KOHN LAPLACIANS ON *CR* MANIFOLDS

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PLAN OF THE TALKS

- **Introduction to CR manifolds**
  Definition, the Cauchy-Riemann complex, the Kohn Laplacian $\Box_b$. A few examples: the Heisenberg group, hypersurfaces in $\mathbb{C}^n$. The Levi form.

- **A few applications**
  Extensions of CR functions.

- **Local solvability, hypoellipticity and subellipticity**
  Definition of these fundamental notions and first analysis of $\overline{\partial}_b$ and $\Box_b$.

- **The Folland-Stein operators on the Heisenberg group**
  Fundamental and relative fundamental solutions. Local solvability and hypoellipticity.

- **The Kohn Laplacian on quadratic CR manifolds**
  Definition of quadratic CR manifolds. Characterization of local solvability and hypoellipticity for $\Box_b$ in this case.

- **The Kohn Laplacian and the condition $Y(q)$ on general CR manifolds**
  This condition is sufficient in general.

1. CR manifolds

Definition 1.1. Let M be a smooth man. of real dim. $2n+k$ with $n, k \geq 1$. We say that M is a *CR* man. of *CR* dim. n and codim. k is there exists a subbundle $\mathcal{L}$ of the complexified tangent bundle $T^{\mathbb{C}}M$ such that:

(1) $\dim_{\mathbb{C}} \mathcal{L} = n$;

(2) $\mathcal{L} \cap \overline{\mathcal{L}} = \{0\}$;

(3) the subbundle $\mathcal{L}$ is integrable, that is if L_1, L_2 are smooth sections of $\mathcal{L}$ then their commutator $[L_1, L_2]$ is also a smooth section of $\mathcal{L}$.

We assume that $\mathcal{L}$ and $\overline{\mathcal{L}}$ are orthogonal. There exists a k-dimensional real subbundle of $T^{\mathbb{C}}M$ denoted by $N(M)$ such that

$$T^{\mathbb{C}}M = \mathcal{L} \oplus \overline{\mathcal{L}} \oplus N(M)\,.$$

Let $\{L_1, \ldots, L_n, \overline{L}_1, \ldots, \overline{L}_n, T_1, \ldots, T_k\}$ be a basis for the smooth sections of the tangent bundle and

$$\{\omega_1, \ldots, \omega_n, \overline{\omega}_1, \ldots, \overline{\omega}_n, \tau_1, \ldots, \tau_n\}\,.$$

be a basis of 1-forms dual to the above basis.

Extend to the exterior algebra

$$\{\omega^I \wedge \tau^K \wedge \overline{\omega}^J :\ |I| + |K| = p,\ |J| = q,\}$$

is an orthonormal basis. Here I, J and K increasing multiindeces, e.g. $I = (i_1, \ldots, i_p)$, $1 \leq i_1 < \cdots < i_p \leq n$, and $q, |I| \leq n$, $|K| \leq k$.

Define

$$\Lambda^{p,q} = \Lambda^p(\mathcal{L}^* \oplus N^*(M)) \widehat{\otimes} \Lambda^q(\overline{\mathcal{L}}^*)$$

and call the space of its sections the space of (p, q)-forms on M.

We now describe an alternative way to introduce *CR* manifolds, as an embedded manifold of some complex space $\mathbb{C}^{n+k}$.

We denote by J the complex structure on $T^{\mathbb{C}}M$. Given a point $z \in M$ we call the *complex tangent space* at z the vector space

$$H_z(M) = T_z(M) \cap JT_z(M)\,.$$

Since $J^2 = -I$, the subspace H_z is even dimensional. Notice that

$$H_z(M) = T_z^{1,0}(\mathbb{C}^{n+k}) \cap T^{\mathbb{C}}(M)\,.$$

We fix an inner product in $T_z(M)$, say the euclidean inner product. We define the *totally real* tangent space at z to be the orthogonal complement of H_z in $T_z(M)$.

Definition 1.2. A submanifold M of $\mathbb{C}^{n+k}$ is called an embedded *CR* manifold if $\dim_{\mathbb{R}} H_z(M)$ is independed of $z \in M$.

Example 1.3. For instance, if $H_z = T_z(M)$ for all z, then M is a complex manifold. On the other hand, if $H_z = \{0\}$, then M is called *totally real*.

In general, in order to avoid trivialities, we will rule out these two cases. That is, we will assume that $0 < \dim_{\mathbb{R}} H_z(M) = n$ and $k > 0$.

Of embedded *CR* manifolds it is useful to have a description in local coordinates.

Lemma 1.4. *Let M be a CR submanifold in $\mathbb{C}^{n+k}$, of codimension k. Then, the following are equivalent:*

(1) $\dim_{\mathbb{R}} H_z(M) = 2n$ *for all* $z \in M$*;*
(2) $T_z(\mathbb{C}^{n+k}) = T_z(M) \oplus J\big(N_z(M)\big)$ *for all* $z \in M$*;*
(3) *for any local defining function system for* M $\{\rho_1, \dots, \rho_k\}$*, we have*

$$\overline{\partial}\rho_1(z) \wedge \cdots \wedge \overline{\partial}\rho_k(z) \neq 0\,.$$

Such a submanifold M in $\mathbb{C}^{n+k}$ will be said to *generic*. In particular, locally, on an open set U, we can represent M as

$$M \cap U = \{z \in U :\ \rho_1(z) = \cdots = \rho_k(z) = 0\}$$

and for $z \in M \cap U$

$$\overline{\partial}\rho_1(z) \wedge \cdots \wedge \overline{\partial}\rho_k(z) \neq 0\,.$$

In this case we can take the subbundle $T^{1,0}(\mathbb{C}^{n+k}) \cap T^{\mathbb{C}}M$ as the subbundle $\mathcal{L}$ in the definition of CR manifold. Here and in what follows, we denote by $T^{1,0}(\mathbb{C}^{N})$ the subbundle of the holomorphic vector fields in $\mathbb{C}^{N}$.

2. The tangential Cauchy-Riemann complex

Define the **tangential Cauchy-Riemann complex**, or **$\overline{\partial}_b$-complex**. Let f be a smooth function on M. Then $\overline{\partial}_b f$ is the $(0,1)$-form on M

$$\langle \overline{\partial}_b f, \overline{L} \rangle = \overline{L}(f)\,,$$

for any smooth section $\overline{L}$ of $\overline{\mathcal{L}}$.

Extend to smooth forms on M, by the standard derivation formula: If ϕ is a $(0,q)$-form on M and $\overline{L}_1, \ldots, \overline{L}_{q+1}$ are smooth sections of $\overline{\mathcal{L}}$, then

$$\begin{aligned}\langle \overline{\partial}_b \phi, (\overline{L}_1, \ldots, \overline{L}_{q+1}) \rangle = \frac{1}{q+1} \Big\{ &\sum_{j=1}^{q+1} (-1)^{j+1} \overline{L}_j \langle \phi, (\overline{L}_1, \ldots, \widehat{\overline{L}_j}, \ldots, \overline{L}_{q+1}) \rangle \\ &+ \sum_{i<j} (-1)^{i+j} \langle \phi, ([\overline{L}_i, \overline{L}_j], \overline{L}_1, \ldots, \widehat{\overline{L}_i}, \ldots, \widehat{\overline{L}_j}, \ldots, \overline{L}_{q+1}) \rangle \Big\}\end{aligned}$$

where $\widehat{\overline{L}_j}$ indicates the fact that the term $\overline{L}_j$ is omitted.

Finally, if $\psi = \phi \wedge \omega^I \wedge \tau^K$, where ϕ is a $(0,q)$-form, then

$$\overline{\partial}_b \psi = \overline{\partial}_b \phi \wedge \omega^I \wedge \tau^K\,.$$

The operator $\overline{\partial}_b$ acts only on the $\Lambda^q(\overline{\mathcal{L}}^*)$-component of a form.

It suffices to consider $\overline{\partial}_b$ acting on $(0,q)$-forms, and we shall do this in these lectures.

On the space of forms on M with coefficients in $L^2(M)$ we have the inner product

$$(\omega, \omega') = \int_M \langle \omega, \omega' \rangle dV\,.$$

The operator $\overline{\partial}_b$ can be viewed as an unbounded operator, with dense domain,

$$\overline{\partial}_b : L^2\Lambda^{0,q}(M) \to L^2\Lambda^{0,q+1}(M)\,.$$

The integrability condition implies the the sequence

$$0 \longrightarrow L^2(M) \xrightarrow{\overline{\partial}_b} L^2\Lambda^{0,1}(M) \xrightarrow{\overline{\partial}_b} \cdots \xrightarrow{\overline{\partial}_b} L^2\Lambda^{0,n-1}(M) \longrightarrow 0$$

forms a complex, that is **on a CR manifold M we have that**

$$\overline{\partial}_b^2 = 0\,.$$

Proof.

