FEFFERMAN–POINCARE INEQUALITY AND REGULARITY FOR QUASILINEAR SUBELLIPTIC EQUATIONS

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1. Introduction

This talk is a survey on some recent results concerning local regularity for weak solutions to some quasilinear degenerate elliptic equations. This topic is a very classical one in the theory of PDE and we start by recalling contributions given by many authors. Early results go back to the outstanding papers by De Giorgi [12], Stampacchia [29], Ladyzhenskaia and Uraltzeva [18] between the end of 50's and the beginning of 60's. Namely, for an elliptic equation of the following kind,

$$-(a_{ij}u_{x_i})_{x_j} + b_i u_{x_i} + cu = f \tag{1.1}$$

they studied the regularity of weak solutions for linear, uniformly elliptic equations assuming the lower order coefficients, b, c, f to belong to some suitable L^p classes. Later, in subsequent papers by Serrin [27], Morrey [22] and Trudinger [30], some results were extended to some nonlinear equations satisfying suitable growth conditions. Despite the sharpness of their results, it appeared that the Lebesgue classes were not the right ones in which to put the lower order terms to obtain regularity. In fact, it is quite easy to find equations (even linear) having regularity properties without lower order terms in any Lebesgue classes. At the beginning of 70's Lewy and Stampacchia [19] showed Hölder continuity properties for the solution of the equation

$$-\Delta u = f \tag{1.2}$$

assuming f to satisfy an L^1 assumption. The assumption was the following

$$\int_{B_R(x_0)} |f(y)|\, dy \le cR^{\lambda}, \quad \forall R < R_0 \tag{1.3}$$

and some $\lambda > 0$ related to the Euclidean dimension n.

After that, Aizenman and Simon, in [1] considered the equation

$$-\Delta u = Vu \tag{1.4}$$

using probabilistic tools and discovered that regularity properties obtainable through Harnack inequality were related to a class of potentials already used for other reasons, the Stummel–Kato class. Functions belonging to this class are required to satisfy the following condition.

Definition 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Assume that*

$$\eta(R) \equiv \sup_{x\in\Omega} \int_{\Omega\cap B_R(x)} \frac{|V(y)|}{|x-y|^{n-2}}\, dy \to 0\,, as\ R \to 0\,. \tag{1.5}$$

Then, we will say that the function V belongs to the Stummel–Kato class, $S(\Omega)$.

An important point is that Aizenman and Simon's proof relies on the following imbedding

$$\int_{B_R} |V|\, u^2\, dx \le c\, \eta(R) \int_{B_R} |\nabla u|^2\, dx\,, \qquad \forall u \in C_0^\infty(B_R)\,, \tag{1.6}$$

and some consequence of it, that hold true if V is in the Stummel–Kato class. The imbedding (1.6) is proved by Aizenman and Simon in [1] and later, using non probabilistic tools, by Schechter (see [26] and [25]). At the same time, Dal Maso and Mosco in [9] developed a pointwise analysis for solutions of equations of the form

$$Lu + \mu\, u = \nu\,, \tag{1.7}$$

where L is a linear second order elliptic operator assumed to be uniformly elliptic in the domain Ω and μ,ν are measure satisfying conditions similar to (1.5). Aizenman and Simon's result were extended to any linear uniformly elliptic operator by Chiarenza, Fabes and Garofalo in 1986 [5]. In their paper, using a representation formula, they proved Harnack inequality and the continuity of weak solutions for equations of the following kind

$$Lu - Vu = 0\,, \tag{1.8}$$

where $L = \partial_{x_j}(a_{ij}\, \partial_{x_i})$. Subsequently, Simader [28] gave also a proof of Harnack inequality and the continuity of weak solutions based on representation formula. At the same time, Hinz and Kalf [17] showed the same results using subsolution estimates. Moreover, Simader shows that, once it is known that the solution u is locally bounded, the continuity of the solution is actually equivalent to the validity of the Harnack inequality. Local regularity properties of weak solutions were studied also in [13] using representation formulas and, in a non linear setting, in Rakotoson and Ziemer [24] (see also [32],[33],[34]).

Now we turn our attention to operators that can be degenerate elliptic. In [16] Gutierrez, using a suitable version of the Stummel–Kato class, adapted to degenerate elliptic equations showed Harnack inequality and regularity for the weak solutions of (1.8) where, L is a linear elliptic operator with degeneracy of A_2 kind, extending to the weighted case, the results in [5](see also [35],[31]).

In [8], Citti,Garofalo and Lanconelli proved the validity of Harnack inequality and the continuity of the solutions for a linear elliptic operator satisfying the Hörmander condition. In [7] the Hölder continuity were obtained for equations of the kind (1.8) assuming the known term in a suitable version of the Morrey class modelled on the level sets of the fundamental solution. Related results are also [4], [20] and [11]. In the same direction but related to the point of view of Dirichlet forms see also [2].

What we want to stress here is that a very useful tool to obtain regularity results is an imbedding like (1.6). After this was recognized, the imbedding (1.6) was generalized and extended in various directions.

In 1982, Aizenman and Simon proved (1.6) and in 1983, C.Fefferman ([15]) proved the imbedding

$$\int_{B_R} |V|\,|u|^p\,dx \le c \int_{B_R} |\nabla u|^p\,dx\,, \qquad \forall u \in C_0^\infty(B_R)\,, \tag{1.9}$$

for $p = 2$, assuming that $V \in L^{r,n-pr}$ – the classical Morrey space. Later, Chiarenza and Frasca in [6] showed the Fefferman imbedding (1.9) for any $1 < p \le n/2$ assuming $V \in L^{r,n-pr}$ with a simpler proof. Subsequently, Danielli in [10], generalized the proof and the result in [6] to the case when the gradient in the right hand side is replaced by the energy of a system of non commuting vector fields. In [14], imbedding (1.6) is proved in a general setting in which the gradient is replaced by the energy of a system of non commuting vector fields. The aim of this talk is to show how imbeddings like (1.6) or (1.9) are useful to obtain regularity for weak solutions of quasilinear subelliptic PDE.

2. Preliminaries and Function spaces

We collect the preliminary assumptions we need in the sequel, so we start with some notations. Let us consider a system $X = (X_1, \ldots, X_m)$ of non commutative vector fields in an open set $\Omega \subseteq \mathbb{R}^n$ with locally Lipschitz continuous coefficients b_{jk}. We write

$$X_j = \sum_{k=1}^{n} b_{jk}\frac{\partial}{\partial x_k}, \quad b_{jk} \in Lip_{loc}(\Omega) \quad j = 1, \ldots, m, \;\; k = 1, \ldots, n, \tag{2.1}$$

and denote by $X_j^* = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(b_{jk}\,\cdot)$ its formal adjoint.

For a given function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, we set

$$Xu = (X_1 u, \ldots, X_m u)\,, \qquad |Xu| = \left(\sum_{j=1}^{m} (X_j u)^2\right)^{\frac{1}{2}} \tag{2.2}$$

where, as usual,

$$X_j u(x) = \langle X_j, \nabla u(x)\rangle\,, \qquad j = 1, \ldots, m\,, \tag{2.3}$$

identifying the X_j's with the first order differential operator that acts on $u \in \mathrm{Lip}(\Omega)$ via formula (2.3).

Using definition (2.2) it is possible to define Sobolev classes replacing the ordinary gradient with the X–gradient. Namely, in the sequel we set

$$W^{1,p}_X(\Omega) = \left\{ u \in L^p(\Omega) \ : \ X_j u \in L^p(\Omega), \ j = 1, \dots, m \right\}, \quad 1 \le p < \infty, \tag{2.4}$$

and for any $u \in W^{1,p}_X(\Omega)$

$$\|u\|_{1,p} \equiv \|u\|_p + \| \, |Xu| \, \|_p \, . \tag{2.5}$$

The completion of the set $C^1_0(\Omega)$ with respect to the norm (2.5) will be denoted by $W^{1,p}_{X,0}(\Omega)$.

