AN INTRODUCTION TO THE ANALYSIS OF THE KOHN LAPLACIAN ON *CR* MANIFOLDS

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Abstract. In these lecture notes we present an introduction to the question of the solvability, the range and the hypoellipticity, subellipticity of the Kohn Laplacian on *CR* manifolds.

These lecture note should not be considered exhaustive of the subject by any means. The only reflect the intersests and kwnoledge of the author.

I wish to thanks the organizers for inviting me to deliver these lectures and for their success in organizing such a nice event.

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Preliminary version[1]

Introduction

The Cauchy Riemann manifolds, in brief *CR* manifolds, arise in a natural way in function theory of several complex variables, as we will try to illustrate in the development of these lectures. The *CR* manifolds have drawn a considerable amount of interest in recent years for their connection with several different research areas in analysis and geometry.

For an account on the hystorical background we refer the reader to the monograph [FK] and the survey paper [AK]. For an extensive account on *CR* manifolds we refer to the monographs [Bo] and [BER].

1. *CR* manifolds

We begin by introducing the setting on which we will be working on.

Definition 1.1. Let M be a smooth manifold of real dimension $2n+k$ with $n, k \geq 1$. We say that M is a *CR* manifold of *CR* dimension n and codimension k is there exists a subbundle $\mathcal{L}$ of the complexified tangent bundle $T^{\mathbb{C}}M$ such that the following conditions hold:

(1) $\dim_{\mathbb{C}} \mathcal{L} = n$;
(2) $\mathcal{L} \cap \overline{\mathcal{L}} = \{0\}$;

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[1]These notes are still in a fairly rough version. The final version will be ready soon and will be available and the site of the school and also on my homepage http://calvino.polito.it/˜peloso

(3) the subbundle $\mathcal{L}$ is integrable, that is if L_1, L_2 are smooth sections of $\mathcal{L}$ then their commutator $[L_1, L_2]$ is also a smooth section of $\mathcal{L}$.

We assume that M is equipped with a Hermitean metric for $T^{\mathbb{C}}M$ so that $\mathcal{L}$ and $\overline{\mathcal{L}}$ are orthogonal. For each $z \in M$ let N_z be the orthogonal complement of $\mathcal{L}_z \oplus \overline{\mathcal{L}}_z$ in $T^{\mathbb{C}}_z M$. This gives rise to a k-dimensional real subbundle of $T^{\mathbb{C}}M$ denoted by $N(M)$. Then

$$T^{\mathbb{C}}M = \mathcal{L} \oplus \overline{\mathcal{L}} \oplus N(M)\,. \tag{1}$$

The pointwise metric on $T^{\mathbb{C}}M$ induces a pointwise dual metric on the space of 1-forms on M, $T^{\mathbb{C}^*}M$.

Let $\{L_1, \dots, L_n, \overline{L}_1, \dots, \overline{L}_n, T_1, \dots, T_k\}$ be a basis for the smooth sections of the tangent bundle $T^{\mathbb{C}}M$, with $L_1, \dots, L_n$ smooth sections of $\mathcal{L}$, $\overline{L}_1, \dots, \overline{L}_n$ smooth sections of $\overline{\mathcal{L}}$ and $T_1, \dots, T_k$ smooth sections of $N(M)$. We can find a basis of 1-forms dual to the above basis; let this basis be

$$\{\omega_1, \dots, \omega_n, \overline{\omega}_1, \dots, \overline{\omega}_n, \tau_1, \dots, \tau_n\}\,.$$

The metric on $T^{\mathbb{C}^*}M$ extend to the exterior algebras of forms in such a way that

$$\{\omega^I \wedge \tau^K \wedge \overline{\omega}^J :\ |I| + |K| = p,\ |J| = q,\}$$

is an orthonormal basis. Here I, J and K increasing multiindeces, e.g. $I = (i_1, \dots, i_p)$, $1 \le i_1 < \cdots < i_p \le n$, and $p, q \le n$, $r \le k$. We also define

$$\Lambda^{p,q} = \Lambda^p(\mathcal{L}^* \oplus N^*(M)) \widehat{\otimes} \Lambda^q(\overline{\mathcal{L}}^*)$$

and call the space of its sections the space of (p, q)-forms on M.

We now introduce the so-called *tangential Cauchy-Riemann complex*, or $\overline{\partial}_b$-complex. Let f be a smooth function on M. Then $\overline{\partial}_b f$ is the $(0, 1)$-form on M defined by

$$\langle \overline{\partial}_b f, \overline{L} \rangle = \overline{L}(f)\,,$$

for any smooth section $\overline{L}$ of $\overline{\mathcal{L}}$. This definition can extended to smooth forms on M, by the standard derivation formula. If ϕ is a $(0, q)$-form on M and $\overline{L}_1, \dots, \overline{L}_{q+1}$ are smooth sections of $\overline{\mathcal{L}}$, then

$$\begin{aligned}\langle \overline{\partial}_b \phi, (\overline{L}_1, \dots, \overline{L}_{q+1}) \rangle = \frac{1}{q+1} \Big\{ &\sum_{j=1}^{q+1} (-1)^{j+1} \overline{L}_j \langle \phi, (\overline{L}_1, \dots, \widehat{\overline{L}_j}, \dots, \overline{L}_{q+1}) \rangle \\ &+ \sum_{i<j} (-1)^{i+j} \langle \phi, ([\overline{L}_i, \overline{L}_j], \overline{L}_1, \dots, \widehat{\overline{L}_i}, \dots, \widehat{\overline{L}_j}, \dots, \overline{L}_{q+1}) \rangle \Big\}\end{aligned}$$

where $\widehat{\overline{L}_j}$ indicates the fact that the term $\widehat{\overline{L}_j}$ is omitted.

Finally, if $\psi = \phi \wedge \omega^I \wedge \tau^K$, where ϕ is a $(0, q)$-form, then

$$\overline{\partial}_b \psi = \overline{\partial}_b \phi \wedge \omega^I \wedge \tau^K\,.$$

Thus, the operator $\overline{\partial}_b$ acts only on the $\Lambda^q(\overline{\mathcal{L}}^*)$-component of a form. It is then sufficient to consider $\overline{\partial}_b$ acting on $(0, q)$-forms, and we shall do this in these lectures.

The pointwise pairing between forms on M can be extended to sections with coefficients in $L^2(M)$, with respect to the given volume form dV:

$$(\omega, \omega') = \int_M \langle \omega, \omega' \rangle dV \,.$$

We denote by $L^2\Lambda^k M$ (or $L^2\Lambda^{p,q}M$) the space of k-forms ((p,q)-forms resp.) with coefficients in $L^2(M)$. Then, the operator $\overline{\partial}_b$ can be viewed as an unbounded operator, with dense domain,

$$\overline{\partial}_b : L^2\Lambda^{0,q}(M) \to L^2\Lambda^{0,q+1}(M)$$

(recall the convention we are adopting).

It is important to notice that, the integrability condition implies the the following sequence

$$0 \longrightarrow L^2(M) \xrightarrow{\overline{\partial}_b} L^2\Lambda^{0,1}(M) \xrightarrow{\overline{\partial}_b} \cdots \xrightarrow{\overline{\partial}_b} L^2\Lambda^{0,n-1}(M) \longrightarrow 0 \tag{2}$$

forms a complex.

Proposition 1.2. *On a CR manifold M we have that $\overline{\partial}_b^2 = 0$.*

Proof. This follows from the observation that, on $\Lambda^{0,q}M$, $\overline{\partial}_b = \pi_{0,q+1} \circ d$, where $\pi_{0,q+1}$ is the projection from $q+1$-forms onto their $(0, q+1)$-components, d is the exterior differentiation and the integrability property of $\overline{\mathcal{L}}$. (more??) □

Definition 1.3. The complex defined in (2) on M is called the Cauchy-Riemann complex, or $\overline{\partial}_b$-complex.

We now define the Kohn Laplacian. Let $\overline{\partial}_b^*$ be the L^2-Hilbert space adjoint of $\overline{\partial}_b$, when acting on $L^2\Lambda^{0,q}$. Then,

$$\overline{\partial}_b^* : L^2\Lambda^{0,q+1} \to L^2\Lambda^{0,q}$$

is a densely defined unbounded operator.

Definition 1.4. The Kohn Laplacian on M is the operator $\Box_b = \Box_{b,q} = \overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b$; then as unbounded operator

$$\Box_b : L^2\Lambda^{0,q} \to L^2\Lambda^{0,q} \,.$$

We now present a few noticeable examples of *CR* manifolds.

1.5. The Heisenberg group. Let H_n be the Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ and product rule given by

$$(z,t)(z',t') = \big(z + z', t + t' - \tfrac{1}{2}\mathrm{Im}\,(z \cdot \overline{z}')\big) \,.$$

Here $z \cdot \overline{z}' = \sum_j z_j \overline{z}'_j$, so that if $z = x + iy$, $z' = x' + iy'$, $-\mathrm{Im}\,(z \cdot \overline{z}') = x \cdot y' - x' \cdot y$ is the symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$. This group law makes H_n into a non-commutative group. The neutral element is $(0,0)$ and the inverse (z,t) is the element $(-z,-t)$. The center of the group is constituted by the elements $(0,t)$.

A detailed analysis of the Kohn Laplacian (and much more) appears in the seminal paper [FS1]. Further information on the Kohn Laplacian on the Heisenberg group can be found in [Ta] and in [Th], for instance.

Define the vector fields

$$X_j = \partial_{x_j} - \tfrac{1}{2} y_j \partial_t,\ Y_j = \partial_{y_j} + \tfrac{1}{2} x_j \partial_t, \quad \text{for } j = 1, \dots, n, \quad \text{and } T = \partial_t \,.$$

Then $\{X_1, \dots, X_n, Y_1, \dots, Y_n, T\}$ form a basis for the left-invariant vector fields on H_n, that is for the tangent bundle of H_n. A basis for the complexified tangent bundle is then given by

$$\{B_1, \dots, B_n,\ \overline{B}_1, \dots, \overline{B}_n,\ T\}$$

where $B_j = \frac{1}{\sqrt{2}}(X_j - iY_j)$ and $\overline{B}_j = \frac{1}{\sqrt{2}}(X_j + iY_j)$, $j = 1, \dots, n$.