This follows from the observation that, on $\Lambda^{0,q}M$, $\overline{\partial}_b = \pi_{0,q+1} \circ d$, where $\pi_{0,q+1}$ is the projection from $q+1$-forms onto their $(0,q+1)$-components, d is the exterior differentiation and the integrability property of $\overline{\mathcal{L}}$. □

The complex defined above on M is called the **Cauchy-Riemann complex, or $\overline{\partial}_b$-complex**.

We now define the **Kohn Laplacian**.

Let $\overline{\partial}_b^*$ be the L^2-Hilbert space adjoint of $\overline{\partial}_b$, when acting on $L^2\Lambda^{0,q}$. Then,

$$\overline{\partial}_b^* : L^2\Lambda^{0,q+1} \to L^2\Lambda^{0,q}$$

is a densely defined unbounded operator.

The Kohn Laplacian on M is the operator

$$\Box_b = \Box_{b,q} = \overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b\,;$$

then as unbounded operator

$$\Box_b : L^2\Lambda^{0,q} \to L^2\Lambda^{0,q}\,.$$

The Heisenberg group.

Let H_n be the Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ and product rule given by

$$(z,t)(z',t') = \Big(z+z', t+t' - \tfrac{1}{2}\mathrm{Im}\,(z\cdot\overline{z}')\Big)\,.$$

Here $z\cdot\overline{z}' = \sum_j z_j \overline{z}'_j$, so that if $z = x+iy$, $z' = x'+iy'$, $-\mathrm{Im}\,(z\cdot\overline{z}') = x\cdot y' - x'\cdot y$ This group law makes H_n into a non-commutative group.

The neutral element is $(0,0)$ and the inverse (z,t) is the element $(-z,-t)$. The center of the group is constituted by the elements $(0,t)$.

Define the vector fields

$$X_j = \partial_{x_j} - \tfrac{1}{2}y_j\partial_t,\ Y_j = \partial_{y_j} + \tfrac{1}{2}x_j\partial_t, \quad \text{for } j=1,\dots,n, \quad \text{and } T = \partial_t\,.$$

A basis for the complexified tangent bundle is then given by

$$\{B_1,\dots,B_n,\,\overline{B}_1,\dots,\overline{B}_n,\,T\}$$

where $B_j = \frac{1}{\sqrt{2}}(X_j - iY_j)$ and $\overline{B}_j = \frac{1}{\sqrt{2}}(X_j + iY_j)$, $j=1,\dots,n$.

Notice that the only non-trivial commutators of these vector fields are

$$[X_j, Y_j] = T, \qquad ([B_j,\overline{B}_j] = iT \text{ resp.}) \quad j=1,\dots,n\,.$$

The Heisenberg group can be also realized as an embedded CR manifold. In fact, if we set

$$\rho(z,\zeta) = \mathrm{Im}\,\zeta - |z|^2$$

for $(z,\zeta)\in\mathbb{C}^n\times\mathbb{C}$, then

$$H_n = \{(z,\zeta)\in\mathbb{C}^n\times\mathbb{C}:\ \rho(z,\zeta)=0\}\,.$$

The dual bases of 1-forms are $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n, \theta\}$ and, for the complexfied bundle

$$\{\beta_1, \dots, \beta_n, \overline{\beta}_1, \dots, \overline{\beta}_n, \theta\}$$

resp., where $\beta_j = \frac{1}{\sqrt{2}}(dx_j + idy_j)$, $\overline{\beta}_j = \frac{1}{\sqrt{2}}(dx_j - idy_j)$, and $\theta = dt - \frac{1}{2}\sum_{j=1}^n (x_j dy_j - y_j dx_j)$.

A $(0,q)$-form ϕ on H_n can be written as

$$\phi = \sum_{|I|=q} \phi_I \overline{\beta}^I$$

and $\overline{\partial}_b$ acts on ϕ as

$$\begin{aligned}\overline{\partial}_b \phi &= \sum_{|I|=q} \sum_{j=1}^n \overline{B}_j \phi_I \overline{\beta}_j \wedge \overline{\beta}^I \\ &= \sum_{|J|=q+1} \left(\sum_{j,I} \varepsilon_J^{jI} \overline{B}_j \phi_I \right) \overline{\beta}^J ,\end{aligned}$$

where ε_J^{jI} equals 0 if $\{j\} \cup I \neq J$ as sets, and equals the sign of the permutation $\begin{pmatrix} jI \\ J \end{pmatrix}$ if $\{j\} \cup I = J$ as sets.

The **adjoint** $\overline{\partial}_b^*$ is defined by the relation

$$\langle \overline{\partial}_b^* \phi, \psi \rangle = \langle \phi, \overline{\partial}_b \psi \rangle$$

and has the expression

$$\begin{aligned}\overline{\partial}_b^* \phi = \overline{\partial}_b^* \left(\sum_{|I|=q} \phi_I \overline{\beta}^I \right) &= - \sum_{j=1}^n \sum_{|I|=q} B_j \phi_I \overline{\beta}_j \lrcorner \overline{\beta}^I \\ &= - \sum_{|K|=q-1} \left(\sum_{j,I} \varepsilon_I^{jK} B_j \phi_I \right) \overline{\beta}^K .\end{aligned}$$

Here $\lrcorner$ denotes the contraction operator of forms.

The **Kohn Laplacian** on H_n can now be calculated.

Proposition 2.1. *Let $\phi = \sum_{|I|=q} \phi_I \overline{\beta}^I$ be a smooth $(0,q)$-form. Then*

$$\Box_b \phi = \sum_{|I|=q} \left(\mathcal{L}_0 + i\left(\frac{n}{2} - q\right)T \right) \phi_I \overline{\beta}^I ,$$

where $\mathcal{L}_0$ is the scalar (left-invariant) differential operator, called the **sublaplacian**

$$\mathcal{L}_0 = -\tfrac{1}{2} \textstyle\sum_{j=1}^n X_j^2 + Y_j^2 = -\tfrac{1}{2} \sum_{j=1}^n B_j \overline{B}_j + \overline{B}_j B_j \,.$$

It is worth noticing that, on the Heisenberg group H_n, the Kohn Laplacian $\Box_b$ acting on $(0,q)$-forms is diagonal. Thus, it can be analyzed by studying a single scalar operator, the sublaplacian $\mathcal{L}_0$. This phenomenon does not appear on more general CR manifolds, as we will see in other examples. However, the main term of $\Box_b$ will remain diagonal in general.

Proof.

It suffices to consider the case of a simple $(0,q)$-form $\phi = f\overline{\beta}^I$. We first compute

$$\begin{aligned} \overline{\partial}_b^* \overline{\partial}_b f \overline{\beta}^I &= \overline{\partial}_b^* \left(\sum_j \overline{B}_j f \overline{\beta}_j \wedge \overline{\beta}^I \right) \\ &= -\sum_k \sum_j B_k \overline{B}_j f \overline{\beta}_k \lrcorner \left(\overline{\beta}_j \wedge \overline{\beta}^I \right) \end{aligned}$$

and

$$\begin{aligned} \overline{\partial}_b \overline{\partial}_b f^* \overline{\beta}^I &= -\overline{\partial}_b \left(\sum_k B_k f \overline{\beta} \lrcorner \overline{\beta}^I \right) \\ &= -\sum_j \sum_k \overline{B}_j B_k f \overline{\beta}_j \wedge \left(\overline{\beta}_k \lrcorner \overline{\beta}^I \right). \end{aligned}$$

Next, notice that for $j \neq k$

$$\overline{\beta}_k \left(\lrcorner \overline{\beta}_j \wedge \overline{\beta}^I \right) = -\overline{\beta}_j \wedge \left(\overline{\beta}_k \lrcorner \overline{\beta}^I \right),$$

and that, for $j = k$

$$\overline{\beta}_j\Big(\lrcorner\overline{\beta}_j \wedge \overline{\beta}^I\Big) = \begin{cases} \beta^I & \text{if } j \notin I \\ 0 & \text{if } j \in I \end{cases}$$

and

$$\overline{\beta}_j \wedge \Big(\overline{\beta}_j \lrcorner \overline{\beta}^I\Big) = \begin{cases} \beta^I & \text{if } j \in I \\ 0 & \text{if } j \notin I \end{cases} .$$

