_{i=1}^{m} v_i X_i(0) \in H\mathbb{G}_0$ we define the halfspaces $S^{\pm}_{\mathbb{G}}(0, v^0)$ as

$$S^{+}_{\mathbb{G}}(0, v^0) := \{x \in \mathbb{G} : \sum_{i=1}^{m} x_i v_i > 0\} \text{ and } S^{-}_{\mathbb{G}}(0, v^0) := \{x \in \mathbb{G} : \sum_{i=1}^{m} x_i v_i < 0\}.$$

Their common boundary is the vertical plane

$$\Pi(0, v^0) := \{x : \sum_{i=1}^{m} x_i v_i = 0\}.$$

If $v = \sum_{i=1}^{m} v_i X_i(y) \in H\mathbb{G}_y$, $S^{\pm}_{\mathbb{G}}(y, v)$ and $\Pi(y, v)$ are the translated sets

$$S^{\pm}_{\mathbb{G}}(y, v) := y \cdot S^{\pm}_{\mathbb{G}}(0, v^0) \quad \text{and} \quad \Pi(y, v) = y \cdot \Pi(0, v^0)$$

where v and v^0 have the same components v_i with respect to the left invariant basis X_i. Hence

$$S^{\pm}_{\mathbb{G}}(y, v) = \{x \in \mathbb{G} : \sum_{i=1}^{m} (x_i - y_i) v_i > 0 (< 0)\}. \tag{61}$$

Clearly, $T_{\mathbb{G}}S(x) = \Pi(x, \nu_E(x))$.

Note also that the class of $\mathbb{G}$-regular hypersurfaces is different from the class of Euclidean C^1 embedded surfaces in $\mathbb{R}^n$. From one side $\mathbb{G}$-regular surfaces can have 'ridges' because continuity of the derivatives of the defining functions f is required only in the horizontal directions; on the other side an Euclidean C^1 surface can have so called characteristic points i.e. points $p \in S$ where the Euclidean tangent plane T_pS contains the horizontal fiber $H\mathbb{G}_p$.

Definition 5.3. If S is an Euclidean $\mathbf{C}^1$ hypersurface in $\mathbb{G}$, we define the characteristic set of S as

$$\mathcal{C}(S) := \{x \in S :\ H\mathbb{G}_x \subseteq T_xS\}. \tag{62}$$

The points of $\mathcal{C}(S)$ are, under many aspects, irregular points of S. Note indeed that the tangent group does not exist in these points. It is also well known that these points are 'few' on smooth hypersurfaces, but only recently V. Magnani [88] has obtained precise estimates of the $\mathcal{H}_c^{Q-1}$ measure of the characteristic sets of C^1 surfaces in general Carnot groups groups $\mathbb{H}^n$, extending previous results of Z. Balogh [9] in the Heisenberg group, of V. Magnani [88] and of B. Franchi, R. Serapioni & F. Serra Cassano [57] in step 2 Carnot groups. Notice that the study of the size of the characteristic set has a long history. We refer to the contributions of M. Derridj [36], B. Franchi & R.L. Wheeden [58], D.Danielli, N.Garofalo & D.M.Nhieu [29]. Magnani's result reads as follows.

Theorem 5.4. *If S is a Euclidean $\mathbf{C}^1$-smooth hypersurface in a Carnot group $\mathbb{G}$ with homogeneous dimension Q. Then*

$$\mathcal{H}_{\mathbb{G}}^{Q-1}(\mathcal{C}(S)) = 0. \tag{63}$$

We can state now our Implicit Function Theorem, holding that a $\mathbb{G}$-regular hypersurface $S = \{f(y) = 0\}$ boundary of the set $E = \{f(y) < 0\}$ can be locally parameterized through a function $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}^n$ so that the $\mathbb{G}$-perimeter of E can be written explicitly in terms of $\nabla_{\mathbb{G}} f$ and Φ.

Theorem 5.5. (Implicit Function Theorem) *Let Ω be an open set in $\mathbb{R}^n$ identified with a Carnot group $\mathbb{G}$, $0 \in \Omega$, and let $f \in \mathbf{C}^1_{\mathbb{G}}(\Omega)$ be such that $f(0) = 0$ and $X_1 f(0) > 0$. Define*

$$E = \{x \in \Omega : f(x) < 0\}, \quad S = \{x \in \Omega : f(x) = 0\},$$

and, for $\delta > 0$, $h > 0$

$$I_\delta = \{\xi = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1}, |\xi_j| \le \delta\}, \qquad J_h = [-h, h].$$

If $\xi = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1}$ and $t \in J_h$, denote now by $\gamma(t, \xi)$ the integral curve of the vector field X_1 at the time t issued from $(0, \xi) = (0, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, i.e.

