One-Side Liouville Theorems
for a Class of Hypoelliptic Ultraparabolic Equations

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1. Introduction

The aim of this paper is to show a one-side Liouville theorem for a class of hypoelliptic ultraparabolic equations and for their “stationary” counterpart. The operators we shall deal with are of the following type:

\[ L = \sum_{i,j=1}^{N} \partial_{x_i} (a_{ij}(x) \partial_{x_j}) + \sum_{i=1}^{N} b_i(x) \partial_{x_i} - \partial_t \quad \text{in} \quad \mathbb{R}^{N+1}, \]

where the coefficients \( a_{ij} \) and \( b_i \) are smooth functions defined in \( \mathbb{R}^N \). The matrix \( A = (a_{ij}) \), \( i, j = 1, \ldots, N \), is supposed to be symmetric and nonnegative definite at any point of \( \mathbb{R}^N \).

Throughout the paper we shall denote by \( z = (x, t) \), \( x \in \mathbb{R}^N \), \( t \in \mathbb{R} \), the point of \( \mathbb{R}^{N+1} \) and by \( Y \) the vector field in \( \mathbb{R}^{N+1} \)

\[ Y := \sum_{i=1}^{N} b_i(x) \partial_{x_i} - \partial_t. \]

Moreover, we shall denote by \( L_0 \) the stationary part of \( L \), i.e.

\[ L_0 = \sum_{i,j=1}^{N} \partial_{x_i} (a_{ij}(x) \partial_{x_j}) + \sum_{i=1}^{N} b_i(x) \partial_{x_i}. \]

We assume the following hypotheses.

(H1) \( L \) is hypoelliptic in \( \mathbb{R}^{N+1} \) and homogeneous of degree two with respect to the group of dilations \( (d_\lambda)_{\lambda > 0} \) given by

\[ d_\lambda(x, t) = (D_\lambda x, \lambda^2 t) \]

\[ D_\lambda x = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N), \]

where \( \sigma = (\sigma_1, \ldots, \sigma_N) \) is an \( N \)-tuple of natural numbers satisfying

\( 1 = \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_N \). \( L \) is \( d_\lambda \)-homogeneous of degree two if

\[ L(u(d_\lambda(x, t))) = \lambda^2 (Lu)(d_\lambda(x, t)) \quad \forall u \in C^\infty(\mathbb{R}^{N+1}). \]
For every \((x, t), (y, \tau) \in \mathbb{R}^{N+1}, t > \tau\), there exists an \(L\)-admissible path \(\eta : [0, T] \rightarrow \mathbb{R}^{N+1}\) such that \(\eta(0) = (x, t), \eta(T) = (y, \tau)\).

An \(L\)-admissible path is any continuous path \(\eta\) which is the sum of a finite number of diffusion and drift trajectories.

A diffusion trajectory is a curve \(\eta\) satisfying, at any points of its domain, the inequality
\[
(\langle \eta'(s), \xi \rangle)^2 \leq \langle \hat{A}(\eta(s))\xi, \xi \rangle \quad \forall \xi \in \mathbb{R}^N.
\]
Here \(\langle , \rangle\) denotes the inner product in \(\mathbb{R}^{N+1}\) and \(\hat{A}(z) = \hat{A}(x, t) = \hat{A}(x)\) stands for the \((N+1) \times (N+1)\) matrix
\[
\hat{A} = \left( \begin{array}{cc}
A & 0 \\
0 & 0
\end{array} \right).
\]

A drift trajectory is a positively oriented integral curve of \(Y\).

Throughout the paper we shall denote by \(Q\) the homogeneous dimension of \(\mathbb{R}^{N+1}\) with respect to the dilations (1.4), i.e.
\[
Q = \sigma_1 + \ldots + \sigma_N + 2
\]
and we assume
\[
Q \geq 5.
\]

Then, the \(D_\lambda\)-homogeneous dimension of \(\mathbb{R}^N\) is \(Q - 2 \geq 3\).

We explicitly remark that the smoothness of the coefficients of \(L\) and the homogeneity assumption in (H1) imply that the \(a_{ij}\)'s and the \(b_i\)'s are polynomial functions (see [L], Lemma 2).