Then, the terms with $j \neq k$ cancel out. The terms with $j = k$ give rise to the sublaplacian

$$\begin{aligned}
\Box_b \phi &= (\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b)\phi \\
&= -\sum_{j\neq k} (B_k \overline{B}_j - \overline{B}_j B_k) f \overline{\beta}_k \Big(\lrcorner\overline{\beta}_j \wedge \overline{\beta}^I\Big) - \Bigg(\sum_{j\in I} \overline{B}_j B_j + \sum_{j\notin I} B_j \overline{B}_j\Bigg) f \overline{\beta}^I \\
&= -\frac{1}{2} \sum_j \Big(\overline{B}_j B_j + B_j \overline{B}_j\Big) f \overline{\beta}^I + \frac{1}{2} \sum_{j\in I} \Big(\overline{B}_j B_j - B_j \overline{B}_j\Big) f \overline{\beta}^I \\
&\qquad\qquad - \frac{1}{2} \sum_{j\notin I} \Big(B_j \overline{B}_j - \overline{B}_j B_j\Big) f \overline{\beta}^I \\
&= (\mathcal{L}_0 f) \overline{\beta}^I - \frac{1}{2} \sum_{j\in I} i(Tf) \overline{\beta}^I + \frac{1}{2} \sum_{j\notin I} i(Tf) \overline{\beta}^I \\
&= \Big(\mathcal{L}_0 + i(\frac{n}{2} - q)\Big) f \overline{\beta}^I .
\end{aligned}$$

From here the statement follows. $\square$

Hypersurfaces in $\mathbb{C}^{n+1}$.

Let $\mathcal{D}$ be a smooth domain in $\mathbb{C}^{n+1}$. Then there exists a neighborhood $\mathcal{U}$ of the boundary $\partial\mathcal{D}$ and a smooth function $\rho : \mathcal{U} \to \mathbb{R}$ such that

$$\mathcal{D} = \{z \in \mathcal{U} : \ \rho(z) < 0\}$$

and $\nabla\rho(z) \neq 0$ on the set $\{\rho(z) = 0\} = \partial\mathcal{D}$.

Such a function ρ can be extended to all of $\mathbb{C}^{n+1}$ in such a way that $\mathcal{D} = \{z \in \mathbb{C}^{n+1} : \ \rho(z) < 0\}$. Then ρ is called a *defining function* for the domain $\mathcal{D}$.

Lemma 2.2. *The boundary of a smooth domain $\mathcal{D}$ as above is a smooth real hypersurface in $\mathbb{C}^{n+1}$ whose tangent space is*

$$T(\partial\mathcal{D}) = \{(\xi,\eta) \in \mathbb{R}^{n+1}\times\mathbb{R}^{n+1} : \ d\rho(z)(\xi,\eta) = \Sigma_j(\partial_{x_j}\rho(z)\xi_j + \partial_{y_j}\rho(z)\eta_j) = 0\} .$$

The complex tangent space *at a point $z \in \partial\mathcal{D}$ is the subset of the tangent space given by*

$$H_z^{1,0} = T_{\mathbb{C},z}(\partial\mathcal{D}) = \{\zeta = \xi + i\eta \in \mathbb{C}^{n+1} : \ \rho(z)(\zeta) = \Sigma_j \partial_{z_j}\rho(z)\zeta_j = 0\} .$$

The subset $\cup_{z\in\partial\mathcal{D}} T_{\mathbb{C},z}(\partial\mathcal{D})$ is a subbundle of (complex)dimension n that makes $\partial\mathcal{D}$ into a CR manifold of CR dimension n and codimension 1.

Proof.

The first part of the statement follows from the fact that, if $\tilde\rho$ is another defining function, then there exists a smooth positive function h defined on a neighborhood $\mathcal{U}$ of $\partial\mathcal{D}$ such that $\tilde\rho = h\cdot\rho$ on $\mathcal{U}$.

The second part follows from the fact that the set $T_{\mathbb{C},z}(\partial\mathcal{D})$ defined in the statement is precisely the subset of the tangent space at z constitued of the tangent vectors $\zeta = \xi + i\eta = (\xi,\eta)$ such that $i\zeta = (-\eta,\xi)$ is also tangent.

Finally, we see that $T_{\mathbb{C}}$ is a subbundle of $T^{\mathbb{C}}(\partial\mathcal{D})$ that forms the CR structure on $\partial\mathcal{D}$. □

We now describe the $\overline{\partial}_b$-complex on the CR manifold $M = \partial\mathcal{D}$. Since M is an embedded manifold, in alternative to the intrisic deinition, valid on any abstract CR manifold, we can describe the $\overline{\partial}_b$-complex extrinsically, using the ambient space complex strucure.

Recall that for a smooth function f, $\overline{\partial} f$ is the $(0,1)$-form $\Sigma_j \overline{\partial}_{z_j} f d\overline{z}_j$, and analogously for ∂. Then, the exterior differentation operator d can be written as $d = \partial + \overline{\partial}$.

Let $\mathcal{U}$ be an open neighborhood of M. Let $I^{p,q}$ be the ideal of $\Lambda^{p,q}(\mathbb{C}^{n+1})$ generated by ρ and $\overline{\partial}\rho$; that is any element of $I^{p,q}$ can be written as

$$\rho\omega_1 + \overline{\partial}\rho \wedge \omega_2$$

where ω_1 is a (p,q)-form and ω_2 a $(p,q-1)$-form. Denote by $\Lambda^{p,q}(\mathbb{C}^{n+1})_{|M}$ and $I^{p,q}{}_{|M}$ resp. the restrictions of $\Lambda^{p,q}(\mathbb{C}^{n+1})$ and $I^{p,q}$ to M, resp.

Then we set $\Lambda^{p,q}(M)$ to be the orthogonal complement of $I^{p,q}{}_{|M}$ in $\Lambda^{p,q}(\mathbb{C}^{n+1})_{|M}$. Let

$$\Pi : \Lambda^{p,q}(\mathbb{C}^{n+1}) \to \Lambda^{p,q}(M)$$

be the mapping obtain by first restrict a (p,q)-form to M and then projecting it to $\Lambda^{p,q}(M)$.

It should be noted that $\Lambda^{p,q}(M)$ is not intrisic to M, that is it is a subspace of the space of complex forms on M, since $\overline{\partial}\rho$ is not orthogonal to the cotangent bundle of M.

For a smooth form in $\Lambda^{p,q}(M)$ we define the tangential Cauchy Riemann operator

$$\overline{\partial}_b : \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)$$

as

$$\overline{\partial}_b(\phi) = \Pi\overline{\partial}\tilde{\phi}\,,$$

where $\tilde{\phi}$ is any smooth (p,q)-form such that $\Pi(\tilde{\phi}) = \phi$. If $\hat{\phi}$ is another such form, then

$$\tilde{\phi} - \hat{\phi} = \rho\omega_1 + \overline{\partial}\rho \wedge \omega_2$$

for some smooth forms ω_1, ω_2. Then,

$$\overline{\partial}(\tilde{\phi} - \hat{\phi}) = \overline{\partial}(\rho\omega_1 + \overline{\partial} \wedge \rho\omega_2) = \overline{\partial} \wedge \rho\omega_1 + \rho\overline{\partial}\omega_1 - \overline{\partial}\rho \wedge \overline{\partial}\omega_2$$

so that

$$\Pi\overline{\partial}(\tilde{\phi} - \hat{\phi}) = 0\,.$$

Hence, the definition is independent of the extension $\tilde{\phi}$. Since $\overline{\partial}^2 = 0$ it follows that also $\overline{\partial}_b^2 = 0$.

This approach gives rise to a complex that is isomorphic to the one defined previously, intrinsically.

Thus, on an embedded CR manifold, the two definitions are equivalent.

The Levi form of M.

Let M be a CR manifold of CR dimension n and codimension k. Let $T^{1,0}M$ be the subbundle of $T^{\mathbb{C}}M$ that defines the CR structure (previously defined by $\mathcal{L}$) and let $T^{0,1}M$ be its complex conjugate. Recall the decomposition

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M \oplus N(M) .$$

Let $N(M)^*$ be the dual bundle of $N(M)$. Then, if $\tau \in N(M)^*$ then τ annihillates $T^{1,0}M \oplus T^{0,1}M$ and it is called the *characteristic bundle*.