Notice that the only non-trivial commutators of these vector fields are

$$[X_j, Y_j] = T, \qquad ([B_j, \overline{B}_j] = iT \text{ resp.}) \quad j = 1, \dots, n \,.$$

Define the subbundle $\mathcal{L}$ as span$\{B_1, \dots, B_n\}$. Then H_n is a *CR* manifold, of *CR* dimension n and codimension 1.

The dual bases of 1-forms are $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n, \theta\}$ and, for the complexfied bundle

$$\{\beta_1, \dots, \beta_n, \overline{\beta}_1, \dots, \overline{\beta}_n, \theta\}$$

resp., where $\beta_j = \frac{1}{\sqrt{2}}(dx_j - i dy_j)$, $\overline{\beta}_j = \frac{1}{\sqrt{2}}(dx_j + i dy_j)$, and $\theta = dt - \frac{1}{2}\sum_{j=1}^n (x_j dy_j - y_j dx_j)$. Then, a $(0, q)$-form ϕ on H_n can be written as

$$\phi = \sum_{|I|=q} \phi_I \overline{\beta}^I$$

and $\overline{\partial}_b$ acts on ϕ as

$$\overline{\partial}_b \phi = \sum_{|I|=q} \sum_{j=1}^n \overline{B}_j \phi_I \overline{\beta}_j \wedge \overline{\beta}^I = \sum_{|J|=q+1} \Big(\sum_{j,I} \varepsilon^{jI}_J \overline{B}_j \phi_I \Big) \overline{\beta}^J \,,$$

where ε^{jI}_J equals 0 if $\{j\} \cup I \neq J$ as sets, and equals the sign of the permutation $\begin{pmatrix} jI \\ J \end{pmatrix}$ if $\{j\} \cup I = J$ as sets.

A simple calculation shows that the adjoint $\overline{\partial}_b^*$ has the expression

$$\begin{aligned} \overline{\partial}_b^* \phi = \overline{\partial}_b^* \Big(\sum_{|I|=q} \phi_I \overline{\beta}^I \Big) &= - \sum_{j=1}^n \sum_{|I|=q} B_j \phi_I \overline{\beta}_j \lrcorner \overline{\beta}^I \\ &= - \sum_{|K|=q-1} \Big(\sum_{j,I} \varepsilon^{jK}_I B_j \phi_I \Big) \overline{\beta}^K \,. \end{aligned}$$

Here $\lrcorner$ denotes the contraction operator of forms.

The Kohn Laplacian on H_n can now be calculated.

Proposition 1.6. *Let* $\phi = \sum_{|I|=q} \phi_I \overline{\beta}^I$ *be a smooth* $(0,q)$*-form. Then*

$$\Box_b \phi = \sum_{|I|=q} \big(\mathcal{L}_0 + i(\tfrac{n}{2} - q)\big) \phi_I \overline{\beta}^I ,$$

where $\mathcal{L}_0$ *is the scalar (left-invariant) differential operator, called the* sublaplacian[2]

$$\mathcal{L}_0 = -\tfrac{1}{2} \sum_{j=1}^n X_j^2 + Y_j^2 = -\tfrac{1}{2} \sum_{j=1}^n B_j \overline{B}_j + \overline{B}_j B_j .$$

It is worth noticing that, on the Heisenberg group H_n, the Kohn Laplacian $\Box_b$ acting on $(0,q)$-forms is diagonal. Thus, it can be analyzed by studying a single scalar operator, the sublaplacian $\mathcal{L}_0$. This phenomenon does not appear on more general *CR* manifolds, as we will see in other examples. However, the main term of $\Box_b$ will remain diagonal in general.

Proof. . It suffices to consider the case of a simple $(0,q)$-form $\phi = f\overline{\beta}^I$. We first compute

$$\begin{aligned} \overline{\partial}_b^* \overline{\partial}_b f \overline{\beta}^I &= \overline{\partial}_b^* \Big(\sum_j \overline{B}_j f \overline{\beta}_j \wedge \overline{\beta}^I \Big) \\ &= -\sum_k \sum_j B_k \overline{B}_j f \overline{\beta}_k \lrcorner \big(\overline{\beta}_j \wedge \overline{\beta}^I \big) \end{aligned}$$

and

$$\begin{aligned} \overline{\partial}_b \overline{\partial}_b f^* \overline{\beta}^I &= -\overline{\partial}_b \Big(\sum_k B_k f \overline{\beta}_\lrcorner \overline{\beta}^I \Big) \\ &= -\sum_j \sum_k \overline{B}_j B_k f \overline{\beta}_j \wedge \big(\overline{\beta}_k \lrcorner \overline{\beta}^I \big) . \end{aligned}$$

Next, notice that for $j \neq k$

$$\overline{\beta}_k \big(\lrcorner \overline{\beta}_j \wedge \overline{\beta}^I \big) = -\overline{\beta}_j \wedge \big(\overline{\beta}_k \lrcorner \overline{\beta}^I \big) ,$$

and that, for $j = k$

$$\overline{\beta}_j \big(\lrcorner \overline{\beta}_j \wedge \overline{\beta}^I \big) = \begin{cases} \beta^I & \text{if } j \notin I \\ 0 & \text{if } j \in I \end{cases}$$

and

$$\overline{\beta}_j \wedge \big(\overline{\beta}_j \lrcorner \overline{\beta}^I \big) = \begin{cases} \beta^I & \text{if } j \in I \\ 0 & \text{if } j \notin I \end{cases} .$$

[2] Although we have already used the symbol $\mathcal{L}$ to denote the integrable subbundle in the definition of a *CR* manifold, it is a classical notation to denote this differential operator by $\mathcal{L}_0$ (or even $\mathcal{L}$). We adopt the former notation $\mathcal{L}_0$ to indicate the sublaplacian. This should cause no confusion.

Therefore, recalling that $[B_k, \overline{B}_j] = 0$ if $j \neq k$, and that $[B_j, \overline{B}_j] = iT$

$$\begin{aligned}
\Box_b \phi &= (\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b)\phi \\
&= -\sum_{j\neq k} (B_k \overline{B}_j - \overline{B}_j B_k) f \overline{\beta}_k \big(\lrcorner \overline{\beta}_j \wedge \overline{\beta}^I\big) - \Big(\sum_{j\in I} \overline{B}_j B_j + \sum_{j\notin I} B_j \overline{B}_j\Big) f \overline{\beta}^I \\
&= -\frac{1}{2}\sum_j \big(\overline{B}_j B_j + B_j \overline{B}_j\big) f \overline{\beta}^I + \frac{1}{2}\sum_{j\in I} \big(\overline{B}_j B_j - B_j \overline{B}_j\big) f \overline{\beta}^I - \frac{1}{2}\sum_{j\notin I} \big(B_j \overline{B}_j - \overline{B}_j B_j\big) f \overline{\beta}^I \\
&= (\mathcal{L}_0 f)\overline{\beta}^I - \frac{1}{2}\sum_{j\in I} i(Tf)\overline{\beta}^I + \frac{1}{2}\sum_{j\notin I} i(Tf)\overline{\beta}^I \\
&= \big(\mathcal{L}_0 + i(\frac{n}{2} - q)\big) f \overline{\beta}^I . \qquad \Box
\end{aligned}$$

We now describe an alternative way to introduce *CR* manifolds, as an embedded manifold of some complex space $\mathbb{C}^{n+k}$.

We denote by J the complex structure on $T^{\mathbb{C}}M$. Given a point $z \in M$ we call the *complex tangent space* at z the vector space

$$H_z(M) = T_z(M) \cap JT_z(M) .$$

Since $J^2 = -I$, the subspace H_z is even dimensional. Notice that

$$H_z(M) = T_z^{1,0}(\mathbb{C}^{n+k}) \cap T^{\mathbb{C}}(\mathbb{C}^{n+k}) .$$

We fix an inner product in $T_z(M)$, say the euclidean inner product. We define the *totally real* tangent space at z to be the orthogonal complement of H_z in $T_z(M)$.

Definition 1.7. A submanifold M of $\mathbb{C}^{n+k}$ is called an embedded *CR* manifold if $\dim_{\mathbb{R}} H_z(M)$ is independed of $z \in M$.

Example 1.8. For instance, if $H_z = T_z(M)$ for all z, then M is a complex manifold. On the other hand, if $H_z = \{0\}$, then M is called *totally real*.

In general, in order to avoid trivialities, we will rule out these two cases. That is, we will assume that $0 < \dim_{\mathbb{R}} H_z(M) = n$ and $k > 0$.

Of embedded *CR* manifolds it is useful to have a description in local coordinates.

Lemma 1.9. *Let M be a CR submanifold in $\mathbb{C}^{n+k}$, of codimension k. Then, the following are equivalent:*

(1) $\dim_{\mathbb{R}} H_z(M) = 2n$ *for all* $z \in M$*;*
(2) $T_z(\mathbb{C}^n) = T_z(M) \oplus J\big(N_z(M)\big)$ *for all* $z \in M$*;*
(3) *for any local defining function system for M $\{\rho_1, \dots, \rho_k\}$, we have*

$$\overline{\partial}\rho_1(z) \wedge \cdots \wedge \overline{\partial}\rho_k(z) \neq 0 .$$

Such a submanifold M in $\mathbb{C}^{n+k}$ will be said to *generic*. In particular, locally, on an open set U, we can represent M as

$$M \cap U = \{z \in U : \rho_1(z) = \cdots = \rho_k(z) = 0\}$$

and for $z \in M \cap U$

$$\overline{\partial}\rho_1(z) \wedge \cdots \wedge \overline{\partial}\rho_k(z) \neq 0\,.$$

In this case we can take the subbundle $T^{1,0}(\mathbb{C}^{n+k}) \cap T^{\mathbb{C}}M$ as the subbundle $\mathcal{L}$ in the definition of *CR* manifold. Here and in what follows, we denote by $T^{1,0}(\mathbb{C}^N)$ the subbundle of the holomorphic vector fields in $\mathbb{C}^N$.