$$\gamma(t, \xi) = \exp(tX_1)(0, \xi).$$

Then there exist $\delta, h > 0$ such that the map $(t, \xi) \to \gamma(t, \xi)$ is a diffeomorphism of a neighborhood of $J_h \times I_\delta$ onto an open subset of $\mathbb{R}^n$, and, if we denote by $\mathcal{U} \subset\subset \Omega$ the image of $\mathrm{Int}(J_h \times I_\delta)$ through this map, we have

$$E \text{ has finite } \mathbb{G}\text{-perimeter in } \mathcal{U}; \tag{i}$$

$$\partial E \cap \mathcal{U} = S \cap \mathcal{U}; \tag{ii}$$

$$\nu_E(x) = -\frac{\nabla_{\mathbb{G}} f(x)}{|\nabla_{\mathbb{G}} f(x)|_x} \text{ for all } x \in S \cap \mathcal{U}, \tag{iii}$$

where ν_E is the generalized inner unit normal defined by (55), that can be identified with a section of $H\mathbb{G}$ with $|\nu(x)|_x = 1$ for $|\partial E|_{\mathbb{G}}$-a.e. $x \in \mathcal{U}$. In particular, ν_E can be identified with a continuous function and $|\nu| \equiv 1$. Moreover, there exists a unique function

$$\varphi = \varphi(\xi) : I_\delta \to J_h$$

such that the following parameterization holds: if $\xi \in I_\delta$, put $\Phi(\xi) = \gamma(\varphi(\xi), \xi)$, then

$$S \cap \tilde{\mathcal{U}} = \{x \in \tilde{\mathcal{U}} : x = \Phi(\xi), \xi \in I_\delta\}; \tag{iv}$$

$$\varphi \text{ is continuous;} \tag{v}$$

the $\mathbb{G}$-perimeter has an integral representation:

$$|\partial E|_{\mathbb{G}}(\tilde{\mathcal{U}}) = \int_{I_\delta} \frac{\sqrt{\sum_{j=1}^m |X_j f(\Phi(\xi))|^2}}{X_1 f(\Phi(\xi))} \, d\mathcal{L}^{n-1}_\xi. \tag{vi}$$

Our next Theorem is a mild regularity result. Roughly speaking, it states that $\mathbb{G}$ regular hypersurfaces do not have cusps or spikes if they are studied with respect to the intrinsic Carnot–Carathéodory distance, while they can be very irregular as Euclidean submanifolds. To make precise the former statement we recall the notion of *essential boundary* (or of *measure theoretic boundary*) $\partial_* F$ of a set $F \subset \mathbb{G}$

$$\partial_* F := \left\{ x \in \mathbb{G} : \limsup_{r\to 0^+} \min \left\{ \frac{\mathcal{L}^n(F \cap U(x,r))}{\mathcal{L}^n(U(x,r))}, \frac{\mathcal{L}^n(F^c \cap U(x,r))}{\mathcal{L}^n(U(x,r))} \right\} > 0 \right\} \tag{64}$$

Notice that the definition above makes sense in any metric measure space and that the essential boundary does not change if, in definition 64), the distance d is substituted by an equivalent distance d'.

Theorem 5.6. *Let $\Omega \subset \mathbb{G}$ be a fixed open set, and E be such that $\partial E \cap \Omega = S \cap \Omega$, where S is a $\mathbb{G}$-regular hypersurface. Then*

$$\partial E \cap \Omega = \partial_* E \cap \Omega. \tag{65}$$

We want now to compare the perimeter measure, on a $\mathbb{G}$-regular hypersurface S, and the intrinsic $(Q-1)$-Hausdorff measure of S. Observe that it makes sense to speak of the perimeter measure of S given that S is locally the boundary of a finite $\mathbb{G}$-perimeter set (as proved in Theorem 5.5). The next theorem gives an explicit form of the density of the perimeter with respect to the intrinsic Hausdorff measure concentrated on S. As a consequence – as it is stated in the following corollary – $\mathbb{G}$-regular hypersurfaces have coherently intrinsic Hausdorff dimension $Q-1$.

Theorem 5.7. *Let ϱ be a distance on $\mathbb{G}$ such that, for all $x, y, z \in \mathbb{G}$ and $\lambda > 0$*

$$\varrho(x\cdot y, x\cdot z) = \varrho(y,z) \quad \text{and} \quad \varrho(\delta_\lambda y, \delta_\lambda z) = \lambda \varrho(y,z), \tag{66}$$

and there exists $c_\varrho > 1$ such that

$$\frac{1}{c}\varrho(y,z) \le d(y,z) \le c\varrho(y,z), \quad \text{for all} \quad y,z \in \mathbb{G}. \tag{67}$$

. *If $\mathfrak{s}_\varrho : H\mathbb{G}_0 \setminus \{0\} \to \mathbb{R}$, is the 1-homogeneous function defined as*

$$\mathfrak{s}_\varrho(v) := \mathcal{L}^{n-1}\left(U_\varrho(0,1) \cap \Pi(0,v)\right),$$

then

$$\begin{aligned} |\partial E|_{\mathbb{G}} \llcorner \Omega &= \mathfrak{s}_\varrho \circ \nu_E \, \mathcal{S}^{Q-1}_{\mathbb{G}} \llcorner (S \cap \Omega) \\ &= \mathcal{L}^{n-1}\left(U_\varrho(0,1) \cap T^g_{\mathbb{G}} S(x)\right) \, \mathcal{S}^{Q-1}_{\mathbb{G}} \llcorner (S \cap \Omega). \end{aligned} \tag{68}$$

Moreover, there is a constant $\alpha_\varrho > 1$, depending only on the distance ϱ, such that

$$0 < \frac{1}{\alpha_\varrho} \le \mathfrak{s}_\varrho(v) \le \alpha_\varrho < \infty.$$

Remark 5.8. If the distance ϱ under consideration is invariant with respect to rotations of $H\mathbb{G}_0 \simeq \mathbb{R}^m$, then the function $\mathfrak{s}_\varrho$ is constant and, with an appropriate choice of the normalization constant in the definition of the Hausdorff measure, (68) takes the particularly neat form

$$|\partial E|_{\mathbb{G}} = \mathcal{S}^{Q-1}_\varrho \llcorner S. \tag{69}$$

We do not know how large is the class of groups whose Carnot–Carathéodory distance enjoys this property. It certainly comprises the Heisenberg groups. For the groups in this class we have

$$|\partial E|_{\mathbb{G}} = \mathcal{S}^{Q-1}_c \llcorner S. \tag{70}$$

Nevertheless, even if ϱ were not rotationally invariant, it always exists another *true metric* invariant, homogeneous, comparable with that is also invariant by rotations of $H\mathbb{G}_0$ (for an example see (47)). If one computes the Hausdorff measure with respect to it, then (69) holds.

