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# NEURONAL OSCILLATIONS IN THE VISUAL CORTEX: Γ-CONVERGENCE TO THE RIEMANNIAN MUMFORD–SHAH FUNCTIONAL*

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**Abstract.** The aim of this paper is to provide a formal link between an oscillatory neural model, whose phase is represented by a difference equation, and the Mumford and Shah functional. A Riemannian metric is induced by the pattern of neural connections, and in this setting the difference equation is studied. Its Euler–Lagrange operator Γ-converges as the dimension of the grid tends to 0 to the Mumford and Shah functional in the same Riemannian space. Correspondingly, the solutions of the phase equation converge to a $BV$ function, which is interpreted as the flow associated with the Mumford and Shah functional. In this way we provide a biological motivation to this celebrated functional.

**Key words.** neural oscillators, Cauchy problem for a difference equation, variational problems, Riemannian metrics, Γ-convergence, Mumford and Shah functional

**AMS subject classifications.** 49J45, 65K10, 39A70, 92C20

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## 1. Introduction.

An intriguing issue that has to be dealt with in the mammalian visual system is how the information distributed in the visual cortex gets bound together into coherent object representations. Along the path going from the physical object to the observer, radiations are completely independent one of the other. The retina is constituted in its turn by a mosaic of histologically separated elements. At the end of this chain, during which the unity of the original object is completely lost, the object shows up again at the perceptual level as a unit. In which way is it possible to reconstruct at the perceptual level the unity of the physical object? This process is known as "binding" or "perceptual grouping," and it has been extensively studied at least from two different points of view: From one side it has been the subject of research in the experimental psychology of Gestalt, oriented to infer the phenomenological laws of perceptual organization [33]. On the other side, neurophysiological studies have been focused on the determination of biological functionalities underlying grouping. In this paper we prove a formal relation between two of these models: a difference equation describing the phase of neuronal oscillators in the visual cortex, and the celebrated Mumford and Shah functional, first introduced as a phenomenological model. The family of discrete Euler–Lagrange functionals associated with the phase equation Γ-converges as the length of the grid tends to 0 to the Mumford and Shah functional in a $BV$ space related to a Riemannian metric.

### 1.1. A phenomenological model.

Mumford and Shah in their celebrated paper [36] proposed to obtain the segmentation of a given image $u_0$ as a minimum of

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the following functional:

$$E(u,K) = \alpha \int_{\mathbb{R}^n \setminus K} |\nabla u|^2 dx + \beta d H^{n-1}(K) + \int_{\mathbb{R}^n} |u - u_0|^2 dx,$$

where $K$ is closed, and $u \in W^{1,2}(\Omega \setminus K)$. This functional has been deeply studied in the weak formulation, provided by De Giorgi, Carriero, and Leaci in [20], who allow $u$ to be a $SBV$ function and $K$ its jump set:

$$MS(u) = \alpha \int_{\mathbb{R}^n} |\nabla u|^2 dx + \beta d H^{n-1}(S(u)) + \int_{\mathbb{R}^n} |u - u_0|^2 dx. \tag{1.1}$$

In the same paper [20] the existence of minima has been proved; their lower semicontinuity has been proved by Ambrosio [1]. The main properties of the minima have been established by Ambrosio and Pallara [3], Ambrosio, Fusco, and Pallara [4], Bonnet [8], David [18], and Bonnet and David [9].

It has also been deeply studied in the problem of Γ-approximation of the functional $MS$, with elliptic functionals. Different families of approximating functionals have been proposed by Ambrosio and Tortorelli [5], Braides and Dal Maso [13], and Gobbino [28], who proved a conjecture of De Giorgi. We are interested in this last result, since it is an approximation of the $MS$ functional with discrete functionals:

$$\frac{1}{\epsilon^{n+1}} \int_{\Omega \times \Omega} arctan \left( \frac{(u(x+\xi) - u(x))^2}{|\xi|} \right) e^{-\frac{|\xi|^2}{\epsilon}} dx d\xi. \tag{1.2}$$

Similar approximation problems have also been studied in [10, 11, 12, 14] in order to investigate the relation between the finite difference expression of the energy of elastic media and its continuous counterpart.

Here we will study the difference equation satisfied by the phase of neural oscillators with a technique similar to the one introduced in [28] and prove that it naturally leads to a nonisotropic version of the Mumford and Shah functional. A different approximation of nonisotropic functionals, analogous to [13], had already been provided by Cortesani [17]. Properties of minima of general anisotropic functionals of $MS$ type have been established by Fonseca and Fusco [26], Trombetti [40], and Fusco, Mingione, and Trombetti [27]. We also refer the reader to Baldi for a degenerate functional of this type [6].