For any \(z = (x, t) \in \mathbb{R}^{N+1}\) we define the \(d_\lambda\)-homogeneous norm \(|\cdot|\) by
\[
|z| = |(x, t)| := (|x|^4 + t^2)^{\frac{1}{4}}
\]
where
\[
|x| = |(x_1, \ldots, x_N)| = \left( \sum_{j=1}^{N} (x_j^2)^{\frac{2}{\sigma_j}} \right)^{\frac{\sigma}{2}}, \quad \sigma = \prod_{j=1}^{N} \sigma_j.
\]

The class of the operators just introduced contains the one recently considered in [KL]. In particular, it contains the heat operators on Carnot groups, the prototype of Kolmogorov operators and the operators obtained by linking the previous ones (see [KL], Example 9.3 and 9.7). An example of operators satisfying our hypotheses (H1) and (H2), and not contained in [KL] is given by \(L = \partial^2_{x_1} + x_1^4 \partial_{x_2} - \partial_t\) in \(\mathbb{R}^3\).

The main result of this paper is the following Liouville-type theorem.

**Theorem 1.1.** Let \(u : \mathbb{R}^{N+1} \rightarrow \mathbb{R}\) be a (smooth) solution to \(Lu = 0\) in \(\mathbb{R}^{N+1}\).

Suppose \(u \geq 0\) and
\[
u(0, t) = O(t^m) \quad \text{as} \quad t \rightarrow \infty
\]
for some \(m \geq 0\). Then
\[
u = \text{const.} \quad \text{in} \quad \mathbb{R}^{N+1}.
\]
Before proceeding we want to note that condition (1.5) cannot be removed in order to get (1.6). Indeed, for example, the function
\[ u(x, t) = \exp(x_1 + x_2 + \ldots + x_N + Nt), \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R}, \]
is nonnegative, non-constant and satisfies the heat equation
\[ \Delta u - \partial_t = 0 \quad \text{in} \quad \mathbb{R}^{N+1}, \quad \Delta = \sum_{j=1}^{N} \partial_{x_j}^2. \]

We stress that \( u \) does not satisfy condition (1.5) since \( u(0, t) = \exp(Nt) \).

From Theorem 1.1 a Liouville type theorem for \( \mathcal{L}_0 \) follows.

**Corollary 1.2.** Let \( v : \mathbb{R}^N \rightarrow \mathbb{R} \) be a (smooth) solution\(^1\) to \( \mathcal{L}_0 v = 0 \) in \( \mathbb{R}^N \).
Then, if \( v \geq 0 \),
\[ v = \text{const.} \quad \text{in} \quad \mathbb{R}^N. \]

**Proof.** The function
\[ u : \mathbb{R}^{N+1} \rightarrow \mathbb{R}, \quad u(x, t) = v(x) \]
satisfies \( \mathcal{L} u = 0 \) in \( \mathbb{R}^{N+1} \). Moreover, \( u \geq 0 \) and
\[ u(0, t) = v(0) \quad \forall t \in \mathbb{R}. \]

Then, by Theorem 1.1, \( u = \text{const.} \) in \( \mathbb{R}^{N+1} \) so that \( v = \text{const.} \) in \( \mathbb{R}^N \). \( \square \)

This Corollary extends to the present class of operators the Liouville Theorem 7.1 in [KL]. A Liouville type theorem for a very wide class of partial differential operators, homogeneous with respect to a group of dilations, was proved by Luo Xuebo in [L]. Luo Xuebo’ s Theorem, which extends previous results by Geller [G] and Rothschild [R], also applies to our operators and, in this context, reads as follows.

**Theorem.** Let \( u \) be a tempered distribution satisfying, in the weak sense of distributions, the equation
\[ \mathcal{L} u = 0 \quad \text{in} \quad \mathbb{R}^{N+1}. \]
Then \( u \) is a polynomial function.

This result reduces the proof of Theorem 1.1 to the proof of the following

**Main Lemma.** Let \( u : \mathbb{R}^{N+1} \rightarrow \mathbb{R} \) be a nonnegative smooth solution to \( \mathcal{L} u = 0 \) in \( \mathbb{R}^{N+1} \) satisfying condition (1.5). Then,
\[ u(z) = O(|z|^n) \quad \text{as} \quad |z| \rightarrow \infty \]
for a suitable \( n > 0 \).