We remark that the Heisenberg group can be also realized as an embedded *CR* manifold. In fact, if we set

$$\rho(z,\zeta) = \operatorname{Im}\zeta - |z|^2$$

for $(z,\zeta) \in \mathbb{C}^n \times \mathbb{C}$, then

$$H_n = \{(z,\zeta) \in \mathbb{C}^n \times \mathbb{C} : \ \rho(z,\zeta) = 0\}\,.$$

1.10. Hypersurfaces in $\mathbb{C}^{n+1}$. The next example of *CR* manifold that we encounter is certainly among the most typical and important ones.

Let $\mathcal{D}$ be a smooth domain in $\mathbb{C}^{n+1}$. This means that there exists a neighborhood $\mathcal{U}$ of the boundary $\partial\mathcal{D}$ and a smooth function $\rho : \mathcal{U} \to \mathbb{R}$ such that

$$\mathcal{D} = \{z \in \mathcal{U} : \ \rho(z) < 0\}$$

and $\nabla\rho(z) \neq 0$ on the set $\{\rho(z) = 0\} = \partial\mathcal{D}$. It is a simple matter to show that such a function ρ can be extended to all of $\mathbb{C}^{n+1}$ in such a way that $\mathcal{D} = \{z \in \mathbb{C}^{n+1} : \ \rho(z) < 0\}$ When globally defined, the function ρ is called a *defining function* for the domain $\mathcal{D}$ (in contrast with a *local* defining function).

Lemma 1.11. *The boundary of a smooth domain $\mathcal{D}$ as above is a smooth real hypersurface in $\mathbb{C}^{n+1}$ whose tangent space is*

$$T(\partial\mathcal{D}) = \{(\xi,\eta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \ d\rho(z)(\xi,\eta) = \textstyle\sum_j (\partial_{x_j}\rho(z)\xi_j + \partial_{y_j}\rho(z)\eta_j) = 0\}\,.$$

The complex tangent space *at a point $z \in \partial\mathcal{D}$ is the subset of the tangent space given by*

$$H^{1,0}_z = T_{\mathbb{C},z}(\partial\mathcal{D}) = \{\zeta = \xi + i\eta \in \mathbb{C}^{n+1} : \ \rho(z)(\zeta) = \textstyle\sum_j \partial_{z_j}\rho(z)\zeta_j = 0\}\,.$$

The subset $\cup_{z\in\partial\mathcal{D}} T_{\mathbb{C},z}(\partial\mathcal{D})$ is a subbundle of (complex)dimension n that makes $\partial\mathcal{D}$ into a CR manifold of CR dimension n and codimension 1.

Proof. The first part of the statement follows from the fact that, if $\tilde\rho$ is another defining function, then there exists a smooth positive function h defined on a neighborhood $\mathcal{U}$ of $\partial\mathcal{D}$ such that $\tilde\rho = h \cdot \rho$ on $\mathcal{U}$.

The second part follows from the fact that the set $T_{\mathbb{C},z}(\partial\mathcal{D})$ defined in the statement is precisely the subset of the tangent space at z constitued of the tangent vectors $\zeta = \xi + i\eta = (\xi,\eta)$ such that $i\zeta = (-\eta,\xi)$ is also tangent.

Finally, we see that $T_{\mathbb{C}}$ is a subbundle of $T^{\mathbb{C}}(\partial\mathcal{D})$ that forms the *CR* structure on $\partial\mathcal{D}$. □

We now describe the $\overline{\partial}_b$-complex on the *CR* manifold $M = \partial\mathcal{D}$. Since M is an embedded manifold, in alternative to the intrisic deinition, valid on any abstract *CR* manifold, we can describe the $\overline{\partial}_b$-complex extrinsically, using the ambient space complex strucure. In general, on the complex manifold $\mathbb{C}^{n+1}$ we have the (Doulbeaut) complex $\overline{\partial}$. For instance, for a smooth function f, $\overline{\partial} f$ is the $(0,1)$-form $\sum_j \overline{\partial}_{z_j} f$, and analogously for ∂. Then, the exterior differentation operator d can be written as $d = \partial + \overline{\partial}$.

Let $\mathcal{U}$ be an open neighborhood of M. Let $I^{p,q}$ be the ideal of $\Lambda^{p,q}(\mathbb{C}^{n+1})$ generated by ρ and $\overline{\partial}\rho$; that is any element of $I^{p,q}$ can be written as

$$\rho\omega_1 + \overline{\partial}\rho \wedge \omega_2$$

where ω_1 is a (p,q)-form and ω_2 a $(p,q-1)$-form. Denote by $\Lambda^{p,q}(\mathbb{C}^{n+1})_{|M}$ and $I^{p,q}{}_{|M}$ resp. the restrictions of $\Lambda^{p,q}(\mathbb{C}^{n+1})$ and $I^{p,q}$ to M, resp. Then we set $\Lambda^{p,q}(M)$ to be the orthogonal complement of $I^{p,q}{}_{|M}$ in $\Lambda^{p,q}(\mathbb{C}^{n+1})_{|M}$. Let

$$\Pi : \Lambda^{p,q}(\mathbb{C}^{n+1}) \to \Lambda^{p,q}(M)$$

be the mapping obtain by first restrict a (p,q)-form to M and then projecting it to $\Lambda^{p,q}(M)$. It should be noted that $\Lambda^{p,q}(M)$ is not intrisic to M, that is it is a subspace of the space of complex forms on M, since $\overline{\partial}\rho$ is not orthogonal to the cotangent bundle of M.

For a smooth form in $\Lambda^{p,q}(M)$ we define the tangential Cauchy Riemann operator

$$\overline{\partial}_b : \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)$$

as

$$\overline{\partial}_b(\phi) = \Pi\overline{\partial}\tilde{\phi}\,,$$

where $\tilde{\phi}$ is any smooth (p,q)-form such that $\Pi(\tilde{\phi}) = \phi$. If $\hat{\phi}$ is another such form, then

$$\tilde{\phi} - \hat{\phi} = \rho\omega_1 + \overline{\partial} \wedge \rho\omega_2$$

for some smooth forms ω_1, ω_2. Then,

$$\overline{\partial}(\tilde{\phi} - \hat{\phi}) = \overline{\partial}(\rho\omega_1 + \overline{\partial} \wedge \rho\omega_2) = \overline{\partial} \wedge \rho\omega_1 + \rho\overline{\partial}\omega_1 - \overline{\partial}\rho \wedge \overline{\partial}\omega_2$$

so that

$$\Pi\overline{\partial}(\tilde{\phi} - \hat{\phi}) = 0\,.$$

Hence, the definition is independent of the extension $\tilde{\phi}$. Since $\overline{\partial}^2 = 0$ it follows that also $\overline{\partial}_b^2 = 0$.

This approach gives rise to a complex that is isomorphic to the one defined previously, intrinsically (for a proof, see [Bo], Sect. 8.3). Thus, on an embedded *CR* manifold, the two definitions are equivalent[3].

We now go back to the operators that we are studying. It is worth mentioning that the Kohn Laplacian $\Box_b$ and the Cauchy-Riemann operator $\overline{\partial}_b$ are not scalar-valued in general. In the top-degree case, and in some particular instances such as the Heisenberg group, we can reduce ourselves to studying scalar-valued operators, but in general these are vector-valued operators, begin differential operators between vector bundles.

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[3]We actullay have described the extrinsic approach only in the case of hypersurfaces, that is in case of codimension 1. We will see that this construction generilizes naturally to the higher codimension cases.

Let M be an embedded *CR* manifold of *CR* dimension n and codimension k. We adopt the notation of Section 1. Locally, a smooth (p, q)-form ϕ can be written as

$$\phi \sum_{|I|=p,\,|J|=q} \phi_{I,J}\omega^I \wedge \overline{\omega}^J \,.$$

Then

$$\overline{\partial}_b\phi = \sum_{I,J}\sum_{j=1}^{n} \overline{L}_j\phi_{I,J}\overline{\omega}_j\omega^I \wedge \overline{\omega}^J + \text{ 0-order terms}\,.$$

Next, a simple integration by parts yield

$$\overline{\partial}_b^*\phi = (-1)^{p-1}\sum_{I,J}\sum_{j=1}^{n} L_j\phi_{I,jK}\overline{\omega}_j\omega^I \wedge \overline{\omega}^K + \text{ 0-order terms}\,.$$

A calculation on the same lines as in the case of the Heisenberg group, we see that

$$\Box_b\phi = \sum_{j=1}^{n}(L_j\overline{L}_j + \overline{L}_jL_j)\phi_I\overline{\omega}^I + \text{lower order terms}\,.$$

2. The Levi form of a *CR* manifold

We now introduce a geometrical invariant on any *CR* manifold, the so-called *Levi form*. This is a quadratic form acting on 2-tensors of elements of the subbundle that gives rise to the *CR* structure and it turns out to be of fundamental importance in the analysis of the Kohn Laplacian and $\overline{\partial}_b$-complex.

