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An invariant Harnack inequality for a class of hypoelliptic ultraparabolic equations

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Abstract. We show an invariant Harnack inequality for a class of hypoelliptic ultraparabolic operators with underlying homogeneous Lie group structures. As a byproduct we prove a Liouville type theorem for the related "stationary" operators. We also introduce a notion of link of homogeneous Lie Groups that allows us to show that our results apply to wide classes of operators.

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1. Introduction

In this paper we are concerned with a class of linear second order ultraparabolic operators of the following type

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0 - \partial_t \qquad \text{in } \mathbb{R}^{N+1}, \tag{1.1}$$

where the X_j' s are smooth vector fields on $\mathbb{R}^N$, i.e. linear first order differential operator in $\mathbb{R}^N$ with coefficients of class C^∞. For our purposes it will be convenient to also consider the X_j' s as vector fields on $\mathbb{R}^{N+1}$. Throughout the paper we shall denote by $z = (x, t)$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, the point of $\mathbb{R}^{N+1}$, and by Y the vector field in $\mathbb{R}^{N+1}$

$$Y := X_0 - \partial_t.$$

Moreover, we shall denote by $\mathcal{L}_0$ the operator in $\mathbb{R}^N$

$$\mathcal{L}_0 := \sum_{j=1}^{m} X_j^2 + X_0.$$

We assume the following conditions are satisfied.

(H1) There exists a homogeneous Lie group in $\mathbb{R}^{N+1}$,

$$\mathbb{L} = (\mathbb{R}^{N+1}, \circ, d_\lambda)$$

such that

(i) $X_1, \dots, X_m, Y$ are left translation invariant on $\mathbb{L}$.

(ii) $X_1, \dots, X_m$ are d_λ-homogeneous of degree one and Y is d_λ-homogeneous of degree two.

(H2) For every $(x, t), (y, \tau) \in \mathbb{R}^{N+1}$, $t > \tau$, there exists an $\mathcal{L}$-*admissible* path $\eta : [0, T] \longrightarrow \mathbb{R}^{N+1}$ such that $\eta(0) = (x, t)$, $\eta(T) = (y, \tau)$. The curve η is called $\mathcal{L}$-admissible if it is absolutely continuous and satisfies

$$\eta'(s) = \sum_{j=1}^{m} \lambda_j(s) X_j(\eta(s)) + \mu(s) Y(\eta(s)) \qquad \text{a.e. in } [0, T]$$

for suitable piecewise constant real functions $\lambda_1, \dots, \lambda_m$, and μ, $\mu \geq 0$.

Before proceeding we would like to list some easy consequences of conditions (H1) and (H2). Their proofs are postponed to the Appendix, Subsection 10.1.

(i) $\mathcal{L}$ and $\mathcal{L}_0$ are hypoelliptic operators in $\mathbb{R}^{N+1}$ and $\mathbb{R}^N$, respectively (see Proposition 10.1).

(ii) The composition law $\circ$ is euclidean in the "time" variable. More explicitly

$$(x, t) \circ (y, \tau) = (S(x, t, y, \tau), t + \tau)$$

for a suitable smooth function S (see Proposition 10.2).

(iii) The dilation d_λ takes the following form

$$d_\lambda(x, t) = (D_\lambda(x), \lambda^2 t)$$

and, denoting by Q the homogeneous dimension of $\mathbb{L}$, one has $Q \geq 3$ (see Remark 10.3).

Throughout the paper, except for Section 2, we shall assume

$$Q \geq 5$$

so that $Q - 2$, the homogeneous dimension of $\mathbb{R}^N$ with respect to D_λ, will be ≥ 3. We shall denote by $|\cdot|$ a fixed d_λ-*homogeneous norm* on $\mathbb{L}$. Precisely, if the dilation d_λ takes the form

$$d_\lambda(x, t) = d_\lambda(x_1, \dots, x_N, t) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_N} x_N, \lambda^2 t), \tag{1.2}$$

then

$$|(x, t)| := \left(\sum_{j=1}^{N} (x_j^2)^{\frac{\sigma}{\sigma_j}} + (t^2)^{\frac{\sigma}{2}} \right)^{\frac{1}{2\sigma}}, \tag{1.3}$$

where $\sigma = 2\prod_{j=1}^{N}\sigma_j$.
For a point $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ we put

$$|x| := \left(\sum_{j=1}^{N}(x_j^2)^{\frac{\sigma}{\sigma_j}}\right)^{\frac{1}{2\sigma}}. \tag{1.4}$$

The aim of this paper is to prove an *invariant* Harnack inequality for the nonnegative solutions to $\mathcal{L}u = 0$.

Parabolic Harnack inequalities have a long history which started with the works by Hadamard [9] and Pini [18] on the heat equation.
For parabolic operators with variable coefficients the Harnack inequality was proved by Moser [17], by using his celebrated iterative argument. Moser' s work has been followed by a huge amount of papers, dealing with linear and non linear parabolic equations, see e.g. [1], [2], [12]; see also the interesting accounts given in [19] and [21]. An elementary proof of the Harnack inequality for solutions to linear parabolic equations with smooth coefficients is contained in [6]. The proof in this paper uses a Mean Value Theorem for the solutions, and it is modelled on the classical proof of the Harnack inequality for harmonic functions. This very elementary approach also works for ultraparabolic equations, see [11] and [7], and can be greatly simplified when the involved operators are invariant with respect to suitable groups of translations and dilations, see [15]. In all these works, however, the knowledge of an explicit fundamental solution, and very sharp estimates of their derivatives, are heavily used. In the present paper we show that the *Mean Value Theorem technique* works only assuming hypotheses (H1) and (H2). Our main result, Theorem 7.1, extends to a wide class of ultraparabolic operators the Harnack inequalities in [7] and [15] (Theorems 5.1-5.1'). Obviously, Theorem 7.1 also applies to the heat operators on Carnot groups, see [25] and the references therein [1]. We would also like to quote the recent paper [4] containing an elementary proof of the Harnack inequality for heat operators that uses Bony's strong maximum principle and basic facts from Abstract Potential Theory.

The present paper is organized as follows. In Section 2 we show the existence of a global fundamental solution Γ for the operators $\mathcal{L}$ satisfying (H1) and (H2) and we prove some of its basic properties. In Section 3 we show that integrating Γ with respect to the t variable one obtains a fundamental solution γ with pole at $x = 0$ of the operator $\mathcal{L}_0$. In Section 4 we briefly show how $\mathcal{L}_0$ acts on some kind of *radial* functions. This result is used in Section 5 in order to obtain an upper gaussian estimate of Γ from which, in Section 6, we obtain the integral identity

$$\int_{\mathbb{R}^N}\Gamma(x,t)\ dx = 1 \qquad \forall t > 0.$$

Section 7 is devoted to the proof of our *Harnack inequality.* We shall follow the procedure already used in [15] (Theorem 5.1). We stress that all the results of

[1]A very interesting account on the parabolic Harnack inequality in several different settings is contained in the more recent monograph [22].

Section 7 come from the properties of Γ proved in Sections 5 and 6. In Section 8 we show a Harnack inequality for the *stationary* operator $\mathcal{L}_0$, easy consequence of the Harnack inequality for $\mathcal{L}$. By using this result we then show a *one-side Liouville-type Theorem for* $\mathcal{L}_0$. Our theorem is strongly related to the one very recently proved in [20] (for other related results, see [8], [16] and [3]). Finally, in Section 9, we show several examples of operators satisfying (H1)-(H2). In particular, we first easily recognize that the H-conditions are satisfied by the Heat operators on Carnot groups, Examples 9.1, 9.2. Then, Example 9.3, we prove (H1)-(H2) for the Kolmogorov-type operators studied in [15]. Example 9.7 deals with a "new" class of ultraparabolic operators $\mathcal{L}$ obtained by *linking* sub-Laplacians on Carnot groups with Kolmogorov-type operators. This example rests on what we call *link of groups*, that we shall introduce, and briefly study, in the Appendix. In the Example 9.7 we link *Carnot groups* $\mathbb{G}$ *with Kolmogorov groups* $\mathbb{K}$, the first ones underlying the sub-Laplacians and the second ones the Kolmogorov operators. The linked groups $\mathbb{G}\triangle\mathbb{K}$ play the role of $\mathbb{L}$ in condition (H1) with respect to the new operators $\mathcal{L}$.
In the Appendix, Section 10, we recall some basic notions on homogeneous Lie groups in $\mathbb{R}^N$ (Subsection 10.1) and we introduce the definition of *link* of groups (Subsection 10.2).

2. Fundamental solution for $\mathcal{L}$

The aim of this section is to prove the existence of a *global* fundamental solution for $\mathcal{L}$, together with some of its basic properties.

We shall follow, with minor changes, the procedure first used in [13] (Theorem 1.1). The same approach was also used in [10] for a class of operators satisfying hypotheses (H1) and (H2).

To begin with we state some well known maximum principles, also giving short proofs of them for reading convenience.

Proposition 2.1. *Let $O \subseteq \mathbb{R}^{N+1}$ be a bounded set and suppose $u \in C^2(O)$ and such that*

$$\mathcal{L}u \geq 0 \quad \text{in } O, \quad \limsup_{z\to\zeta} u(z) \leq 0 \quad \forall \zeta \in \partial O.$$

Then $u \leq 0$.

Proof. The function $w(x,t) = e^t$ satisfies

$$\mathcal{L}w < 0 \text{ in } \quad O \quad \text{and} \quad \inf_O w > 0.$$

Then, the assertion follows from the classical Picone' s maximum principle. □

Proposition 2.2. *Let O be a bounded open subset of $\mathbb{R}^{N+1}$ and let $z_0 = (x_0, t_0) \in O$. Define*

$$O_{z_0} := \{z = (x,t) \in O : t < t_0\}, \quad \partial_{z_0} O := \{z = (x,t) \in \partial O : t \leq t_0\}.$$

Let $u \in C^2(O)$ be such that

$$\mathcal{L}u \geq 0 \quad e \quad \limsup_{z \to \zeta} u(z) \leq 0 \quad \forall \zeta \in \partial_{z_0} O.$$

Then $u \leq 0$ in O_{z_0}.