This Lemma, together with Luo Xuebo’ s Theorem, immediately gives the

\(^1\)Obviously, \( \mathcal{L}_0 \) is hypoelliptic in \( \mathbb{R}^N \) since \( \mathcal{L} \) is hypoelliptic in \( \mathbb{R}^{N+1} \). Then, every distributional solution to \( \mathcal{L}_0 v = 0 \) is smooth.
Proof of Theorem 1.1. Let $u$ be a solution to $\mathcal{L}u = 0$ satisfying the hypotheses of Theorem 1.1. By the Main Lemma, $u$ is a tempered distribution so that, by Luo Xuebo’s Theorem, $u$ is a polynomial function. Then, $u = u_0 + \ldots + u_m$, where $u_k$ ($k = 0, 1, \ldots, m$) is a polynomial function $d_\lambda$-homogeneous of degree $k$ and $u_m \geq 0$, since $u \geq 0$. On the other hand, being $\mathcal{L}u = 0$ and $\mathcal{L}u_k$ $d_\lambda$-homogeneous of degree $k - 2$, if $k \geq 2$, we have $\mathcal{L}u_k = 0$ for every $k = 0, 1, \ldots, m$. In particular $\mathcal{L}u_m = 0$. Since $u_m$ is nonnegative and $d_\lambda$-homogeneous of degree $m \geq 0$, there exists $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$ such that

$$u_m(z_0) = \inf_{\mathbb{R}^{N+1}} u_m.$$  

By the strong maximum principle (see next section, Proposition 2.2) we then have

$$u_m(x, t) = u_m(x_0, t_0) \quad \forall (x, t) \in \mathbb{R}^N \times ]-\infty, t_0[.$$  

Since $u_m$ is a polynomial function, this obviously implies

$$u_m(x, t) = u_m(x_0, t_0) \quad \forall (x, t) \in \mathbb{R}^{N+1}.$$  

Then $m = 0$ and $u \equiv u_0$, i.e. $u$ is a constant function.  

2. A Harnack Inequality

In this section we shall prove the following Harnack inequality for nonnegative solutions to $\mathcal{L}u = 0$.

Theorem 2.1. Let $u : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ be a nonnegative solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{N+1}$. Then, there exist two positive constants $C = C(\mathcal{L})$ and $\theta = \theta(\mathcal{L})$ such that

$$\sup_{C_r} u \leq Cu(0, r^2) \quad \forall r > 0,$$

where, for $\rho > 0$, $C_\rho$ denotes the $d_\lambda$-symmetric ball

$$C_\rho := \{z \in \mathbb{R}^{N+1} \mid |z| < \rho \}.$$  

In order to prove this result, our main tool is a Mean-Value Theorem for the $\mathcal{L}$-harmonic functions, i.e. for the solutions to $\mathcal{L}u = 0$.

From hypotheses (H1) and (H2), by easily adapting the procedure already used in [LP1], [BLU] and [KL], we can prove the existence of a fundamental solution $\Gamma(z, \zeta)$ of $\mathcal{L}$ with the following properties.

(i) $\Gamma$ is smooth in $\{(z, \zeta) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \mid z \neq \zeta \},$

(ii) $\Gamma(\cdot, \zeta) \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ and $\mathcal{L}\Gamma(\cdot, \zeta) = -\delta_\zeta$ for every $\zeta \in \mathbb{R}^{N+1},$

(iii) $\Gamma(z, \cdot) \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ and $\mathcal{L}^*\Gamma(z, \cdot) = -\delta_z$ for every $z \in \mathbb{R}^{N+1},$

(iv) $\limsup_{\zeta \to z} \Gamma(z, \zeta) = \infty$ for every $z \in \mathbb{R}^{N+1},$

(v) $\Gamma(0, \zeta) \to 0$ as $\zeta \to \infty,$ $\Gamma(0, d_\lambda(\zeta)) = \lambda^{-\frac{Q+2}{2}}\Gamma(0, \zeta),$

(vi) $\Gamma((x, t), (\zeta, \tau)) \geq 0$ if $t > \tau,$

(vii) $\Gamma((x, t), (\zeta, \tau)) = \Gamma((x, 0), (\zeta, \tau - t)).$

In (iii) $\mathcal{L}^*$ denotes the formal adjoint of $\mathcal{L}$. We would like to stress that property (vi) follows from the invariance of $\mathcal{L}$ with respect to the translations parallel to the $t$-axis. The second part of property (vi) can be proved as in [KL], Section 2, by using the following strong maximum principle.