Using the vector fields, $X_1, \dots, X_m$, it is possible to define a distance. A piecewise C^1 curve $\gamma : [0, T] \to \mathbb{R}^n$ is called X–sub-unit, if whenever $\gamma'(t)$ exists one has

$$\langle \gamma'(t), \xi \rangle^2 \le \sum_{j=1}^m \langle X_j(\gamma(t)), \xi \rangle^2 \qquad \forall \xi \in \mathbb{R}^n. \tag{2.6}$$

The X–sub-unit lenght of γ is by definition $l_S(\gamma) = T$. Given $x, y \in \mathbb{R}^n$, we denote by $\Phi(x, y)$ the collection of all X–sub-unit curves connecting x to y. Then

$$d(x, y) = \inf \{ l_S(\gamma) \ : \ \gamma \in \Phi(x, y) \} \tag{2.7}$$

defines a distance, usually called the Carnot–Caratheodory distance generated by the system X. We will denote $B(x, r) = \{ y \in \mathbb{R}^n \ : \ d(x, y) < r \}$ the metric ball centered at x of radius r and whenever x is not relevant we shall write B_r. We shall denote with $d_e(x, y) = |x - y|$ the usual Euclidean distance in $\mathbb{R}^n$.

The following assumptions will be used in all the sequel.

(A1) The identity map $i : (\mathbb{R}^n, d_e) \to (\mathbb{R}^n, d)$ is continuous;

(A2) (Doubling condition) For every bounded set $\Omega \subset \mathbb{R}^n$ there exist costants C_D, $R_D > 0$ such that for $x_0 \in \Omega$ and $0 < r < R_D$ one has

$$|B(x_0, 2r)| \le C_D |B(x_0, r)|;$$

(A3) (Weak-L^1 Poincarè type inequality) Given Ω as in (A2), there exist positive constants C_P and $\alpha \ge 1$ such that for any $x_0 \in \Omega$, $0 < r < R_D$ and $u \in C^1(B(x_0, \alpha r))$, one has

$$\sup_{\lambda > 0} [\lambda |\{ x \in B(x_0, r) \ : \ |u(x) - u_{B(x_0,r)}| > \lambda \}|] \le C_P R \int_{B(x_0, \alpha r)} |Xu| dx,$$

where $u_{B(x_0,r)}$ denotes the integral average $\fint_{B(x_0,r)} u(y)\, dy$.

The number $Q = \log_2 C_D$, will be called the homogeneous dimension of Ω.

Definition 2.1. *Let $V \in L^1_{loc}(\Omega)$, $r > 0$ and $1 < p < Q$. We set*

$$\phi_V(r) \equiv \sup_{x \in \Omega} \left(\int_{\Omega \cap B(x,r)} \frac{d(x, y)}{|B(x, d(x, y))|} \left(\int_{\Omega \cap B(x,r)} |V(z)| \frac{d(z, y)}{|B(z, d(z, y))|} dz \right)^{\frac{1}{p-1}} dy \right)^{p-1}. \tag{2.8}$$

We say that a function $V \in L^1_{loc}(\Omega)$ belongs to the space $(\tilde{M}_X)_p(\Omega)$ if and only if $\phi_V(r)$ is finite for any $r > 0$.

Definition 2.2 (Stummel-Kato class). *We say that a function $V \in L^1_{loc}(\Omega)$ belongs to the space $(M_X)_p(\Omega)$ if and only if $\phi_V(r)$ defined in (2.1) is finite for any $r > 0$ and, in addition, $\lim_{r\to 0+} \phi_V(r) = 0$.*

We will call $\phi_V(r)$ the $(M_X)_p$ – *modulus* of V. It is not difficult to prove that $\phi_V(r)$ is a continuous function.

Definition 2.3. *Let $1 < p < Q$. We say that $V \in (M_X)_p(\Omega)$ belongs to the class $(M_X)'_p(\Omega)$ if*

$$\exists \delta > 0 \quad : \quad \int_0^\delta \frac{\phi_V(t)^{\frac{1}{p}}}{t}\, dt < +\infty\,. \tag{2.9}$$

Remark 2.4. *We explicitely note that:*

1. *When $X = \nabla$ the spaces $(\tilde{M}_X)_p$, $(M_X)_p$, and $(M_X)'_p$ gives back the usual Stummel–Kato classes;*
2. *The function $\phi_V(r)$ is doubling, i.e.:*

$$\exists c > 1 \quad : \quad \phi_V(2r) \le c\phi_V(r)\,, \quad r > 0\,.$$

Now we define some spaces very closely connected to Stummel classes.

Definition 2.5 (Morrey spaces). *Let $p \in [1, +\infty[$ and $\lambda > 0$. We say that $V \in L^p_{loc}(\Omega)$ belongs to the intrinsic Morrey space with respect to the system $X = (X_1, \ldots, X_m)$, $L^{p,\lambda}_X(\Omega)$, if*

$$\|V\|_{L^{p,\lambda}_X(\Omega)} = \sup_{\substack{x\in\Omega \\ 0<r<d_0}} \left(\frac{r^\lambda}{|B(x,r)\cap\Omega|} \int\limits_{B(x,r)\cap\Omega} |V(y)|^p\, dy \right)^{\frac{1}{p}} < +\infty, \tag{2.10}$$

where $d_0 = \min(\operatorname{diam}(\Omega), R_D)$.

It is quite easy to compare Stummel–Kato classes and Morrey spaces. Indeed,

Proposition 2.6. *Let $1 < p < Q$ and $0 < \varepsilon < p$. If $V \in L^{1,p-\varepsilon}_X(\Omega)$ we have*

$$\phi_V(r) \le C\,(C_D, p, \varepsilon)\,\|V\|_{L^{1,p-\varepsilon}_X} r^\varepsilon \tag{2.11}$$

for any $0 < r < R_D$ and then

$$L^{1,p-\varepsilon}_X(\Omega) \subseteq (M_X)'_p(\Omega). \tag{2.12}$$

Proof. For $x \in \Omega$ and $0 < r_1 < r_2$ we set

$$A(r_1, r_2, x) = \{y \in \Omega \;:\; r_1 \le d(x,y) < r_2\}\,, \quad A_j = A\left(\frac{r}{2^{j+1}}, \frac{r}{2^j}, x\right)\,. \tag{2.13}$$

Then, we have

$$\int_{\{y\in\Omega\,:\,d(x,y)<r\}} \frac{d(x,y)}{|B(x,d(x,y))|}\left(\int_{\{z\in\Omega\,:\,d(z,x)<r\}} |V(z)|\frac{d(z,y)}{|B(z,d(z,y))|}\,dz\right)^{\frac{1}{p-1}} dy \tag{2.14}$$

$$= \sum_{j=1}^{+\infty}\int_{A_j} \frac{d(x,y)}{|B(x,d(x,y))|}\left(\sum_{k=0}^{j}\int_{A_j} |V(z)|\frac{d(z,y)}{|B(z,d(z,y))|}\,dz\right)^{\frac{1}{p-1}} dy$$

$$\le C(C_D,p)\sum_{j=1}^{+\infty}\int_{B_{\frac{r}{2^j}}} \frac{\frac{r}{2^j}}{|B(x,\frac{r}{2^j})|}\left(\sum_{k=0}^{j}\frac{\frac{r}{2^{j-k}}}{|B(x,\frac{r}{2^{j-k}})|}\int_{B_{\frac{r}{2^{j-k}}}} |V(z)|\,dz\right)^{\frac{1}{p-1}} dy$$

$$\le C(C_D,p)\|V\|^{\frac{1}{p-1}}_{L_X^{1,p-\varepsilon}}\sum_{j=1}^{+\infty}\frac{r}{2^j}\left(\sum_{k=0}^{j}\left(\frac{r}{2^{j-k}}\right)^{\varepsilon-p+1}\right)^{\frac{1}{p-1}}$$

$$= C(C_D,p,\varepsilon)\|V\|^{\frac{1}{p-1}}_{L_X^{1,p-\varepsilon}} r^{\frac{\varepsilon}{p-1}}\,.$$

□

3. Fefferman–Poincare inequality

In this section we state and prove a version of Fefferman–Poincare inequality suitable for our purposes.