**1.2. A neurophysiological model.** From the neurological point of view there is a large amount of experimental evidence that grouping is represented in the brain with a temporal coding, meaning that semantically homogeneous areas in the image would be encoded in the synchronization (phase locking) of oscillatory neural responses [22]. Shuster and Wagner [38, 39] described the emergence of oscillations in the visual cortex by modelling every cortical column by densely connected Wilson–Cowan neurons [41]. The appropriate mean field equations for the cluster of neurons show that every column can be interpreted as an oscillator. The visual cortex is then modelled as a collection of oscillators coupled with long range sparse interactions, represented by the reduced phase equation, on a grid of length 1:

$$\partial_t u(t) = \Delta_{-\xi} \Big( \phi \big( \Delta_\xi u \big) \Big)(x), \tag{1.3}$$

where $\Delta_\xi$ is the difference operator which acts as follows on each function $f$:

$$\Delta_\xi f(x) = f(x+\xi) - f(x).$$

The function $\phi$ is continuous, odd, and periodic of period $2\pi$ so that $\phi(\pi) = \phi(-\pi) = 0$.

The same equation can be adapted to a grid of arbitrary length. Since the function $u$ represents the phases of the oscillators, we can assume that $\Delta_\xi u$ takes its values in the interval $[-\pi, \pi]$ and $\phi = 0$ in $\mathbb{R}\backslash[-\pi, \pi]$. If $p > 0$, we call

$$\phi_{|\xi|}(z) = \frac{1}{|\xi|^{1-1/p}}\phi\big(|\xi|^{1/p} z\big), \quad |\xi| \neq 0, \tag{1.4}$$

and a suitable rescaling of the function $u$ is a solution of the equation

$$\partial_t u(t) = \frac{1}{|\xi|}\Delta_{-\xi}\left(\phi_{|\xi|}\left(\frac{\Delta_\xi u}{|\xi|}\right)\right)(x).$$

This finite difference degenerate parabolic equation has been extensively studied in one dimension in [34, 35]. Its ability to reach phase locking solutions and to present phase discontinuities has been outlined.

In higher dimension Shuster and Wagner also proposed to convolve with a Gaussian kernel, which expresses the probability that an oscillator is connected to another. They obtain the equation

$$\partial_t u(t) = \int_{\mathbb{R}^n} e^{-\frac{|\xi|}{\epsilon}}\frac{1}{|\xi|}\Delta_{-\xi}\left(\phi_{|\xi|}\left(\frac{\Delta_\xi u}{|\xi|}\right)\right)\frac{d\xi}{\epsilon^n} \tag{1.5}$$

with the change of variable $\eta = \xi/\epsilon$

$$= \int_{\mathbb{R}^n}\frac{1}{\epsilon|\eta|}\Delta_{-\epsilon\eta}\left(e^{-|\eta|}\phi_{\epsilon|\eta|}\left(\frac{\Delta_{\epsilon\eta} u}{\epsilon|\eta|}\right)\right)d\eta.$$

In this study we consider (1.5) in the $n$-dimensional space and with space variant anisotropic connections. Indeed, several neurophysiological studies show that the association field between cortical columns are space variant and strongly anisotropic [25]. Riemannian metric is directly induced by the coupling strength between cortical columns.

A Riemannian metric is defined in $\mathbb{R}^n$ if at every point there is defined a matrix $g_{ij}$ positive defined and continuous. In this case we call the Riemannian norm $|\eta|_g = g_{ij}\eta_i\eta_j$ and the Riemannian difference quotient

$$D^\epsilon_{g\eta}u(x) := \begin{cases} \frac{(\Delta_{\epsilon\eta}u(x))mod(2\pi)}{\epsilon|\eta|_g} & \text{if } \epsilon|\eta|_g \neq 0, \\ 0 & \text{if } \epsilon|\eta|_g = 0. \end{cases} \tag{1.6}$$

If $g$ is the identity, this difference quotient reduces to the standard one, and we denote it $D^\epsilon_\eta u$.

The resulting equation is then

$$\partial_t u(t) = \int_{\mathbb{R}^n} D^{-\epsilon}_\eta\left(\frac{|\eta|}{|\eta|_g}e^{-|\eta|_g}\phi_{\epsilon|\eta|_g}\left(D^{-\epsilon}_{g\eta}u\right)h\right)d\eta, \tag{1.7}$$

for a continuous function $h$, where $\phi_{\epsilon|\eta|_g}$ is defined in (1.4).