Definition 2.3. Let $z \in M$. The Levi form is defined to be the hermitian form Φ on $T^{1,0}M$ taking values in $N(M)$ given by

$$\Phi(L_1, L_2) = i\Theta([L_1, \overline{L}_2]) ,$$

for $L_1, L_2 \in T^{1,0}M$ and where Θ is the projection of $T^{\mathbb{C}}M$ onto $N(M)$. The Levi form *in the direction* τ is the quadratic form

$$\langle \Phi(L_1, L_2), \tau \rangle = i\langle [L_1, \overline{L}_2]), \tau \rangle ,$$

again for $L_1, L_2 \in T^{1,0}M$.

If $M = \partial\mathcal{D}$ is the boundary of smooth domain in $\mathbb{C}^{n+1}$ with defining function ρ, then the Levi form can be described as follows. Let $a, b \in T^{\mathbb{C}}(\mathbb{C}^{n+1}) \cap T^{1,0}(\partial\mathcal{D})$, then

$$\Phi(a, b) = \left(\sum_{j,k=1}^{n+1} \frac{\partial^2 \rho(z)}{\partial z_j \partial \overline{z}_k} a_j \overline{b}_k \right) J(\nabla\rho) .$$

Definition 2.4. A real hypersurface $M \subseteq \mathbb{C}^{n+1}$ is called pseudoconvex if the Levi form is either positive semidefinite or negative semidefinite at every point on M. It is said to be strictly pseudoconvex if it is either positive definite or negative definite at every point.

The previous example can be generalized to the case of CR manifolds of higher codimensions.

Let $M \subseteq \mathbb{C}^{n+k}$ be an embedded generic CR manifold of codimension $k > 1$. There exist k smooth real-valued functions $\rho_1, \dots, \rho_k$ such that

$$M = \{z \in \mathbb{C}^{n+k} : \ \rho_1(z) = \cdots = \rho_k(z) = 0\}$$

and $\overline{\partial}\rho_1(z) \wedge \cdots \wedge \overline{\partial}\rho_k(z) \neq 0$. Without loss of generality, we may assume that $\nabla\rho_1, \dots, \nabla\rho_k$ form an orthonormal basis for $N(M)$ at every point of M.

Proposition 2.5. *With the above hypotheses, the Levi form* Φ *of* M *is given by*

$$\Phi(W, W') = \sum_{\ell=1}^{k} \left(\sum_{i,j=1}^{n+k} \frac{\partial^2 \rho_\ell(z)}{\partial z_i \partial \overline{z}_j} w_i \overline{w}'_j \right) J(\nabla\rho_\ell)(z)\,.$$

where $W = (w_1, \dots, w_n)$, $W' = (w'_1, \dots, w'_n)$

The tangential Cauchy-Riemann equations

Let $\mathcal{D}$ be a smoothly bounded domain in $\mathbb{C}^{n+1}$ with boundary $M = \partial\mathcal{D}$. Let ω be a smooth (p,q)-form on M. We wish to answer the following two questions about extensions of ω to the ambient space $\mathbb{C}^{n+1}$:

(1) Does there exist a smooth form $\tilde{\omega}$ in $\mathbb{C}^{n+1}$ such that $\Pi\tilde{\omega} = \omega$ on M?
(2) Does there exist a smooth form ϕ in $\mathbb{C}^{n+1}$ such that $\phi_{|M} = \omega$ and $\overline{\partial}\phi = 0$ on $\mathcal{D}$?

Recall that the holomorphic degree p of the form is irrelevant, so we will put $p = 0$. Recall also that $0 \leq q \leq n$.

Consider question (2) first. If ϕ is a $\overline{\partial}$-closed form, then $\overline{\partial}_b\phi_{|M}\Pi\overline{\partial} = 0$. Then $\overline{\partial}_b\overline{\omega} = 0$ is a necessary condition in order for (2) to be satisfied.

We say that a function on M is a *CR* function if $\overline{\partial}_b f = 0$. In order to state the condition on the Levi form, we need the following definition.

Definition 2.6. A smooth domain in $\mathbb{C}^N$

$$\mathcal{D} = \{z \in \mathbb{C}^N : \ \rho(z) < 0\}$$

(or more generally $\mathcal{C}^2$-smoothness of the boundary suffices) is said to be (Levi) pseudoconvex if the quadradic form

$$\sum_{j,k=1}^{N} \frac{\partial^2\rho(z)}{\partial z_j \overline{\partial} z_k}\zeta_j\overline{\zeta}_k \geq 0$$

for all $\zeta \in T^{1,0}(\partial\mathcal{D})$.

Notice that the above quadratic form (which is called the *Levi form of the domain* $\mathcal{D}$ corrispond to the Levi form of the *CR* manifold $\partial\mathcal{D}$ in the direction $J\overline{\partial}\rho$. Recall also that $T^{1,0}(\partial\mathcal{D}) = \{\zeta \in \mathbb{C}^N : \ \sum_j \zeta_j\partial_{z_j}\rho(z) = 0, \ z \in \partial\mathcal{D}\}$.

Theorem 2.7. *Let $\mathcal{D}$ be a smooth domain in $\mathbb{C}^{n+1}$. Let ω be a smooth (p,q)-form on $M = \partial\mathcal{D}$, $0 \leq p \leq n$, $1 \leq q \leq n$. Then, there exists a form $\tilde{\omega} \in \mathcal{C}^\infty_{(p,q)}(\overline{\mathcal{D}})$ such that $\Pi\tilde{\omega} = \omega$ and $\overline{\partial}\tilde{\omega} = 0$ in $\mathcal{D}$ if and only if*

$$\int_M \omega \wedge \psi = 0 \qquad \textit{for every } \psi \in \mathcal{C}^\infty_{(n-p,n-q-1)}(\overline{\mathcal{D}}) \cap Ker(\overline{\partial}).$$

Moreover, if $1 \leq q < n$, the above condition is equivalent to

$$\overline{\partial}_b \omega = 0 \quad \textit{on } M.$$

Local solvability, hypoellipticity, subellipticity.

Let $P = \Sigma_{|\alpha|\le m} a_\alpha(x)(i^{-1}\partial)_x^\alpha$ be a differential operator with smooth coeffients in a given open set Ω. We say that P is *locally solvable* at a point $x_0 \in \Omega$ if there exists an open neighborhood U of x_0 such that for every $f \in \mathcal{C}_0^\infty(\Omega)$ there exists a distribution $u \in \mathcal{D}'(\Omega)$ such that

$$Pu = f \qquad \text{in } U\,.$$

Given the operator P as above, we call *principal symbol* of P the smooth function $p_m(x,\xi) = \Sigma_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ defined on the cotangent bundle $T^*(\Omega)$. The *characteristic variety* of P is the set

$$\Sigma_P = \big\{(x,\xi) \in T^*(\Omega) : p_m(x\xi) = 0\big\}\,.$$

The operator P is called of *principal type* if $(x_0,\xi_0) \in \Sigma_P$ implies that $dp_m(x_0,\xi_0) \neq 0$.

An operator that is not of pricipal type is called of *multiple characteristic*. In this generality we have the following criterion of Hörmander.

Theorem 2.8. *Let P be as above. Then P is locally solvable at $x_0 \in \Omega$ if and only if there exists an open neighborhood V of x_0 and a positive integer k such that*

$$\|v\|_{H^{-k}} \le C\|{}^tPv\|_{H^k}$$

for every $v \in \mathcal{C}_0^\infty(U)$. (Here $\|\cdot\|_{H^s}$ denotes the norm in the Sobolev space, $s \in \mathbb{R}$.)

We will see next time that the operator $\overline{B}$ in H_1 is not locally solvable (Lewy unsolvable operator) and this will give rise to a series of questions concerning the solvability of $\overline{\partial}_b$.

3. Realtions between $\overline{\partial}_b$-complex and $\Box_b$

Since $\overline{\partial}_b$ and $\overline{\partial}_b^*$ form complexes, we see that

$$\begin{aligned}\overline{\partial}_b\Box_b &= \overline{\partial}_b(\overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b) = \overline{\partial}_b\overline{\partial}_b^*\overline{\partial}_b \\ &= (\overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b)\overline{\partial}_b = \Box_b\overline{\partial}_b\,.\end{aligned}$$

Hence, $\overline{\partial}_b$ and $\Box_b$ commute:

$$\overline{\partial}_b\Box_b = \Box_b\overline{\partial}_b\,.$$

This actually means

$$\overline{\partial}_{b,q}\Box_{b,q} = \Box_{b,q+1}\overline{\partial}_{b,q}$$

The same is true for $\overline{\partial}_b^*$ and $\Box_b$:

$$\overline{\partial}_{b,q}^*\Box_{b,q} = \Box_{b,q-1}\overline{\partial}_{b,q}^*\,.$$

Now, suppose $\Box_{b,q}$ is invertible. Let $\mathcal{G}_q$ be its inverse. Let f be a $(0,q)$-form such that $\overline{\partial}_b f = 0$. Then set

$$u = \mathcal{G}_q\overline{\partial}_b^* f\,.$$

Then u satisfies

$$\begin{aligned}\overline{\partial}_b u &= \overline{\partial}_b\mathcal{G}_q\overline{\partial}_b^* f = \mathcal{G}_q\overline{\partial}_b\overline{\partial}_b^* f \\ &= \mathcal{G}_q(\overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b)f = \mathcal{G}_q\Box_{b,q}f \\ &= f\,.\end{aligned}$$

Therefore, from the solvability of $\Box_{b,q}$ we obtain the solvabilty of the Cauchy-Riemann equation for a given $\overline{\partial}_b$-closed $(0,q+1)$-form f.