Theorem 3.1. *Let Ω be a bounded open set $\Omega\subset\mathbb{R}^n$, with homogeneous dimension Q and let $1<p<Q$. Suppose (A1) - (A3) hold true, and $V\in(M_X)_p(\Omega)$. Then it exists a positive constant c independent of u such that*

$$\int_\Omega |V(x)|\,|u(x)|^p\,dx \le\; c\,\phi_V(2r)\int_\Omega |Xu(x)|^p\,dx \tag{3.1}$$

for any smooth function u compactly supported in B_r.

Proof. The main point in the proof is the use of a suitable representation formula. We use Theorem 1 in [21] and then,

$$\int_B |V(x)|\,|u(x)|^p\,dx \le c\int_B |V(x)|\,|u(x)|^{p-1}\left(\int_B |Xu(y)|\frac{d(x,y)}{|B(x,d(x,y))|}\,dy\right)dx$$

$$\le c\int_B |Xu(y)|\left(\int_B |V(x)|\,|u(x)|^{p-1}\frac{d(x,y)}{|B(x,d(x,y))|}\,dx\right)dy$$

$$\le c\left(\int_B |Xu(y)|^p\,dy\right)^{\frac{1}{p}}\cdot\left[\int_B\left(\int_B |V(x)|\,|u(x)|^{p-1}\frac{d(x,y)}{|B(x,d(x,y))|}\,dx\right)^{\frac{p}{p-1}} dy\right]^{\frac{p-1}{p}}\,.$$

We also have

$$\int_B\left(\int_B |V(x)|\,|u(x)|^{p-1}\frac{d(x,y)}{|B(x,d(x,y))|}\,dx\right)^{\frac{p}{p-1}}\,dy$$
$$\leq \int_B\left(\int_B |V(z)|\frac{d(z,y)}{|B(z,d(z,y))|}\,dz\right)^{\frac{1}{p-1}}\cdot\left(\int_B |V(x)|\,|u(x)|^{p}\frac{d(x,y)}{|B(x,d(x,y))|}\,dx\right)dy =$$
$$= \int_B |V(x)|\,|u(x)|^p\int_B\frac{d(x,y)}{|B(x,d(x,y))|}\cdot\left(\int_B |V(z)|\frac{d(z,y)}{|B(z,d(z,y))|}\,dz\right)^{\frac{1}{p-1}}\,dy\,dx$$
$$\leq \phi^{\frac{1}{p-1}}(2r)\int_B |V(x)|\,|u(x)|^p\,dx\,.$$

Merging the previous inequalities we obtain the desired conclusion. □

The next corollary is an easy consequence of the previous Theorem. It can be obtained via a standard partition of unity.

Corollary 3.2. *Under the same assumptions of Theorem 3.1 we have that for any* $\sigma > 0$ *there exists a positive function* $K(\sigma) \sim \frac{\sigma}{[\phi_V^{-1}(\sigma)]^{Q+p}}$ *such that*

$$\int_\Omega |V(x)|\,|u(x)|^p\,dx \leq \sigma\int_\Omega |Xu(x)|^p\,dx + K(\sigma)\int_\Omega |u(x)|^p\,dx, \tag{3.2}$$

for all $u \in W^{1,p}_{X,0}(\Omega)$.

Proof. Let $\varepsilon > 0$. Let r be a positive number that we will be choosen later. Let $\left\{\alpha_i^p\right\}$, $i = 1, 2, \ldots, N(r)$, be a finite partition of unity of $\overline{\Omega}$, such that supp $\alpha_i \subseteq B_r(x_i)$, with $x_i \in \overline{\Omega}$. We apply Theorem 3.1 to the functions $\alpha_j\,u$ and we get

$$\int_\Omega |V(x)||u(x)|^p\,dx = \int_\Omega |V(x)||u(x)|^p\sum_{i=1}^{N(r)}\alpha_i^p(x)\,dx =$$
$$= \sum_{i=1}^{N(r)}\int_\Omega |V(x)||u(x)|^p\alpha_i^p(x)\,dx$$
$$\leq \sum_{i=1}^{N(r)} c\phi_V(2r)\left(\int_\Omega |Xu(x)|^p\alpha_i^p(x)\,dx + \int_\Omega |X\alpha_i(x)|^p|u(x)|^p\,dx\right)$$
$$\leq c\,\phi_V(2r)\left(\int_\Omega |Xu(x)|^p\,dx + \frac{N(r)}{r^p}\int_\Omega |u(x)|^p\,dx\right).$$

Now, to obtain the result it suffices to choose r such that $c\,\phi_V(2r) = \varepsilon$. After that we note that $N(r) \simeq r^{-Q}$ and the corollary follows. □

4. Some lemmas

In this section we collect some lemmas which will be useful in the sequel. We give proofs for sake of completeness.

Lemma 4.1 (see [23])**.** *Let* $\mu\ :]0,+\infty[$ *a continuous increasing function such that* $\lim\limits_{r\to 0}\mu(r)=0$ *and let* $0<\theta<1$*. The series*

$$\sum_{i=0}^{+\infty}\theta^i log\,\mu^{-1}\left(\theta^{qi}\right), \tag{4.1}$$

where $q>0$*, is convergent if and only if there exists* $\rho>0$ *such that*

$$\int_0^\rho \frac{\mu^{\frac{1}{q}}(t)}{t}\,dt<+\infty. \tag{4.2}$$

Proof. We claim that (4.2) is actually equivalent to the convergence of the following series

$$\sum_{j=0}^{+\infty}(\theta a_i-a_{i+1}),$$

where $a_j=\theta^j\log\mu^{-1}\left(\mu(\rho)\theta^{qj}\right)$.

Indeed

$$\begin{aligned}\int_0^\rho \frac{\mu^{\frac{1}{q}}(t)}{t}\,dt &= \int_0^{\mu(\rho)}\frac{s^{\frac{1}{q}}}{\mu^{-1}(s)}\frac{1}{\mu'(\mu^{-1}(s))}\,ds\\ &=\sum_{j=0}^{+\infty}\int_{\mu(\rho)\theta^{q(j+1)}}^{\mu(\rho)\theta^{qj}}\frac{s^{\frac{1}{q}}}{\mu^{-1}(s)}\frac{1}{\mu'(\mu^{-1}(s))}\,ds\\ &<\sum_{j=0}^{+\infty}\Big\{\mu^{\frac{1}{q}}(\rho)\theta^j\log\mu^{-1}\left(\mu(\rho)\theta^{qj}\right)-\\ &\quad-\frac{1}{\theta}\mu^{\frac{1}{q}}(\rho)\theta^{j+1}\log\mu^{-1}\left(\mu(\rho)\theta^{q(j+1)}\right)\Big\}\\ &=\frac{\mu^{\frac{1}{q}}(\rho)}{\theta}\sum_{i=0}^{+\infty}(\theta a_j-a_{j+1}).\end{aligned}$$

Exactly in the same way it is possible to show that

$$\int_0^\rho \frac{\mu^{\frac{1}{q}}(t)}{t}\,dt>\sum_{i=0}^{+\infty}(\theta a_i-a_{i+1}) \tag{4.3}$$

and the claim is proved.

It is trivial to recognize that the two series $\sum_{j=0}^{+\infty}(\theta a_j-a_{j+1}),\sum_{j=0}^{+\infty}a_j$ have the same behaviour. □

Lemma 4.2. *Let* $0<\gamma<1$ *and let* $h:]0,+\infty[\ \to\ ]0,+\infty[$ *be a non decreasing function such that* $\lim\limits_{t\to 0}h(t)=0$ *and*

$$h(t)\le c\,h(t/2)\quad \forall t>0, \tag{4.4}$$

for some constant $c > 1$. Moreover, let $\omega :]0,+\infty[\to]0,+\infty[$ be a non decreasing function. Assume that there exists $\overline{\rho} > 0$ such that,

$$\omega(\rho) \le \gamma\omega(4\rho) + h(\rho) \qquad \forall\rho < \rho_0 < 1\,. \tag{4.5}$$

Then there exist $\overline{\rho} \le \rho_0$, $0 < \sigma \le 1$ and $K > 0$ such that

$$\omega(\rho) \le K h^{\sigma}(\rho) \qquad \forall\rho < \overline{\rho}\,. \tag{4.6}$$

Proof. Let $\tilde{\rho} > 0$ be such that $h(\rho) < 1$ for $\rho < \tilde{\rho}$. Set $\overline{\rho} = \min(\tilde{\rho}, \rho_0)$, we choose $R > 0$ such that $R < \overline{\rho}$.