**1.3. Relation between the stated models.** In this paper we prove a first relation between the stated models, and we provide a biological motivation for the Mumford and Shah functional. We prove the existence of a solution $u_\epsilon$ of the Cauchy problem associated with (1.7), defined for all $t \geq 0$, and we prove that it $\Gamma$-converges as $\epsilon$ goes to 0 to the gradient flow relative to the Mumford and Shah functional in the Riemannian space with metric $g_{ij}$.

Precisely the Euler–Lagrange functional associated with (1.7) is

$$F_\epsilon(u) = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{-|\eta|_g} \varphi_{\epsilon|\eta|_g} \left( D^\epsilon_{g\eta} u \right) h(x) dx \right) d\eta, \tag{1.8}$$

where $\varphi_{\epsilon|\xi|}$ is a primitive of the function $\phi_{\epsilon|\xi|}$ defined in (1.4) and the following theorem holds.

THEOREM 1.1. *Assume as before that $\phi$ is continuous, it is odd, $\phi > 0$ in $[0, \pi[$, and $\phi = 0$ on $[\pi, \infty[$. Let us call $\beta$ the constant value assumed by the primitive $\varphi$ of $\phi$ on the interval $[\pi, \infty[$, and assume that there exist constants $\alpha > 0$ and $p > 1$ such that*

$$\frac{\varphi(z)}{z^p} \to \alpha \neq 0 \quad as \quad z \to 0^+. \tag{1.9}$$

*Then the family $F_\epsilon$ defined in* (1.8) *$\Gamma$-converges in $L^1_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$ to the Mumford and Shah functional*

$$MS(u, \mathbb{R}^n) = \alpha\, c_{np} \int_{\mathbb{R}^n} |\nabla_g u|^p_g \frac{h(x)}{\sqrt{g(x)}} dx + \beta\, c_{n1} \int_{S(u)} |\nu_g|_g \frac{h(x)}{\sqrt{g(x)}} dH^{n-1} \tag{1.10}$$

*if $u \in SBV$, $MS(u, \mathbb{R}^n) = +\infty$ otherwise. $S(u)$ is the jump set of $u$, $\nu_g$ is the normal to $S(u)$ in the Riemannian metrics, $g = \det(g_{ij})$, and $c_{np}$ and $c_{n1}$ are dimensional constants, defined in* (2.2). *(We refer the reader to section* 2*, where the formal definitions of the jump set and the metric are recalled).*

REMARK 1.1. *The Riemannian Mumford and Shah functional is obtained for $h = g$ and $p = 2$:*

$$MS(u, \mathbb{R}^n) = \alpha\, c_{n2} \int_{\mathbb{R}^n} |\nabla_g u|^2_g \sqrt{g} dx + \beta\, c_{n1} \int_{S(u)} |\nu_g|_g \sqrt{g} dH^{n-1}.$$

*In the limit case $p = 1$, the functional $MS$ becomes the total variation functional, and an approximation result can be obtained with a modification of the technique used here as in* [30].

The functional $F_\epsilon$ is a generalization of a Riemannian setting of the functional studied in [28]. The proof in this last paper is based on the slicing method and uses in full strength the isotropy of the functional. The main idea of our proof is the adaptation of the known technique to an anisotropic setting. Indeed, we first note that any Riemannian metric admits a representation of the form

$$g^{ij} \xi_i \xi_j = c_{n2} \int_{\mathbb{R}^n} e^{-|\eta|_g} \frac{(\langle \xi, \eta \rangle)^2}{|\eta|^2_g} \sqrt{g} d\eta, \tag{1.11}$$

where $c_{n2}$ is a constant, depending on the dimension of the space (see Proposition 2.5). This representation allows us to write $g$ in terms of an isotropic scalar product and to extend to an anisotropic situation a convergence result known in the isotropic case.

As an application of the Γ-convergence Theorem 1.1, we prove an approximation result for minima of the *MS* functional.