Corollary 5.9. *If S is a $\mathbb{G}$-regular hypersurface then the Hausdorff dimension of S, with respect to the Carnot–Carathéodory metric d or any other metric d' comparable with it, is $Q-1$.*

Corollary 5.9 combined with Theorem 5.4 yields the following comparison result between Euclidean $\mathbf{C}^1$-smooth hypersurfaces and $\mathbb{G}$-regular hypersurfaces. We have

Theorem 5.10. *If S is a Euclidean $\mathbf{C}^1$-smooth hypersurface in a Carnot group $\mathbb{G}$ with homogeneous dimension Q, then the Hausdorff dimension of S, with respect to the Carnot–Carathéodory metric d or any other metric d' comparable with it, is $Q-1$.*

The reverse assertion is false: there exist $\mathbb{G}$-regular hypersurfaces in $\mathbb{G} \equiv \mathbb{R}^n$ that have Euclidean Hausdorff dimension greater than $n-1$: indeed, recently B. Kirchheim and F. Serra Cassano ([72]) have shown that there exist $\mathbb{G}$-regular hypersurfaces in the Heisenberg group $\mathbb{H}^1$ ($Q=4$, $n=3$) with Euclidean Hausdorff dimension 2.5.

5.2. **Rectifiability in Carnot groups.** The following results are the core of [57] (see also [56]). We remind that De Giorgi's celebrated structure theorem in Euclidean spaces ([31], [32]) states that if $E \subset \mathbb{R}^n$ is a set of locally finite perimeter, then the associated perimeter measure $|\partial E|$ is concentrated on a portion of the topological boundary ∂E, the so-called reduced boundary $\partial^* E \subset \partial E$. In addition, $\partial^* E$ is $\mathcal{H}^{d-1}$-rectifiable, i.e. $\partial^* E$, up to a set of $(d-1)$-Hausdorff measure zero, is a countable union of compact subsets of $\mathbf{C}^1$ submanifolds and the perimeter measure is the $(n-1)$-Hausdorff measure of the reduced boundary. Roughly speaking, this says that the perimeter measure in supported on a portion of the topological boundary ∂E, that can be expressed – after removing a negligible set of "bad points" – as the countable union of compact subsets of "good hypersurfaces".

If in the spirit of De Giorgi's theorem we want to describe the structure of sets of finite intrinsic perimeter in a Carnot group $\mathbb{G}$, we need a natural notion of rectifiable subsets, and in this perspective, the correct definition of "good hypersurfaces", i.e. of intrinsic C^1-regular submanifold of $\mathbb{G}$ given in the previous Section provides a key tool. Keeping in mind this notion, the following definition is the natural counterpart of the corresponding Euclidean definition.

Definition 5.11. $\Gamma \subset \mathbb{G}$ is said to be *($(Q-1)$-dimensional) $\mathbb{G}$-rectifiable* if there exists a sequence of $\mathbb{G}$-regular hypersurfaces $(S_j)_{j\in\mathbb{N}}$ such that

$$\mathcal{H}_c^{Q-1}\left(\Gamma \setminus \bigcup_{j\in\mathbb{N}} S_j\right) = 0. \tag{71}$$

Before we enter the study of the rectifiability of the reduced boundary (whatever this means, as we shall see below), let us point out the relationships between our definition in Carnot groups and the standard Euclidean notion. The following result proved in [57] yields that " negligible" subsets of codimension 1 in a Carnot group with respect to the Euclidean distance are " negligible" subsets of codimension 1 with respect to Carnot–Carathédory distance.

Proposition 5.12. *Let $\mathbb{G}$ be a Carnot group. For any $\alpha \geq 0$ and $R > 0$ there is a constant $c(\alpha, R) > 0$ such that for any set $E \subset \mathbb{G} \cap U(0,R)$*

$$\mathcal{H}_c^{\alpha+Q-n}(E) \leq c(\alpha, R)\mathcal{H}^{\alpha}(E), \qquad \alpha \geq 0. \tag{72}$$

In particular, for all $E \subset \mathbb{G}$,

$$\mathcal{H}^{\alpha}(E) = 0 \implies \mathcal{H}_c^{\alpha+Q-n}(E) = 0, \qquad \alpha \geq 0. \tag{73}$$

Proposition 5.12 combined with Theorem 5.4 yields

Theorem 5.13. *Let $\mathbb{G} = \mathbb{R}^n$ be a Carnot group. Then, if S is a $(n-1)$-dimensional Euclidean rectifiable set of $\mathbb{R}^n$ then S is also $(Q-1)$-dimensional $\mathbb{G}$-rectifiable.*

On the other hand, there are $(Q-1)$-dimensional $\mathbb{G}$-rectifiable sets in a Carnot groups $\mathbb{G}$ identified with $\mathbb{R}^n$ that are not $(n-1)$-dimensional Euclidean rectifiable. Indeed, in [10] a set $N \subset \mathbb{R}^3$ is constructed, such that for an appropriate $\varepsilon > 0$,

$$\mathcal{H}_c^3(N) = 0 \quad \text{and} \quad \mathcal{H}^{2+\varepsilon}(N) > 0.$$