2.1. Let M be a *CR* manifold of *CR* dimension n and codimension k. Let $T^{1,0}M$ be the subbundle of $T^{\mathbb{C}}M$ that defines the *CR* structure (previously defined by $\mathcal{L}$) and let $T^{0,1}M$ be its complex conjugate. Recall the decomposition (1)

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M \oplus N(M)\,.$$

Let $N(M)^*$ be the dual bundle of $N(M)$. Then, if $\tau \in N(M)^*$ then τ annihillates $T^{1,0}M \oplus T^{0,1}M$ and it is called the *characteristic bundle.* Notice in fact that the operator $\Box_b$ is not elliptic and that its characteristic variety is given by

$$\Sigma_{\Box_b} = N^*(M)\,.$$

Definition 2.2. Let $z \in M$. The Levi form is defined to be the hermitian form Φ on $T^{1,0}M$ taking values in $N(M)$ given by

$$\Phi(L_1, L_2) = i\Theta([L_1, \overline{L}_2])\,,$$

for $L_1, L_2 \in T^{1,0}M$ and where Θ is the projection of $T^{\mathbb{C}}M$ onto $N(M)$. The Levi form *in the direction* τ is the quadratic form

$$\langle\Phi(L_1, L_2), \tau\rangle = i\langle[L_1, \overline{L}_2]), \tau\rangle\,,$$

again for $L_1, L_2 \in T^{1,0}M$.

Example 2.3. When M has codimension 1, then $N(M)^*$ is also 1-dimensional and there exist only two oriented directions τ and $-\tau$. Moreover, if in particular $M = \partial\mathcal{D}$ is the boundary of smooth domain in $\mathbb{C}^{n+1}$ with defining function ρ, then the Levi form can be described as follows. Let $a, b \in T^{\mathbb{C}}(\mathbb{C}^{n+1}) \cap T^{1,0}(\partial\mathcal{D})$, then

$$\Phi(a,b) = \Big(\sum_{j,k=1}^{n+1} \frac{\partial^2 \rho(z)}{\partial z_j \partial \overline{z}_k} a_j \bar{b}_k \Big) J(\nabla\rho)\,.$$

It is a simple exercise that this definition is (essentially) independent of the choiche of the defining function for $\mathcal{D}$. Indeed, if $\tilde{\rho} = h \cdot \rho$ is another defining function for $\mathcal{D}$, then $\Phi_{\tilde{\rho}} = h\Phi_\rho$, where $\Phi_{\tilde{\rho}}$ and Φ_ρ denote the expressions for the Levi form obtained using the definying function $\tilde{\rho}$ and ρ resp. Thus, the Levi form turns out to be defined modulo the multiplication of a positive smooth function on M.

Moreover, M can be thought as the boundary of the interior of ${}^cM$. This ammounts to replace ρ with $-\rho$, thus to replace the direction τ with $-\tau$.

The next definition introduces a concept of fundamental importance in the analysis of holomorphic functions in domain in complex space.

Definition 2.4. A real hypersurface $M \subseteq \mathbb{C}^{n+1}$ is called pseudoconvex if the Levi form is either positive semidefinite or negative semidefinite at every point on M. It is said to be strictly pseudoconvex if it is either positive definite or negative definite at every point.

The previous example can be generalized to the case of *CR* manifolds of higher codimensions.

Example 2.5. Let $M \subseteq \mathbb{C}^{n+k}$ be an embedded generic *CR* manifold of codimension $k > 1$. There exist k smooth real-valued functions $\rho_1, \dots, \rho_k$ such that

$$M = \{ z \in \mathbb{C}^{n+k} : \ \rho_1(z) = \cdots = \rho_k(z) = 0 \}$$

and $\overline{\partial}\rho_1(z) \wedge \cdots \wedge \overline{\partial}\rho_k(z) \neq 0$. Without loss of generality, we may assume that $\nabla\rho_1, \dots, \nabla\rho_k$ form an orthonormal basis for $N(M)$ at every point of M.

Proposition 2.6. *With the above hypotheses, the Levi form Φ of M is given by*

$$\Phi(W,W') = \sum_{\ell=1}^{k} \Big(\sum_{i,j=1}^{n+k} \frac{\partial^2 \rho_\ell(z)}{\partial z_i \partial \overline{z}_j} w_i \overline{w}'_j \Big) J(\nabla\rho_\ell)(z)\,.$$

Proof. By definition $\Phi(W,W') = i\Theta([W, \overline{W}'])$. Since $\{\nabla\rho_1, \dots, \nabla\rho_k\}$ is an orthonormal system it is a basis for orthogonal complement of $T^{\mathbb{C}}M$ in $T^{\mathbb{C}}(\mathbb{C}^{n+k})$. Let J denote the complex structure in $T^{\mathbb{C}}(\mathbb{C}^{n+k})$. Obviously, $T^{1,0} \oplus T^{0,1}$ is J-invariant. Then $\Theta([L_1, \overline{L}_2]) = \tilde{\Theta}(J[L_1, \overline{L}_2])$, where $\tilde{\Theta}$ is the orthogonal projection onto $\tilde{N}(M)$ at every point of M, the projection Θ is given by

$$\Theta(V) = \sum_{\ell=1}^{k} \langle d\rho_\ell, V \rangle \nabla\rho_\ell\,.$$

3. The tangential Cauchy-Riemann equations

In this section we present some applications of the differential operators $\overline{\partial}_b$ and $\Box_b$. Our goal is just to illustrate some problems that lead to studying the above operators. We will not be able to present most of the proofs of the results we present, since the techniques involved would require a background that goes beyond the scope of these lectures. However, we hope this part would serve as a motivation and a suggestion for further reading.

Let $\mathcal{D}$ be a smoothly bounded domain in $\mathbb{C}^{n+1}$ with boundary $M = \partial\mathcal{D}$. Let ω be a smooth (p, q)-form on M. We wish to answer the following two questions about extensions of ω to the ambient space $\mathbb{C}^{n+1}$:

(1) Does there exist a smooth form $\tilde{\omega}$ in $\mathbb{C}^{n+1}$ such that $\Pi\tilde{\omega} = \omega$ on M?
(2) Doest there exist a smooth form ϕ in $\mathbb{C}^{n+1}$ such that $\phi_{|M} = \omega$ and $\overline{\partial}\phi = 0$ on $\mathcal{D}$?

Recall that the holomorphic degree p of the form is irrelevant, so we will put $p = 0$. Recall also that $0 \leq q \leq n$.

Consider question (2) first. If ϕ is a $\overline{\partial}$-closed form, then $\overline{\partial}_b\phi_{|M}\Pi\overline{\partial} = 0$. Then $\overline{\partial}_b{\equiv}0$ is a necessary condition in order for (2) to be satisfied. We now show that if M is compact (i.e. if the domain $\mathcal{D}$ is bounded) and its Levi form is semidefinite in a given direction, then (2) is also sufficient in the case $q = 0$ (i.e. when ω is a function). We say that a function on M is a *CR* function if $\overline{\partial}_b f = 0$. In order to state the condition on the Levi form, we need the following definition.

Definition 3.1. A smooth domain in $\mathbb{C}^N$

$$\mathcal{D} = \{z \in \mathbb{C}^N : \ \rho(z) < 0\}$$

(or more generally $\mathcal{C}^2$-smoothness of the boundary suffices) is said to be (Levi) pseudoconvex if the quadradic form

$$\sum_{j,k=1}^{N} \frac{\partial^2\rho(z)}{\partial z_j \overline{\partial} z_k}\zeta_j\overline{\zeta}_k \geq 0$$

for all $\zeta \in T^{1,0}(\partial\mathcal{D})$.

Notice that the above quadratic form (which is called the *Levi form of the domain* $\mathcal{D}$ corrispond to the Levi form of the *CR* manifold $\partial\mathcal{D}$ in the direction $J\overline{\partial}\rho$. Recall also that $T^{1,0}(\partial\mathcal{D}) = \{\zeta \in \mathbb{C}^N : \sum_j \zeta_j\partial_{z_j}\rho(z) = 0,\, z \in \partial\mathcal{D}\}$.

Theorem 3.2. *Let $\mathcal{D}$ be a smooth domain in $\mathbb{C}^{n+1}$. Let ω be a smooth (p, q)-form on $M = \partial\mathcal{D}$, $0 \leq p \leq n$, $1 \leq q \leq n$. Then, there exists a form $\tilde{\omega} \in \mathcal{C}^{\infty}_{(p,q)}(\overline{\mathcal{D}})$ such that $\Pi\tilde{\omega} = \omega$ and $\overline{\partial}\tilde{\omega} = 0$ in $\mathcal{D}$ if and only if*

$$\int_M \omega \wedge \psi = 0 \qquad \textit{for every } \psi \in \mathcal{C}^{\infty}_{(n-p,n-q-1)}(\overline{\mathcal{D}}) \cap Ker(\overline{\partial})\,.$$

Moreover, if $1 \leq q < n$, the above condition is equivalent to

$$\overline{\partial}_b\omega = 0 \quad \textit{on } M.$$

Proof. The proof is based on an appropriate choice of the coordinates in $\mathbb{C}^{n+1}$. □

Since $\overline{\partial}_b$ and $\overline{\partial}_b^*$ form complexes, we see that

$$\overline{\partial}_b \Box_b = \overline{\partial}_b(\overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b) = \overline{\partial}_b\overline{\partial}_b^*\overline{\partial}_b$$
$$= (\overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b)\overline{\partial}_b = \Box_b\overline{\partial}_b \,.$$

Hence, $\overline{\partial}_b$ and $\Box_b$ commute. The same is true for $\overline{\partial}_b^*$ and $\Box_b$.

Now, suppose $\Box_{b,q}$ is invertible. Let $\mathcal{G}_q$ be its inverse. Let f be a $(0,q)$-form such that $\overline{\partial}_b f = 0$. Then set

$$u = \mathcal{G}_q\overline{\partial}_b^* f \,.$$

Then u satisfies

$$\overline{\partial}_b u = \overline{\partial}_b\mathcal{G}_q\overline{\partial}_b^* f = \mathcal{G}_q\overline{\partial}_b\overline{\partial}_b^* f$$
$$= \mathcal{G}_q(\overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b)f = \mathcal{G}_q\Box_{b,q} f$$
$$= f \,.$$

Therefore, from the solvability of $\Box_{b,q}$ we obtain the solvabilty of the Cauchy-Riemann equation for a given $\overline{\partial}_b$-closed $(0,q+1)$-form f.