Proof. Let $\varepsilon,\ \delta > 0$ be fixed and consider the function

$$u_\varepsilon(z) = u(z) - \frac{\varepsilon}{(t_0 - \delta - t)}, \qquad z \in O_{(x_0, t_0 - \delta)}.$$

We have

$$\mathcal{L}u_\varepsilon(z) = \mathcal{L}u(z) + \frac{\varepsilon}{(t_0 - \delta - t)^2} > 0 \qquad \forall z \in O_{(x_0, t_0 - \delta)}$$

and

$$\limsup_{z \to \zeta} u_\varepsilon(z) \leq 0 \qquad \forall \zeta \in \partial O_{(x_0, t_0 - \delta)}.$$

By the previous theorem we get $u_\varepsilon(z) \leq 0$ in $O_{(x_0, t_0 - \delta)}$. Letting ε and δ go to zero we obtain the assertion. □

Corollary 2.3. *Let O be a bounded open subset of $\mathbb{R}^{N+1}$ and let $\varphi \in C_0^\infty(O)$ be such that*

$$\text{supp } \varphi \subseteq \mathbb{R}^N \times] - \lambda_0, \lambda_0[, \qquad \lambda_0 > 0.$$

If $u \in C^\infty(O)$ satisfies

$$\begin{cases} \mathcal{L}u = -\varphi & \text{in } O \\ \lim_{z \to \zeta} u(z) = 0 & \forall \zeta \in \partial O \end{cases},$$

then

$$\sup_O |u| \leq 2\lambda_0 \sup_O |\varphi|.$$

Proof. Let $0 < \lambda_1 < \lambda_0$ be such that supp$\varphi \subseteq \mathbb{R}^N \times] - \lambda_1, \lambda_1[$ and choose a smooth function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ such that $0 \leq \psi \leq 1$, $\psi' \geq \frac{1}{2\lambda_0}$ in $] - \lambda_1, \lambda_1[$. Then, if we define

$$M = 2\lambda_0 \sup_O |\varphi| \quad \text{and} \quad v(x,t) = u(x,t) - M\psi(t)$$

we have

$$\mathcal{L}v \geq 0 \quad \text{in } O \quad \text{and} \quad \lim_{z \to \zeta} v(z) \leq 0 \quad \forall \zeta \in \partial O.$$

Therefore, by Theorem 2.1, $v \leq 0$ in O. This means $u \leq M\psi$. The same estimate holds for $-u$. Thus, since $0 \leq \psi \leq 1$, $\sup_O |u| \leq M$. □

Hypothesis (H2) immediately implies the following strong maximum principle

Proposition 2.4. *Let u be a nonnegative smooth solution to $\mathcal{L}u = 0$ in the strip $\mathbb{R}^N \times]T_1, T_2[,\ T_1 < T_2$.*

Suppose $u(x_0, t_0) = 0$ for a suitable $x_0 \in \mathbb{R}^N$ and $T_1 < t_0 < T_2$. Then

$$u(x,t) = 0 \qquad \forall (x,t) \in \mathbb{R}^N \times]T_1, t_0[.$$

Proof. It is enough to use hypothesis (H2) and Bony maximum principle ([5], Theorem 3.1). □

By proceeding as in [4], pag.1161, we can prove the existence of an open neighbourhood O_1 of the origin in $\mathbb{R}^{N+1}$ such that

(i) $d_\lambda(O_1) \subseteq O_1$ for $0 < \lambda \leq 1$,
(ii) $\bigcup_{\lambda>0} d_\lambda(O_1) = \mathbb{R}^{N+1}$.

Here d_λ stands for the λ-dilation on $\mathbb{L}$. Moreover, O_1 can be choosen in such a way that the following proposition holds.

Proposition 2.5. *There exists a function $G_1 \in C^\infty(\{(z,\zeta) \in O_1 \times O_1 | z \neq \zeta\})$, $G_1 \geq 0$ such that:*

(i) *for any fixed $z \in O_1$, $G_1(z,\cdot) \in L^1_{\text{loc}}(O_1)$,*
(ii) *for any fixed $\zeta \in O_1$, $G_1(z,\zeta) \longrightarrow 0$ as $z \longrightarrow z_0$, $\forall z_0 \in \partial O_1$,*
(iii) *$G_1(x,t,\xi,\tau) = 0$ if $t \leq \tau$.*
(iv) *For any $f \in C_0^\infty(O_1)$ the function*

$$u(z) = \mathcal{G}_1 f(z) := \int_{O_1} G_1(z,\zeta) f(\zeta) d\zeta, \ z \in O_1$$

is smooth in O_1 and solves the problem

$$\begin{cases} \mathcal{L}u = -f & \text{in } O_1 \\ \lim_{z\to\zeta} u(z) = 0 & \forall \zeta \in \partial O_1 \end{cases} .$$

We shall call G_1 a Green function for $\mathcal{L}$ related to O_1. A Green function for $\mathcal{L}^*$ related to O_1 is given by

$$G^*(z,\zeta) := G(\zeta,z).$$

$\mathcal{L}^*$ denotes the formal adjoint of $\mathcal{L}$.

Since the X_j's are d_λ-homogeneous of degree one, then $X_j^* = -X_j$. It follows that $\mathcal{L}^* = \sum_{j=1}^m X_j^2 - X_0 + \partial_t$. The proposition can be proved proceeding as in [4], Theorem 3.2. We just explicitly mention that property (iii) is a consequence of the "parabolic" maximum principle of Proposition 2.2.

For every $\lambda > 0$ let us now define $O_\lambda := d_\lambda(O_1)$ and

$$G_\lambda(z,\zeta) := \lambda^{-Q+2} G_1(d_{\lambda^{-1}} z, d_{\lambda^{-1}} \zeta) \tag{2.1}$$

where Q is the homogeneous dimension of $\mathbb{L}$ (see $(H1)$). We also put $G^*_\lambda(z,\zeta) = G_\lambda(\zeta,z)$. It is quite easy to recognize that $G_\lambda(G^*_\lambda)$ is a *Green function for* $\mathcal{L}(\mathcal{L}^*)$. Next lemma will show that G_λ is increasing with respect to λ. It can be proved exactly as Lemma 3.4 in [4].

Lemma 2.6. *If $0 < \lambda_1 \leq \lambda_2$ then*

$$G_{\lambda_1}(z,\zeta) \leq G_{\lambda_2}(z,\zeta) \qquad \forall z, \ \zeta \in O_{\lambda_1}, z \neq \zeta.$$

With Proposition 2.5 and Lemma 2.6 in hands we can define

$$\Gamma(z,\zeta) := \sup_{\lambda>0} G_\lambda(z,\zeta) = \lim_{\lambda\to\infty} G_\lambda(z,\zeta) \qquad z,\zeta \in \mathbb{R}^{N+1},\ z \neq \zeta.$$

Theorem 2.7. Γ *is a fundamental solution for* $\mathcal{L}$*, smooth out of the diagonal, nonnegative and with support in a halfspace. More precisely*

(i) *For any fixed* $z \in \mathbb{R}^{N+1}$, $\Gamma(\cdot,z)$ *and* $\Gamma(z,\cdot)$ *belong to* $L^1_{\mathrm{loc}}(\mathbb{R}^{N+1})$.

(ii) *For every* $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ *and* $z \in \mathbb{R}^{N+1}$,

$$\mathcal{L}\int_{\mathbb{R}^{N+1}} \Gamma(z,\zeta)\varphi(\zeta)\ d\zeta \ = \ \int_{\mathbb{R}^{N+1}} \Gamma(z,\zeta)\mathcal{L}\varphi(\zeta)\ d\zeta \ = \ -\varphi(z).$$

(iii) $\Gamma \in C^\infty\Big(\{(z,\zeta) \in \mathbb{R}^{N+1}\times\mathbb{R}^{N+1} \mid z \neq \zeta\}\Big)$.

(iv) $\mathcal{L}(\Gamma(\cdot,\zeta)) = -\delta_\zeta$ *for every* $\zeta \in \mathbb{R}^{N+1}$.

(v) $\Gamma \geq 0$ *and* $\Gamma(x,t,\xi,\tau) > 0$ *if and only if* $t > \tau$.

(vi) *If we define* $\Gamma^*(z,\zeta) := \Gamma(\zeta,z)$ *then* Γ^* *is a fundamental solution for* $\mathcal{L}^*$ *satisfying the dual properties of* (ii) *and* (iv).

Proof.

(i) For Corollary 2.3 we get

$$\int_{O_\lambda} G_\lambda(z,\zeta)\varphi(\zeta)\ d\zeta \leq 2\lambda_0^2 \sup|\varphi|. \tag{2.2}$$

for every $\varphi \in C_0^\infty(\mathbb{R}^N\times(-\lambda_0^2,\lambda_0^2))$ and $\lambda > \lambda_0$. Letting λ go to infinity we obtain

$$\int_{\mathbb{R}^{N+1}} \Gamma(z,\zeta)\varphi(\zeta)\ d\zeta \leq 2\lambda_0^2 \sup|\varphi|$$

which prove that $\Gamma(z,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^{N+1})$. In a similar way, by using G^*_λ instead of G_λ, we can prove that $\Gamma(\cdot,z) \in L^1_{\mathrm{loc}}(\mathbb{R}^{N+1})$.

(ii) For every fixed $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ let us define

$$u(z) := \int_{\mathbb{R}^{N+1}} \Gamma(z,\zeta)\varphi(\zeta)\ d\zeta, \quad z \in \mathbb{R}^{N+1}.$$

From inequality (2.2) it follows that $u \in L^\infty(\mathbb{R}^{N+1})$. Moreover, by the definition of Γ, for any $\psi \in C_0^\infty(\mathbb{R}^{N+1})$, we have

$$\begin{aligned} <\mathcal{L}(u),\psi> \quad &= \quad <u,\mathcal{L}^*\psi> = \lim_{\lambda\to\infty} <\mathcal{G}_\lambda\varphi,\mathcal{L}^*\psi> \\ &= \quad \lim_{\lambda\to\infty} <\mathcal{L}(\mathcal{G}_\lambda\varphi),\psi> = \ -<\varphi,\psi>. \end{aligned}$$

Then $\mathcal{L}u = -\varphi$ in the weak sense of distributions. The hypoellipticity of $\mathcal{L}$ gives now that $u \in C^\infty(\mathbb{R}^{N+1})$. Finally

$$\int_{\mathbb{R}^{N+1}} \Gamma(z,\zeta)\mathcal{L}\varphi(\zeta)\ d\zeta \ = \ \lim_{\lambda\to\infty} \mathcal{G}_\lambda(\mathcal{L}\varphi)(z) \ = \ -\varphi(z).$$

This completes the proof of (ii).

(iii) This property follows from the hypoellipticity of $\mathcal{L}$ by using a standard device (see, e.g. [5], Theorem 6.1).

(iv) For every fixed $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ and for any $\zeta \in \mathbb{R}^{N+1}$ we have

$$\begin{aligned}
\int_{\mathbb{R}^{N+1}} \mathcal{L}\Gamma(z,\zeta)\varphi(z)\, dz &= \int_{\mathbb{R}^{N+1}} \Gamma(z,\zeta)\mathcal{L}^*\varphi(z)\, dz \\
&= \lim_{\lambda\to\infty} \int_{\mathbb{R}^{N+1}} G_\lambda(z,\zeta)\mathcal{L}^*\varphi(z)\, dz \\
&= \lim_{\lambda\to\infty} \int_{\mathbb{R}^{N+1}} G_\lambda^*(\zeta,z)\mathcal{L}^*\varphi(z)\, dz \;=\; -\varphi(\zeta).
\end{aligned}$$

This proves (iv).