Proposition 2.2. Let $u$ be a nonnegative solution to the equation $\mathcal{L}u = 0$ in the halfspace

$$S := \mathbb{R}^N \times ]-\infty, t_0[, \ t_0 \in \mathbb{R}.$$
Suppose there exists a point \( z_1 = (x_1, t_1) \in S \) such that
\[ u(x_1, t_1) = 0. \]

Then \( u = 0 \) in \( \mathbb{R}^N \times ]-\infty, t_1[. \)

**Proof.** Let us denote by \( P_{z_1}(S) \) the propagation set of \( z_1 \) in \( S \), i.e. the set
\[ P_{z_1}(S) = \{ z \in S : \text{there exists an } L\text{-admissible path } \eta : [0, T] \rightarrow S \text{ s. t. } \eta(0) = z_1, \eta(T) = z \}. \]
The hypothesis (H2) implies \( P_{z_1}(S) = \mathbb{R}^N \times ]-\infty, t_1[. \) On the other hand since \( z_1 \) is a minimum point of \( u \) and the minimum spreads all over \( P_{z_1} \) (see [A]), we get
\[ u(z) = u(z_1) \quad \forall \ z \in \mathbb{R}^N \times ]-\infty, t_1[. \]
Then, the assertion follows since \( u(z_1) = 0. \)

For every \( (0, T) \in \mathbb{R}^{N+1} \) and \( r > 0 \) we define the \( L \)-ball centered at \( (0, T) \) and with radius \( r \), as follows
\[ \Omega_r(0, T) := \left\{ \zeta \in \mathbb{R}^{N+1} : \Gamma((0, T), \zeta) > \left( \frac{1}{r} \right)^{Q-2} \right\}. \]
Then, if \( \mathcal{L}u = 0 \) in \( \mathbb{R}^{N+1} \), the following Mean Value formula holds
\[ u(0, T) = \left( \frac{1}{r} \right)^{Q-2} \int_{\Omega_r(0, T)} K(T, \zeta) u(\zeta) \ d\zeta, \]
where
\[ K(T, \zeta) = \frac{< A(\xi) \nabla_\xi \Gamma, \nabla_\xi \Gamma >}{\Gamma^2}, \quad \zeta = (\xi, \tau), \]
and \( \Gamma \) stands for \( \Gamma((0, T), (\xi, \tau)) \). Moreover, \( <, > \) denotes the inner product in \( \mathbb{R}^N \) and \( \nabla_\xi \) is the gradient operator \( \partial_{\xi_1}, \ldots, \partial_{\xi_N} \).

Formula (2.2) is one of the numerous extensions of the classical Gauss Mean Value Theorem for harmonic functions. For a proof of it we directly refer to [LP2], Theorem 1.5.

The following lemmas will be crucial for our purposes.

**Lemma 2.3.** Let \( U \) be an open connected subset of \( \mathbb{R}^{N+1} \). Let \( u : U \rightarrow \mathbb{R} \) be a smooth function such that
\[ A(x) \nabla_x u(x, t) = 0, \quad Y u(x, t) = 0 \quad \forall \ (x, t) \in U. \]
Then \( u \) is constant in \( U \).

**Proof.** Let us denote by \( X_k \) the vector field
\[ X_k := \sum_{j=1}^N a_{kj} \partial_{x_j}. \]
Since \( \mathcal{L} \) is hypoelliptic and its coefficients are polynomial functions, the following rank condition holds (see [D])
\[ \text{rank } \text{Lie}(X_1, \ldots, X_N, Y)(x, t) = N + 1 \quad \forall \ (x, t) \in \mathbb{R}^{N+1}. \]

On the other hand, by hypothesis (2.3),
\[ Zu = 0 \quad \text{in } U \quad \forall \ Z \in \text{Lie}(X_1, \ldots, X_N, Y). \]
Then, by the rank condition (2.4), $\nabla_z u(z) = 0$ at any point $z \in U$, and $u$ is constant.

**Lemma 2.4.** The closed set

$$U := \{ \zeta = (\xi, \tau) : K(T, \zeta) = 0, \ \tau < T \}$$

does not contain interior points.

**Proof.** We argue by contradiction and assume $K(T, \zeta) = 0$ for every $\zeta$ in a nonempty connected open set $U \subseteq \mathbb{R}^N \times ]-\infty, T[$. Then, letting $h(\zeta) := \Gamma((0, T), \zeta)$, we have

$$A(\xi) \nabla_\xi h(\xi, \tau) = 0 \quad \forall (\xi, \tau) \in U,$$

div$(A \nabla h) \equiv 0$ in $U$. The $\mathcal{L}^*$-harmonicity of $h$ now gives $Y h \equiv 0$ in $U$. This is absurd because $h(\zeta) = h(\xi, \tau) = \Gamma((0, 0), (\xi, \tau - T))$ and $z \mapsto \Gamma((0, z)$ is $d_\lambda$-homogeneous of degree $2 - Q \neq 0$. □