Hence N is (trivially) $3 = (Q-1)$-dimensional $\mathbb{H}^1$-rectifiable, but it is not 2-dimensional Euclidean rectifiable because its Euclidean Hausdorff dimension is strictly larger than 2. As we mentioned above, a sharper result in this direction is contained in [72]: there exist $\mathbb{G}$-regular hypersurfaces in the Heisenberg group $\mathbb{H}^1$ ($Q = 4$, $n = 3$) with Euclidean Hausdorff dimension 2.5. We recall here that relationships between Euclidean and intrinsic Hausdorff measure in Heisenberg groups have been deeply investigated in [10], where also sharp results were obtained.

Thus we are left with the notion of reduced boundary for subsets of a Carnot group. The definition we give is a simple translation of the Euclidean one, as follows.

Definition 5.14. (Reduced boundary) Let E be a $\mathbb{G}$-Caccioppoli set; we say that $x \in \partial^*_{\mathbb{G}} E$ if

$$(i) \qquad |\partial E|_{\mathbb{G}}(U(x,r)) > 0 \qquad \text{for any } r > 0;$$

$$(ii) \qquad \text{there exists} \quad \lim_{r\to 0} \fint_{U(x,r)} \nu_E \, d|\partial E|_{\mathbb{G}};$$

$$(iii) \qquad \left\| \lim_{r\to 0} \fint_{U(x,r)} \nu_E \, d|\partial E|_{\mathbb{G}} \right\|_{\mathbb{R}^{m_1}} = 1.$$

The limits in Definition 5.14 should be understood as a convergence of the averages of the coordinates of ν_E with respect to the chosen moving base of the fibers.

Definition 5.14 is a straightforward extention of its Euclidean counterpart, but its utility is not obvious. Indeed, in the Euclidean setting, it is immediate to show that the perimeter measure is concentrated on the reduced boundary, since, by Lebesgue–Besicovitch Differentiation Lemma, given a Radon measure μ, for any $f \in L^1_{\text{loc}}(d\mu)$

$$\lim_{r\to 0} \fint_{|y-x|<r} f(y) \, d\mu_E \to f(x)$$

as $r \to 0$ for μ-a.e. x. This implies that $|\partial E| = |\partial E| \llcorner \partial^*_{\mathbb{G}}$.

Unfortunately, Besicovitch covering lemma, that is the main tool of the proof of Lebesgue–Besicovitch Differentiation Lemma, fails to hold in Carnot groups, see [73] and [113].

We do not know whether nevertheless Lebesgue–Besicovitch Differentiation Lemma still holds in Carnot groups, but at least it holds when μ is the perimeter measure, thanks to a deep asymptotic estimate proved by Ambrosio in [1]. The corresponding differentiation lemma reads as follows.

Lemma 5.15. (Differentiation Lemma) *Assume E is a $\mathbb{G}$-Caccioppoli set, then*

$$\lim_{r\to 0} \fint_{U(x,r)} \nu_E \, d|\partial E|_{\mathbb{G}} = \nu_E(x), \qquad \text{for } |\partial E|_{\mathbb{G}}\text{-a.e. } x,$$

that is $|\partial E|_{\mathbb{G}}$-a.e. $x \in \mathbb{G}$ belongs to the reduced boundary $\partial^*_{\mathbb{G}} E$.

From now on the group $\mathbb{G}$ will be a step 2 Carnot group.

Indeed, the keystep for the main result of this paper, i.e. the so-called *Blow-up Theorem* stated below, fails to be true for general groups of step greater than 2, as we can see from the Example 1.

Specializing our notations, in step 2 Carnot groups, we have

$$\mathfrak{g} = V_1 \oplus V_2, \quad [V_1, V_1] = V_2, \quad [V_1, V_2] = \{0\},$$

and

$$Q = m_1 + 2(n - m_1).$$

We can prove now the following results.
i) At each point of the reduced boundary of a $\mathbb{G}$-Caccioppoli set there is a (generalized) tangent group;
ii) Both the reduced boundary and the measure-theoretic boundary are $(Q-1)$-dimensional $\mathbb{G}$-rectifiable sets;
iii) $|\partial E|_{\mathbb{G}} = c\mathcal{S}_\infty^{Q-1} \llcorner \partial^* E$, i.e. the perimeter measure equals a constant times the spherical $(Q-1)$-dimensional Hausdorff measure restricted to the reduced boundary.
iv) An intrinsic divergence theorem holds for $\mathbb{G}$-Caccioppoli sets.

The precise meaning of statement i) is the content of the following Blow-up Theorem. It is precisely point i) that can be false in a general Carnot group. Indeed we provide an example of a $\mathbb{G}$-regular hypersurface $S = \partial E$ in a step 3 group (the so-called Engel group, see e.g. [64], [96]) such that $0 \in \partial^*_{\mathbb{G}} E$ but E has not generalized tangent group at that point.