(v) Since $G_\lambda((x,t),(\xi,\tau)) \geq 0$, $= 0$ if $t \leq \tau$, then

$$\Gamma((x,t),(\xi,\tau)) \geq 0, \quad = 0 \quad \text{if } t \leq \tau.$$

Let us now prove that $\Gamma((x,t),(\xi,\tau)) > 0$ if $t > \tau$.
We argue by contradiction and suppose $\Gamma((x_0,t_0),(\xi_0,\tau_0)) = 0$ for some (x_0,t_0), $(\xi_0,\tau_0) \in \mathbb{R}^{N+1}$, $t_0 > \tau_0$.
Define

$$u(z) = u(x,t) := -\Gamma((x,t),(\xi_0,\tau_0)), \qquad x \in \mathbb{R}^N, t > \tau_0.$$

By property (iv), $\mathcal{L}u = 0$ in $\mathbb{R}^N \times]\tau_0,\infty[$. Moreover $u \leq 0$ and $u(x_0,t_0) = 0$. Then, by the strong maximum principle of Proposition 2.4, $u(x,t) = 0$ in the strip $\mathbb{R}^N \times]\tau_0, t_0[$. Thus, since $\Gamma((x,t),(\xi_0,\tau_0)) = 0$ if $t < \tau_0$, we have

$$\int_{\mathbb{R}^{N+1}} \Gamma(z,\zeta_0)\varphi(z)\, dz = 0 \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^N \times]-\infty, t_0[).$$

This contradicts (iv), and completes the proof of (v).

(vi) This statement is quite obvious. □

Next Proposition will show some other important properties of Γ.

Proposition 2.8. *The fundamental solution* Γ *has the following properties*

(i) $\Gamma(d_\lambda(z), d_\lambda(\zeta)) = \lambda^{-Q+2}\Gamma(z,\zeta)$.

(ii) *There exists* $C > 0$ *such that*

$$0 \leq \Gamma(z,\zeta) \leq \frac{C}{|z|^{Q-2}} \qquad \text{if} \quad |z| \geq 2|\zeta| \tag{2.3}$$

and

$$0 \leq \Gamma(z,\zeta) \leq \frac{C}{|\zeta|^{Q-2}} \qquad \text{if} \quad |\zeta| \geq 2|z|. \tag{2.4}$$

Proof.

(i) From the definition of G_λ we obtain

$$G_n\left(d_\lambda z, d_\lambda \zeta\right) = n^{-Q+2} G_1\left(d_{\frac{\lambda}{n}} z, d_{\frac{\lambda}{n}} \zeta\right) \;=\; \lambda^{-Q+2} G_{\frac{n}{\lambda}}(z,\zeta).$$

The assertion follows letting n go to infinity.

(ii) Since $|\cdot|$ is a homogeneous norm on $\mathbb{L}$, by the previous property we have

$$\Gamma(z,\zeta) = \; |z|^{-Q+2}\Gamma(d_{\frac{1}{|z|}} z, d_{\frac{1}{|z|}}\zeta)$$

so that, if $|z| > 2|\zeta|$,

$$0 \le \Gamma(z,\zeta) \le \frac{1}{|z|^{Q-2}} \max_{|z'|=1, |\zeta'|\le\frac{1}{2}} \Gamma(z',\zeta') =: C|z|^{-Q+2}.$$

From these inequalities assertion (2.3) follows. (2.4) can be proved as (2.3).

□

From this proposition we easily obtain the following important Corollary.

Corollary 2.9. *Let Γ be the fundamental solution of $\mathcal{L}$ constructed above. Then*

(i) $\Gamma(z,\zeta) = \Gamma(\zeta^{-1}\circ z)$ *with*

$$\Gamma(\cdot) := \Gamma(\cdot, 0).$$

(ii) $\limsup_{z\to\zeta} \Gamma(z,\zeta) = \infty$ *for every* $\zeta \in \mathbb{R}^{N+1}$.

Proof.

(i) Let $\zeta \in \mathbb{R}^{N+1}$ be arbitrarily fixed and define

$$u(z) = \Gamma(\zeta^{-1}\circ z) - \Gamma(z,\zeta), \qquad z \in \mathbb{R}^{N+1}.$$

Since $\mathcal{L}$ is left translation invariant on $\mathbb{L}$, by Theorem 2.7-(iv) we have: $\mathcal{L}u = -\delta_\zeta + \delta_\zeta = 0$. Moreover, by Proposition 2.8-(ii), $u(z) \longrightarrow 0$ as $z \longrightarrow \infty$. Then, by the maximum principle of Proposition 2.1, $u \equiv 0$ and (i) follows.

(ii) By the previous property,

$$\limsup_{z\longrightarrow\zeta} \Gamma(z,\zeta) = \limsup_{z\longrightarrow 0} \Gamma(z).$$

On the other hand, since Γ is the d_λ-homogeneous of degree $-Q+2$, Proposition 2.8-(i),

$$\Gamma(0,t) = t^{\frac{2-Q}{2}}\Gamma(0,1) \qquad \text{for every } t>0.$$

Being $\Gamma(0,1) > 0$ (see Theorem 2.7-(v)) and $Q \ge 3$, this identity obviously implies $\Gamma(0,t) \longrightarrow \infty$ as $t \to 0$, and the assertion follows.

□

3. Fundamental solution for $\mathcal{L}_0$

In this section we show that the operator

$$\mathcal{L}_0 := \sum_{j=1}^{m} X_j^2 - X_0 \tag{3.1}$$

has a fundamental solution γ with pole at $x = 0$ given by

$$\gamma(x) := \int_0^\infty \Gamma(x,t)\ dt \tag{3.2}$$

where Γ is the fundamental solution of $\mathcal{L}$, with pole at $(x,t) = (0,0)$ found in Section 2.

First of all we remark that the integral to the right hand side of (3.2) is convergent for every $x \in \mathbb{R}^N \backslash \{0\}$. Indeed, it is enough to observe that, from inequality (2.3),

$$\Gamma(x,t) = O\left(t^{-\frac{Q-2}{2}}\right) \qquad \text{as } t \longrightarrow +\infty$$

and remind that $Q - 2 \geq 3$.
It is easy to see that γ is D_λ-homogeneous of degree $-Q+4$. Indeed for every $x \neq 0$ we have

$$\begin{aligned} \gamma(D_\lambda(x)) &= \int_0^\infty \Gamma\left(D_\lambda(x), t\right)\ dt \\ &= \lambda^{-Q+2} \int_0^\infty \Gamma(x, \lambda^{-2}t)\ dt = \lambda^{-Q+4} \int_0^\infty \Gamma(x,t)\ dt \\ &= \lambda^{-Q+4}\gamma(x). \end{aligned}$$

A trivial application of Lebesgue Dominated Convergence Theorem shows that γ is continuous out of the origin. Moreover, since $\Gamma(x,t) > 0$ for $t > 0$, $\gamma(x) > 0$ for any $x \neq 0$.

From these properties of γ we immediately get the following estimates: there exists a constant $C > 0$ such that

$$\frac{1}{C}|x|^{4-Q} \leq \gamma(x) \leq C|x|^{4-Q} \tag{3.3}$$

where $|\cdot|$ stands for the D_λ homogeneous norm (1.4).
From the bounds (3.3) it follows that $\gamma \in L^1_{\text{loc}}(\mathbb{R}^N \backslash \{0\})$.

Proposition 3.1. *In the weak sense of distributions we have*

$$\mathcal{L}_0\gamma = -\delta, \qquad \delta := \textit{Dirac measure at } x = 0.$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ and define

$$\phi_k(x,t) = \varphi(x)\psi\left(\frac{t}{k}\right), \qquad k \in \mathbb{N},$$

where $\psi \in C_0^\infty(\mathbb{R})$, $\psi(s) = 0$ if $|s| \geq 2$, $\psi(s) = 1$ if $|s| \leq 1$.

Since $\Gamma(x,t)$ is the fundamental solution of $\mathcal{L}$ with pole at $(x,t)=(0,0)$, we have

$$-\varphi(0) = -\phi_k(0,0) = \int_{\mathbb{R}^{N+1}} \Gamma(x,t)\mathcal{L}^*\phi_k(x,t)\ dxdt$$

so that, as $k \longrightarrow \infty$,

$$\begin{aligned} -\varphi(0) &= \int_{\mathbb{R}^{N+1}} \Gamma(x,t)\mathcal{L}_0^*\varphi(x)\ dxdt \\ &= \int_0^\infty \left(\int_{\mathbb{R}^N} \Gamma(x,t)dt\right)\mathcal{L}_0^*\varphi(x)\ dx = \int_{\mathbb{R}^N} \gamma(x)\mathcal{L}_0^*\varphi(x)\ dx. \end{aligned}$$

□

Corollary 3.2. $\gamma \in C^\infty(\mathbb{R}\backslash\{0\})$.

Proof. It immediately follows from the previous proposition and the hypoellipticity of $\mathcal{L}_0$. □

4. Radial solution to $\mathcal{L}_0 u = 0$

Let γ be the fundamental solution of $\mathcal{L}_0$ found in the previous Section. Define

$$\rho(x) := (\gamma(x))^{-\frac{1}{Q-4}}.$$

The results of Section 3 show that ρ is D_λ-homogeneous of degree one, continuous in $\mathbb{R}^N$, smooth and strictly positive in $\mathbb{R}^N\backslash\{0\}$. Moreover, for a suitable $C>0$,

$$\frac{1}{C} \le \frac{\rho(x)}{|x|} \le C \qquad \forall x \neq 0.$$

We say that $u : \mathbb{R}^N\backslash\{0\} \longrightarrow \mathbb{R}$ is a *radial* function if there exists $f :]0,\infty[\longrightarrow \mathbb{R}$ such that $u(x) = f(\rho(x))$ for any $x \neq 0$.

Proposition 4.1. *Let $u = f(\rho)$ be a radial function. Then, if f is smooth,*

$$\mathcal{L}_0 u = |\nabla_{\mathcal{L}}\rho|^2 \left(\frac{Q-3}{\rho} f'(\rho) + f''(\rho)\right) \qquad \text{in } \mathbb{R}^N\backslash\{0\}, \tag{4.1}$$

where $|\nabla_{\mathcal{L}}\rho|^2 := \sum_{j=1}^m (X_j\rho)^2$.

Proof. A direct easy computation shows that

$$\mathcal{L}_0 u = f''(\rho)|\nabla_{\mathcal{L}}\rho|^2 + f'(\rho)\mathcal{L}_0\rho. \tag{4.2}$$

On the other hand $\rho^{4-Q} = \gamma$ is $\mathcal{L}_0$-harmonic in $\mathbb{R}^N\backslash\{0\}$. Then, using (4.2) with $f(s) = s^{4-Q}$, we get

$$\mathcal{L}_0\rho = (Q-3)\frac{|\nabla_{\mathcal{L}}\rho|^2}{\rho}.$$

Replacing this in (4.2) we obtain (4.1). □

5. A gaussian estimate for Γ

In this section we show the following upper estimate of Γ.

$$\Gamma(x,t) \leq \frac{C}{t^{\frac{Q-2}{2}}} \exp\left(-\frac{\rho^2(x)}{Ct}\right) \qquad \forall x \in \mathbb{R}^N,\ \forall t > 0, \tag{5.1}$$

being C a positive constant.