4. Local solvability, hypoellipticity, subellipticity

We recall some well-known facts and definitions about scalar differential operators with smooth coefficients in some open set Ω in the real space $\mathbb{R}^N$.

Let $P = \sum_{|\alpha|\le m} a_\alpha(x)(i^{-1}\partial)_x^\alpha$ be a differential operator with smooth coeffients in a given open set Ω. We say that P is *locally solvable* at a point $x_0 \in \Omega$ if there exists an open neighborhood U of x_0 such that for every $f \in \mathcal{C}_0^\infty(\Omega)$ there exists a distribution $u \in \mathcal{D}'(\Omega)$ such that

$$Pu = f \qquad \text{in } U \,.$$

Moreover, we say that P is *hypoelliptic* in Ω if $Pu \in \mathcal{C}^\infty(\Omega')$, with $\Omega' \subseteq \Omega$ implies that also $u \in \mathcal{C}^\infty(\Omega')$.

In the previous section we have mentioned that the Lewy operator is not locally solvable at the origin in $\mathbb{R}^3$. It came as big surprise when H. Lewy showed his example of a (so simple) partial differential operator which is not locally solvable. Considerable research was made after this discovery. We will not even make any attempt to describe this fertile area of research and the extremely vast bibliography. We mention though that, in a series of fundamental results, Eskin, Niremberg/Trevés, Beals/Fefferman characterized the local solvability for operators of *principal type*.

Given the operator P as above, we call *principal symbol* of P the smooth function $p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ defined on the cotangent bundle $T^*(\Omega)$. The *characteristic variety* of P is the set

$$\Sigma_P = \{(x,\xi) \in T^*(\Omega) : p_m(x\xi) = 0\} \,.$$

The operator P is called of *principal type* if $(x_0,\xi_0) \in \Sigma_P$ implies that $dp_m(x_0,\xi_0) \neq 0$.

An operator that is not of pricipal type is called of *multiple characteristic.* In this generality we have the following criterion of Hörmander.

Theorem 4.1. *Let P be as above. Then P is locally solvable at $x_0 \in \Omega$ if and only if there exists an open neighborhood V of x_0 and a positive integer k such that*

$$\|v\|_{H^{-k}} \leq C\|{}^tPv\|_{H^k}$$

for every $v \in \mathcal{C}_0^\infty(U)$. (Here $\|\cdot\|_{H^s}$ denotes the norm in the Sobolev space, $s \in \mathbb{R}$.)

Proof. See ???. □

We conclude this section by recalling Hörmander theorem on hypoellipticity of sum of squares.

Let $\{X_1, \dots, X_n\}$ be smooth real vector fields defined on an open set Ω in $\mathbb{R}^N$. Define $\mathcal{G}_1$ to the collection $\{X_1, \dots, X_n\}$ and, inductively, $\mathcal{G}_j$ to be the collection of the commutators of the form $[X, Y]$ with $X \in \mathcal{G}_1$ and $Y \in \mathcal{G}_{j-1}$.

We say that $\{X_1, \dots, X_n\}$ satisfies the *Hörmander condition* in Ω if there exists an integer k such that the vector fields $\mathcal{G}_k$ span the tangent space of $\mathbb{R}^N$ at every point in Ω. In other words if$\{X_1, \dots, X_n\}$ and their commutators up to order k span the tangent space of $\mathbb{R}^N$ at every point in Ω.

Theorem 4.2. (Hörmander) *If P is an differential operator of the form*

$$P = \sum_{j=1}^{n} X_j^2 + X_0 + b(x)\,,$$

where the vector fields X_j are real, $j = 0, \dots, n$, B is a smooth complex-valued functions and $\{X_1, \dots, X_n\}$ satisfies the Hörmander condition in Ω, then P is hypoelliptic in Ω.

More precisely, there exists $\varepsilon > 0$ such that, given any compact set $K \subset \Omega$ and $s \geq 0$, there exists a constant $C(K, s) = C > 0$ such that for all $u \in \mathcal{C}_0^\infty(K)$ we have the estimate

$$\|u\|_{H^{s+\varepsilon}} + \leq C\big(\|Pu\|_{H^s} + \|u\|\big)\,.$$

An estimate like the one above is called a *subelliptic estimate.* For a proof we refer, among the many available, to the original paper by Hörmander [Hö1] or the excellent tractise [T1].

Main goal of these lectures if to describe some results on the solvability and hypolellipticity of $\Box_b$ in some classes of *CR* manifolds. The conditions characterizing these classes stem from the signature of the scalar components of the Levi form of M. We begin by the simplest case, the Heisenberg group. In this case, the codimension is 1, and the explicit coordinates and the group structure allow one to obtain rather explicit formulas for the inverse (and partial inverse) of the operators involved.

5. The operators of Folland-Stein on the Heisenberg group

On the Heisenberg group H_n there a 1-parameter family of non-isotropic dilations that are group automorphisms. These dilations are

$$D_r(z, t) = (rz, r^2t)\,,$$

for $r > 0$. We recall that also the rotations

$$U(z,t) = (Uz, t)$$

for U in the unitary group of $\mathbb{C}^n$ are automorphisms of H_n.

In Section 1 we saw that for a smooth $(0,q)$-form $\phi = \sum_{|J|=q} \phi_J \overline{\beta}^J$ we have

$$\square_b(\phi) = \sum_J \Big(\sum_{j=1}^n \big(\mathcal{L}_0 + +i(\tfrac{n}{2} - q)T\big)\phi_J \Big) \overline{\beta}^J .$$

Thus, it becomes natural to study the second order left-invariant differential operators

$$\mathcal{L}_\alpha = \mathcal{L}_0 + i\frac{\alpha}{2}T = -\frac{1}{2}\sum_{j=1}^n (B_j\overline{B}_j + \overline{B}_jB_j) + i\frac{\alpha}{2}T$$

for $\alpha \in \mathbb{C}$.

We say that a differential operator $\mathcal{P}$ on the Heisenberg group is homogenous of degree d if

$$\mathcal{P}(f(D_r)(z,t)) = r^d \mathcal{P}(f)(D_r(z,t)) .$$

It is possible to show that on H_n the only differential operators that are left-invariant, homogeneous of degree 2 and invariant under rotations are the operators $\mathcal{L}_\alpha$ above. (A proof of this fact requires the notion of *group Fourier transform* and it is postponed to Section ??.)

We begin our analysis of the $\mathcal{L}_\alpha$. Notice that we cannot immediatey apply Hörmander's theorem since the operator of order 1 does not have real coefficients. The first step is to construct a fundamental solution for $\mathcal{L}_\alpha$, for the *admissible values* of α. In the case of the sublaplacian, that is when $\alpha = 0$, the fundamental solution was determined by Folland. Later, this calculation was extended to the considerably more complicated case of non-zero α by Folland and Stein [FS1].

Proposition 5.1. (Folland-Stein) *For $\alpha \in \mathbb{C}$ define the locally integrable function on H_n*

$$E_\alpha(z,t) = \big(|z|^2 + it\big)^{-(n-\alpha)/2}\big(|z|^2 - it\big)^{-(n+\alpha)/2} .$$

Let

$$\gamma_\alpha = 2^{2-2n}\pi^{n+1}\Gamma\big((n+\alpha)/2\big)^{-1}\Gamma\big((n-\alpha)/2\big)^{-1} .$$

Then, in the sense of distributions,

$$\mathcal{L}_\alpha E_\alpha = \gamma_\alpha \delta_0 ,$$

where δ_0 denotes the Dirac delta at the origin.

Proof. The proof of this fact is somewhat straightforward. One consider the regularized version of the E_α:

$$E_{\alpha,\varepsilon}(z,t) = \big(|z|^2 + \varepsilon^2 + it\big)^{-(n-\alpha)/2}\big(|z|^2 + \varepsilon^2 - it\big)^{-(n+\alpha)/2} .$$

Now $E_{\alpha,\varepsilon} \in \mathcal{C}^\infty$ and by computing $\mathcal{L}_\alpha E_{\alpha,\varepsilon}$ one obtains that

$$\mathcal{L}_\alpha E_{\alpha,\varepsilon}(z,t) = F_{\alpha,\varepsilon}(z,t)$$

where

$$F_{\alpha,\varepsilon}(z,t) = \varepsilon^2(n^2-\alpha^2)\big(|z|^2+\varepsilon^2+it\big)^{-(n+2-\alpha)/2}\big(|z|^2+\varepsilon^2-it\big)^{-(n+2+\alpha)/2}$$
$$= \varepsilon^{-2n-n}F_{\alpha,1}\big(D_{\varepsilon^{-1}}(z,t)\big).$$

Since ε^{-2n-n} is the Jacobian of the automorphism $D_{\varepsilon^{-1}}$, if f is any test function on H_n

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle f, F_{\alpha,\varepsilon}\rangle &= \lim_{\varepsilon\to 0}\int_{H_n} f(z,t)F_{\alpha,\varepsilon}(z,t)dV(z,t)\\ &= \lim_{\varepsilon\to 0}\varepsilon^{-2n-n}\int_{H_n} f(z,t)F_{\alpha,1}\big(D_{\varepsilon^{-1}}(z,t)\big)dV(z,t)\\ &= \lim_{\varepsilon\to 0}\int_{H_n} f(\varepsilon z,\varepsilon^2 t)F_{\alpha,1}(z,t)dV(z,t)\\ &= \left(\int_{H_n} F_{\alpha,1}(z,t)dV(z,t)\right)\langle f,\delta_0\rangle\,.\end{aligned}$$

Thus, it suffices to show that

$$\int_{H_n} F_{\alpha,1}(z,t)dV(z,t) = \gamma_\alpha$$

where γ_α is as in the statement. This is a clever but, at this stage for us, unenlighting calculation, (for which we refer to [FS1]). □

At this point we notice that the constant γ_α is non-zero if and only if

$$\alpha \neq (n \pm 2k) \qquad k = 0,1,2,\dots\,.$$

We call these values the *admissible values*.