Statement iii) fits in the general problem of comparing different geometric measures in Carnot groups. A good reference for this problem, in Euclidean spaces, is Matilla's book [92]. In the setting of the Heisenberg group, in [29] it is proved that the perimeter of an Euclidean $C^{1,1}$-hypersurface is equivalent to its $(Q-1)$-dimensional intrinsic Hausdorff measure, whereas in [54] it is proved that on the boundary of sets of finite intrinsic perimeter the $(Q-1)$-dimensional intrinsic spherical Hausdorff measure coincide – after a suitable normalization – with the perimeter measure. In the setting of general Carnot groups the problem is essentially open. The equivalence of the intrinsic perimeter and of the $(Q-1)$-dimensional intrinsic Hausdorff measure for $\mathbf{C}^1_{\mathbb{G}}$-hypersurfaces in a general Carnot groups has been proved in the previous subsection. In addition, the perimeter measure of a smooth set in general subriemannian spaces equals the intrinsic Minkowski content, as it is proved in Theorem 3.17. In Ahlfors-regular metric spaces, a general representation theorem of the perimeter measure of sets of finite perimeter in terms of the Hausdorff measure is proved in [1] (see also the refined result for subriemannian manifolds in [2]), showing that the intrinsic perimeter admits a density ϑ with respect to the Hausdorff measure that is locally summable and bounded away from zero. Statement iii) says precisely that, thanks to i) and ii), in step 2 Carnot groups the function ϑ is constant.

To state our result, let us fix few notations. For any set $E \subset \mathbb{G}$, $x_0 \in \mathbb{G}$ and $r > 0$ we consider the translated and dilated sets E_{r,x_0} defined as

$$E_{r,x_0} = \{x : x_0 \cdot \delta_r(x) \in E\} = \delta_{\frac{1}{r}} \tau_{x_0^{-1}} E.$$

If x_0 is fixed and there is no ambiguity, we shall write simply E_r, and in addition we set $E_{x_0} = E_{1,x_0}$. Moreover if $v \in H\mathbb{G}_{x_0}$ we define the halfspaces $S^+_{\mathbb{G}}(v)$ and $S^-_{\mathbb{G}}(v)$ as

$$\begin{aligned} S^+_{\mathbb{G}}(v) &:= \{x : \langle \pi_{x_0} x, v\rangle_{x_0} \geq 0\} \\ S^-_{\mathbb{G}}(v) &:= \{x : \langle \pi_{x_0} x, v\rangle_{x_0} \leq 0\}. \end{aligned} \tag{74}$$

The common topological boundary $T^g_{\mathbb{G}}(v)$ of $S^+_{\mathbb{G}}(v)$ and of $S^-_{\mathbb{G}}(v)$ is the subgroup of $\mathbb{G}$

$$T^g_{\mathbb{G}}(v) := \{x : \langle \pi_{x_0} x, v\rangle_{x_0} = 0\}.$$

Theorem 5.16. (Blow-up Theorem) *If E is a $\mathbb{G}$-Caccioppoli set, $x_0 \in \partial^*_{\mathbb{G}} E$ and $\nu_E(x_0) \in H\mathbb{G}_{x_0}$ is the inward normal as defined in (55) then*

$$\lim_{r\to 0} \mathbf{1}_{E_{r,x_0}} = \mathbf{1}_{S^+_{\mathbb{G}}(\nu_E(x_0))} \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{G}) \tag{75}$$

and for all $R > 0$

$$\lim_{r\to 0} |\partial E_{r,x_0}|_{\mathbb{G}}(U(0,R)) = |\partial S^+_{\mathbb{G}}(\nu_E(x_0))|_{\mathbb{G}}(U(0,R)). \tag{76}$$

Notice that, by Proposition 3.2,

$$|\partial S^+_{\mathbb{G}}(\nu_E(x_0))|_{\mathbb{G}}(U(0,R)) = \mathcal{H}^{n-1}(T^g_{\mathbb{G}}(\nu_E(0)) \cap U(0,R)).$$

As we have already pointed out, Theorem 5.16 fails to hold in general Carnot groups of step $k > 2$. In fact, the core of the following example consists in showing that in Carnot groups of step greater than 2 can exists cones (i.e. dilation-invariant sets) that are not flat (they are not of the form $S^{\pm}_{\mathbb{G}}(v)$ for some horizontal vector v) but nevertheless with vertex belonging to the reduced boundary.

The following counterexample was inspired by Martin Reimann, and then Roberto Monti found a preliminary form of the counterexample itself.

Example 1. *Let us recall the definition of Engel algebra and group. Let* $\mathbb{E} = (\mathbb{R}^4, \cdot)$ *be the Carnot group whose Lie algebra is* $\mathfrak{g} = V_1 \oplus V_2 \oplus V_3$ *with* $V_1 = \mathrm{span}\,\{X_1, X_2\}$, $V_2 = \mathrm{span}\,\{X_3\}$, *and* $V_3 = \mathrm{span}\,\{X_4\}$, *the only non zero commutation relations being*

$$[X_1, X_2] = -X_3 \quad , \quad [X_1, X_3] = -X_4.$$

In exponential coordinates the group law takes the form

$$x \cdot y = H(\sum_{i=1}^4 x_i X_i, \sum_{i=1}^4 y_i X_i),$$

where H is given by the Campbell-Hausdorff formula

$$H(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] - [Y,[X,Y]]\right). \tag{77}$$

In exponential coordinates an explicit representation of the vector fields is

$$X_1 = \partial_1 + \frac{x_2}{2}\partial_3 + (\frac{x_3}{2} - \frac{x_1 x_2}{12})\partial_4 \quad , \quad X_2 = \partial_2 - \frac{x_1}{2}\partial_3 + \frac{x_1^2}{12}\partial_4$$
$$X_3 = \partial_3 - \frac{x_1}{2}\partial_4 \quad , \quad X_4 = \partial_4.$$