If $\rho \in \left[\frac{R}{4}, R\right]$, letting

$$M = \sup_{[R/4,R]} \frac{\omega(\rho)}{h(\rho)}, \tag{4.7}$$

we have

$$\omega(\rho) \le M h(\rho)\,. \tag{4.8}$$

If $\rho \in \left[\frac{R}{4^2}, \frac{R}{4}\right]$, by (4.5) and and the fact that $h(4\rho) < 1$, we have

$$\omega(\rho) \le \gamma M h(4\rho) + h(\rho) \le \gamma M h^{\sigma}(4\rho) + h^{\sigma}(\rho) \tag{4.9}$$

for every $\sigma : 0 < \sigma \le 1$. Now we fix σ such that

$$\gamma\, c^{2\sigma} = a < 1 \tag{4.10}$$

where c is the constant in (4.4), we obtain

$$\omega(\rho) \le (Ma + 1) h^{\sigma}(\rho)\,. \tag{4.11}$$

Iterating this procedure, if $\rho \in \left[\frac{R}{4^{i+1}}, \frac{R}{4^i}\right]$, we have

$$\omega(\rho) \le \left(Ma^i + \sum_{k=0}^{i-1} a^k\right) h^{\sigma}(\rho) \le \left[M + \frac{1}{1-a}\right] h^{\sigma}(\rho)\,, \tag{4.12}$$

and then we get (4.6) taking $K = M + \frac{1}{1-a}$. □

5. Application to quasilinear subelliptic PDE: Harnack inequality

In this section we show how Fefferman–Poincare inequality is used to obtain regularity results for a class of subelliptic quasilinear PDE. We shall follow the classical Moser procedure. We will show that any weak solution is locally bounded and for any non negative weak solution, a Harnack inequality hold true. In the next section we will show how to deduce regularity from Harnack inequality.

Let Ω be a bounded domain having homogeneous dimension Q and let

$$A(x,u,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m\,, \quad B(x,u,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R} \tag{5.1}$$

be measurable functions satisfying the following structural assumptions

$$
\left\{\begin{array}{ll} |A(x,u,\xi)| & \le a|\xi|^{p-1} + b|u|^{p-1} + e \\ |B(x,u,\xi)| & \le c|\xi|^{p-1} + d|u|^{p-1} + f \\ \xi \cdot A(x,u,\xi) & \ge |\xi|^p - d|u|^p - g \end{array}\right. \tag{5.2}
$$

for a.e. $x \in \Omega \subset \mathbb{R}^n$, $\forall u \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^n$. All the results in this section will be referred to equations of the following kind:

$$
\sum_{j=1}^{m} X_j^* A_j(x,u(x),Xu(x)) + B(x,u(x),Xu(x)) = 0\,. \tag{5.3}
$$

The first thing to make precise is what is meant to be a weak solution of (5.3).

Definition 5.1. *A function $u \in W^{1,p}_{X,loc}(\Omega)$ is said to be a weak solution of (5.3) in Ω if*

$$
\sum_{j=1}^{m} \int_\Omega A_j(x,u(x),Xu(x)) X_j\,\varphi(x)\,dx + \int_\Omega B(x,u(x),Xu(x))\,\varphi(x)\,dx = 0\,, \tag{5.4}
$$

for every $\varphi \in W^{1,p}_{X,0}(\Omega)$.

We explicitely note that definition (5.1) can be meaningful according to suitable assumptions made on the coefficients in the structure assumptions (5.2) by using Theorem 3.1.

Our first result is the following local boundedness result.

Theorem 5.2 (Local Boundedness)**.** *Suppose (A1)-(A3) hold true. Let Ω be a bounded domain having local homogeneous dimension Q and $u \in W^{1,p}_{X,loc}(\Omega)$, with $1 < p < Q$, be a weak solution of (5.3). Let us assume that the structure conditions (5.2) hold true with*

$$
a \in \mathbb{R}\,,\ b^{p/p-1}\,, c^p\,,\ d\,,\ e^{p/p-1}\,,\ f\,, g\,,\ \in (M_X)'_p(\Omega)\,. \tag{5.5}
$$

Then, there exists a positive constant c, independent of u, such that, for any $B_r = B(x_0,r)$ for which $B(x_0,4r) \subset \Omega$ and $r < R_D$, we have

$$
\|u\|_{L^\infty(B_r)} \le C \left\{ \left(\fint_{B_{2r}} |u|^p\,dx \right)^{\frac{1}{p}} + h(r) \right\} \tag{5.6}
$$

where

$$
h(r) = \left[\phi_{e^{\frac{p}{p-1}}}(2r) + \phi_g(2r) \right]^{\frac{1}{p}} + \left[\phi_f(2r) \right]^{\frac{1}{p-1}}\,. \tag{5.7}
$$

Proof. The first thing to do is to simplify the structure assumptions (5.2). Setting

$$
v = |u| + h(r)\,, \tag{5.8}
$$

from (5.2) we easily get

$$
\left\{\begin{array}{l} |A(x,u,\xi)| \le\ a|\xi|^{p-1} + b_1|v|^{p-1} \\ |B(x,u,\xi)| \le\ c|\xi|^{p-1} + d_1|v|^{p-1} \\ \xi{\cdot}A(x,u,\xi) \ge |\xi|^p - d_1|v|^p \end{array}\right. \tag{5.9}
$$

where,

$$b_1 = b + h^{1-p}e \quad , \quad d_1 = d + h^{1-p} f + h^{-p} g\,. \tag{5.10}$$

Now, the functions $b_1^{\frac{p}{p-1}}$ and d_1 belong to the class $(M_X)'_p(B_{4r})$ and moreover,

$$\phi_{b_1^{\frac{p}{p-1}}}(\rho) \le C(p)\left[\phi_{b^{\frac{p}{p-1}}}(\rho) + h^{-p}\phi_{e^{\frac{p}{p-1}}}(\rho)\right] \le C(p)\left[\phi_{b^{\frac{p}{p-1}}}(\rho) + 1\right]$$
$$\phi_{d_1}(\rho) \le C(p)\left[\phi_d(\rho) + h^{1-p}\phi_f(\rho) + h^{-p}\phi_g(\rho)\right] \le C(p)\left[\phi_d(\rho) + 2\right]\,,\ 0 < \rho < 2r\,.$$

This means that, under our assumptions (5.5), the reduced structure assumptions (5.9) are of the same kind of the general structure assumptions (5.2). We now build a test function to be used in the definition of weak solution (5.4). Fix $q \ge 1$ and $l > h$ and let

$$F(v) = \begin{cases} v^q & \text{if} \quad h \le v \le l \\ q l^{q-1}(v-l) + l^q & \text{if} \quad l \le v \end{cases} \tag{5.11}$$

and

$$G(u) = \operatorname{sign} u\left(F(v)[F'(v)]^{p-1} - q^{p-1}h^{\beta}\right) \quad u \in\,]-\infty, +\infty[\ , \tag{5.12}$$

where β such that $pq = p + \beta - 1$.

Finally we declare which test function we are going to use in (5.4). We shall take

$$\varphi(x) = \eta^p(x)\, G(u), \tag{5.13}$$

where $\eta(x)$ is a smooth function such that $0 \le \eta \le 1$, identically 1 in B_r, compactly supported in B_{2r}.

Now we follow the classical pattern in [27] and substitute our test function $\varphi(x)$ in the definition (5.4). Using the structure conditions (5.9), we obtain

$$\int_{B_{2r}} \eta^p |Xw|^p\, dx \le ap \int_{B_{2r}} |(X\eta)w||\eta(Xw)|^{p-1}\, dx + q^{p-1}p \int_{B_{2r}} b_1|(X\eta)w||\eta w|^{p-1}\, dx$$
$$+ \int_{B_{2r}} c|\eta w||\eta(Xw)|^{p-1}\, dx + (1+p)q^{p-1}\int_{B_{2r}} d_1|\eta w|^p\, dx$$

where $w = w(x) = F(v)$.