Notice that when α is not purely imaginary, Hörmander theorem does not apply. It turns out instead that $\mathcal{L}_\alpha = \mathcal{L}_0 + i\frac{\alpha}{2}T$ is a family of operators with $\mathcal{L}_0$ hypoelliptic (since in this case Hörmander theorem does apply) that we will show are all hypoelliptic but for a discrete infinite subset of values of the parameter α.

We now are ready to characterize the local solvability and th hypoellipticity of the operators $\mathcal{L}_\alpha$. We remark that, due to the invariance by translation, local solvability at one point is equivalent to local solvability at any other point in H_n. We therefore restric our attention to the origin.

Theorem 5.2. *Let $\mathcal{L}_\alpha$ be as above. Then the following conditions are equivalent:*

(1) *$\mathcal{L}_\alpha$ is locally solvable at the origin (or at any other point);*
(2) *$\mathcal{L}_\alpha$ is hypoelliptic;*
(3) *α is an admissible value.*

Corollary 5.3. *Consider the Kohn Laplacian $\square_b = \square_{b,q}$ on H_n acting on $(0,q)$-forms. Then the following are equivalent:*

(1) *$\square_{b,q}$ is locally solvable;*
(2) *$\square_{b,q}$ is hypoelliptic;*
(3) *$q \neq 0, n$.*

Proof of the theorem. Suppose first that α is an admissible value. Then $\gamma_\alpha \neq 0$ and $\gamma_\alpha^{-1} E_\alpha$ is a fundamental solution for $\mathcal{L}_\alpha$; fact that of course implies that $\mathcal{L}_\alpha$ is locally solvable.

Moreveor, this fundamental solution is a distribution that coincides with a $\mathcal{C}^\infty$ away from the origin, and it is homogenous of degree $-2n$ (with respect to the automoriphic dilations D_r). Thus, $\gamma_\alpha^{-1} E_\alpha$ is a locally integrable function, smooth away from the origin. This easily implies that $\mathcal{L}_\alpha$ is hypoelliptic in this case. For, suppose that $\mathcal{L}_\alpha u = f$ is $\mathcal{C}^\infty$ in an open set U. By multiplying f by test function which is identically 1 on U, we may assume that F has compact support. Then $u = f * \gamma_\alpha^{-1} E_\alpha$ (is a well-defined distribution and) satisfies

$$\mathcal{L} u = f \qquad \text{on} \quad U$$

and is smooth on U.

Therefore, (3) implies (1) and (2).

Next, suppose that (2) holds. Then ${}^t\mathcal{L}_{-\alpha}$ is locally solvable, by Corollary **??**.

Finally, to finish the proof we show that if α is not admissible then $\mathcal{L}_\alpha$ is not locally solvable (notice that α is admissible if and only if $-\alpha$ is).

Notice that, if α is not admissible

$$\mathcal{L}_\alpha E_\alpha = 0\,.$$

We apply Corwin-Rothschild criterion. Let ψ be a Schwarzt function in $H_n(\equiv \mathbb{R}^{2n+1})$ such that

$$\int_{H_n} \psi(z,t) p(z,t)\, dV(z,t) = 0$$

for all polynomials p, where dV represents the Lebesgue measure on H_n. Now define

$$\phi = \psi * E_{-\alpha}\,.$$

It can be shown that ϕ is a Schwartz function on H_n (for details see [DPR] Theorem 3.3). Indeed, the convolution is well defined and gives a smooth function. To show that ϕ and all its derivatives decay faster than $(1+|(z,t)|)^{-N}$ for all N, we use the moment condition of ψ.

Hence, ϕ is a Schwartz function on H_n that satisfies

$${}^t\mathcal{L}_\alpha(\phi) = \mathcal{L}_{-\alpha}(\psi * E_{-\alpha}) = \psi * (\mathcal{L}_{-\alpha} E_{-\alpha}) = 0\,.$$

Hence, $\mathcal{L}_\alpha$ is not locally solvable. □

Proof of the corollary. Notice that this result in particular says that $\Box_{b,q}$ is invertible if and only if $0 < q < n$.

The proof follows immediately from the theorem. We only need to show that for $q = 0, \ldots, n$,

$$2\big(\frac{n}{2} - q\big) \qquad \text{is admissible if and only if} \quad q = 0, n\,. \qquad \Box$$

We now discuss the operators $\mathcal{L}_\alpha$ for α non-admissible. We saw that these operators are neither hypoelliptic nor locally solvable. Theorem 5.1 gives us a way to obtain a *relative* fundamental solution.

Theorem 5.4. *Let $\alpha = \pm(n+k)$, $k = 0, 1, \dots,$ be a non-admissible value. Then there exist a distribution F_α and an $L^2(H_n, dV)$-Hilbert space orthogonal projection S_α such that*

$$\mathcal{L}_\alpha F_\alpha = \delta_0 - S_\alpha\,.$$

The projection S_α is given by convolution with a locally integrable function K_α. When $\alpha = n$ i.e. when $q = 0$, $S_\alpha = S$ is called the Szegö projection.

Proof. Consider the equation

$$\mathcal{L}_\alpha E_\alpha = \gamma_\alpha \delta_0\,,$$

given by Proposition 5.1. This is valid in $\alpha \in \mathbb{C}$ and in fact it is analytic in α. Moreover, the function γ_α has simple zeros at the non-admissible values. Differentiating the above equality and evaluating at a non-admissible value α_0 we obtain

$$iTE_{\alpha_0} + \mathcal{L}_{\alpha_0} F_{\alpha_0} = \gamma'_{\alpha_0}\delta_0\,,$$

where γ'_{α_0} can be calculated using the explicit expression of γ_α given by Proposition 5.1. Moreover,

$$\begin{aligned} F_{\alpha_0} &= \frac{d}{d\alpha}\Big(\big(|z|^2 + it\big)^{-(n-\alpha)/2} \big(|z|^2 - it\big)^{-(n+\alpha)/2} \Big)_{|\alpha_0} \\ &= \frac{1}{2}\Big(\big(-\log(|z|^2 - it) + \log(|z|^2 + it)\big)\big(|z|^2 + it\big)^{-(n-\alpha)/2}\big(|z|^2 - it\big)^{-(n+\alpha)/2} \Big)_{|\alpha_0} \\ &= \frac{1}{2}\log\Big(\frac{|z|^2 + it}{|z|^2 - it}\Big)\big(|z|^2 + it\big)^{-(n-\alpha_0)/2}\big(|z|^2 - it\big)^{-(n+\alpha_0)/2}\,. \end{aligned}$$

Finally, we (would) have to show that convolution with iTE_{α_0} is an orthogonal L^2 projection. This fact, and the explicit expression for the Szegö projection $S_n = S$ can be found in [S], Chapter XIII. □

We remark that the L^2-kernel of $\Box_b$ is given by those $(0,q)$-forms ω such that

$$\overline{\partial}_b \omega = \overline{\partial}_b^* \omega = 0\,.$$

For, these forms are certainly in the kernel of $\Box_b$. On the other hand, if $\Box_b \omega = 0$, then

$$\begin{aligned} 0 &= \langle \Box_b \omega, \omega \rangle \\ &= \langle (\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b)\omega, \omega \rangle \\ &= \langle \overline{\partial}_b^* \omega, \overline{\partial}_b^* \omega \rangle + \langle \overline{\partial}_b \omega, \overline{\partial}_b \omega \rangle \\ &= \|\overline{\partial}_b^* \omega\|^2 + \|\overline{\partial}_b \omega\|^2\,. \end{aligned}$$

By the equality $\Box_{b,q} F = \delta_0 - S$ we see that S is exactly the Hilbert space projection onto $\ker \Box_{b,q}$. Therefore, when $q = 0$

$$\ker \Box_b = \ker \overline{\partial}_b \cap \ker \overline{\partial}_b^* = \ker \overline{\partial}_b$$

since $\overline{\partial}_b^*$ is identically zero on functions. Thus, S is the orthogonal projection onto the subspace of *CR* functions.

Corollary 5.5. *On H_1 the operator $\overline{B}$ is not locally solvable.*

Proof. Notice that on H_1, $\overline{\partial}_b f = (\overline{B} f)\beta$, so that $\overline{\partial}_b f = 0$ if and only if $\overline{B} f = 0/$

Now, since $\ker \overline{\partial}_b = \ker \Box_{b,q}$, we know that a distribution ϕ is in $f \in \ker \overline{B}$ if and only if $\Box_{b,q} f = 0$ with $q = 0$. We know that there exists Schwartz function in the kernel of $\Box_{b,q}$ in this case. Hence there are Schwartz functions in the kernel of $\overline{B}$. This shows that $\overline{B}$ is not locally solvable. □

Now we go back to the Kohn Laplacian $\Box_{b,q}$ and discuss the relation on the Cauchy-Riemann equations.

Suppose that $0 < q < n$, so that $\Box_{b,q}$ is invertible. Let $\mathcal{G}_q$ be the inverse operator of $\mathcal{L}_{n-2q}$. Let f be a $(0, q)$-form such that $\overline{\partial}_b f = 0$. Then set

$$u = \mathcal{G}_q \overline{\partial}_b^* f .$$

Then., using the fact that f is a *CR* form, we see that u satisfies

$$\begin{aligned} \overline{\partial}_b u &= \overline{\partial}_b \mathcal{G}_q \overline{\partial}_b^* f = \mathcal{G}_q \overline{\partial}_b \overline{\partial}_b^* f \\ &= \mathcal{G}_q (\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b) f = \mathcal{G}_q \Box_{b,q} f \\ &= f , \end{aligned}$$

i.e. u satisfies the Cauchy-Riemann equation $\overline{\partial}_b f = u$.