Let $E = \{x \in \mathbb{R}^4 \, : \, f(x) \geq 0\}$, *where*

$$f(x) = \frac{1}{6}x_2(x_1^2 + x_2^2) - \frac{1}{2}x_1 x_3 + x_4.$$

Since $\partial E = \{x \in \mathbb{R}^4 \, : \, f(x) = 0\}$ *is a smooth Euclidean manifold, then E is a* $\mathbb{G}$*-Caccioppoli set (see Proposition 3.2). Moreover,*

$$\nabla_{\mathbb{E}} f(x) = \big(0, \frac{1}{2}(x_1^2 + x_2^2)\big),$$

and, by the Implicit Function Theorem (Theorem 5.5),

$$\nu_E(x) = -\frac{\nabla_{\mathbb{E}} f(x)}{|\nabla_{\mathbb{E}} f(x)|} = (0, -1)$$

for all $x \in \partial E \setminus N$, *where* $N = \{x \in \mathbb{E} \, : \, x_1 = x_2 = 0\}$. *Since* $|\partial E|_{\mathbb{E}}(N) = 0$, *then the origin belongs to the reduced boundary of E. On the other hand, since* $f(\delta_\lambda x) = \lambda^3 f(x)$ *for* $\lambda > 0$, *it follows that* $E_{\lambda,0} = \delta_\lambda E = E$, *so that (75) fails to be true since E is not a vertical halfspace.*

Even if we do not enter into the details of the proof of Theorem 5.16, we want to stress the technical point where the assumption on the step of $\mathbb{G}$ is used. In the Euclidean setting an elementary statement says that $\frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_n} = 0$ implies $f = f(x_1)$. In Carnot groups the corresponding statement should be that the vanishing of $X_2 f$ to $X_{m_1} f$ yields that f is a function of just one variable. But this is false as simple examples in the Heisenberg group $\mathbb{H}^1$ show. What is possible to prove in step 2 groups is that if $Y_1, \ldots, Y_{m_1}$ are left invariant smooth orthonormal (horizontal) sections, if $Y_2 f = \cdots = Y_{m_1} f = 0$ and if $Y_1 f$ is positive, then f is an increasing function of one variable. Example 1 shows that in groups of step 3 or larger, even this last weaker statement is false.

Lemma 5.17. *Let $\mathbb{G}$ be a step 2 group and let $Y_1, \dots, Y_{m_1}$ be left invariant smooth orthonormal sections of $H\mathbb{G}$. Assume that $g : \mathbb{G} \to \mathbb{R}$ satisfies*

$$Y_1 g \ge 0 \quad \textit{and} \quad Y_j(g) = 0 \quad \textit{if} \quad j = 2, \dots, m_1. \tag{78}$$

Then the level lines of g are "vertical hyperplanes orthogonal to Y_1" that is sets that are group translations of

$$S(Y_1) := \{p : \langle \pi_0 p, Y_1(0)\rangle = 0\}.$$

We can state now our main structure theorem for $\mathbb{G}$-Caccioppoli sets.

Theorem 5.18. (Structure of $\mathbb{G}$-Caccioppoli sets) *If $E \subseteq \mathbb{G}$ is a $\mathbb{G}$-Caccioppoli set, then*

$$\partial^*_{\mathbb{G}} E \textit{ is } (Q-1)\textit{-dimensional } \mathbb{G}\textit{-rectifiable,} \tag{i}$$

that is $\partial^*_{\mathbb{G}} E = N \cup \bigcup_{h=1}^{\infty} K_h$, where $\mathcal{H}_c^{Q-1}(N) = 0$ and K_h is a compact subset of a $\mathbb{G}$-regular hypersurface S_h;

$$\nu_E(p) \textit{ is the } \mathbb{G}\textit{-normal to } S_h \textit{ at } p, \textit{ for all } p \in K_h; \tag{ii}$$

$$|\partial E|_{\mathbb{G}} = \vartheta_c \mathcal{S}_c^{Q-1} \llcorner \partial^*_{\mathbb{G}} E, \tag{iii}$$

where

$$\vartheta_c(x) = \frac{1}{\omega_{Q-1}} \mathcal{H}^{n-1}\left(\partial S^+_{\mathbb{G}}(\nu_E(x)) \cap U(0,1)\right).$$

As usual ω_k is the k-dimensional measure of the k-dimensional ball in $\mathbb{R}^k$. If we replace the $\mathcal{S}_c$-measure by the $\mathcal{S}_\infty$-measure, the corresponding density ϑ_∞ turns out to be a constant. More precisely

$$|\partial E|_{\mathbb{G}} = \vartheta_\infty \, \mathcal{S}_\infty^{Q-1} \llcorner \partial^*_{\mathbb{G}} E, \tag{iv}$$

where

$$\vartheta_\infty = \frac{\omega_{m_1-1}\omega_{m_2}\varepsilon_2^{m_2}}{\omega_{Q-1}} = \frac{1}{\omega_{Q-1}} \mathcal{H}^{n-1}\left(\partial S^+_{\mathbb{G}}(\nu_E(0)) \cap U_\infty(0,1)\right).$$

Here ε_2 is the constant appearing in (47) and ω_k is the k-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^k$.

Finally, the following divergence theorem is then an easy consequence of Theorem 5.18, but we stress the fact that the measure theoretic boundary appears in the identity (ii). As in the Euclidean space, the corresponding statement for the reduced boundary holds straightforwardly. However, the interest of the statement for the measure theoretic boundary comes not only from the fact that – as in the Euclidean setting – the last one is sometimes easier to deal with, but mainly from the fact that the measure theoretic boundary – unlike the reduced boundary – is independent of the choice of the metric.