With the aid of the elementary inequality

$$ab^{p-1} \le \frac{1}{p}\varepsilon^{1-p}a^p + \left(1 - \frac{1}{p}\right)\varepsilon b^p, \quad \forall \varepsilon > 0$$

we can greatly simplify the previous one to get

$$\int_{B_{2r}} \eta^p |Xw|^p\, dx \le C(p,a) q^{p-1}\left\{\int_{B_{2r}} |w(X\eta)|^p\, dx + \int_{B_{2r}} V|\eta w|^p\, dx\right\}, \tag{5.14}$$

where we put

$$V = b_1^{\frac{p}{p-1}} + c^p + d_1\,. \tag{5.15}$$

The function V belongs to the class $(M_X)'_p(B_{4r})$ and we can estimate its Stummel modulus. Namely we have,

$$\begin{aligned}\phi_V(\rho) &\le C(p)\left\{\phi_{b_1^{\frac{p}{p-1}}}(\rho)+\phi_{c^p}(\rho)+\phi_{d_1}(\rho)\right\}\\ &\le C(p)\left\{\phi_{b^{\frac{p}{p-1}}}(\rho)+\phi_{c^p}(\rho)+\phi_d(\rho)+3\right\}, \quad 0<\rho<2r\,.\end{aligned}$$

Now we use the Fefferman–Poincare inequality, or to be precise, a consequence of it. We are going to use corollary 3.2 in order to move to the left hand side one of the integrals appearing in the right hand side. We obtain,

$$\begin{aligned}\int_{B_{2r}}\eta^p|Xw|^p\,dx \le Cq^{p-1}\Big\{(1+\sigma)\int_{B_{2r}}|w(X\eta)|^p\,dx+\\ +\sigma\int_{B_{2r}}\eta^p|Xw|^p\,dx+K(\sigma)\int_{B_{2r}}\eta^pw^p\,dx\Big\} \quad \forall\sigma>0\,,\end{aligned}$$

where

$$K(\sigma)\sim\frac{\sigma}{\left[\phi_V^{-1}(\sigma)\right]^{Q+p}}, \tag{5.16}$$

and C is a positive constant independent of the function w.

To get our goal we fix $\sigma=\dfrac{1}{2Cq^{p-1}}$ and then,

$$\int_{B_{2r}}\eta^p|Xw|^p\,dx\le C\left\{q^{p-1}\int_{B_{2r}}|X\eta|^pw^p\,dx+q^{p-1}K\left(\frac{1}{2Cq^{p-1}}\right)\int_{B_{2r}}\eta^pw^p\,dx\right\}.$$

Use of the Sobolev inequality yields

$$\left(\int_{B_{2r}}|\eta w|^{p^*}\,dx\right)^{\frac{p}{p^*}}\le C\frac{r^p}{|B_r|^{\frac{p}{Q}}}\left\{q^{p-1}\int_{B_{2r}}|X\eta|^pw^p\,dx+q^{p-1}K\left(\frac{1}{2Cq^{p-1}}\right)\int_{B_{2r}}\eta^pw^p\,dx\right\}, \tag{5.17}$$

where $p^*=\frac{pQ}{Q-p}$ and C is a positive constant independent of w. Now, let $\eta(x)$ be identically 1 in $B_{r_1}=B(x_0,r_1)$, $0\le\eta(x)\le 1$ in B_{r_2} r_1 and r_2 satisfy $r\le r_1<r_2\le 2r$ and $|X\eta|\le\dfrac{C}{r_2-r_1}$. Using the properties of the function η in (5.17) we have

$$\left(\int_{B_{r_1}}w^{p^*}\,dx\right)^{\frac{p}{p^*}}\le C\frac{r^p}{|B_r|^{\frac{p}{Q}}}\frac{1}{(r_2-r_1)^p}q^{p-1}K\left(\frac{1}{2Cq^{p-1}}\right)\int_{B_{r_2}}w^p\,dx, \tag{5.18}$$

that is

$$\left(\int_{B_{r_1}}w^{p\chi}\,dx\right)^{\frac{1}{\chi}}\le C\frac{r^p}{|B_r|^{\frac{p}{Q}}}\frac{1}{(r_2-r_1)^p}\frac{1}{\left[\phi_V^{-1}\left(\frac{1}{Cq^{p-1}}\right)\right]^{Q+p}}\int_{B_{r_2}}w^p\,dx, \tag{5.19}$$

where $\chi=\frac{p^*}{p}=\frac{Q}{Q-p}$.

Letting now $l \to +\infty$ we get $w \to v^q$ and then

$$\left(\int_{B_{r_1}} v^{pq\chi}\,dx\right)^{\frac{1}{pq\chi}} \le C^{\frac{1}{pq}} \frac{r^{\frac{1}{q}}}{|B_r|^{\frac{1}{qQ}}} \frac{1}{(r_2-r_1)^{\frac{1}{q}}} \left[\frac{1}{\phi_V^{-1}\left(\frac{1}{Cq^{p-1}}\right)}\right]^{\frac{Q+p}{pq}} \left(\int_{B_{r_2}} v^{pq}\,dx\right)^{\frac{1}{pq}}.$$

If we set $\gamma = pq$, we have

$$\|v\|_{L^{\chi\gamma}(B_{r_1})} \le C^{\frac{1}{\gamma}} \frac{r^{\frac{p}{\gamma}}}{|B_r|^{\frac{p}{\gamma Q}}} \left(\frac{1}{r_2-r_1}\right)^{\frac{p}{\gamma}} \left[\frac{1}{\left(\phi_V^{-1}\left(\frac{1}{C\left(\frac{\gamma}{p}\right)^{p-1}}\right)\right)^{Q+p}}\right]^{\frac{1}{\gamma}} \|v\|_{L^{\gamma}(B_{r_2})}.$$

Now we want to iterate this last inequality. Set

$$\gamma_i = p\chi^i\,,\ r_i = r + \frac{r}{2^i}\,, \quad i = 1,2,\dots,$$

We have

$$\|v\|_{L^{\gamma_{i+1}}(B_{r_{i+1}})} \le C^{\frac{1}{p\chi^i}} \left(\frac{2^{i+1}}{|B_r|^{\frac{1}{Q}}}\right)^{\frac{1}{\chi^i}} \left[\frac{1}{\left(\phi_V^{-1}\left(\frac{1}{C\chi^{(p-1)i}}\right)\right)^{Q+p}}\right]^{\frac{1}{p\chi^i}} \|v\|_{L^{\gamma_i}(B_{r_i})}.$$

Now the inequality is ready to be iterated. We obtain

$$\|v\|_{L^{\infty}(B_r)} \le C|B_r|^{-\frac{1}{p}} \prod_{j=0}^{+\infty} \left[\frac{1}{\left(\phi_V^{-1}\left(\frac{1}{C\chi^{(p-1)j}}\right)\right)^{Q+p}}\right]^{\frac{1}{p\chi^j}} \|v\|_{L^p(B_{2r})}\,.$$

We stress that

$$\prod_{j=0}^{+\infty} \left[\frac{1}{\left(\phi_V^{-1}\left(\frac{1}{C\chi^{(p-1)j}}\right)\right)^{Q+p}}\right]^{\frac{1}{p\chi^j}} < +\infty$$

if and only if the series

$$\sum_{j=0}^{+\infty} \frac{1}{\chi^j} \log\,\phi_V^{-1}\left(\frac{1}{\chi^{(p-1)j}}\right)$$

is convergent. The conclusion thus follows by lemma (4.1). □

Theorem 5.3 (Harnack inequality)**.** *Suppose (A1)-(A3) hold true. Let Ω be a bounded domain having local homogeneous dimension Q and $u \in W^{1,p}_{X,loc}(\Omega)$, with $1 < p < Q$, be a nonnegative weak solution of (5.3). Let us assume that the structure conditions (5.2) hold true with*

$$a \in \mathbb{R}\,,\ b^{p/p-1}\,, c^p\,,\ d\,,\ e^{p/p-1}\,,\ f\,, g\,, \in (M_X)'_p(\Omega)\,. \tag{5.20}$$