Following an idea in [4], let us put

$$A := \{x \in \mathbb{R}^N \mid \rho(x) \geq 1\} \quad \text{and} \quad B = A \times]0,1[$$

and define

$$w(x,t) := \exp(-\sigma(1-t)\rho^2(x)),$$

where $\sigma > 0$ will be fixed later.

Using Proposition 4.1 we obtain

$$\begin{aligned} \mathcal{L}w &= (\mathcal{L}_0 - \partial_t)w \\ &= w|\nabla_{\mathcal{L}}\rho|^2(-2(Q-3)\sigma(1-t) + 4(\sigma\rho(1-t))^2 - 2\sigma(1-t)) - w\sigma\rho^2, \end{aligned}$$

so that, in B,

$$\mathcal{L}w \leq \rho^2 w\sigma(-1 + 4\sigma|\nabla_{\mathcal{L}}\rho|^2). \tag{5.2}$$

On the other hand, since $|\nabla_{\mathcal{L}}\rho|$ is D_λ-homogeneous of degree zero and continuous in $\mathbb{R}^N \backslash \{0\}$,

$$\sup_{\mathbb{R}^N\backslash\{0\}} |\nabla_{\mathcal{L}}\rho|^2 = \sup_{|x|=1} |\nabla_{\mathcal{L}}\rho(x)|^2 =: C_0 < \infty.$$

Then, if we choose $0 < \sigma < \frac{1}{4C_0}$, from (5.2) we get

$$\mathcal{L}w \leq 0 \qquad \text{in } B.$$

Let us now put

$$C_1 := \sup\left\{\frac{\Gamma(x,t)}{w(x,t)} \;\Big|\; \rho(x) = 1,\ 0 \leq t \leq 1\right\}$$

Then $\mathcal{L}(\Gamma - C_1 w) \geq 0$ in B, $\Gamma - C_1 w \leq 0$ on ∂B and $\Gamma - Cw$ goes to zero as $z \longrightarrow \infty$ in B. From the *parabolic* maximum principle of Proposition 2.2, we thus obtain $\Gamma \leq C_1 w$ in B. In particular,

$$\Gamma\left(x, \frac{1}{2}\right) \leq C_1 \exp\left(-\sigma\frac{\rho^2(x)}{2}\right) \qquad \text{if} \quad \rho(x) \geq 1,$$

so that, by the d_λ-homogeneity of Γ,

$$\begin{aligned} \Gamma(x,t) &= (2t)^{-\frac{Q-2}{2}}\Gamma\left(D_{\frac{1}{\sqrt{2t}}}(x), \frac{1}{2}\right) \\ &\leq C_2 t^{-\frac{Q-2}{2}} \exp\left(-\sigma\frac{\rho^2(x)}{4t}\right) \qquad \text{if} \quad \frac{\rho(x)}{\sqrt{2t}} \geq 1. \end{aligned}$$

On the other hand for a suitable constant $C_3 > 0$ the inequality

$$\Gamma(x,t) \le C_3 t^{-\frac{Q-2}{2}} \exp\left(-\sigma \frac{\rho^2(x)}{4t}\right)$$

trivially holds if $\frac{\rho(x)}{\sqrt{2t}} \le 1$, due to the d_λ-homogeneity of Γ. This complete the proof of (5.1).

6. An integral identity for Γ

The aim of this section is to prove the identity

$$\int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\, d\xi = 1 \qquad \forall x \in \mathbb{R}^N,\ \forall t > \tau. \tag{6.1}$$

We first remark that

$$\int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\, d\xi \le 1 \qquad \forall x \in \mathbb{R}^N,\ \forall t > \tau. \tag{6.2}$$

This inequality can be proved just proceeding as in the proof of Lemma 4.1 and Lemma 4.2 in [13], and by using the gaussian upper estimate of Γ found in the previous section.

Next lemma, together with (6.2), is crucial to show (6.1).

Lemma 6.1. *For every $(x_0, t_0) \in \mathbb{R}^N$ and $t_1 < t_0$ we have*

$$\int_{t_1}^{t_0} \left(\int_{\mathbb{R}^N} \Gamma(x_0,t_0,\xi,\tau)\, d\xi \right) d\tau = t_0 - t_1 \tag{6.3}$$

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^N)$ be such that $\phi(\xi) = 1$ if $|\xi| \le 1$ and $\phi(\xi) = 0$ if $|\xi| \ge 2$. Moreover let $(\psi_k)_{k\in\mathbb{N}}$ be a sequence of smooth functions in $C_0^\infty(\mathbb{R})$ such that $0 \le \psi_k \le 1$, $\psi_k(s) = 0$ if $s \le t_1$, $\psi_k(t_0) = 1$, $\psi_k(s) = 0$ if $s \ge 2t_0$, $|\psi_k'(s)| \le 2$ and

$$\psi_k'(s) \longrightarrow \frac{1}{t_0 - t_1} \qquad \text{as } k \longrightarrow \infty, \quad \text{if } t_1 < s < t_0.$$

Then, since Γ is a fundamental solution of $\mathcal{L}$,

$$\begin{aligned}
\phi\left(\frac{x_0}{k}\right)\psi_k(t_0) &= -\int_{-\infty}^{+\infty}\int_{\mathbb{R}^N} \Gamma(x_0,t_0,\xi,\tau)\mathcal{L}\left(\phi\left(\frac{\xi}{k}\right)\psi_k(\tau)\right) d\xi\, d\tau \\
&= -\int_{t_1}^{t_0}\left(\int_{k\le|\xi|\le 2k} \Gamma(x_0,t_0,\xi,\tau)\mathcal{L}_0\left(\phi\left(\frac{\xi}{k}\right)\right)\psi_k(\tau)\, d\xi\right) d\tau \\
&+ \int_{t_1}^{t_0}\left(\int_{\mathbb{R}^N} \Gamma(x_0,t_0,\xi,\tau)\phi\left(\frac{\xi}{k}\right)\psi_k'(\tau)\, d\xi\right) d\tau.
\end{aligned}$$

Letting k go to infinity, and using again the gaussian estimate of Section 5, we easily obtain

$$1 = \int_{t_1}^{t_0}\int_{\mathbb{R}^N} \Gamma(x_0,t_0,\xi,\tau)\frac{1}{t_0-t_1}\, d\xi\, d\tau$$

Thus, (6.3) holds.
The identity (6.3) clearly implies

$$\int_{t_1}^{t} \left(1 - \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\ d\xi\right)\ d\tau = 0$$

for every t, $t_1 \in \mathbb{R}$, $t_1 < t$, so that, by (6.2),

$$1 = \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\ d\xi \qquad \text{for a.e. } \tau < t.$$

Then, the assertion follows by the continuity of the right hand side with respect to τ. □

7. Harnack inequality for $\mathcal{L}$

In this section we shall prove a Harnack inequality for the nonnegative solutions to $\mathcal{L}u = 0$. Our result extends the Harnack inequality proved in [10]. To begin with we introduce the notation needed to state the inequality.

For $r > 0$ and $z_0 \in \mathbb{R}^{N+1}$ we define

$$C_r(z_0) := z_0 \circ d_r(C_1) \quad \text{and} \quad S_r(z_0) = z_0 \circ d_r(S_1)$$

where

$$C_1 = \left\{ z \in \mathbb{R}^{N+1} \ : \ |z| \le 1 \right\} \quad \text{and} \quad S_1 = \left\{ z \in C_1 \ : \ \frac{1}{4} \le -t \le \frac{3}{4} \right\}.$$

Then, the following theorem holds.

Theorem 7.1. *Let O be an open set of $\mathbb{R}^{N+1}$ containing $C_r(z_0)$ for some $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$. Then, there exist two positive constants $\theta = \theta(\mathcal{L})$ and $C = C(\mathcal{L})$, $0 < \theta < 1$, such that*

$$\sup_{S_{\theta r}(z_0)} u \le C u(z_0) \tag{7.1}$$

for every $u \ge 0$ solution to $\mathcal{L}u = 0$ in O.

We shall prove this theorem by using a *mean value* property of the $\mathcal{L}$-harmonic functions.

Given $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$ we define the *$\mathcal{L}$-ball of center z_0 and radius r* as follows

$$\Omega_r(z_0) := \left\{ z \in \mathbb{R}^{N+1} \ : \ \Gamma(z^{-1} \circ z_0) > \left(\frac{1}{r}\right)^{Q-2} \right\}.$$

Obviously

$$\Omega_r(z_0) = z_0 \circ \Omega_r \quad \text{where} \quad \Omega_r := \Omega_r(0).$$

From Proposition 2.8 and Corollary 2.9 it follows that $\Omega_r(z_0)$ is bounded and non empty for every $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$. Then $\overline{\Omega}_1 \subseteq C_{R_0}$ where

$$R_0 = \max_{z \in \overline{\Omega}_1} |z|.$$

Thus

$$\Omega_{\frac{r}{R_0}}(z_0) \subseteq C_r(z_0) \qquad \forall z_0 \in \mathbb{R}^{N+1},\ \forall r > 0. \tag{7.2}$$

We also note that

$$\bigcup_{r>0} \Omega_r = \mathbb{R}^N \times] - \infty, 0[. \tag{7.3}$$

Indeed, if $z = (x,t) \in \Omega_r$ then $\Gamma((x,t)^{-1}) > (\frac{1}{r})^{Q-2}$. On the other hand $(x,t)^{-1} = (y,-t)$ for a suitable $y \in \mathbb{R}^N$. It follows that $\Gamma(y,-t) > (\frac{1}{r})^{Q-2}$, hence $-t > 0$ so that $z = (x,t) \in \mathbb{R}^N \times] - \infty, 0[$.
Viceversa, given $z = (x,t) \in \mathbb{R}^N \times] - \infty, 0[$, for a suitable $y \in \mathbb{R}^N$, we have $\Gamma((x,t)^{-1}) = \Gamma(y,-t) > 0$. This implies $\Gamma((x,t)^{-1}) > (\frac{1}{r})^{Q-2}$ for a suitable $r > 0$, hence $(x,t) \in \Omega_r$. This completes the proof of (7.3).

Then, since S_1 is a compact subset of $\mathbb{R}^N \times] - \infty, 0[$, there exists $R_1 > 0$ such that $S_1 \subseteq \Omega_{R_1}$. As a consequence

$$S_r \subseteq \Omega_{rR_1} \qquad \forall r > 0.$$

This inclusion, together with (7.2), gives

$$S_{\theta r}(z_0) \subseteq \Omega_{\frac{r}{R_0}}(z_0) \subseteq C_r(z_0) \qquad \forall r > 0. \tag{7.4}$$

where $\theta = \frac{1}{R_0 R_1}$. The properties of Γ showed in the previous sections imply the following Mean Value Theorem for $\mathcal{L}$-harmonic functions.