THEOREM 1.2. *Let $1 \le q < +\infty$, and let $g \in L^q(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$. Then for every $\epsilon > 0$ there exists a solution $(u_\epsilon)$ of the minimum problem*

$$m_\epsilon = \min\left\{F_\epsilon(u) + \int_{\mathbb{R}^n} |u-g|^q dx : u \in BV(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z}), |Du|(\mathbb{R}^n) \le \frac{1}{\epsilon}\right\}.$$

*Moreover, for every sequence $(\epsilon_j)$ with $\epsilon_j \to 0$ the family $(u_{\epsilon_j})$ has a subsequence converging in $L^1_{loc}$ to a solution of the minimum problem*

$$m_0 = \min\left\{MS(u, \mathbb{R}^n) + \int_{\mathbb{R}^n} |u-g|^q dx, u \in SBV(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})\right\}. \tag{1.12}$$

*Finally, $m_\epsilon \to m_0$ as $\epsilon \to 0$.*

The proof is mainly based on a compactness result for a family of functions $u_\epsilon$ such that $F_\epsilon(u_\epsilon)$ is bounded. Indeed, since for every $\epsilon$ the functional $F_\epsilon$ has a minimum, by the compactness result, all the minima belong to the same compact subset of $BV$. Once this is established, the existence of the minimum point for $MS$ follows from a general property of the Γ-convergence.

Then we apply the Γ-convergence result to the difference equation (1.7). For a fixed function $u_0 \in BV(\mathbb{R}, \mathbb{R}/2\pi\mathbb{Z})$, we consider a piecewise constant approximating family $(u_{0\epsilon})$ and for every $\epsilon > 0$ the problem

$$\begin{cases} \partial_t u_\epsilon(t) = -\nabla F_\epsilon(u_\epsilon(t)), \quad t \ge 0, \\ \\ u_\epsilon(0) = u_{0\epsilon}. \end{cases} \tag{1.13}$$

We prove that the solution $(u_\epsilon)$ is defined for every $t > 0$ and belongs to $C([0, +\infty[; L^p_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z}))$.

It converges in $BV$ to a function $u \in C([0, +\infty[; L^p_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z}))$, which will then be interpreted as the flow associated with the Mumford and Shah functional, with initial datum $u_0$. This function $u$ is a natural candidate for the flow associated with the Mumford and Shah functional. By now we can give only a characterization for $u$ under the additional assumption that $p = 2$ and out of the jump set (see Corollary 5.5 below). The problem of the behavior of the jump set is still open, even in the Euclidean situation.

This paper is organized as follows. In section 2 we give some preliminary definitions of Γ-convergence and of Riemannian manifold. In sections 3 and 4, respectively, we prove Theorems 1.1 and 1.2. Finally, in section 5 we describe the behavior of the flow.

## 2. Preliminary definitions and notations.

**2.1. *BV* functions and Γ-convergence.** In this section we recall the definitions of functions of bounded variation and of Γ-convergence of functionals.

The class of $BV$ functions is a class of functions whose distributional derivative is a nonnegative measure. We recall here the definition and refer the reader to [24] or

[23], where these notions are presented in full details. See also [21], where the set of $BV$ functions with values in $\mathbb{R}/2\pi\mathbb{Z}$ is studied.

DEFINITION 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote $M(\Omega)$ the set of all signed Radon measures on $\Omega$ with bounded total variation. We say that a function $u \in L^1(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ is a function of bounded variation, and we write $u \in BV(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ if all its distributional derivatives $D_i u$, $i = 1, \ldots, n$, belong to $M(\Omega)$. It is well known that the following relation is satisfied almost everywhere:*

$$\lim_{\rho \to 0} \rho^{-n} \int_{B_\rho(x)} |u(y) - z| dy = 0$$

*for some $z \in \mathbb{R}$, and all points $x$ satisfying this relation are called Lebesgue points. The jump set $S(u)$ is the complementary of the set of Lebesgue points of $u$. If $u \in BV(\Omega, \mathbb{R}/2\pi\mathbb{Z})$, then the set $S(u)$ has Hausdorff measure at most $n-1$. Moreover, for $H^{n-1}$ in almost every $x \in S(u)$ it is possible to find $a$, $b \in \mathbb{R}/2\pi\mathbb{Z}$ and a unitary vector $\nu$ such that*

$$\lim_{\rho \to 0} \rho^{-n} \int_{B^\nu_\rho(x)} |u(y) - a| dy = 0, \quad \lim_{\rho \to 0} \rho^{-n} \int_{B^{-\nu}_\rho(x)} |u(y) - b| dy = 0,$$

*where $B^\nu_r(x)$ is the half sphere $\{y \in B_r(x) : \langle y - x, \nu \rangle > 0\}$. The triplet $(a, b, \nu)$ is uniquely determined up to a change of sign, and it will be denoted $(u^+(x), u^-(x), \nu_u(x))$.*

The distributional derivative $Du$ admits the following decomposition:

$$Du = D^a u + D^j u + D^c u,$$

where $D^a u = \nabla u \mathcal{L}_n$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}_n$,

$$D^j u = \big(u^+(x) - u^-(x)\big) \nu_u H^{n-1} \lfloor S(u)$$

is the jump part, and $D^c u$ is the Cantor part of $Du$.