Then, there exists a positive constant c, independent of u, such that, for any $B_r = B(x_0, r)$ for which $B(x_0, 4r) \subset \Omega$ and $r < R_D$, we have

$$\max_{B_r} u \le c \left\{ \min_{B_r} u + h(r) \right\} \tag{5.21}$$

where

$$h(r) = \left[\phi_{e^{\frac{p}{p-1}}}(2r) + \phi_g(2r) \right]^{\frac{1}{p}} + \left[\phi_f(2r) \right]^{\frac{1}{p-1}} . \tag{5.22}$$

Proof. We start as in theorem 5.2, setting $v = |u| + h$, with h defined by (5.22). From this it follows that conditions (5.9) are verified. Now let η be a non negative smooth function compactly supported in B_{3r}. Taking as test function in (5.4)

$$\varphi(x) = \eta^p(x) v^\beta(x), \quad \beta \in \mathbb{R}\,, \tag{5.23}$$

we have

$$\int_{B_{3r}} |Xv|^p \eta^p v^{\beta-1}\, dx \le C_1(p, a)(1 + |\beta|^{-1})^p \left\{ \int_{B_{3r}} |X\eta|^p v^{p+\beta-1}\, dx + \right. \\ \left. + \int_{B_{3r}} V \eta^p v^{p+\beta-1}\, dx \right\} , \tag{5.24}$$

where $V = b_1^{\frac{p}{p-1}} + c^p + d_1$. Setting

$$w(x) = \begin{cases} v^q(x) & \text{where} \quad pq = p + \beta - 1 \quad \text{if} \quad \beta \ne 1 - p \\ \log v(x) & \text{if} \quad \beta = 1 - p \end{cases} \tag{5.25}$$

by (5.24) we have

$$\int_{B_{3r}} \eta^p |Xw|^p\, dx \le C_1 |q|^p (1 + |\beta|^{-1})^p \left\{ \int_{B_{3r}} |X\eta|^p w^p\, dx + \int_{B_{3r}} V \eta^p w^p\, dx \right\} , \beta \ne 1 - p \tag{5.26}$$

while

$$\int_{B_{3r}} \eta^p |Xw|^p\, dx \le C_1 \left\{ \int_{B_{3r}} |X\eta|^p\, dx + \int_{B_{3r}} V \eta^p\, dx \right\} \text{ if } \beta = 1 - p\,. \tag{5.27}$$

We start considering (5.27). By theorem 3.1, we have

$$\int_{B_{3r}} V \eta^p\, dx \le C_2 \phi_V(1) \int_{B_{3r}} |X\eta|^p\, dx$$

and then, from (5.27), we have

$$\int_{B_{3r}} \eta^p |Xw|^p\, dx \le C_3(p, a, \phi_V, \operatorname{diam} \Omega) \int_{B_{3r}} |X\eta|^p\, dx\,.$$

Let B_h an arbitrary open ball contained in B_{2r}. Choosing $\eta(x)$ so that $\eta(x) = 1$ in B_h, $0 \le \eta \le 1$ in $B_{3r} \setminus B_h$ and $|X\eta| \le \dfrac{3}{h}$, we get

$$\|Xw\|_{L^p(B_h)} \le C_4(p, a, \phi_V, \operatorname{diam} \Omega) \frac{|B_h|^{\frac{1}{p}}}{h}.$$

Therefore, by Poincaré inequality and John–Nirenberg lemma (see [3]) we have $w(x) = \log v(x) \in BMO_X$. Then there exist two positive constants p_0 and C_5 depending on the same arguments of C_4, such that

$$\left(\fint_{B_{2r}} e^{p_0 w}\,dx\right)^{\frac{1}{p_0}}\left(\fint_{B_{2r}} e^{-p_0 w}\,dx\right)^{\frac{1}{p_0}} \le C_5\,. \tag{5.28}$$

Set

$$\Phi(p,h) = \left(\int_{B_h} |v|^p\,dx\right)^{\frac{1}{p}} \tag{5.29}$$

for any real number $p \neq 0$; by (5.28), recalling that $w = \log v$ we have

$$\frac{1}{|B_{2r}|^{\frac{1}{p_0}}}\Phi(p_0,2r) \le C_5|B_{2r}|^{\frac{1}{p_0}}\Phi(-p_0,2r). \tag{5.30}$$

We consider now the case (5.26). By corollary 3.2 we obtain

$$\begin{aligned}\int_{B_{3r}} |Xw|^p\eta^p\,dx \le C\Bigg\{(|q|^p+1)\left(1+\frac{1}{|\beta|}\right)^p\int_{B_{3r}}|X\eta|^p w^p\,dx \\ +\left[\frac{1}{\phi_V^{-1}\left(|q|^{-p}\left(1+\frac{1}{|\beta|}\right)^{-p}\right)}\right]^{Q+p}\int_{B_{3r}}\eta^p w^p\,dx\Bigg\}\,.\end{aligned} \tag{5.31}$$

By Sobolev inequality we have

$$\begin{aligned}\left(\int_{B_{3r}} |\eta w|^{p^*}\,dx\right)^{\frac{p}{p^*}} \le C\frac{r^p}{|B_r|^{\frac{p}{Q}}}\Bigg\{(|q|^p+2)\left(1+\frac{1}{|\beta|}\right)^p\int_{B_{3r}}|X\eta|^p w^p\,dx \\ +\left[\frac{1}{\phi_V^{-1}\left(|q|^{-p}\left(1+\frac{1}{|\beta|}\right)^{-p}\right)}\right]^{Q+p}\int_{B_{3r}}\eta^p w^p\,dx\Bigg\}\,,\end{aligned} \tag{5.32}$$

where $p^* = \frac{pQ}{Q-p} = p\chi$ and C is a positive constant independent of w.

Let r_1 and r_2 be real numbers such that $r \le r_1 < r_2 \le 2r$. Let the function η be chosen so that $\eta(x) = 1$ in B_{r_1}, $0 \le \eta(x) \le 1$ in B_{r_2}, $\eta(x) = 0$ outside B_{r_2}, $|X\eta| \le \frac{C}{r_2-r_1}$. We have

$$\left(\int_{B_{r_1}} w^{p^*}\,dx\right)^{\frac{p}{p^*}} \le C\frac{r^p}{|B_r|^{\frac{p}{Q}}}\frac{1}{(r_2-r_1)^p}(|q|^p+2)$$

$$\left(1+\frac{1}{|\beta|}\right)^p\left[\frac{1}{\phi_V^{-1}\left(|q|^{-p}\left(1+\frac{1}{|\beta|}\right)^{-p}\right)}\right]^{Q+p}\int_{B_{r_2}} w^p\,dx\,.$$

Putting $\gamma = pq = p + \beta - 1$ and recalling that $w(x) = v^q(x)$, we have
(5.33)

$$\Phi(\chi\gamma, r_1) \leq C^{\frac{1}{\gamma}} \left(\frac{r}{|B_r|^{\frac{1}{Q}}}\right)^{\frac{1}{q}} (|q|^p+2)^{\frac{1}{\gamma}} \left(1 + \frac{1}{|\beta|}\right)^{\frac{1}{q}} \left[\frac{1}{\phi_V^{-1}\left(|q|^{-p}\left(1+\frac{1}{|\beta|}\right)^{-p}\right)}\right]^{\frac{Q+p}{\gamma}} \frac{1}{(r_2-r_1)^{\frac{1}{p}}} \Phi(\gamma, r_2)\,,$$

for positive $\gamma \neq p - 1$, and

$$\Phi(\chi\gamma, r_1) \geq C^{\frac{1}{\gamma}} \left(\frac{r}{|B_r|^{\frac{1}{Q}}}\right)^{\frac{1}{q}} (|q|^p+2)^{\frac{1}{\gamma}} \left[\frac{1}{\phi_V^{-1}(|q|^{-p})}\right]^{\frac{Q+p}{\gamma}} \frac{1}{(r_2-r_1)^{\frac{1}{p}}} \Phi(\gamma, r_2)\,, \tag{5.34}$$

for negative γ. These are the inequalities which we wish to iterate. In order that (5.33) be applicable at each stage, we choose an initial value $p_0' \leq p_0$ in such a way that the point $p = 1$ lies midway between two consecutive iterates of p_0' and for $i = 0, 1, \ldots,$ we let

$$p_i = \chi^i p_0' \qquad r_i = r + \frac{r}{2^i}\,. \tag{5.35}$$

Thus we also obtain

$$|\beta| \geq \frac{\chi - 1}{1 + \chi}\,. \tag{5.36}$$

Iterating (5.33) and using lemma 4.1 to prove the convergence of the iteration procedure, we have

$$\Phi(\infty, r) \leq C(p, a, \phi_V, \operatorname{diam}\Omega)|B_r|^{\frac{-1}{p_0}} \Phi(p_0, 2r)\,. \tag{5.37}$$