4. A non-solvability criterion

Recall Hörmander criterion

Theorem 4.1. *Given a partial diff. oper. with smooth coefficients P, then P is locally solvable at $x_0 \in \Omega$ if and only if there exists an open neighborhood V of x_0 and a positive integer k such that*

$$\|v\|_{H^{-k}} \leq C\|{}^t P v\|_{H^k}$$

for every $v \in \mathcal{C}_0^\infty(U)$. (Here $\|\cdot\|_{H^s}$ denotes the norm in the Sobolev space, $s \in \mathbb{R}$.)

In the case of homogenoues Lie groups (i.e. nilpotent Lie groups with a 1-parameter family of dilations) this can be used to give the following necessary criterion for local solvability.

Theorem 4.2. *(**Corwin-Rothschild**) Let $\mathcal{G}$ be a homogenous Lie group, let P be a left-invariant partial diff. oper. with smooth coefficients. Suppose that there exists a non-zero Schwartz function f such that*

$${}^t Pf = 0$$

then P cannot be locally solvable.

We remark that since P is invariant by left translation, locally solvability at the origin is equivalent to local solvability at any other point. Thus, we simply talk of local solvability for such operators.

5. The operators of Folland-Stein on the Heisenberg group

These are the differential operators

$$\mathcal{L}_\alpha = \mathcal{L}_0 + i\frac{\alpha}{2}T = -\frac{1}{2}\sum_{j=1}^{n}(B_j\overline{B}_j + \overline{B}_j B_j) + i\frac{\alpha}{2}T$$

for $\alpha \in \mathbb{C}$.

These operators are related to $\Box_{b,q}$ by the formula

$$\Box_b(\phi) = \sum_J \left(\sum_{j=1}^{n}\Big(\mathcal{L}_0 + +i(\tfrac{n}{2} - q)T\Big)\phi_J\right)\overline{\beta}^J .$$

On the Heisenberg group H_n there a 1-parameter family of non-isotropic dilations that are group automorphisms. These dilations are

$$D_r(z,t) = (rz, r^2 t)\,,$$

for $r > 0$. We recall that also the rotations

$$U(z,t) = (Uz, t)$$

for U in the unitary group of $\mathbb{C}^n$ are automorphisms of H_n.

We say that a differential operator $\mathcal{P}$ on the Heisenberg group is homogenous of degree d if

$$\mathcal{P}(f(D_r)(z,t)) = r^d\mathcal{P}(f)(D_r(z,t))\,.$$

It is possible to show that on H_n the only differential operators that are left-invariant, homogeneous of degree 2 and invariant under rotations are the operators $\mathcal{L}_\alpha$ above.

Theorem 5.1. (Folland-Stein) *For $\alpha \in \mathbb{C}$ define the locally integrable function on H_n*

$$E_\alpha(z,t) = \Big(|z|^2 + it\Big)^{-(n-\alpha)/2}\Big(|z|^2 - it\Big)^{-(n+\alpha)/2}.$$

Let

$$\gamma_\alpha = 2^{2-2n}\pi^{n+1}\Gamma\Big((n+\alpha)/2\Big)^{-1}\Gamma\Big((n-\alpha)/2\Big)^{-1}.$$

Then, in the sense of distributions,

$$\mathcal{L}_\alpha E_\alpha = \gamma_\alpha \delta_0\,,$$

where δ_0 denotes the Dirac delta at the origin.

Proof.
Let

$$E_{\alpha,\varepsilon}(z,t) = \Big(|z|^2 + \varepsilon^2 + it\Big)^{-(n-\alpha)/2}\Big(|z|^2 + \varepsilon^2 - it\Big)^{-(n+\alpha)/2}.$$

Now $E_{\alpha,\varepsilon} \in \mathcal{C}^\infty$ and we compute

$$\mathcal{L}_\alpha E_{\alpha,\varepsilon}$$

We obtain

$$\mathcal{L}_\alpha E_{\alpha,\varepsilon}(z,t) = F_{\alpha,\varepsilon}(z,t)$$

where

$$\begin{aligned} F_{\alpha,\varepsilon}(z,t) &= \varepsilon^2(n^2-\alpha^2)\Big(|z|^2+\varepsilon^2+it\Big)^{-(n+2-\alpha)/2}\Big(|z|^2+\varepsilon^2-it\Big)^{-(n+2+\alpha)/2} \\ &= \varepsilon^{-2n-n}F_{\alpha,1}\Big(D_{\varepsilon^{-1}}(z,t)\Big). \end{aligned}$$

In the sense of distributions

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle f, F_{\alpha,\varepsilon}\rangle &= \lim_{\varepsilon\to 0}\int_{H_n} f(z,t)F_{\alpha,\varepsilon}(z,t)dV(z,t)\\ &= \lim_{\varepsilon\to 0}\varepsilon^{-2n-n}\int_{H_n} f(z,t)F_{\alpha,1}\Big(D_{\varepsilon^{-1}}(z,t)\Big)dV(z,t)\\ &= \lim_{\varepsilon\to 0}\int_{H_n} f(\varepsilon z,\varepsilon^2 t)F_{\alpha,1}(z,t)dV(z,t)\\ &= \left(\int_{H_n} F_{\alpha,1}(z,t)dV(z,t)\right)\langle f,\delta_0\rangle\,.\end{aligned}$$

Thus, it suffices to show that

$$\int_{H_n} F_{\alpha,1}(z,t)dV(z,t) = \gamma_\alpha$$

where γ_α is as in the statement. $\square$

Notice that the constant γ_α is non-zero if and only if

$$\alpha \neq (n \pm 2k) \qquad k = 0, 1, 2, \ldots\,.$$

We call these values the *admissible values*.

Theorem 5.2. *Let $\mathcal{L}_\alpha$ be as above. Then the following conditions are equivalent:*

(1) *$\mathcal{L}_\alpha$ is locally solvable at the origin (or at any other point);*

(2) *$\mathcal{L}_\alpha$ is hypoelliptic;*

(3) *α is an admissible value.*

Corollary 5.3. *Consider the Kohn Laplacian $\Box_b = \Box_{b,q}$ on H_n acting on $(0,q)$-forms. Then the following are equivalent:*

(1) *$\Box_{b,q}$ is locally solvable;*

(2) *$\Box_{b,q}$ is hypoelliptic;*

(3) *$q \neq 0, n$.*

Proof of the theorem.

Suppose first that α is an admissible value. Then $\gamma_\alpha \neq 0$ and $\gamma_\alpha^{-1} E_\alpha$ is a fundamental solution for $\mathcal{L}_\alpha$; then $\mathcal{L}_\alpha$ is locally solvable.

Moreveor,

$$\gamma_\alpha^{-1} E_\alpha \in \mathcal{C}^\infty \setminus \{0\}$$

and it is homogenous of degree $-2n$ (with respect to the automoriphic dilations D_r).

Thus, $\gamma_\alpha^{-1} E_\alpha$ is a locally integrable function, smooth away from the origin.

Then $\mathcal{L}_\alpha$ is hypoelliptic in this case.

For, suppose that $\mathcal{L}_\alpha u = f$ is $\mathcal{C}^\infty$ in an open set U. By multiplying f by test function which is identically 1 on U, we may assume that F has compact support. Then $u = f * \gamma_\alpha^{-1} E_\alpha$ (is a well-defined distribution and) satisfies

$$\mathcal{L}u = f \qquad \text{on} \quad U$$

and is smooth on U.

Therefore, (3) implies (1) and (2).

Next, suppose that (2) holds. Then ${}^t\mathcal{L}_{-\alpha}$ is locally solvable.

Finally, to finish the proof we show that if α is not admissible then $\mathcal{L}_\alpha$ is not locally solvable (notice that α is admissible if and only if $-\alpha$ is).