Proposition 7.2. *Let u be a (smooth) solution to $\mathcal{L}u = 0$ in the open set $O \in \mathbb{R}^{N+1}$. Then, for every $\mathcal{L}$-ball $\Omega_r(z_0)$ with closure contained in O, we have*

$$u(z_0) = \left(\frac{1}{r}\right)^{Q-2} \int_{\Omega_r(z_0)} u(\zeta) K(z_0, \zeta) \ \ d\zeta, \tag{7.5}$$

where

$$K(z,\zeta) = \frac{|\nabla_{\mathcal{L}} \Gamma(z,\zeta)|^2}{\Gamma^2(z,\zeta)}, \qquad \nabla_{\mathcal{L}} = (X_1, \ldots, X_m).$$

Here $\nabla_{\mathcal{L}}$ acts on the variable ζ.

Proof. Let us write the X_j' s as follows:

$$X_j = \sum_{k=1}^{N} a_j^{(k)} \partial_{x_k}, \qquad j = 0, 1, \ldots, m.$$

Since these vector fields are d_λ-homogeneous of a strictly positive degree, the coefficient $a_j^{(k)}$ is independent of x_k, $k \in \{1, \ldots, N\}$, $j \in \{0, 1, \ldots, m\}$. Then, the

X_j's are divergence free, i.e. $\text{div} X_j \equiv 0$ in $\mathbb{R}^N$ for every $j \in \{0, 1, \ldots, m\}$.
As a consequence, for a suitable $N \times N$ matrix A, the operator $\mathcal{L}$ takes the following form

$$\mathcal{L} = \text{div}(A\nabla) + Y, \qquad \nabla = (\partial_{x_1}, \ldots, \partial_{x_N}), \tag{7.6}$$

and Y is divergence free in $\mathbb{R}^{N+1}$. Moreover

$$< A(x)\xi, \xi >= \sum_{j=1}^{m} < X_j(x), \xi >^2 \qquad \text{for every } x, \xi \in \mathbb{R}^N.$$

Then, in order to prove identity (7.5) it is now enough to proceed exactly as in [14], pages 308-313, by using the properties of Γ showed in Sections 5 and 6, together with inequality (5.1) and identity (6.1) . $\square$

For the proof of Harnack inequality, we need the following lemma

Lemma 7.3. *Let $z = (x, t) \in \mathbb{R}^{N+1}$ be fixed. Then the set*

$$\Sigma := \ \{\zeta = (\xi, \tau) \in \mathbb{R}^{N+1} \mid \tau < t, \ K(z, \zeta) = 0\}$$

does not contain interior points.

Proof. By contradiction, assume $K(z, \zeta) = 0$ for every $\zeta \in U$, with U an open subset of $\mathbb{R}^N \times] - \infty, t[$. Then $X_j\Gamma(z, \cdot) \equiv 0$ in U for any $j = 1, \ldots, m$, hence $\sum_{j=1}^{m} X_j^2 (\Gamma(z, \cdot)) \equiv 0$ in the same open set. Since $\mathcal{L}^*\Gamma(z, \cdot) \equiv 0$ in $\mathbb{R}^N \times] - \infty, t[$, this implies

$$(X_0 - \partial_t)\Gamma(z, \cdot) \equiv 0 \qquad \text{in } U.$$

As a consequence, by the rank condition, the euclidean gradient of $\Gamma(z, \cdot)$ is identically zero in U. Then $\Gamma(\zeta^{-1} \circ z) = C_0$ for any $\zeta \in U$. The change of variable $\zeta^{-1} \circ z = z'$ gives $\Gamma(z') = C_0$ for any $z' \in U' = U^{-1} \circ z$. Writing $z' = (x', t')$ and reminding that the composition law in $\mathbb{L}$ restricted to the time axis is the euclidean one, we have $t' > 0$ (the time component of $\zeta^{-1} \circ z$ is $t - \tau$ and $\tau < t$). Therefore $C_0 > 0$. This is absurd because Γ is d_λ-homogeneous of degree $-Q + 2$ and $-Q + 2 \neq 0$. $\square$

We now state a convergence theorem, easy consequence of a weak Harnack inequality due to Bony.

Proposition 7.4. *Let (u_n) be a sequence of $\mathcal{L}$-harmonic functions in a open set $O \subseteq \mathbb{R}^N$. Suppose (u_n) monotone increasing and such that*

$$u := \sup_{n \in \mathbb{N}} u_n < \infty$$

in a dense subset T of O. Then $u < \infty$ everywhere, $u \in C^\infty(O)$ and satisfies $\mathcal{L}(u) = 0$ in O.

Proof. By Theorem 7.1 in [5], for every fixed compact set $K \subseteq O$, there exist $x_1, \ldots, x_p \in T$ and a constant $C > 0$ such that

$$\sup_K (u_n - u_m) \le C \sum_{j=1}^{p} (u_n(x_j) - u_m(x_j)) \qquad \forall n \ge m.$$

Then, since $(u_n - u_m)(x_j) \longrightarrow 0$ as $n, m \longrightarrow \infty$, for any $j \in \{1, \ldots, p\}$, the sequence (u_n) is locally uniformly convergent in O, so that u is finite everywhere. Moreover, since $\mathcal{L} u_n = 0$ for any $n \in \mathbb{N}$, it follows that $\mathcal{L} u = 0$ in O in the weak sense of distributions. The hypoellipticity of $\mathcal{L}$ implies that $u \in C^\infty(O)$ and satisfies the equation in the classical sense. □

Proof. (of Theorem 7.1) Since $\mathcal{L}$ is left translation invariant on $\mathbb{L}$ and d_λ-homogeneous (of degree two), it is enough to prove inequality (7.1) for $z_0 = 0$ and $r = 1$. We argue by contradiction and assume that (7.1), with $z_0 = 0$ and $r = 1$, is false. Then, for every $n \in \mathbb{N}$ there exists $u_n \in C^\infty(O)$, $u \ge 0$, such that $\mathcal{L} u_n = 0$ and

$$\sup_{S_\theta(z_0)} u_n > 4^n u_n(z_0). \tag{7.7}$$

Let us now use the second inclusion in (7.4) and the mean value property of Proposition 7.2 to state that

$$u_n(z_0) = \left(\frac{1}{\rho}\right)^{Q-2} \int_{\Omega_\rho(z_0)} K(z_0, \zeta) u_n(\zeta)\, d\zeta, \qquad \rho = \frac{1}{R_0}.$$

Since K is nonnegative and strictly positive in a dense open subset of $\Omega_\rho(z_0)$, see Lemma 7.3, this identity and inequality (7.7) imply $u_n(z_0) > 0$.
Let us now define

$$w_n = \frac{u_n}{u_n(z_0)} \qquad \text{and} \qquad w = \sum_{n=1}^{\infty} \frac{w_n}{2^n}.$$

Then $w(z_0) = 1$ and

$$1 = w(z_0) = \left(\frac{1}{\rho}\right)^{Q-2} \int_{\Omega_\rho(z_0)} K(z_0, \zeta) w(\zeta)\, d\zeta.$$

As a consequence, by the positivity property of K (Lemma 7.3), we get $w < \infty$ in a dense subset of $\Omega_\rho(z_0)$. By Proposition 7.4, $w \in C^\infty(\Omega_\rho(z_0))$ and $\mathcal{L} w = 0$ in $\Omega_\rho(z_0)$. In particular, since $S_\theta(z_0)$ is a compact subset of $\Omega_\rho(z_0)$ (see (7.4)) we have

$$\sup_{S_\theta(z_0)} w < \infty. \tag{7.8}$$

On the other hand, by inequality (7.7)

$$\sup_{S_\theta(z_0)} w \geq \sup_{S_\theta(z_0)} \frac{w_n}{2^n} \geq 2^n \qquad \text{for any } n \in \mathbb{N}.$$

This contradicts (7.8) and completes the proof. □

8. A Harnack inequality for $\mathcal{L}_0$

In this Section we show a kind of Harnack inequality for nonnegative solutions to $\mathcal{L}_0 u = 0$. For $r > 0$ we shall denote by B_r the *ball*

$$B_r := \{x \in \mathbb{R}^N \ : \ |x| < r\}.$$

Our result reads as follows.

Theorem 8.1. *There exist two positive constants C and $\lambda > 0$ such that*

$$\sup_{B_r} u \leq C \inf_{B_r} u \tag{8.1}$$

for every nonnegative (smooth) solution to $\mathcal{L}_0 u = 0$ in an open set $O \supseteq B_{\lambda r}$.

In order to prove this theorem we need the following lemma, in which we shall use the notations of the previous section.

Lemma 8.2. *There exists $R > 0$ such that*

$$(x, 0) \in S_R\left(\left(y, \frac{R^2}{2}\right)\right) \tag{8.2}$$

for every $x, y \in \mathbb{R}^N$ such that $|x|, |y| \leq 1$.

Proof. The inclusion (8.2) is equivalent to the following one

$$(x, 0) \circ \left(y, \frac{R^2}{2}\right)^{-1} = d_R(\xi, \tau) \tag{8.3}$$

for some $(\xi, \tau) \in \mathbb{R}^{N+1}$ satisfying $|\xi| \leq 1$ and $\frac{1}{4} \leq -\tau \leq \frac{3}{4}$.
Since $d_R(\xi, \tau) = (C_R(\xi), R^2\tau)$ and d_R is an automorphism of $\mathbb{L}$, identity (8.3) means

$$\left(C_{\frac{1}{R}}(x), 0\right) \circ \left(C_{\frac{1}{R}}(y), \frac{1}{2}\right)^{-1} = (\xi, \tau) \tag{8.4}$$

Now, as $R \longrightarrow \infty$ the left hand side of (8.4) converges to

$$(0, 0) \circ \left(0, \frac{1}{2}\right)^{-1} = \left(0, -\frac{1}{2}\right)$$

uniformly with respect to $x, y \in \{(x, y) \ : \ |x|, |y| \leq 1\}$. Then the assertion follows. □

Proof. (of Theorem 8.1) Since $\mathcal{L}_0$ is D_λ-homogeneous it is enough to prove (8.1) for $r = 1$.

Let $x, y \in \mathbb{R}^N$ be such that $|x|, |y| \leq 1$. By the previous lemma there exists $R > 0$ such that

$$\{(x, 0) \ : \ |x| \leq 1\} \subseteq S_R\left(\left(y, \frac{R^2}{2}\right)\right).$$

Now, let u be a nonnegative solution to $\mathcal{L}_0 u = 0$ in O.

Since $U(x, t) := u(x)$ is independent of t, we have

$$\mathcal{L}U = 0 \qquad \text{in } O \times \mathbb{R}.$$

Let us denote by λ a real positive constant such that

$$\bigcup_{|y|\leq 1} C_{\frac{R}{\theta}}\left(\left(y, \frac{R^2}{2}\right)\right) \subseteq B_\lambda \times \mathbb{R}.$$

Here θ denotes the constant appearing in Theorem 7.1.