Now if $\gamma_i = \chi^i p_0$ and $r_i = r + \frac{r}{2^i}$, the iteration of (5.34) yields

$$\Phi(-\infty, r) \geq C(p, a, \phi_V, \operatorname{diam}\Omega)|B_r|^{\frac{1}{p_0}} \Phi(-p_0, 2r)\,. \tag{5.38}$$

Therefore, collecting together all the previous inequalities and noting that from Hölder's inequality

$$\Phi(p_0', 2r) \leq \Phi(p_0, 2r)|B_r|^{\frac{1}{p_0'} - \frac{1}{p_0}}\,, \tag{5.39}$$

we obtain

$$\Phi(\infty, r) \leq C\Phi(-\infty, r) \tag{5.40}$$

where $C \equiv C(p, a, \phi_V, \operatorname{diam}\Omega)$, that is

$$\max_{B_r} u \leq C\left\{\min_{B_r} u + h\right\}\,. \tag{5.41}$$

□

Remark 5.4. *We wish to note that the proof of Theorem 5.3 works also with weak subsolutions of (5.3) and the proof of Theorem 5.3 provides a weak Harnack inequality for non negative weak supersolutions.*

6. Application to quasilinear subelliptic PDE: Regularity

This section is devoted to obtain regularity result directly from Harnack inequalities proved in the previous section. By a standard argument Harnack's inequality implies that weak solutions of (5.3) are continuous with respect to the Carnot Caratheodory metric $d(x,y)$. In fact we have

Theorem 6.1. *Suppose (A1)-(A3) hold true. Let Ω be a bounded domain having local homogeneous dimension Q and $u \in W^{1,p}_{X,loc}(\Omega)$, $1 < p < Q$, be a weak solution of (5.3). Let us assume that structure conditions (5.2) hold true with*

$$a \in \mathbb{R}\,,\ b^{p/p-1}\,, c^p\,,\ d\,,\ e^{p/p-1}\,,\ f\,, g\,, \in (M_X)'_p(\Omega)\,. \tag{6.1}$$

Then u is continuous in Ω.

Proof. Let $\Omega' \subset\subset \Omega$. By theorem 5.2 we have

$$|u(x)| \le L \tag{6.2}$$

where L is a positive constant depending on n, p, a, $\phi_{e^{\frac{p}{p-1}}}(1)$, $\phi_f(1)$, $\phi_g(1)$ and Ω'. Let B_r be a metric ball contained in Ω'. Then the functions

$$M(r) = \max_{B_r} u \qquad , \qquad m(r) = \min_{B_r} u \tag{6.3}$$

are then well defined in B_r and $\overline{u} = M(r) - u$, is a non negative weak solution in B_r of equation

$$\sum_{j=1}^{m} X_j^* \tilde{A}_j(x, \overline{u}, X\overline{u}) + \tilde{B}(x, \overline{u}, X\overline{u}) = 0\,. \tag{6.4}$$

We note that $\tilde{A}(x,\overline{u},\overline{\xi})$ and $\tilde{B}(x,\overline{u},\overline{\xi})$ are defined by

$$\tilde{A}(x,\overline{u},\overline{\xi}) = -A(x, M - \overline{u}, -\overline{\xi})$$

$$\tilde{B}(x,\overline{u},\overline{\xi}) = B(x, M - \overline{u}, -\overline{\xi}),$$

and satisfy

$$\begin{cases} |\tilde{A}(x,\overline{u},\overline{\xi})| \le a|\overline{\xi}|^{p-1} + \overline{b}|\overline{u}|^{p-1} + \overline{e} \\ |\tilde{B}(x,\overline{u},\overline{\xi})| \le c|\overline{\xi}|^{p-1} + \overline{d}|\overline{u}|^{p-1} + \overline{f} \\ \overline{\xi}\tilde{A}(x,\overline{u},\overline{\xi}) \ge |\overline{\xi}|^p - \overline{d}|\overline{u}|^p - \overline{g} \end{cases} \tag{6.5}$$

where $\overline{b}(x)$, $\overline{d}(x)$, $\overline{e}(x)$, $\overline{f}(x)$ and $\overline{g}(x)$ are measurable functions belonging to $(M_X)'_p$ defined by

$$\begin{cases} \overline{b}(x) = & 2^p b(x) \\ \overline{d}(x) = & 2^p d(x) \\ \overline{e}(x) = & 2^p b(x) L^{p-1} + e(x) \\ \overline{f}(x) = & 2^p d(x) L^{p-1} + f(x) \\ \overline{g}(x) = & 2^p d(x) L^{p-1} + g(x)\,. \end{cases} \tag{6.6}$$

Since $\overline{u}$ is a nonnegative weak solution, we can apply theorem 5.3 to get

$$\max_{B_{\frac{r}{3}}} \overline{u}(x) \leq C\left(\min_{B_{\frac{r}{3}}} \overline{u}(x) + \overline{h}\right) \tag{6.7}$$

where

$$C \equiv C\left(p, a, \phi_{b^{\frac{p}{p-1}}}(1), \phi_{c^p}(1), \phi_d(1)\right), \overline{h} \equiv \overline{h}(r) = \left[\phi_{\overline{e}^{\frac{p}{p-1}}}(r) + \phi_{\overline{g}}(r)\right]^{\frac{1}{p}} + \left[\phi_{\overline{f}}(r)\right]^{\frac{1}{p-1}} . \tag{6.8}$$

Note that $\overline{h}$ is a positive non decreasing function with $\lim_{r\to 0} \overline{h}(r) = 0$, such that

$$\overline{h}\left(\frac{r}{2}\right) \geq K\overline{h}(r) \quad , \quad 0 < K < 1.$$

We have

$$M(r) - m\left(\frac{r}{3}\right) \leq C\left[M(r) - M\left(\frac{r}{3}\right) + \overline{h}(r)\right] . \tag{6.9}$$

In the same way, setting

$$\overline{\overline{u}} = u - m(r) \tag{6.10}$$

we obtain

$$M\left(\frac{r}{3}\right) - m(r) = \max_{B_{\frac{r}{3}}} \overline{\overline{u}} \leq C\left(\min_{B_{\frac{r}{3}}} \overline{\overline{u}} + \overline{h}\right) = C\left[m\left(\frac{r}{3}\right) - m(r) + \overline{h}(r)\right] . \tag{6.11}$$

Adding the previous inequalities we have

$$M\left(\frac{r}{3}\right) - m\left(\frac{r}{3}\right) \leq \frac{C-1}{C+1}[M(r) - m(r)] + \frac{2C}{C+1}K^2\overline{h}\left(\frac{r}{4}\right) . \tag{6.12}$$

Set, for $\rho > 0$

$$\omega(\rho) = M(\rho) - m(\rho), \gamma = \frac{C-1}{C+1} \; h(r) = \frac{2C}{C+1}K^2\overline{h}(r), \tag{6.13}$$

then

$$\omega\left(\frac{r}{4}\right) \leq \omega\left(\frac{r}{3}\right) \leq \gamma\omega(r) + h\left(\frac{r}{4}\right) . \tag{6.14}$$

From lemma 4.2 it follows

$$\omega\left(\frac{r}{4}\right) \leq Kh^{\sigma}\left(\frac{r}{4}\right) \tag{6.15}$$

which is the continuity of the solution u. □

Our next result concerns hölder continuity of weak solutions. It is clear that, if we want to improve the result we have to restrict our assumptions. So we will assume that the coefficients in the structure will belong to some Morrey classes. Namely we are going to prove the following