Notice that, if α is not admissible

$$\mathcal{L}_\alpha E_\alpha = 0 \,.$$

We apply Corwin-Rothschild criterion. Let ψ be a Schwarzt function in $H_n (\equiv \mathbb{R}^{2n+1})$ such that

$$\int_{H_n} \psi(z,t) p(z,t)\, dV(z,t) = 0$$

for all polynomials p, where dV represents the Lebesgue measure on H_n. Now define

$$\phi = \psi * E_{-\alpha} \,.$$

It can be shown that ϕ is a Schwartz function on H_n

Hence, ϕ is a Schwartz function on H_n that satisfies

$${}^t\mathcal{L}_\alpha(\phi) = \mathcal{L}_{-\alpha}(\psi * E_{-\alpha}) = \psi * (\mathcal{L}_{-\alpha} E_{-\alpha}) = 0 \,.$$

Hence, $\mathcal{L}_\alpha$ is not locally solvable. $\square$

Proof of the corollary.

Notice that this result in particular says that $\Box_{b,q}$ is invertible if and only if $0 < q < n$.

The proof follows immediately from the theorem. We only need to show that for $q = 0, \dots, n$,

$$2\Big(\frac{n}{2} - q\Big) \qquad \text{is admissible if and only if} \quad q = 0, n\,. \qquad \square$$

We now discuss the operators $\mathcal{L}_\alpha$ for α non-admissible.

We saw that these operators are neither hypoelliptic nor locally solvable. Theorem 5.1 gives us a way to obtain a *relative* fundamental solution.

Theorem 5.4. *Let $\alpha = \pm(n+k)$, $k = 0, 1, \dots,$ be a non-admissible value. Then there exist a distribution F_α and an $L^2(H_n, dV)$-Hilbert space orthogonal projection S_α such that*

$$\mathcal{L}_\alpha F_\alpha = \delta_0 - S_\alpha\,.$$

The projection S_α is given by convolution with a locally integrable function K_α. When $\alpha = n$ i.e. when $q = 0$, $S_\alpha = S$ is called the Szegö projection.

Proof.

Consider the equation

$$\mathcal{L}_\alpha E_\alpha = \gamma_\alpha \delta_0\,,$$

given by Proposition 5.1. This is valid in $\alpha \in \mathbb{C}$ and in fact it is analytic in α.

Moreover, the function γ_α has simple zeros at the non-admissible values.

Differentiating the above equality and evaluating at a non-admissible value α_0 we obtain

$$iTE_{\alpha_0} + \mathcal{L}_{\alpha_0} F_{\alpha_0} = \gamma'_{\alpha_0} \delta_0 \,,$$

where

- γ'_{α_0} can be calculated using the explicit expression of γ_α;
- F_{α_0} is calculated as follows:

$$\begin{aligned}
F_{\alpha_0} &= \frac{d}{d\alpha}\Big(\big(|z|^2+it\big)^{-(n-\alpha)/2}\big(|z|^2-it\big)^{-(n+\alpha)/2}\Big)\Big|_{\alpha_0}\\
&= \frac{1}{2}\Big(\big(-\log(|z|^2-it)+\log(|z|^2+it)\big)\big(|z|^2+it\big)^{-(n-\alpha)/2}\big(|z|^2-it\big)^{-(n+\alpha)/2}\Big)\Big|_{\alpha_0}\\
&= \frac{1}{2}\log\Big(\frac{|z|^2+it}{|z|^2-it}\Big)\big(|z|^2+it\big)^{-(n-\alpha_0)/2}\big(|z|^2-it\big)^{-(n+\alpha_0)/2}\,.
\end{aligned}$$

Finally, we would have to show that convolution with iTE_{α_0} is an orthogonal L^2 projection. $\square$

We remark that the L^2-kernel of $\Box_b$ is given by those $(0,q)$-forms ω such that

$$\overline{\partial}_b \omega = \overline{\partial}_b^* \omega = 0\,.$$

For, these forms are certainly in the kernel of $\Box_b$. On the other hand, if $\Box_b \omega = 0$, then

$$\begin{aligned}
0 &= \langle \Box_b \omega, \omega\rangle\\
&= \langle (\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b)\omega, \omega\rangle\\
&= \langle \overline{\partial}_b^*\omega, \overline{\partial}_b^*\omega\rangle + \langle \overline{\partial}_b\omega, \overline{\partial}_b\omega\rangle\\
&= \|\overline{\partial}_b^*\omega\|^2 + \|\overline{\partial}_b\omega\|^2\,.
\end{aligned}$$

By the equality $\Box_{b,q}F = \delta_0 - S$ we see that S is exactly the Hilbert space projection onto $\ker \Box_{b,q}$. Therefore, when $q = 0$

$$\ker \Box_b = \ker \overline{\partial}_b \cap \ker \overline{\partial}_b^* = \ker \overline{\partial}_b$$

since $\overline{\partial}_b^*$ is identically zero on functions. Thus, S is the orthogonal projection onto the subspace of CR functions.

Corollary 5.5. *On H_1 the operator $\overline{B}$ is not locally solvable.*

Proof.
Notice that on H_1, $\overline{\partial}_b f = (\overline{B} f)\beta$, so that $\overline{\partial}_b f = 0$ if and only if $\overline{B} f = 0$.

Now, since $\ker \overline{\partial}_b = \ker \Box_{b,q}$, we know that a distribution ϕ is in $f \in \ker \overline{B}$ if and only if $\Box_{b,q} f = 0$ with $q = 0$.

We know that there exists Schwartz function in the kernel of $\Box_{b,q}$ in this case. Hence there are Schwartz functions in the kernel of $\overline{B}$. This shows that $\overline{B}$ is not locally solvable. $\Box$

Now we go back to the Kohn Laplacian $\Box_{b,q}$ and discuss the relation on the Cauchy-Riemann equations.

Suppose that $0 < q < n$, so that $\Box_{b,q}$ is invertible. Let $\mathcal{G}_q$ be the inverse operator of $\mathcal{L}_{n-2q}$. Let f be a $(0,q)$-form such that $\overline{\partial}_b f = 0$. Then set

$$u = \mathcal{G}_q \overline{\partial}_b^* f \,.$$

Then., using the fact that f is a CR form, we see that u satisfies

$$\begin{aligned}
\overline{\partial}_b u &= \overline{\partial}_b \mathcal{G}_q \overline{\partial}_b^* f = \mathcal{G}_q \overline{\partial}_b \overline{\partial}_b^* f \\
&= \mathcal{G}_q (\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b) f = \mathcal{G}_q \Box_{b,q} f \\
&= f \,,
\end{aligned}$$

i.e. u satisfies the Cauchy-Riemann equation $\overline{\partial}_b f = u$.

6. QUADRATIC CR MANIFOLDS

We now consider a class of higher codimension CR manifolds that can be viewed as generalization of the Heisenberg group. These are called *quadratic CR manifolds*.

Let Φ be a hermitean form on $\mathbb{C}^n \times \mathbb{C}^n$ having value in some $\mathbb{C}^k$:

$$\Phi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^k\,,$$

where $\Phi(z, z') = \overline{\Phi(z', z)}$. The associated *quadratic manifold* is

$$M = \big\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^k :\ \operatorname{Im} w = \Phi(z, z)\big\}\,.$$

Notice that, if $k = 1$ and $\Phi(z, z') = z\overline{z}'$, then M is just the Heisenberg group. In fact, also in this generality, M has an underlying Lie group structure.

For $(z', w') \in M$ the complex-affine transformation of $\mathbb{C}^n \times \mathbb{C}^k$

$$\tau_{(z',w')}(z, w) = \Big(z + z', w + w' + 2i\Phi(z, z')\Big)$$

maps M onto itself, and

$$\begin{aligned} \tau_{(z',w')}\tau_{(z'',w'')} &= \tau_{(z'+z'',w'+w''+2i\Phi(z',z''))} \\ {\tau_{(z',w')}}^{-1} &= \tau_{(-z',-w'+2i\Phi(z',z'))}. \end{aligned}$$

Under the identification of $\tau_{(z',w')}$ with $(z', w') \in M$, this composition law defines a Lie group structure on M.

This defines a group multiplication on M

$$(z, t)(z', t') = \Big(z + z', t + t' + 2\Im\Phi(z, z')\Big).$$

We call G_Φ this group.

The Levi form on G_Φ.