Then, if $O \supseteq B_\lambda$, it follows that $C_{\frac{R}{\theta}}\left(\left(y, \frac{R^2}{2}\right)\right) \subseteq B_\lambda \times \mathbb{R} \subseteq O \times \mathbb{R}$ for every $y \in \mathbb{R}^N$, $|y| \leq 1$. Thus, by Theorem 7.1, denoting by z the point $\left(y, \frac{R^2}{2}\right)$, we have

$$u(x) = U(x, 0) \leq \sup_{S_R(z)} U \leq CU(z) = Cu(y).$$

Hence

$$u(x) \leq Cu(y) \qquad \forall x, y \in \mathbb{R}^N, \ |x|, |y| \leq 1.$$

This completes the proof. □

From Theorem 8.1 a one-side Liouville-type theorem easily follows.

Corollary 8.3. *Let u be a (smooth) nonnegative solution to*

$$\mathcal{L}_0 u = 0 \qquad \text{in } \ \mathbb{R}^N.$$

Then $u \equiv \text{const}$.

Proof. Let $m := \inf_{\mathbb{R}^N} u$ and put $v := u - m$. Then $v \geq 0$ and $\mathcal{L}_0(v) = 0$ in $\mathbb{R}^N$. From Theorem 8.1 we get

$$0 \leq \sup_{B_r} u \leq C \inf_{B_r} u \qquad \forall r > 0.$$

As $r \longrightarrow \infty$ we obtain

$$0 \leq \sup_{\mathbb{R}^N} u \leq C \inf_{\mathbb{R}^N} u = 0$$

so that $v \equiv 0$, i.e. $u \equiv m$. □

9. Examples

Example 9.1 (Heat operators on Carnot groups).

Let $(\mathbb{R}^N, \circ)$ be a Lie group in $\mathbb{R}^N$. Assume that $\mathbb{R}^N$ can be split as follows

$$\mathbb{R}^N = \mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_m}$$

and that the dilations

$$\begin{aligned} D_\lambda : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad D_\lambda(x) &= D_\lambda(x^{(N_1)}, \ldots, x^{(N_m)}) \\ &:= (\lambda x^{(N_1)}, \ldots, \lambda^m x^{(N_m)}) \end{aligned}$$

$$x^{(N_i)} \in \mathbb{R}^{N_i}, \quad i = 1, \ldots, m, \quad \lambda > 0,$$

are automorphisms of $(\mathbb{R}^N, \circ)$.
We also assume

$$\text{rank Lie}\{X_1, \ldots, X_{N_1}\}(x) = N \quad \forall x \in \mathbb{R}^N \tag{9.1}$$

where the X_j's are left invariant on $(\mathbb{R}^N, \circ)$ and

$$X_j(0) = \frac{\partial}{\partial x_j^{(N_1)}}, \qquad j = 1, \ldots, N_1.$$

Then $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ is said to be a *Carnot group* whose *homogeneous dimension* Q_0 is the natural number

$$Q_0 := N_1 + 2N_2 + \ldots + mN_m.$$

The vector fields $X_1, \ldots, X_{N_1}$ are the *generators* of $\mathbb{G}$,

$$\Delta_{\mathbb{G}} := \sum_{j=1}^{N_1} X_j^2$$

is the *canonical sub-Laplacian* on $\mathbb{G}$ and the parabolic operator

$$\mathcal{L} = \Delta_{\mathbb{G}} - \partial_t \qquad \text{in } \mathbb{R}^{N+1} \tag{9.2}$$

is called the *canonical heat operator* on $\mathbb{G}$. Obviously $\mathcal{L}$ is an operator of the type (1.1) with $X_0 = 0$.
If we define

$$\mathbb{L} = (\mathbb{R}^{N+1}, \circ, d_\lambda)$$

with $d_\lambda(x,t) = (D_\lambda(x), \lambda^2 t)$ and the composition law $\circ$ given by

$$(x,t) \circ (x', t') = (x \circ x', t + t')$$

then $\mathbb{L}$ is a homogeneous group, and the operator $\mathcal{L}$ in (9.2) satisfies condition (H1) in the Introduction. We explicitly remark that the homogeneous dimension of $\mathbb{L}$ is $Q := Q_0 + 2$.
Let us now show that $\mathcal{L}$ also satisfies (H2). Let $(x,t), (y,\tau) \in \mathbb{R}^{N+1}$ be such that $\tau < t$. By the rank condition (9.1) we can apply Chow' s Theorem to state the existence of a piecewise regular path $\eta : [0,T] \longrightarrow \mathbb{R}^N$, whose regular components

are integral curves of a vector fields in the family $\{\pm X_1, \dots, \pm X_m\}$, such that $\eta(0) = x$, $\eta(T) = y$. Then, the path $\widehat{\eta} : [0, T + t - \tau] \longrightarrow \mathbb{R}^{N+1}$,

$$\widehat{\eta}(s) = \begin{cases} (\eta(s), t) & \text{if} \quad 0 \le s \le T \\ \\ (y, t + T - s) & \text{if} \quad T \le s \le T + t - \tau \end{cases},$$

is an $\mathcal{L}$-admissible curve connecting (x, t) and (y, τ).

Example 9.2 ("Parabolic" operators on Carnot group).

Let $\mathbb{G} = (\mathbb{R}^N, \circ, D_\lambda)$ be a Carnot group with generators

$$X_1, \dots, X_{N_1}.$$

Assume $N_1 < N$ and choose a vector field X_0 in the second layer of the Lie algebra of $\mathbb{G}$. More clearly,

$$X_0 \in \text{span}\{[X_j, X_k] \ : \ j, k = 1, \dots, N_1\}.$$

Then

$$\mathcal{L} = \sum_{j=1}^{N_1} X_j^2 + X_0 - \partial_t$$

is an operator of the type (1.1) trivially satisfying condition (H1) with respect to the homogeneous group

$$\mathbb{L} = (\mathbb{R}^{N+1}, \circ, d_\lambda)$$

of the previous example.

Let us now show that $\mathcal{L}$ also satisfies (H2). Let $(x, t), (y, \tau) \in \mathbb{R}^N, \tau < t$, be arbitrarily given. Let $\tilde{\eta} = \tilde{\eta}(s) = (\eta(s), s + \tau),\ s \ge 0$, be the integral curve of $-X_0 + \partial_t$ such that $\tilde{\eta}(0) = (y, \tau)$. $\tilde{\eta}$ is defined for every $s \ge 0$. Denote by $\overline{z}$ the point

$$\overline{z} := \tilde{\eta}(t - \tau) = (\eta(t - \tau), t)$$

Since rank Lie$\{X_1, \dots, X_{N_1}\}(x) = N$ for every $x \in \mathbb{R}^N$, by Chow' s Theorem there exists a piecewise regular path $\hat{\eta} : [0, \hat{T}] \longrightarrow \mathbb{R}^N$, whose regular components are integral curve of a vector field in $\{\pm X_1, \dots, \pm X_{N_1}\}$, such that $\hat{\eta}(0) = x,\ \hat{\eta}(1) = \overline{x}$. Define

$$\bar{\eta} : [0, \hat{T}] \longrightarrow \mathbb{R}^N, \quad \bar{\eta}(s) = (\hat{\eta}(s), t).$$

Then $\bar{\eta} + (-\tilde{\eta})$ is an $\mathcal{L}$-admissible path connecting (x, t) and (y, τ). This shows that conditions (H2) is satisfied.

Example 9.3 (Kolmogorov operators).

Let us split $\mathbb{R}^N$ as follows

$$\mathbb{R}^N = \mathbb{R}^p \times \mathbb{R}^r$$

and denote by $x = (x^{(p)}, x^{(r)})$ its points. Let B a $N \times N$ real matrix taking the following block form

$$B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ B_1 & 0 & 0 & \dots & 0 \\ 0 & B_2 & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & B_k & 0 \end{pmatrix}$$

where B_j is a $r_j \times r_{j-1}$ matrix with rank r_j, and $r_0 = p \geq r_1 \geq \ldots \geq r_k \geq 1$, $r_0 + r_1 + \ldots + r_k = N$. Denote

$$E(t) = \exp(-tB)$$

and introduce in $\mathbb{R}^{N+1}$ the following composition law

$$(x,t) \circ (y,\tau) := (y + E(-\tau)x, t + \tau). \tag{9.3}$$

In [15] it is proved that

$$\mathbb{K} = (\mathbb{R}^{N+1}, \circ, d_\lambda)$$

is a homogeneous Lie group with respect to the dilations

$$\begin{aligned} d_\lambda(x,t) &= d_\lambda(x^{(p)}, x^{(r_1)}, \ldots, x^{(r_k)}, t) \\ &= (\lambda x^{(p)}, \lambda^3 x^{(r_1)}, \ldots, \lambda^{2k+1} x^{(r_k)}, \lambda^2 t) \end{aligned}$$

The homogeneous dimension of $\mathbb{K}$ is

$$Q = p + 3r_1 + \ldots + (2k+1)r_k + 2.$$

We call $\mathbb{K}$ a *Kolmogorov-type group.*

Let us now consider the operator

$$\mathcal{K} \;=\; \Delta_{\mathbb{R}_p} + < Bx, D > -\partial_t,$$

where $\Delta_{\mathbb{R}_p}$ denotes the usual Laplace operator in $\mathbb{R}^p$, $<,>$ is the inner product in $\mathbb{R}^N$ and $D = (\partial_{x_1}, \ldots, \partial_{x_N})$. It is easy to see that $\mathcal{K}$ can be written as in (1.1) with $m = p$, $X_j = \partial_{x_j}$, $1 \leq j \leq p$, and $X_0 =< Bx, D >$. The first order partial differential operator

$$Y =< Bx, D > -\partial_t$$

will be called the *total derivative operator* on $\mathbb{K}$. By Proposition 2.2 in [15], Y is d_λ-homogeneous of degree two. Moreover, the operator $\mathcal{K}$ satisfies condition (H1) with $\mathbb{L}$ replaced by the group $\mathbb{K}$.

Let us now prove that $\mathcal{K}$ also satisfies (H2). Let $(x_0, t_0) \in \mathbb{R}^{N+1}$ be arbitrarily fixed and define

$$\begin{aligned} F = \{(x,t) \in \mathbb{R}^{N+1} \quad | \quad & \text{there exists a } \mathcal{K}\text{-admissible path} \\ & \eta : [0,T] \longrightarrow \mathbb{R}^{N+1} \;:\; \eta(0) = (x_0,t_0),\; \eta(T) = (x,t)\}. \end{aligned}$$

The following claim will show (H2).

Claim 9.4. $F = \mathbb{R}^N \times]-\infty, t_0[$.