6. The case of quadratic *CR* manifolds

We now consider a class of higher codimension *CR* manifolds that can be viewed as generalization of the Heisenberg group. These are called *quadratic CR manifolds*, (also called hyperquadric in the very nice introductory paper [T2]). The relevant properties of the Kohn Laplacian and $\overline{\partial}_b$-complex all have a precise characterization in terms of the signatures of the components of the Levi form. For this reason mainly but not only, they are of great interests in this type of analysis.

Let Φ be a hermitean form on $\mathbb{C}^n \times \mathbb{C}^n$ having value in some $\mathbb{C}^k$:

$$\Phi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^k ,$$

where $\Phi(z, z') = \overline{\Phi(z', z)}$. The associated *quadratic manifold* is

$$M = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^k : \ \operatorname{Im} w = \Phi(z, z)\} .$$

Notice that, if $k = 1$ and $\Phi(z, z') = z\overline{z}'$, then M is just the Heisenberg group. In fact, also in this generality, M has an underlying Lie group structure.

For $(z', w') \in M$ the complex-affine transformation of $\mathbb{C}^n \times \mathbb{C}^k$

$$\tau_{(z',w')}(z, w) = \big(z + z', w + w' + 2i\Phi(z, z')\big)$$

maps M onto itself, and

$$\begin{aligned} \tau_{(z',w')}\tau_{(z'',w'')} &= \tau_{(z'+z'',w'+w''+2i\Phi(z',z''))} \\ {\tau_{(z',w')}}^{-1} &= \tau_{(-z',-w'+2i\Phi(z',z'))} . \end{aligned}$$

Under the identification of $\tau_{(z',w')}$ with $(z',w') \in M$, this composition law defines a Lie group structure on M.

We introduce coordinates $(z,t) \in \mathbb{C}^n \times \mathbb{C}^k$ to denote the element $\big(z, t + i\Phi(z,z)\big) \in M$. Once pulled back to M the group multiplication takes the form

$$(z,t)(z',t') = \big(z + z', t + t' + 2\Im\Phi(z,z')\big).$$

We call G_Φ this group.

6.1. The Levi form on G_Φ. We now describe the Levi form on G_Φ. Notice that the manifold M can be describe by the k defining functions

$$\rho_j(z,w) = \operatorname{Im} w_j - \Phi_j(z,z) \qquad j = 1, \dots, k;$$

where Φ_j denotes the j-th component of Φ in the given fixed basis. Then Φ_j is an $n \times n$ hermitean matrix. We first fix our attention to the origin. We wish to compute

$$\frac{\partial^2 \rho_j(z,w)}{\partial\zeta_\ell \overline{\partial}\zeta_m} \zeta_\ell \overline{\zeta}_m$$

where we write momentarly $\zeta = (z,w) \in \mathbb{C}^{n+k}$.

The tangent bundle $T^{1,0}$ is spanned by the vector fields $B_1, \dots, B_n$. Hence, we immediately see that the j-th component of the Levi form of M is exactly the j-th component of the form Φ, that is the Levi form of M is just Φ, thought as taking values in the normal bundle $N(M)$.

For $\tau \in N^*(M)$ we denote by Φ^τ the scalar-valued form $\tau(\Phi(\cdot,\cdot))$.

It should be noted that we do not require that the Levi form is non-degerate. Moreover, it is possible that all the Φ^τ are degenerate, even though there is no common radical that can be factored out to decompose G_Φ as the product of a nilpotent and an abelian group.

For example let, $\Phi : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^2$, $\Phi = (\Phi_1, \Phi_2)$, with $\Phi_j(z,z') = z'^* A_j z$ and

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In the following will be significant the following definition.

Definition 6.2. *Let $n^+(\tau)$, resp. $n^-(\tau)$, the number of positive, resp. negative, eigenvalues of Φ^τ. We define Ω_q to be the cone*

$$\Omega_q = \{\tau : n^+(\tau) = q, n^-(\tau) = n - q\}.$$

6.3. The Kohn Laplacian on G_Φ. For $v \in \mathbb{C}^n$, denote by $\partial_v f$ the directional derivative of a function f on $\mathbb{C}^n \times \mathbb{C}^k$ in the direction v and let X_v be the left-invariant vector field on G_Φ that coincides with ∂_v at the origin. It is easy to check that

$$X_v f(z,t) = \partial_v f(z,t) + 2\Im\Phi(z,v) \cdot \nabla_t f(z,t).$$

Take the standard basis $\{v_1,\dots,v_n\}$ of $\mathbb{C}^n$ and define

$$B_j = \tfrac{1}{\sqrt{2}}(X_{v_j} - iX_{Jv_j}),$$
$$\overline{B}_j = \tfrac{1}{\sqrt{2}}(X_{v_j} + iX_{Jv_j})\,,$$

for $j=1,\dots,n$; where J denotes the complex structure in $\mathbb{C}^n$.

We denote by $\overline{\beta}^I$ the $(0,q)$-form $\overline{\beta}_{i_1}\wedge\cdots\wedge\overline{\beta}_{i_q}$, where $I=(i_1,\dots,i_q)$ is a strictly increasing multi-index. Given a $(0,q)$-form $\phi=\sum_{|I|=q}\phi_I\overline{\beta}^I$ with smooth coefficients, we set

$$\bar\partial_b\phi = \sum_{|I|=q}\sum_{k=1}^n \overline{B}_k(\phi_I)\overline{\beta}_k\wedge\overline{\beta}^I = \sum_{|J|=q+1}\sum_{k,|I|=q}\varepsilon^J_{kI}\overline{B}_k(\phi_I)\overline{\beta}^J\,, \tag{3}$$

where we adopt the previous convention on ε^J_{kI}'s.

Let $dz\,dt$ denote the left-invariant Haar measure on G_Φ. On the space $L^2(G_\Phi)\otimes\Lambda_q$ of $(0,q)$-forms with coefficients in $L^2(G_\Phi)$ we consider the inner product

$$\langle\phi,\psi\rangle = \int_{G_\Phi}\langle\phi(z,t),\psi(z,t)\rangle\,dV(z,t)\,.$$

The formal adjoint $\bar\partial_b^*$ of $\bar\partial_b$ can be easily computed to yield

$$\overline{\partial}_b^*\big(\sum_{|I|=q}\phi_I\overline{\beta}^I\big) = \sum_{|J|=q-1}\big(-\sum_{k,|I|=q}\varepsilon^I_{kJ}B_k\phi_I\big)\overline{\beta}^J. \tag{4}$$

We now compute the Kohn Laplacian $\Box_b^{(q)}=\bar\partial_b\bar\partial_b^*+\bar\partial_b^*\bar\partial_b$.

Proposition 6.4. *With respect to the selected basis, the operator $\Box_b^{(q)}$ is represented by a matrix $\big(\Box_{LK}\big)$ of scalar left-invariant differential operators on G_Φ as*

$$\Box_b^{(q)}\big(\sum_K\phi_K\overline{\beta}^K\big) = \sum_L\big(\sum_K\Box_{LK}\phi_K\big)\overline{\beta}^L.$$

Then,

$$\Box_{LK} = \delta_{LK}\mathcal{L}_0 + M_{LK}$$

where δ_{LK} is the Kronecker delta, $\mathcal{L}_0=-\frac12\sum_{k=1}^n(\overline{B}_kB_k+B_k\overline{B}_k)$ and

$$M_{LK} = \begin{cases} \frac12\big(\sum_{k\in K}[B_k,\overline{B}_k]-\sum_{k\notin K}[B_k,\overline{B}_k]\big) & \textit{if } K=L,\\ \varepsilon(K,L)[B_k,\overline{B}_\ell] & \textit{if } |\{K\cap L\}|=q-1,\\ 0 & \textit{otherwise.}\end{cases}$$

Here, given two multi-indices K and L such that $|K|=|L|=q$ and $|\{K\cap L\}|=q-1$, we set

$$\varepsilon(K,L)=(-1)^m$$

where m is the number of elements in $K\cap L$ between the unique element $k\in K\setminus L$ and the unique element $\ell\in L\setminus K$.

Notice that, even in this relatively fairly simple situation, the Kohn Laplacian is far from being diagonal.

Proof. The proof follows similar lines to the ones in the case of the Heisenberg group. One can easily see that

$$\bar\partial_b(\bar\partial_b^*\phi) = -\sum_{|L|=q}\Big(\sum_{k,\ell,|J|=q-1,|K|=q}\varepsilon^K_{kJ}\varepsilon^L_{\ell J}\overline{B}_\ell B_k\phi_K\Big)\overline{\beta}^L$$

and that

$$\bar\partial_b^*(\bar\partial_b\phi) = -\sum_{|L|=q}\Big(\sum_{i,j,|H|=q+1,|K|=q}\varepsilon^H_{jK}\varepsilon^H_{iL}B_i\overline{B}_j\phi_K\Big)\overline{\beta}^L\,.$$

Hence,

$$\Box^{(q)}_b(\phi) = -\sum_{|L|=q}\sum_{|K|=q}\Big(\sum_{\ell,k,|J|=q-1}\varepsilon^K_{kJ}\varepsilon^L_{\ell J}\overline{B}_\ell B_k + \sum_{i,j,|H|=q+1}\varepsilon^H_{jK}\varepsilon^H_{iL}B_i\overline{B}_j\Big)\phi_K\overline{\beta}^L.$$

Then,

$$\Box_{LK} = -\sum_{\ell,k,|J|=q-1}\varepsilon^K_{kJ}\varepsilon^L_{\ell J}\overline{B}_\ell B_k - \sum_{i,j,|H|=q+1}\varepsilon^H_{jK}\varepsilon^H_{iL}B_i\overline{B}_j. \tag{5}$$

When $K=L$ the indices k and ℓ are forced to be equal, as well as i and j. Hence,

$$\begin{aligned}\Box_{LL} &= -\big(\sum_{k\in L}\overline{B}_k B_k + \sum_{j\notin L}B_j\overline{B}_j\big)\\ &= -\frac12\sum_{k=1}^n(\overline{B}_kB_k + B_k\overline{B}_k) - \frac12\big(\sum_{k\in L}[\overline{B}_k,B_k] + \sum_{k\notin L}[B_k,\overline{B}_k]\big).\end{aligned}$$

This proves the statement for the terms along the diagonal.