Notice that the manifold M can be describe by the k defining functions

$$\rho_j(z,w) = \operatorname{Im} w_j - \Phi_j(z,z) \qquad j = 1,\dots,k;$$

where Φ_j denotes the j-th component of Φ in the given fixed basis. Then Φ_j is an $n \times n$ hermitean matrix. We first fix our attention to the origin. We wish to compute

$$\frac{\partial^2 \rho_j(z,w)}{\partial \zeta_\ell \overline{\partial} \zeta_m} \zeta_\ell \overline{\zeta}_m$$

where we write momentarly $\zeta = (z,w) \in \mathbb{C}^{n+k}$.

We immediately see that the j-th component of the Levi form of M is exactly the j-th component of the form Φ, that is the Levi form of M is just Φ, thought as taking values in the normal bundle $N(M)$.

For $\tau \in N^*(M)$ we denote by Φ^τ the scalar-valued form $\tau(\Phi(\cdot,\cdot))$.

It should be noted that we do not require that the Levi form is non-degerate. Moreover, it is possible that all the Φ^τ are degenerate, even though there is no common radical that can be factored out to decompose G_Φ as the product of a nilpotent and an abelian group.

For example let, $\Phi : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^2$, $\Phi = (\Phi_1, \Phi_2)$, with $\Phi_j(z,z') = z'^* A_j z$ and

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Definition 6.1. *Let $n^+(\tau)$, resp. $n^-(\tau)$, the number of positive, resp. negative, eigenvalues of Φ^τ. We define Ω_q to be the cone*

$$\Omega_q = \{\tau : n^+(\tau) = q, n^-(\tau) = n - q\} \ .$$

The Kohn Laplacian on G_Φ.

For $v \in \mathbb{C}^n$, denote by $\partial_v f$ the directional derivative of a function f on $\mathbb{C}^n \times \mathbb{C}^k$ in the direction v and let X_v be the left-invariant vector field on G_Φ that coincides with ∂_v at the origin. It is easy to check that

$$X_v f(z,t) = \partial_v f(z,t) + 2\Im\Phi(z,v) \cdot \nabla_t f(z,t).$$

Take the standard basis $\{v_1, \ldots, v_n\}$ of $\mathbb{C}^n$ and define

$$B_j = \tfrac{1}{\sqrt{2}}(X_{v_j} - iX_{Jv_j}),$$
$$\overline{B}_j = \tfrac{1}{\sqrt{2}}(X_{v_j} + iX_{Jv_j}) \,,$$

for $j = 1, \ldots, n$; where J denotes the complex structure in $\mathbb{C}^n$.

We denote by $\overline{\beta}^I$ the $(0,q)$-form

$$\overline{\beta}_{i_1} \wedge \cdots \wedge \overline{\beta}_{i_q},$$

Given a $(0,q)$-form $\phi = \Sigma_{|I|=q} \phi_I \overline{\beta}^I$ with smooth coefficients, we set

$$\bar\partial_b \phi = \sum_{|I|=q} \sum_{k=1}^{n} \overline{B}_k(\phi_I)\overline{\beta}_k \wedge \overline{\beta}^I = \sum_{|J|=q+1} \sum_{k,|I|=q} \varepsilon^J_{kI} \overline{B}_k(\phi_I)\overline{\beta}^J \,,$$

The formal adjoint $\bar\partial_b^*$ of $\bar\partial_b$ can be easily computed to yield

$$\overline{\partial}_b^* \Big(\sum_{|I|=q} \phi_I \overline{\beta}^I \Big) = \sum_{|J|=q-1} \Big(- \sum_{k,|I|=q} \varepsilon^I_{kJ} B_k \phi_I \Big) \overline{\beta}^J.$$

We now compute the Kohn Laplacian $\Box_{b,q}$

Proposition 6.2. *With respect to the selected basis, the operator* $\Box_b^{(q)}$ *is represented by a matrix* $\big(\Box_{LK}\big)$ *of scalar left-invariant differential operators on* G_Φ *as*

$$\Box_b^{(q)}\Big(\sum_K \phi_K \overline{\beta}^K\Big) = \sum_L \Big(\sum_K \Box_{LK}\phi_K\Big)\overline{\beta}^L.$$

Then,

$$\Box_{LK} = \delta_{LK}\mathcal{L}_0 + M_{LK}$$

where δ_{LK} *is the Kronecker delta,*

$$\mathcal{L}_0 = -\tfrac{1}{2}\textstyle\sum_{k=1}^n(\overline{B}_k B_k + B_k\overline{B}_k)$$

and

$$M_{LK} = \begin{cases} \frac{1}{2}\Big(\sum\limits_{k\in K}[B_k,\overline{B}_k] - \sum\limits_{k\notin K}[B_k,\overline{B}_k]\Big) & \text{if } K = L, \\ \varepsilon(K,L)[B_k,\overline{B}_\ell] & \text{if } |\{K\cap L\}| = q-1, \\ 0 & \text{otherwise.} \end{cases}$$

Here, given two multi-indices K and L such that $|K| = |L| = q$ and $|\{K\cap L\}| = q-1$, we set

$$\varepsilon(K,L) = (-1)^m$$

where m is the number of elements in $K\cap L$ between the unique element $k\in K\setminus L$ and the unique element $\ell\in L\setminus K$.

Notice that, even in this relatively fairly simple situation, the Kohn Laplacian is far from being diagonal.

 Proof.

One can easily see that

$$\bar\partial_b(\bar\partial_b^*\phi) = -\sum_{|L|=q}\Big(\sum_{k,\ell,|J|=q-1,|K|=q} \varepsilon^K_{kJ}\varepsilon^L_{\ell J}\overline{B}_\ell B_k\phi_K\Big)\overline{\beta}^L$$

and that

$$\bar\partial_b^*(\bar\partial_b\phi) = -\sum_{|L|=q}\Big(\sum_{i,j,|H|=q+1,|K|=q} \varepsilon^H_{jK}\varepsilon^H_{iL} B_i\overline{B}_j\phi_K\Big)\overline{\beta}^L\,.$$

Hence,

$$\Box_b^{(q)}(\phi) = -\sum_{|L|=q}\sum_{|K|=q}\Big(\sum_{\ell,k,|J|=q-1}\varepsilon^K_{kJ}\varepsilon^L_{\ell J}\overline{B}_\ell B_k + \sum_{i,j,|H|=q+1}\varepsilon^H_{jK}\varepsilon^H_{iL}B_i\overline{B}_j\Big)\phi_K\overline{\beta}^L.$$

Then,

$$\Box_{LK} = -\sum_{\ell,k,|J|=q-1}\varepsilon^K_{kJ}\varepsilon^L_{\ell J}\overline{B}_\ell B_k - \sum_{i,j,|H|=q+1}\varepsilon^H_{jK}\varepsilon^H_{iL}B_i\overline{B}_j. \tag{1}$$

When $K = L$ the indices k and ℓ are forced to be equal, as well as i and j. Hence,

$$\begin{aligned}\Box_{LL} &= -\Big(\sum_{k\in L}\overline{B}_k B_k + \sum_{j\notin L} B_j\overline{B}_j\Big)\\ &= -\frac{1}{2}\sum_{k=1}^{n}(\overline{B}_k B_k + B_k\overline{B}_k) - \frac{1}{2}\Big(\sum_{k\in L}[\overline{B}_k, B_k] + \sum_{k\notin L}[B_k,\overline{B}_k]\Big).\end{aligned}$$

This proves the statement for the terms along the diagonal.

When $K \neq L$, the coefficient $\varepsilon^K_{kJ}\varepsilon^L_{\ell J}$ is different from 0 if only if $K = J\cup\{k\}$ and $L = J\cup\{\ell\}$. Notice that, given K and L such that $|\{K\cap L\}| = q-1$, they uniquely determine J, k and ℓ. Analogously, $\varepsilon^H_{jK}\varepsilon^H_{iL}\neq 0$ if and only if $H = K\cup\{j\} = L\cup\{i\}$. Then, necessarily, $|\{K\cap L\}| = q-1$ as before, and if k and ℓ are as above, $j=\ell$ and $i = k$.

It follows that $\Box_{LK} = 0$ unless $|\{K \cap L\}| = q - 1$. In this case, each of the sums in (1) reduces to one single term, and

$$\Box_{LK} = -\varepsilon^K_{kJ}\varepsilon^L_{\ell J}\overline{B}_\ell B_k - \varepsilon^H_{\ell K}\varepsilon^H_{kL}B_k\overline{B}_\ell,$$

with $J = K \cap L$ and $H = K \cup L$. Moreover,

$$\varepsilon^K_{kJ}\varepsilon^L_{\ell J} = -\varepsilon^H_{\ell K}\varepsilon^H_{kL} = \varepsilon(K, L)\,.$$

Thus,

$$\Box_{LK} = \varepsilon(K, L)[B_k, \overline{B}_\ell],$$

which proves the proposition. $\Box$

The main results on G_Φ.