Proof. We split the proof of this claim in two steps.
Step 1. Since $X_j = \partial_{x_j}$, $j \in \{1, \dots, p\}$, it is quite obvious that

$$(x^{(p)}, \overline{x}^{(r)}, \bar{t}) \in F \qquad \forall x^{(p)} \in \mathbb{R}^p \tag{9.4}$$

if $(\overline{x}^{(p)}, \overline{x}^{(r)}, \bar{t}) \in F$ for some $\overline{x}^{(p)} \in \mathbb{R}^p$. In particular

$$(x^{(p)}, {x_0}^{(r)}, t_0) \in F \qquad \forall x^{(p)} \in \mathbb{R}^p \tag{9.5}$$

Step 2. Let us now use the integral curve of Y. By the sake of simplicity we assume that B is as follows

$$B = \begin{pmatrix} 0 & 0 & 0 \\ B_1 & 0 & 0 \\ 0 & B_2 & 0 \end{pmatrix}$$

where B_i is a $r_i \times r_{i-1}$ matrix with rank r_i, $i = 1, 2,\ r_0 = p \geq r_1 \geq r_2,\ r_1 + r_2 = r$. The proof in the general case uses the same argument we are going to use. The integral curves of Y are given by

$$\begin{aligned} x^{(p)} &= \alpha^{(p)}, \quad x^{(r_1)} = \alpha^{(r_1)} + sB_1\alpha^{(p)} \\ x^{(r_2)} &= \alpha^{(r_2)} + sB_2\alpha^{(r_1)} + \frac{s^2}{2}B_2B_1\alpha^{(p)}, \\ t &= \tau - s, \end{aligned}$$

with $(\alpha^{(p)}, \alpha^{(r_1)}, \alpha^{(r_2)}, \tau) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^r \times \mathbb{R}$. Then, also using what proved in Step 1,

$$(y^{(p)},\ y_0^{(r_1)},\ y_0^{(r_2)},\ t_0 - s) \in F, \qquad \forall y^{(p)} \in \mathbb{R}^p,\ \forall s > 0, \tag{9.6}$$

and

$$y_0^{(r_1)} = x_0^{(r_1)} + sB_1x^{(p)}\ ,\ y_0^{(r_2)} = x_0^{(r_2)} + sB_2x_0^{(r_1)} + \frac{s^2}{2}B_2B_1x^{(p)}$$

being $x^{(p)} \in \mathbb{R}^p$, arbitrarily fixed.
Starting from the point (9.6), again with integral curves of Y, we obtain

$$(z^{(p)},\ z_0^{(r_1)},\ z_0^{(r_2)},\ t_0 - 2s) \in F, \qquad \forall z^{(p)} \in \mathbb{R}^p,\ \forall s > 0 \tag{9.7}$$

and

$$\begin{aligned} z_0^{(r_1)} &= y_0^{(r_1)} + sB_1y^{(p)} = x_0^{(r_1)} + sB_1(x^{(p)} + y^{(p)}), \\ z_0^{(r_2)} &= y_0^{(r_2)} + sB_2y_0^{(r_1)} + \frac{s^2}{2}B_2B_1y^{(p)} \\ &= x_0^{(r_2)} + sB_2x_0^{(r_1)} + \frac{s^2}{2}B_2B_1x^{(p)} \\ &+ sB_2(x_0^{(r_1)} + sB_1(x^{(p)}) + \frac{s^2}{2}B_2B_1y^{(p)} \\ &= x_0^{(r_2)} + 2sB_2x_0^{(r_1)} + \frac{s^2}{2}B_2B_1(3x^{(p)} + y^{(p)}), \end{aligned}$$

being $x^{(p)}, y^{(p)} \in \mathbb{R}^p$ arbitrarily fixed. Given $t < t_0$ let us now choose $s = \frac{t_0-t}{2}$, so that

$$t_0 - 2s = t.$$

Moreover, for every fixed $(x^{(r_1)}, x^{(r_2)}) \in \mathbb{R}^{r_1} \times \mathbb{R}^{r_2}$, let v, $w \in \mathbb{R}^p$ be such that

$$\begin{aligned} sB_1 v &= x^{(r_1)} - x_0^{(r_1)} \\ \frac{s^2}{2} B_2 B_1 w &= x^{(r_2)} - (x_0^{(r_2)} + 2sB_2 x_0^{(r_1)}). \end{aligned}$$

We explicitly remark that these linear equations are solvable since B_1 and B_2B_1 are $r_1 \times p$ and $r_2 \times p$ matrices with rank r_1 and r_2, respectively.
Then, if $x^{(p)}$, $y^{(p)} \in \mathbb{R}^p$ satisfy

$$x^{(p)} + y^{(p)} = v \qquad 3x^{(p)} + y^{(p)} = w,$$

from (9.7) we obtain

$$(z^{(p)}, x^{(r_1)}, x^{(r_2)}, t) \in F$$

for every $(z^{(p)}, x^{(r_1)}, x^{(r_2)}, t) \in \mathbb{R}^N \times]0, t_0[$.
This proves the Claim. □

Remark 9.5. The matrix $E(t)$ in (9.3) takes the following triangular form

$$E(t) = \begin{pmatrix} I_p & 0 \\ E_1(t) & I_r \end{pmatrix}$$

where I_p and I_r are the identity matrix in $\mathbb{R}^p$ and $\mathbb{R}^r$, respectively. Then, the composition law in $\mathbb{K}$ has the following structure:

$$(x^{(p)}, x^{(r)}, t) \circ (y^{(p)}, y^{(r)}, \tau) = (x^{(p)} + y^{(p)}, x^{(r)} + y^{(r)} + E_1(\tau) x^{(p)}, t + \tau)$$

Remark 9.6. For what we need in the sequel it is crucial to note that, for any fixed $(x, t) \in \mathbb{R}^{N+1}$,

$$\partial_\tau \left((x,t) \circ (y,\tau) \right) |_{(y,\tau)=(0,0)} = -Y.$$

Example 9.7 (Sub-Kolmogorov operators).

Let $G = (\mathbb{R}^p \times \mathbb{R}^q, \circ, d_\lambda^{(1)})$ be a Carnot group with first layer $\mathbb{R}^p$ (see Example 9.1). Moreover, let $\mathbb{K} = (\mathbb{R}^p \times \mathbb{R}^r \times \mathbb{R}, \circ, d_\lambda^{(2)})$ be a Kolmogorov group (see Example 9.2). Finally, let $\mathbb{L} = (\mathbb{R}^{N+1}, \circ, d_\lambda)$, $N = p + q + r$,

$$\mathbb{L} = \mathbb{G} \triangle \mathbb{K},$$

be the *link* of $\mathbb{G}$ and $\mathbb{K}$ (see Appendix, Subsection 10.2).

Consider the operator

$$\mathcal{L} = \Delta_{\mathbb{G}} + Y,$$

where

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{p} X_j^2 \qquad \text{and} \qquad Y$$

are, respectively, the canonical sub-laplacian on $\mathbb{G}$ and the *total derivative operator* on $\mathbb{K}$. We call $\mathcal{L}$ a *sub-Kolmogorov operator.*

Due to the structure of the composition law in $\mathbb{K}$ (see Remark 9.5) and thank to the Proposition 10.4 in the Appendix, the vector fields $X_1, X_2, \dots, X_p$ are left translation invariant on $\mathbb{L}$ and d_λ homogeneous of degree one. On the other hand, due to the Remark 9.6 and the Proposition 10.5 in the Appendix, Y is left translation invariant on $\mathbb{L}$ and d_λ homogeneous of degree two. Then, $\mathcal{L}$ satisfies condition (H1).

By following the same lines of the proof of Claim 9.4 in Example 9.3, it can be proved that $\mathcal{L}$ also satisfies condition (H2). We only need to remark that in the first step of the proof we have to use the rank condition

$$\text{rank Lie}\{X_1, \dots, X_p\}(x^{(p,q)}) = p + q \qquad \forall x^{(p,q)} \in \mathbb{R}^{(p,q)}$$

in order to state the $\{X_1, \dots, X_p\}$ connectivity of $\mathbb{R}^{(p,q)}$.

10. Appendix

10.1. Homogeneous Lie groups in $\mathbb{R}^N$: some reminds

We say that a Lie group $\mathbb{G} = (\mathbb{R}^N, \circ)$ is a homogeneous group if there exists an N-tuple of natural numbers $\sigma = (\sigma_1, \dots, \sigma_N)$, $\sigma_1 = 1$, such that the *dilation*

$$d_\lambda : \mathbb{R}^N \longrightarrow \mathbb{R}^N,\ d_\lambda(x_1, \dots, x_N) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_N} x_N)$$

is an automorphism of $\mathbb{G}$ for every $\lambda > 0$. One usually assumes $\sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_N$.

In what follows, we shall denote by g the Lie algebra of a homogeneous Lie group $\mathbb{G}$. As usual, we agree to identify the vector field $X = \sum_{j=1}^N a_j \partial_{x_j}$ with the vector value function $(a_1, \dots, a_N)$.

It is quite easy to recognize that the coefficients of any vector field $X \in g$ are polynomial functions. Moreover if $X_1, \dots, X_m \in g$, then

$$\dim(\text{span}\{X_1(x), \dots, X_m(x)\})$$

is independent of $x \in \mathbb{R}^N$. This last remark, together with Frobenius Theorem, immediately gives the following Proposition.

Proposition 10.1. *Let $X = \{X_1, \dots, X_m\} \subseteq g$. Assume $\mathbb{R}^N$ is X-connected, i.e.: for every $x, y \in \mathbb{R}^N$ there exists an absolutely continuous path $\eta : [0, T] \longrightarrow \mathbb{R}^N$ such that $\eta(0) = x$, $\eta(T) = y$ and*

$$\eta'(s) = \sum_{j=1}^m \lambda_j(s) X_j(\eta(s)) \qquad \textit{a.e. in } [0, T],$$

where the λ_j' s are piecewise constant real functions. Then

$$\text{rank Lie}\{X_1, \dots, X_m\}(x) = N \qquad \forall x \in \mathbb{R}^N. \tag{10.1}$$

Proof. By contradiction, assume there exists $x_0 \in \mathbb{R}^N$ such that

$$\text{rank Lie}\{X_1, \dots, X_m\}(x_0) = k < N.$$

Then, by what previously noticed,

$$\text{rank Lie}\{X_1, \dots, X_m\}(x) = k \qquad \forall x \in \mathbb{R}^N,$$

so that, by Frobenius Integrability Theorem, $\mathbb{R}^N$ is foliated in disjoint k-dimensional leafs (see [23], Vol. I, pag. 264; see also [24]). This contradicts our connectivity assumption. □

Given $X \in g$ we denote by $\exp(sX)(x)$, $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$, the solution to the Cauchy problem

$$\eta' = X(\eta), \quad \eta(0) = x.$$

The map $(s, x) \longmapsto \exp(sX)(x)$ is everywhere defined and smooth in $\mathbb{R} \times \mathbb{R}^N$. We know that

$$\text{Exp} : g \longrightarrow \mathbb{G}, \quad \text{Exp}(X) = \exp(X)(0)$$

is a global diffeomorphism whose inverse is denote by Log. Moreover, for every $x, y \in \mathbb{R}^N$,

$$x \circ y = \exp(\text{Log}(y))(x). \tag{10.2}$$

From this result we immediately get the following proposition.