When $K\neq L$, the coefficient $\varepsilon^K_{kJ}\varepsilon^L_{\ell J}$ is different from 0 if only if $K=J\cup\{k\}$ and $L=J\cup\{\ell\}$. Notice that, given K and L such that $|\{K\cap L\}|=q-1$, they uniquely determine J,k and ℓ. Analogously, $\varepsilon^H_{jK}\varepsilon^H_{iL}\neq 0$ if and only if $H=K\cup\{j\}=L\cup\{i\}$. Then, necessarily, $|\{K\cap L\}|=q-1$ as before, and if k and ℓ are as above, $j=\ell$ and $i=k$.

It follows that $\Box_{LK}=0$ unless $|\{K\cap L\}|=q-1$. In this case, each of the sums in (5) reduces to one single term, and

$$\Box_{LK} = -\varepsilon^K_{kJ}\varepsilon^L_{\ell J}\overline{B}_\ell B_k - \varepsilon^H_{\ell K}\varepsilon^H_{kL}B_k\overline{B}_\ell,$$

with $J=K\cap L$ and $H=K\cup L$. Moreover,

$$\varepsilon^K_{kJ}\varepsilon^L_{\ell J} = -\varepsilon^H_{\ell K}\varepsilon^H_{kL} = \varepsilon(K,L)\,.$$

Thus,

$$\Box_{LK} = \varepsilon(K,L)[B_k,\overline{B}_\ell],$$

which proves the proposition. □

6.5. The main results on G_Φ. We now present the main theorems on the Kohn Laplacian and the $\overline{\partial}_b$-equations on G_Φ. References for these results are [PR1, PR2].

We begin with the local solvability for $\Box_{b,q}$.

Theorem 6.6. *The Kohn Laplacian $\Box_{b,q}$ is locally solvable if and only if there is no $\tau \in N*(M)$ such that $n^+(\tau) = q$ and $n^-(\tau) = n - q$.*

More precisely, the following conditions are equivalent.

(1) Ω_q *is non-empty;*
(2) $\Box_{b,q}$ *is not locally solvable;*
(3) *ker* $\Box_{b,q} \cap L^2\Lambda^{0,q}(G_\Phi)$ *is non-empty;*

When $\Box_{b,q}$ is not solvable, the orthogonal projection onto its L^2-null-space is given by convolution on G_Φ with an operator-valued distribution S_q for which it is possible to give an explicit formula.

Next we discuss the hypoellipticity of the Kohn Laplacian.

Definition 6.7. We say that a *CR* manifold M with Levi form Φ satisfies condition $Y(q)$ at a point $z \in M$ is for every $\tau \in N^*(M)$, Φ^τ_z has at least $\max(q+1, n-q+1)$ eigenvalues with the same sign, or at least $\min(q+1, n-q+1)$ pairs of eigenvalues with opposite signs.

Theorem 6.8. *The following conditions are equivalent:*

(1) $\mathrm{span}_{\mathbb{R}}\{\Phi(z,z)\} = N(M)$ *and there exists $C > 0$ such that for each ϕ in the Schwartz space*
$$\|(\mathcal{L}_0 \otimes I)\phi\|_{L^2} \le C\|\Box_{b,q}\phi\|_{L^2};$$
(2) $\Box_{b,q}$ *is hypoelliptic;*
(3) *there exists no non-zero $\tau \in N^*(M)$ such that $n^+(\tau) \le n - q$ and $n^-(\tau) \le q$;*
(4) Φ *satisfies condition $Y(q)$.*

We remark that condition (3) and (4) are both equivalent to the following condition: There exists no non-zero $\tau \in N^*(M)$ such that
$$\begin{cases} \min\big(n^+(\tau), n^-(\tau)\big) \le \min(q, n-q) \\ \max\big(n^+(\tau), n^-(\tau)\big) \le \max(q, n-q)\,. \end{cases}$$

Proofs of these theorems are based on the group Fourier transform, introduction and discussion of which would require space and effort that go beyond the scope of these lectures. We refer to the paper [PR1] for some details.

7. Sufficiency of the $Y(q)$ condition for hypoellipticity

In this final section we return to the case of a general *CR* manifold. The result we present are due to Shaw and Wang [ShW].

Recall the decomposition (1) of the complexified tangent space of M
$$T^{\mathbb{C}}M = \mathcal{L} \oplus \overline{\mathcal{L}} \oplus N(M)\,.$$

Let $\{L_1, \dots, L_n, \overline{L}_1, \dots, \overline{L}_n, T_1, \dots, T_k\}$ be a basis for the smooth sections of the tangent bundle $T^{\mathbb{C}}M$, with $L_1, \dots, L_n$ smooth sections of $\mathcal{L}$,

Lemma 7.1. *Assume that M satisfies condition $Y(q)$ at a point $z \in M$. Then there exists an open neighborhood U of z on which the vector fields $\{X_1, \dots, X_n\}$ satisfy Hörmander's condition, where*

$$X_j = \Re L_j \qquad j = 1, \dots, n; \qquad X_j = \Im L_j \qquad j = n+1, \dots, 2n\,.$$

Proof. In fact, it suffices to consider the first order commutators in order to span the tangent space $\mathbb{R}^{2n+k}$. For, let τ_ℓ be a given direction in $N(M)^*$. Since $\mathcal{P}_z$ satisfies condition $Y(q)$, Phi^{τ_ℓ} is such that

$$\min(q, n-q) \le n^+(\tau_\ell), n^-(\tau_\ell) \le \max(q, n-q)\,.$$

In particular Phi^{τ_ℓ} has at least a non-zero eigenvalue. Since Phi^{τ_ℓ} is the matrix whose entries with respect to the basis $\{L_1, \dots, L_n\}$ are

$$\delta_{jk}[L_j, \overline{L}_k]$$

we see that there exists at least one j such that $[L_j, \overline{L}_j]$ has non-trivial component in the direction $J(\tau_\ell)$. □

This lemma alone does not guarantees that $\Box_{b,q}$ is hypoelliptic, since ($\Box_{b,q}$ is not a scalar operator and) the lower order terms are not real.

Theorem 7.2. *Suppose that M is a CR manifold of CR dimension n and codimension $k \ge 1$. Assume that satisfies condition $Y(q)$ at a point $z \in M$. Then there exists an open neighborhood U of z on which the Kohn Laplacian $\Box_{b,q}$ satisfies the subelliptic estimates*

$$\|\eta_1 \phi\|_{H^{s+1}} \le C\big(\|\eta_2 \Box_{b,q} \phi\|_{H^s} + \|\phi\|\big)\,,$$

where η_1, η_2 are $\mathcal{C}^\infty$ cut-off fucntions supported in U, $\eta_2 = 1$ on $supp\, \eta_1$.

Proof. For a complete proof we refer the reader to [ShW]. Here we sketch the argument.

Define

$$Q_b(\phi, \phi) = \|\overline{\partial}_b \phi\|^2 + \|dbb^* \phi^2\|^2 + \|\phi\|^2$$

One begins by showing that, by setting

$$\sum_{j=1}^n \|L_j \phi\|^2 = \|\phi\|^2_{\mathcal{L}}\,, \qquad \sum_{j=1}^n \|\overline{L}_j \phi\|^2 = \|\phi\|^2_{\overline{\mathcal{L}}}\,,$$

we have

$$\|\phi\|^2_{\mathcal{L}} + \|\phi\|^2_{\overline{\mathcal{L}}} + \sum_{\ell=1}^k \sum_{I,j} |\Re(T_\ell \phi_{IJ}, \phi_{IJ})| \le C Q_b(\phi, \phi)\,. \tag{6}$$

From this, using the Hörmander condition on the vector fields $\{X_1, \dots, X_{2n}\}$ and the corresponding subelliptic estimates, it follows that

$$\|\phi\|_{H^{1/2}} \le C Q_b(\phi, \phi)\,, \tag{7}$$

which in turns implies the desired estimate.

In order to prove (6) one manipulates the energy form $Q_b(\phi, \phi)$ to obtain the estimate from below (here we assume $k = 1$ for simplicity of notation)

$$Q_b(\phi, \phi) \ge \varepsilon \|\phi\|^2_{\overline{\mathcal{L}}} + \sum_{I,J} a_{IJ} \Re(T_\ell \phi_{IJ}, \phi_{IJ}) - \delta\big(\|\phi\|^2_{\mathcal{L}} + \|\phi\|^2\big)\,,$$

where

$$a_{IJ} = \sum_{j\in J\setminus\sigma I,J} \lambda_j - (1-\varepsilon) \sum_{j\in\sigma I,J\setminus J} \lambda_j + \varepsilon \sum_{j\in\sigma I,J\cap\setminus J} \lambda_j \,,$$

the λ_j's are the eigeinvalues of Φ (that we are assume to be scalar-valued for simplicitynow) and

$$\sigma I, J = \{ j : \lambda_j < 0 \text{ if } \Re(T_\ell \phi_{IJ}, \phi_{IJ}) > 0 \text{ and } \lambda_j > 0 \text{ if } \Re(T_\ell \phi_{IJ}, \phi_{IJ}) < 0 \} \,.$$

Since M satisfies the condition $Y(q)$, we can select $\varepsilon > 0$ and small so that

$$a_{IJ} > 0 \,.$$

From this the result follows. □

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