We begin with the local solvability for $\Box_{b,q}$.

Theorem 6.3. *The Kohn Laplacian $\Box_{b,q}$ is locally solvable if and only if there is no $\tau \in N^*(M)$ such that $n^+(\tau) = q$ and $n^-(\tau) = n - q$.*

More precisely, the following conditions are equivalent.

(1) Ω_q *is non-empty;*

(2) $\Box_{b,q}$ *is not locally solvable;*

(3) $\ker \Box_{b,q} \cap L^2\Lambda^{0,q}(G_\Phi)$ *is non-empty;*

When $\Box_{b,q}$ is not solvable, the orthogonal projection onto its L^2-null-space is given by convolution on G_Φ with an operator-valued distribution S_q for which it is possible to give an explicit formula.

Next we discuss the hypoellipticity of the Kohn Laplacian.

Definition 6.4. We say that a CR manifold M with Levi form Φ satisfies condition $Y(q)$ at a point $z \in M$ is for every $\tau \in N^*(M)$, Φ_z^τ has at least $\max(q+1, n-q+1)$ eigenvalues with the same sign, or at least $\min(q+1, n-q+1)$ pairs of eigenvalues with opposite signs.

Theorem 6.5. *The following conditions are equivalent:*

(1) $\operatorname{span}_{\mathbb{R}}\{\Phi(z,z)\} = N(M)$ *and there exists* $C > 0$ *such that for each* ϕ *in the Schwartz space*

$$\|\Big(\mathcal{L}_0 \otimes I\Big)\phi\|_{L^2} \leq C\|\Box_{b,q}\phi\|_{L^2};$$

(2) $\Box_{b,q}$ *is hypoelliptic;*

(3) *there exists no non-zero* $\tau \in N^*(M)$ *such that* $n^+(\tau) \leq n-q$ *and* $n^-(\tau) \leq q$*;*

(4) Φ *satisfies condition* $Y(q)$*.*

We remark that condition (3) and (4) are both equivalent to the following condition: There exists no non-zero $\tau \in N^*(M)$ such that

$$\begin{cases} \min\Big(n^+(\tau), n^-(\tau)\Big) \leq \min(q, n-q) \\ \max\Big(n^+(\tau), n^-(\tau)\Big) \leq \max(q, n-q)\,. \end{cases}$$

Sufficiency of the $Y(q)$ condition for hypoellipticity

In this final section we return to the case of a general CR manifold. The result we present are due to Shaw and Wang.

Recall the decomposition of the complexified tangent space of M

$$T^{\mathbb{C}}M = \mathcal{L} \oplus \overline{\mathcal{L}} \oplus N(M) .$$

Let $\{L_1, \ldots, L_n, \overline{L}_1, \ldots, \overline{L}_n, T_1, \ldots, T_k\}$ be a basis for the smooth sections of the tangent bundle $T^{\mathbb{C}}M$, with $L_1, \ldots, L_n$ smooth sections of $\mathcal{L}$,

Lemma 6.6. *Assume that M satisfies condition $Y(q)$ at a point $z \in M$. Then there exists an open neighborhood U of z on which the vector fields $\{X_1, \ldots, X_n\}$ satisfy Hörmander's condition, where*

$$X_j = \Re L_j \qquad j = 1, \ldots, n; \qquad X_j = \Im L_j \qquad j = n+1, \ldots, 2n .$$

Proof.

In fact, it suffices to consider the first order commutators in order to span the tangent space $\mathbb{R}^{2n+k}$. For, let τ_ℓ be a given direction in $N(M)^*$.

Since $\mathcal{P}_z$ satisfies condition $Y(q)$, Phi^{τ_ℓ} is such that

$$\min(q, n-q) \leq n^+(\tau_\ell), n^-(\tau_\ell) \leq \max(q, n-q) .$$

In particular Phi^{τ_ℓ} has at least a non-zero eigenvalue. Since Phi^{τ_ℓ} is the matrix whose entries with respect to the basis $\{L_1, \ldots, L_n\}$ are

$$\delta_{jk}[L_j, \overline{L}_k]$$

we see that there exists at least one j such that $[L_j, \overline{L}_j]$ has non-trivial component in the direction $J(\tau_\ell)$. □

This lemma alone does not guarantees that $\Box_{b,q}$ is hypoelliptic, since ($\Box_{b,q}$ is not a scalar operator and) the lower order terms are not real.

Theorem 6.7. *Suppose that M is a CR manifold of CR dimension n and codimension $k \geq 1$. Assume that satisfies condition $Y(q)$ at a point $z \in M$. Then there exists an open neighborhood U of z on which the Kohn Laplacian $\Box_{b,q}$ satisfies the subelliptic estimates*

$$\|\eta_1 \phi\|_{H^{s+1}} \leq C\Big(\|\eta_2 \Box_{b,q}\phi\|_{H^s} + \|\phi\|\Big),$$

where η_1, η_2 are $\mathcal{C}^\infty$ cut-off fucntions supported in U, $\eta_2 = 1$ on $supp\, \eta_1$.

Proof.
Define

$$Q_b(\phi,\phi) = \|\overline{\partial}_b \phi\|^2 + \|dbb^* \phi^2\|^2 + \|\phi\|^2$$

One begins by showing that, by setting

$$\sum_{j=1}^{n} \|L_j \phi\|^2 = \|\phi\|^2_{\mathcal{L}}, \qquad \sum_{j=1}^{n} \|\overline{L}_j \phi\|^2 = \|\phi\|^2_{\overline{\mathcal{L}}},$$

we have

$$\|\phi\|^2_{\mathcal{L}} + \|\phi\|^2_{\overline{\mathcal{L}}} + \sum_{\ell=1}^{k} \sum_{I,j} |\Re(T_\ell \phi_{IJ}, \phi_{IJ})| \leq C Q_b(\phi,\phi).$$

From this, using the Hörmander condition on the vector fields $\{X_1, \ldots, X_{2n}\}$ and the corresponding subelliptic estimates, it follows that

$$\|\phi\|_{H^{1/2}} \leq C Q_b(\phi,\phi),$$

which in turns implies the desired estimate.

In order to prove the estimate above, one manipulates the energy form $Q_b(\phi,\phi)$ to obtain the estimate from below (here we assume $k=1$ for simplicity of notation)

$$Q_b(\phi,\phi) \geq \varepsilon\|\phi\|_{\mathcal{L}}^2 + \sum_{I,J} a_{IJ}\Re(T_\ell\phi_{IJ},\phi_{IJ}) - \delta\Big(\|\phi\|_{\mathcal{L}}^2 + \|\phi\|^2\Big)\,,$$

where

$$a_{IJ} = \sum_{j\in J\setminus\sigma I,J} \lambda_j - (1-\varepsilon) \sum_{j\in\sigma I,J\setminus J} \lambda_j + \varepsilon \sum_{j\in\sigma I,J\cap\setminus J} \lambda_j\,,$$

the λ_j's are the eigeinvalues of Φ (that we are assume to be scalar-valued for simplicity now) and

$$\sigma I, J = \{j : \lambda_j < 0 \text{ if } \Re(T_\ell\phi_{IJ},\phi_{IJ}) > 0 \text{ and } \lambda_j > 0 \text{ if } \Re(T_\ell\phi_{IJ},\phi_{IJ}) < 0\}\,.$$

Since M satisfies the condition $Y(q)$, one of the following cases must hold:

(1) if the Levi form has $\max(n+1-q,q+1)$ positive eigenvalues of the same sign, then there exists a $j\in J$ and an $\ell\notin J$ such that λ_j and λ_ℓ have the same sign (which we may assume to be positive by replacing T with $-T$);

(2) if the Levi form has $\min(n+1-q,q+1)$ pairs of eigenvalues of the opposite sign, then there exist $j,\ell\notin J$ so that λ_j and λ_ℓ have opposite signs;

(3) if the Levi form has $\min(n+1-q,q+1)$ pairs of eigenvalues of the opposite sign, then there exist $j,\ell\in J$ so that λ_j and λ_ℓ have opposite signs;

From this, we can select $\varepsilon > 0$ and small so that

$$a_{IJ} > 0\,,$$

if

$$\Re(T_\ell \phi_{IJ}, \phi_{IJ}) > 0\}\,.$$

and

$$a_{IJ} < 0\,,$$

if

$$\Re(T_\ell \phi_{IJ}, \phi_{IJ}) < 0\}\,.$$

From this the result follows. $\square$