Theorem 6.2. *Suppose (A1)-(A3), hold true. Let Ω be a bounded domain with local homogeneous dimension Q. Let $u \in W^{1,p}_X(\Omega)$, $1 < p < Q$, be a weak solution of (5.3). Let us assume structure conditions (5.2) where*

$$a \in \mathbb{R}\,,\; b^{p/p-1}\,, c^p\,,\; d\,,\; e^{p/p-1}\,,\; f\,, g\,, \in L^{1,p-\epsilon}_X(\Omega)\,. \tag{6.16}$$

Then the weak solution u is locally hölder continuous in Ω with respect to the Carnot–Caratheodory metric, namely for any $\Omega' \subset\subset \Omega$ there exist $c > 0$ and $\alpha > 0$ depending on the Morrey modulus of the coefficients of equation (5.3) such that

$$|u(x) - u(y)| \leq c\, d(x,y)^{\alpha} \quad \forall x, y \in \Omega'\,. \tag{6.17}$$

Proof. We note that our assumptions (6.16) are more restrictive than (6.1) and then, the result of our previous theorem hold true. To improve our previous result, it is sufficient to observe that thanks to our lemma 2.6 the h function appearing in the continuity result is now a power of the distance $d(x,y)$ and then the modulus of continuity of the solution has an algebraic decay. This ensures hölder continuity. □

Remark 6.3. *We want to compare the assumptions in the papers* [10] *and* [11]. *In* [10] *the author assumed the following*

$$\begin{cases} a = constant, \\ b,\, e \in L^{q,q(p-1)}_X(\Omega), \quad \frac{p}{p-1} < q < \frac{Q}{p-1}, \\ c \in L^{q,q(1-\varepsilon)}_X(\Omega), \quad p < q < \frac{Q}{1-\varepsilon}, \\ d,\; f,\; g \in L^{q,q(p-\varepsilon)}_X(\Omega), \quad 1 < q < \frac{Q}{p-\varepsilon}\,. \end{cases} \tag{6.18}$$

In [11] *the authors assumed the following*

$$\begin{cases} a = constant \\ b,\, e \in L^{\frac{Q}{p-1}}_{loc}(\Omega) \\ c \in L^{p,p-\varepsilon}_X(\Omega) \\ d,\; f,\; g \in L^{l,l(p-\varepsilon)}_X(\Omega), \quad 1 < l < \frac{Q}{p-\varepsilon}\,. \end{cases} \tag{6.19}$$

Using Hölder inequality it is easy to prove that our assumptions are more general than those contained both in [10] *and in* [11].

Remark 6.4. *We also point out that our results extend the sharp elliptic case to the non-commutative setting.*

References

[1] M.Aizenman,B.Simon Brownian motion and Harnack's inequality for Schrödinger operators. *Comm.Pure Appl. Math.*, 209–271.

[2] M.Biroli, U.Mosco Kato spaces and Dirichlet forms. *Pot. Analysis*, 10:327–345,(1999).

[3] S.M.Buckley Inequalities of John–Nirenberg type in doubling spaces. J.Anal. Math. 79:215–240 (1999)

[4] L. Capogna,D. Danielli and N. Garofalo An embedding theorem and the Harnack inequality for nonlinear subelliptic equations. *Comm. P.D.E.*, 18 n.9-10:1765–1794, (1993)

[5] F. Chiarenza,E. Fabes and N. Garofalo Harnack's inequality for Schrödinger operators and the continuity of the solutions. *Proc. A.M.S.*, 98:415–425 (1986)

[6] F. Chiarenza,M.Frasca A remark on a paper by C.Fefferman. *Proc. A.M.S.*, 108:407–409, (1990)

[7] G.Citti,G.Di Fazio Hölder continuity of the solutions for operators which are a sum of squares of vector fields plus a potential, *Proc.Amer.Math.Soc.*, 122:741–750 (1994),

[8] G.Citti, N. Garofalo and E.Lanconelli Harnack's inequality for sum of squares of vector fields plus a potential. *Amer. J.Math.*, 115:699–734 (1994)

[9] U.Mosco G.Dal Maso Wiener criteria and energy decay for relaxed Dirichlet problems. *Arch. for Rational Mech. and Analysis*, 95:345–387, (1986)

[10] D. Danielli A Fefferman-Phong type inequality and applications to quasilinear subelliptic equations. *Potential Analysis*, 115:387–413, (1999).

[11] D. Danielli, N. Garofalo and D. Nhieu Trace inequalities for Carnot-Caratheodory spaces and applications. *Ann. Scuola Norm. Sup. Pisa*, 4:195–252, (1998).

[12] E.De Giorgi Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat.*, 3:25–43,(1957).

[13] G. Di Fazio Poisson equation in Morrey spaces. *J.Math.Anal.Appl.*, 163:157–167,(1992).

[14] G. Di Fazio and P. Zamboni A Fefferman-Poincaré type inequality for Carnot-Carathéodory vector fields. *Proc. A.M.S.*, 130:2655–2660, (2002).

[15] C.Fefferman The uncertainty principle. *Bull. Amer. Math. Soc. (N.S.)*, 9:129–206, (1983).

[16] C.E.Gutierrez Harnack's inequality for degenerate Schrödinger operators. *Trans.Amer.Soc.*, 403–419,(1994).

[17] A.Hinz.,H.Kalf Subsolutions estimates and Harnack's inequality for Schrödinger operators˙ *J. Reine Angew. Math.*, 404:118–134,(1990)

[18] O.A. Ladizhenskaia and N. Ural'ceva *Linear and quasilinear elliptic equations*. Academic Press, (1968).

[19] H.Lewy,G.Stampacchia On the smoothness of superharmonics which solve a minimum problem. *J.Analyse Math.*, 23:227–236, (1970).

[20] G.Lu On Harnack's Inequality for a class of strongly degenerate Schrödinger operators formed by vector fields *Diff. Integral equations*, 7: 73–100(1994).

[21] G.Lu,R.L.Wheeden An optimal representation formula for Carnot–Caratheodory vector fields. *Bull.London Math.Soc.*, 30:578–584, (1998).

[22] C.B.Morrey *Multiple integrals in the calculus of variations*. Springer Verlag, (1966).

[23] M.A.Ragusa and P.Zamboni *Local regularity of solutions to quasilinear elliptic equations with general structure*. Commun. Appl. Anal. 3 (1999), no. 1, 131–147.

[24] J.M. Rakotoson and W.P. Ziemer Local behavior of solutions of quasilinear elliptic equations with general structure. *Trans. A.M.S.*, 319:747–764, (1990).

[25] M.Schechter Imbedding estimates involving new norms and applications. *Bull.Amer.Math.Soc.*, 11:163–166, (1984).

[26] M.Schechter *Spectra of Partial differential operators*. North Holland, (1986).

[27] J. Serrin Local behavior of solutions of quasilinear equations. *Acta Math.*, 111:247–302, (1964).

[28] C. Simader An elementary proof of Harnack's inequality for Schrödinger operators and related topics *Math.Z.*,203:129–152 (1990)

[29] G.Stampacchia Le probleme de Dirichlet pour les equations elliptiques du second ordre a coefficients discontinus. *Ann.Ist.Fourier*, XV:189–258, (1965).

[30] N.S.Trudinger On Harnack type inequalities and their applications to quasilinear elliptic equations. *Comm. Pure Applied Math.*, 20:721–747, (1967).

[31] C.Vitanza,P.Zamboni Necessary and sufficient conditions for Hölder continuity of solutions of degenerate Schrödinger operators *Le Matematiche*, 52:393–409 (1997).

[32] P.Zamboni Local boundedness of solutions of quasilinear elliptic equations with coefficients in Morrey spaces. *Boll. UMI*, 8-B:985–997, (1994).

[33] P.Zamboni Local behaviour of solutions of quasilinear elliptic equations with coefficients in Morrey spaces. *Rendiconti di Matematica*, 15(VII):251–262, (1995).

[34] P. Zamboni The Harnack inequality for quasilinear elliptic equations under minimal assumptions. *Manuscripta Math.*, 102:311–323, (2000).

[35] P. Zamboni Hölder continuity for solutions of linear degenerate elliptic equations under minimal assumptions *J.of Differential equations*, 182:121–140, (2002).

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