Theorem 5.19. (Divergence Theorem) *Let E be a $\mathbb{G}$-Caccioppoli set, then*

$$|\partial E|_{\mathbb{G}} = \vartheta_\infty \, \mathcal{S}_\infty^{Q-1} \llcorner \partial_{*,\mathbb{G}} E, \tag{i}$$

and the following version of the divergence theorem holds

$$-\int_E div_{\mathbb{G}} \varphi \, d\mathcal{L}^n = \vartheta_\infty \int_{\partial_{*,\mathbb{G}} E} \langle \nu_E, \varphi \rangle \, d\mathcal{S}_\infty^{Q-1}, \qquad \forall \varphi \in \mathbf{C}^1_0(\mathbb{G}, H\mathbb{G}). \tag{ii}$$

6. The Grushin plane

In this Section we discuss some problems related to Poincaré inequality associated with non-smooth vector fields. As we have already mentioned, fairly general results in this direction can be found in [46], [42], [74] and [94]. However, here we restrict ourselves to the case of $n = 2$, where the results take a simpler form, however full of interesting features. In [59] it is proved that, after a change of variables, we can assume that the vector fields X_1, X_2 have the form

$$X_1 = \partial_1 \quad , \quad X_2 = \lambda(x_1, x_2)\partial_2,$$

where λ is Lipschitz continuous and nonnegative. For sake of simplicity we assume that λ is independent of x_2, i.e. $\lambda(x_1, x_2) \equiv \lambda(x_1)$. Moreover, we write $x_1 = x$, $x_2 = y$. The plane $\mathbb{R}^2_{(x,y)}$ endowed with the Carnot–Carathéodory metric associated with $X_1 = \partial_x$ and $X_2 = \lambda(x)\partial_y$ is called sometimes the *Grushin plane.*

In [42], Theorem 2.3 we proved the following characterization of the metric balls of the Grushin plane.

Proposition 6.1. *If $z_0 = (x_0, y_0)$ and $t > 0$, set*
 (i) $\Lambda(z_0, t) = \sup_{|x-x_0|<t} \lambda(x)$;
 (ii) $F(z_0, t) = t\Lambda(z_0, t)$,
 (iii) $Q(z_0, t) = (x_0 - t, x_0 + t) \times (y_0 - F(z_0, t), y_0 + F(z_0, t))$.
If $\Lambda(z, t) > 0$ for $t > 0$ and for any $z \in \mathbb{R}^2$, then there exists $b > 1$ such that

$$Q(z_0, t/b) \subset B(z_0, t) \subset Q(z_0, bt),\ t > 0.$$

Corollary 6.2. *If $\Lambda(z, t) > 0$ for $t > 0$ and for any $z \in \mathbb{R}^2$, then the Carnot–Carathéodory metric in the Grushin plane is locally doubling with respect to Lebesgue measure if and only if the map $t \to \Lambda(z, t)$ is locally uniformly doubling with respect to z, i.e. if and only if for any compact set K there exist $C_K > 0$, $t_K > 0$ such that*

$$\Lambda(z, 2t) \leq C_K\Lambda(z, t) \qquad \text{if } z \in K \text{ and } 0 < t < t_K. \tag{79}$$

In particular, if (79) holds, then

$$|B(z_0, t)| \approx t^2\Lambda(z_0, t),$$
$$\varrho(z_1, z_2) \approx |x_1 - x_2| + F^{-1}(z_1, |y_1 - y_2|),$$

where $F^{-1}(z_1, t) = (F(z_1, \cdot))^{-1}(t)$ (notice the map $F(z_1, \cdot)$ is strictly increasing).

Proof. Suppose (79) holds. If $z \in K$ and we have

$$\begin{aligned}|B(z, 2t)| &\leq |Q(z, 2bt)| = 16b^2t^2\Lambda(z, 2bt)\\ &\leq C_{b,K}(t/b)^2\Lambda(z, t/b) = C_{b,K}|Q(z, t/b)| \leq C_{b,K}|B(z, t)|.\end{aligned}$$

Suppose now d is doubling. Then

$$\begin{aligned}\Lambda(z, 2t) &= \frac{|Q(z, 2t)|}{16t^2} \leq \frac{|B(z, 2bt)|}{16t^2}\\ &\leq C_{b,K}\frac{|B(z, t/b)|}{4t^2} \leq C_{b,K}\frac{|Q(z, t)|}{4t^2} = C_{b,K}\Lambda(z, t).\end{aligned}$$

□

Let us remind now the RH_∞ condition introduced in [44]. Let X be a metric space endowed with a metric ϑ and a doubling measure μ. Then, if $\omega \geq 0$ belongs to $L^1_{\text{loc}}(X)$, we say that $\omega \in RH_\infty$ if

$$\fint_B \omega d\mu \approx \text{ess sup}_B\omega$$

for all ϑ-balls B.

Proposition 2.3 in [44] reads as follows.