Proposition 10.2. *Let us denote by π the projection*

$$\pi : \mathbb{R}^N \longrightarrow \mathbb{R}, \quad \pi(x_1, \dots, x_N) = x_N.$$

Assume that the N-th component of every $X \in g$ is a constant function i.e. $\pi(X)$ =const., for any $X \in g$. Then,

$$\pi(x \circ y) = \pi(x) + \pi(y). \tag{10.3}$$

Proof. Due to the hypothesis,

$$\pi(\exp(X)(z)) = \pi(z) + \pi(X). \tag{10.4}$$

Then

$$\pi(\text{Exp}(X)) = \pi(X) \qquad \forall X \in g,$$

so that,

$$\pi(\text{Log}(y)) = \pi(y) \qquad \forall y \in \mathbb{R}^N. \tag{10.5}$$

Thus, by (10.2), (10.4) and (10.5)

$$\begin{aligned} \pi(x \circ y) &= \pi(\exp(\text{Log}(y))(x)) = \pi(x) + \pi(\text{Log}(y)) \\ &= \pi(x) + \pi(y). \end{aligned}$$

□

Remark 10.3. Let $\mathbb{L} = (\mathbb{R}^{N+1}, \circ, d_\lambda)$ be the homogeneous Lie group of condition (H1) in the Introduction. For suitable natural numbers $\sigma_1, \dots, \sigma_N, \sigma_{N+1}$ we have

$$d_\lambda(x,t) = d_\lambda(x_1, \dots, x_N, t) = (\lambda^{\sigma_1}, \dots, \lambda^{\sigma_N}, \lambda^{\sigma_{N+1}} t)$$

Since $Y = X_0 - \partial_t$ is supposed to be d_λ-homogeneous of degree two, for every smooth function $u = u(t)$ only dependent on the variable t we have

$$\mathcal{L}(u(\lambda^{\sigma_{N+1}} t)) = \lambda^2 (\mathcal{L}u)(\lambda^{\sigma_{N+1}} t),$$

so that

$$\partial_t(u(\lambda^{\sigma_{N+1}} t)) = \lambda^2 (\partial_t u)(\lambda^{\sigma_{N+1}} t).$$

This obviously implies $\sigma_{N+1} = 2$. As a consequence

$$Q := \sigma_1 + \ldots + \sigma_N + \sigma_{N+1} \geq \sigma_1 + \sigma_{N+1} \geq 1 + 2 = 3.$$

10.2. Link of groups

In this section we split $\mathbb{R}^N, N \geq 3$, as follows

$$\mathbb{R}^N = \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r$$

and denote its points by

$$x = (x^{(p)}, x^{(q)}, x^{(r)}),$$

where $x^{(p)} \in \mathbb{R}^p, x^{(q)} \in \mathbb{R}^q$ and $x^{(r)} \in \mathbb{R}^r$. We shall also use the notation

$$x^{(p,q)} := (x^{(p)}, x^{(q)}) \quad \text{and} \quad x^{(p,r)} := (x^{(p)}, x^{(r)})$$

Accordingly, for consistency of notations, we shall write $\mathbb{R}^{(p,q)}$ and $\mathbb{R}^{(p,r)}$ instead of $\mathbb{R}^p \times \mathbb{R}^q$ and $\mathbb{R}^p \times \mathbb{R}^r$, respectively. Let

$$\mathbb{G}_1 = (\mathbb{R}^{(p,q)}, \circ, d_\lambda^{(1)}) \quad \text{and} \quad \mathbb{G}_2 = (\mathbb{R}^{(p,r)}, \circ, d_\lambda^{(2)})$$

be homogeneous Lie groups.[2] Assume the composition laws in $\mathbb{G}_1$ and $\mathbb{G}_2$ have the following structure

$$x^{(p,q)} \circ y^{(p,q)} = (x^{(p)}, x^{(q)}) \circ (y^{(p)}, y^{(q)}) := (x^{(p)} + y^{(p)}, Q(x^{(p,q)}, y^{(p,q)})) \quad (10.6)$$

$$x^{(p,r)} \circ y^{(p,r)} = (x^{(p)}, x^{(r)}) \circ (y^{(p)}, y^{(r)}) := (x^{(p)} + y^{(p)}, R(x^{(p,r)}, y^{(p,r)})) \quad (10.7)$$

where Q and R take their values in $\mathbb{R}^q$ and $\mathbb{R}^r$, respectively. We also assume that the dilations $d_\lambda^{(1)}$ and $d_\lambda^{(2)}$ take the following form

$$d_\lambda^{(1)}(x^{(p)}, x^{(q)}) = (\lambda x^{(p)}, \rho_\lambda^{(1)}(x^{(q)}))$$

$$d_\lambda^{(2)}(x^{(p)}, x^{(r)}) = (\lambda x^{(p)}, \rho_\lambda^{(2)}(x^{(r)}))$$

We define the *link* of $\mathbb{G}_1$ and $\mathbb{G}_2$ as follows. Given $x^{(p,q)} \in \mathbb{R}^{(p,q)}$ and $x^{(p,r)} \in \mathbb{R}^{(p,r)}$ we put

$$(x^{(p)}, x^{(q)}) \bigtriangleup (x^{(p)}, x^{(r)}) := (x^{(p)}, x^{(q)}, x^{(r)})$$

[2] We use the same notation $\circ$ to denote the composition law in in $\mathbb{G}_1$ and in $\mathbb{G}_2$. The contest will avoid confusion.

and define in $\mathbb{R}^N$ the following composition law

$$\begin{aligned} x \circ y &= (x^{(p)}, x^{(q)}, x^{(r)}) \circ (y^{(p)}, y^{(q)}, y^{(r)}) \\ &:= (x^{(p,q)} \circ y^{(p,q)}) \triangle (x^{(p,r)} \circ y^{(p,r)}) \\ &\equiv (x^{(p)} + y^{(p)}, Q(x^{(p,q)}, y^{(p,q)}), R(x^{(p,r)}, y^{(p,r)})). \end{aligned}$$

We also define a family of dilations d_λ by

$$\begin{aligned} d_\lambda : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad d_\lambda(x) &= d_\lambda(x^{(p)}, x^{(q)}, x^{(r)}) \\ &:= d_\lambda^{(1)}(x^{(p,q)}) \triangle d_\lambda^{(2)}(x^{(p,r)}) \\ &\equiv (\lambda x^{(p)}, \rho_\lambda^{(1)}(x^{(q)}), \rho_\lambda^{(2)}(x^{(r)})). \end{aligned}$$

It is quite easy to recognize that

$$\mathbb{G} := (\mathbb{R}^N, \circ, d_\lambda)$$

is a *homogeneous group*, that we call the *link of* $\mathbb{G}_1$ *and* $\mathbb{G}_2$. In the sequel we shall use the following

Agreement. Given a vector field in $\mathbb{R}^{(p,q)}$:

$$X = \sum_{i=1}^{p} a_i^{(p)} \partial_{x_i^{(p)}} + \sum_{i=1}^{q} a_i^{(q)} \partial_{x_i^{(q)}}$$

where the coefficients $a_i^{(p)}$ and $a_i^{(q)}$ are smooth functions of the variable $x^{(p,q)}$, we consider X as a vector field in $\mathbb{R}^N$. Analogously, every vector field in $\mathbb{R}^{(p,r)}$,

$$Y = \sum_{i=1}^{p} b_i^{(p)} \partial_{x_i^{(p)}} + \sum_{i=1}^{r} a_i^{(r)} \partial_{x_i^{(r)}}$$

will be also viewed as a vector field in $\mathbb{R}^N$.

The following proposition is crucial for our purposes.

Proposition 10.4. *Assume the function R in (10.7) is independent of $y_i^{(p)}$ for some $i \in \{1, \ldots, p\}$. Then the vector fields*

$$X\left(x^{(p,q)}\right) := \frac{\partial}{\partial y_i^{(p)}} \tau_{x^{(p,q)}}(y^{(p,q)})\big|_{y^{(p,q)}=0}, \quad x^{(p,q)} \in \mathbb{R}^{(p,q)} \tag{10.8}$$

is left translation invariant on $\mathbb{G}$, and d_λ-homogeneous of degree one.

Note. Hereafter $\tau_{x^{(p,q)}}$ denotes the left translation on $\mathbb{G}_1$

$$y^{(p,q)} \longmapsto \tau_{x^{(p,q)}}\left(y^{(p,q)}\right) := x^{(p,q)} \circ y^{(p,q)}$$

We shall use similar notation for the left translations on $\mathbb{G}_2$ and $\mathbb{G}$. From the general theory of Lie groups and taking into account (10.6), the vector field X in (10.8) is left translation invariant on $\mathbb{G}_1$ and $d_\lambda^{(1)}$-homogeneous of degree one.

Proof. We first prove the left translation invariance of X on $\mathbb{G}$.
Given $x=(x^{(p)},x^{(q)},x^{(r)})\in\mathbb{R}^N$, we have

$$\begin{aligned}\frac{\partial}{\partial y_i^{(p)}}\,\tau_x(y) &= \frac{\partial}{\partial y_i^{(p)}}(x^{(p)}+y^{(p)},Q(x^{(p,q)},y^{(p,q)}),R(x^{(p,r)},y^{(p,r)}))\\ &= \frac{\partial}{\partial y_i^{(p)}}(x^{(p)}+y^{(p)},Q(x^{(p,q)},y^{(p,q)}),0^{(r)})\end{aligned}$$

where $0^{(r)}$ denotes the null vector of $\mathbb{R}^r$.

Then, by (10.8) and the previous Agreement on the vector fields,

$$\frac{\partial}{\partial y_i^{(p)}}\,\tau_x(y)|_{y=0} = X(x^{(p,q)}) \tag{10.9}$$

This identity shows that X is left translation invariant on $\mathbb{G}$. The same identity also shows the remaining part of the Proposition, since, from the general theory of homogeneous Lie groups the left-hand side of (10.9) is $d_\lambda^{(1)}$ homogeneous of degree one. □

The previous proof can be trivially adapted to prove the following Proposition.

Proposition 10.5. *Let*

$$Y\left(x^{(p,r)}\right) := \frac{\partial}{\partial y_i^{(r)}}\tau_{x^{(p,r)}}(y^{(p,r)})|_{y^{(p,r)}=0}$$

for some $i\in\{1,\dots,r\}$.
Then Y is left translation invariant on $\mathbb{G}$. Moreover, if Y is $d_\lambda^{(2)}$ homogeneous of degree n, then it also is d_λ-homogeneous of the same degree.

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