**Lemma 2.5.** There exists a positive constant $\theta = \theta(L)$ such that

$$C_\theta \subseteq \Omega r_0(0, 1).$$

**Proof.** By the property (vi) of $\Gamma$, it is $\Gamma((0, 1), (0, 0)) > 0$. Then, for a suitable positive constant $r_0$ and $\theta_0$, we have

$$\Gamma((0, 1), \zeta) > \left( \frac{1}{r_0} \right)^{Q-2} \quad \forall \zeta \in C_\theta.$$

This means that

$$C_{\theta_0} \subseteq \Omega r_0(0, 1)$$

and the assertion is proved. □

We are now in the position to give the proof of Theorem 2.1.

Next Lemma easily follows from Theorem 7.1 in [B].

**Lemma 2.6.** Let $(u_n)$ be a sequence of $\mathcal{L}$-harmonic function in an open set $\Omega \subseteq \mathbb{R}^{N+1}$:

$$\mathcal{L}u_n = 0 \quad \text{in } \Omega \quad \forall n \in \mathbb{N}.$$  

Suppose $(u_n)$ is monotone increasing and convergent in a dense subset of $\Omega$. Then $(u_n)$ converges at any point of $\Omega$ to a smooth function $u$ such that $\mathcal{L}u = 0$ in $\Omega$.

**Proof of Theorem 2.1.** Since $\mathcal{L}$ is $d_\lambda$-homogeneous of degree two, it is enough to prove inequality (2.1) for $r = 1$. We argue by contradiction and assume that (2.1), with $r = 1$, is false. Then, there exists a sequence $(u_n)$ of nonnegative $\mathcal{L}$-harmonic functions such that

$$(2.5) \quad \sup_{C_\theta} u_n \geq 4^* u_n(0, 1).$$

By the Mean Value formula (2.2),

$$(2.6) \quad u_n(0, 1) = \left( \frac{1}{r_0} \right)^{Q-2} \int_{\Omega r_0(0, 1)} K(1, \zeta) u_n(\zeta) \, d\zeta, \quad n \in \mathbb{N},$$

so that, since $\Omega r_0(0, 1) \supseteq C_\theta$, see Lemma 2.5,

$$u_n(0, 1) \geq \left( \frac{1}{r_0} \right)^{Q-2} \int_{C_\theta} K(1, \zeta) u_n(\zeta) \, d\zeta.$$
On the other hand, by inequality (2.5) and Lemma 2.4, $u_n$ and $K(1, \cdot)$ are strictly positive in a non-empty open subset of $C_\theta$. It follows that $u_n(0,1) > 0$ for every $n \in \mathbb{N}$. Let us now put

$$v_n = \frac{u_n}{u_n(0,1)} \quad \text{and} \quad v = \sum_{n=1}^{\infty} \frac{v_n}{2^n}.$$ 

From the Mean Value formulas (2.6) we obtain

$$1 = v(0) = \left(\frac{1}{r_0}\right)^{Q-2} \int_{\Omega_{r_0}(0,1)} K(1, \zeta) \ v(\zeta) \ d\zeta,$$

so that $v < \infty$ at any point of

$$T := \{\zeta \in \Omega_{r_0}(0,1) : K(1, \zeta) > 0\}.$$

By Proposition 2.2 the closure of $T$ contains $\Omega_{r_0}(0,1)$. Then, by Lemma 2.6, $v$ is finite and smooth in $\Omega_{r_0}(0,1)$. In particular $v$ is continuous in $C_\theta$. Then,

$$\sup_{C_\theta} v < \infty.$$

On the other hand, by inequality (2.5),

$$\sup_{C_\theta} \frac{v_n}{2^n} = \frac{1}{2^n} \sup_{C_\theta} \frac{u_n}{u_n(0)} \geq 2^n.$$

Hence $\sup_{C_\theta} v \geq 2^n$ for every $n \in \mathbb{N}$. This contradicts (2.7) and proves the Theorem.

With Theorem 2.1 at hand, the Main Lemma stated in the Introduction easily follows.

**Proof of Main Lemma.** Let $u$ be a nonnegative $L$-harmonic function in $\mathbb{R}^{N+1}$ satisfying the growth condition (2.2). Then, by Theorem 2.1,

$$\sup_{|z| \leq 6r} u(z) \leq C u(0, r^2) \leq C_1(1 + r^{2n}).$$

This obviously implies

$$u(z) \leq C_2(1 + |z|^{2n}) \quad \forall z \in \mathbb{R}^N.$$

□

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