Proposition 6.3. *Let (X, ϑ, μ) be a homogeneous space and let $\omega \in L^1_{\mathrm{loc}}$ and $\omega > 0$ μ-a.e. Then*

(i) *$\omega \in RH_\infty$ iff $\omega^\beta \in RH_\infty$ for $\beta > 0$;*
(ii) *if $\omega \in RH_\infty$, then $\omega \in A_\infty$, and hence $\omega\mu$ is a doubling measure;*
(iii) *if $\omega \in RH_\infty$ and $u \in A_\infty$, then $\omega u \in A_\infty$.*

We can state now a necessary and sufficient condition in the Grushin plane in order that the Carnot–Carathéodory distance is locally doubling and a $(1,1)$-Poincaré inequality holds. In turn, this implies a (p,q)-Poincaré inequality, as pointed out in Remark 2.16.

Theorem 6.4. *Let $\lambda \geq 0$ be a Lipschitz continuous function. If $\lambda \in RH_\infty$, then the Carnot–Carathéodory distance d is doubling and a $(1,1)$-Poincaré inequality holds, i.e. for any Lipschitz continuous function f and for any Carnot–Carathéodory ball B*

$$\int_B |f - f_B|\, d\mathcal{L}^2 \leq C\, r(B) \int_B |Xf|\, d\mathcal{L}^2, \tag{80}$$

where $r(B)$ is the radius of B, and C is independent of B and f.

Conversely, if the Carnot–Carathéodory distance d is doubling and (80) holds, then $\lambda \in RH_\infty$.

Proof. Suppose $\lambda \in RH_\infty$. Then, by Proposition 6.3, (ii), $\lambda\mathcal{L}^2$ is a doubling measure, and hence, by the very definition of RH_∞, $\Lambda(z, .)$ is uniformly doubling, too. On the other hand, (80) follows by [42], Example 2 of Section 6.

Suppose now the Carnot–Carathéodory distance d is doubling and (80) holds. Then, arguing as in Theorem 3.6, we can conclude that, if $E \subset \mathbb{R}^2$ is an open set with C^1-boundary, then for any Carnot–Carathéodory ball B we have

$$\min\{|E \cap B|, |B \setminus E|\} \leq Cr(B) \int_{B\cap\partial E} \left(n_x^2 + \lambda(x)^2 n_y^2\right)^{1/2} d\mathcal{H}^1, \tag{81}$$

where $n = (n_x, n_y)$ is the outward unit normal to ∂E, and $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure supported by ∂E. For sake of simplicity take now $B = B(0, br)$, and choose

$$E = \{(x, y) \in \mathbb{R}^2 \,:\, y < \Lambda_0(x)\}, \text{ where } \Lambda_0(x) = \int_0^x \lambda(t)dt.$$

Since $Q := Q(0, r) \subset B$, then in (81) we can replace $\min\{|E\cap B|, |B\setminus E|\}$ by $\min\{|E\cap Q|, |Q\setminus E|\}$. Analogously, the integral on $B\cap\partial E$ at the right hand side of (81) can be replaced by the integral on $\tilde{Q} \cap \partial E$, where $\tilde{Q} = Q(0, b^2 r)$, i.e. we get

$$\min\{|E \cap Q|, |Q \setminus E|\} \leq Cr \int_{\tilde{Q}\cap\partial E} \left(n_x^2 + \lambda(x)^2 n_y^2\right)^{1/2} d\mathcal{H}^1, \tag{82}$$

In addition, when $|x| \leq b^2 r$ we have $|\Lambda_0(x)| \leq b^2 r \Lambda(0, b^2 r) = F(0, b^2 r)$, and analogously $|\Lambda_0(x)| \leq F(0, r)$ when $|x| \leq r$, so that

$$Q \cap E = \{(x, y) \in \mathbb{R}^2 \,:\, |x| < r\,, -F(0, r) < y < \Lambda_0(x)\}, \tag{83}$$

and

$$\tilde{Q} \cap \partial E = \{(x, y) \in \mathbb{R}^2 \,:\, |x| < b^2 r\,, y = \Lambda_0(x)\}. \tag{84}$$

Since $\Lambda_0(x) \geq 0$ for $x \geq 0$, and $\Lambda_0(x) \leq 0$ for $x \leq 0$, then, by (83), $(0, r) \times (-F(0, r), 0) \subset Q \cap E$, and $(-r, 0) \times (0, F(0, r)) \subset Q \setminus E$. Thus

$$\min\{|E \cap Q|, |Q \setminus E|\} \geq rF(0, r).$$

Finally, by (84), a parametrization of $\tilde{Q} \cap \partial E$ is given by $\gamma(t) = (t, \Lambda_0(t))$, with $|t| < b^2 r$. Replacing in (82) we get

$$rF(0,r) \leq Cr \int_{-b^2 r}^{b^2 r} \lambda(t) dt. \tag{85}$$

Dividing now both sides in (85) by r^2 and keeping in mind that $\Lambda(0,r) \approx \Lambda(0, b^2 r)$ by doubling ($\Lambda(0,\cdot)$ is doubling by Corollary 6.2) we get eventually that $\lambda \in RH_\infty$. $\square$

If $\lambda = |\varphi|$, φ being a smooth function, then it is possible to prove that Poincaré inequality (80) holds if the associated Carnot–Carathéodory distance is doubling (with respect to Lebesgue measure). This follows from Theorem 6.4 by the final Remark in [42], Section 6 that reads as follows.

Proposition 6.5. *If $\lambda = |\varphi|$, where $\varphi \in C^\infty(\mathbb{R}^2)$, then $\Lambda(z,\cdot)$ is doubling if and only if $\lambda \in RH_\infty$.*

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Bruno Franchi: Dipartimento di Matematica, Università di Bologna, Piazza di porta San Donato,5, 40126 Bologna, Italy

E-mail address: franchib@dm.unibo.it