A $BV$ function $u$ is a special function of bounded variation if $D^c u = 0$ and the set of these functions is denoted $SBV(\Omega)$. A function $u$ belongs to $SBV_{loc}(\Omega, \mathbb{R}/2\pi\mathbb{Z})$ if $u \in SBV(A, \mathbb{R}/2\pi\mathbb{Z})$ for all $A \subset\subset \Omega$.

Let us now recall the De Giorgi definition of $\Gamma$-convergence.

DEFINITION 2.2. *If $(X, d)$ is a metric space, a family $F_j : X \to \mathbb{R}$ of functionals $\Gamma$-converges to $F$ as $j \to \infty$ if the following two conditions are satisfied:*

(i) *for every $u$ in $X$ and any sequence $(u_j)$ converging to $u$ in $X$,*

$$F(u) \le \liminf_j F_j(u_j);$$

(ii) *for every $u \in X$ there exists a sequence $(u_j)$ converging to $u$ in $X$ such that*

$$F(u) \ge \limsup_j F_j(u_j).$$

This notion of convergence captures the behavior of minimizers in the sense of the following theorem.

THEOREM 2.3. *Let us suppose that the family $F_j$ of functionals $\Gamma$-converges to $F$ as $j \to +\infty$ and that there exists a compact set $K$ such that $F_j$ takes its minimum on $K$ for every $j \in \mathbb{N}$ . Then $F$ has a minimum.*

We also refer the reader to [19], where these notions are introduced and described.

**2.2. Riemannian metrics.** In this subsection we recall the definition of Riemannian metric and refer the reader to [32] for a detailed presentation.

DEFINITION 2.4. *A Riemannian metric on a differentiable manifold $M$ is given by a scalar product on each tangent space $T_qM$, $q \in M$, which depends smoothly on the point $q$.*

Thus, if $M$ has dimension $n$ and $x = (x^1, \dots, x^n)$ are local coordinates of $M$, then a metric can be represented by a positive definite, symmetric matrix $G(x) = (g_{ij}(x))_{i,j}$ whose coefficients depend smoothly on $x$. Besides, the scalar product of two tangent vectors $v, w \in T_qM$ is $\langle v, w\rangle_g = g_{ij}(x)v^i w^j$, and the norm is $|v|_g^2 = g_{ij}(x)v^i v^j$. We remark that a Riemannian metric induces a metric on the cotangent bundle $T^*M = \cup_{q\in M} T_q^*M$ defined as follows: if $\zeta, \eta \in T_q^*M$, then

$$\langle \zeta, \eta\rangle_g = g^{ij}(x(q))\eta_i\zeta_j,$$

where $G^{-1} = (g^{ij})_{ij}$ is the inverse matrix of $G$. If a metric $g_{ij}$ is defined on an open set $\Omega$ in $\mathbb{R}^n$ and $u \in BV(\Omega, \mathbb{R}/2\pi\mathbb{Z})$, the Riemannian gradient is the vector

$$\nabla_g u = G^{-1}\nabla u,$$

and its norm in the metric $(g_{ij})$ is

$$|\nabla_g u|_g = (g^{ij}\partial_i u \partial_j u)^{1/2}.$$

Analogously, if $\nu_u$ is the normal to the set $S(u)$, defined at the end of Definition 2.1, the normal vector with respect to the metric $g$ is

$$\nu_g = G^{-1}\nu_u, \tag{2.1}$$

and its norm is $|\nu_g|_g = (g^{ij}(\nu_u)_i(\nu_u)_j)^{1/2}$ (see [7]).

Finally, we prove a duality relation between the norm on the tangent space and the cotangent.

PROPOSITION 2.5. *Let $v \in \mathbb{R}^n$, and let us call $v_g = G^{-1}v$, as in the definition of the Riemannian gradient or Riemannian normal vector. Then*

$$(|v_g|_g)^p = c_{np}\int_{\mathbb{R}^n} e^{-|\eta|_g}\frac{|\langle v, \eta\rangle|^p}{|\eta|_g^p}\sqrt{g}d\eta \tag{2.2}$$

*and*

$$\langle v_g, w_g\rangle_g = c_{n2}\int_{\mathbb{R}^n} e^{-|\eta|_g}\frac{\langle v, \eta\rangle\langle w, \eta\rangle}{|\eta|_g^2}\sqrt{g}d\eta \tag{2.3}$$

*for suitable constants $c_{np}$, depending on the dimension of the space and $p$.*

*Proof.* We fix a vector $w$ of Euclidean length 1 and note that

$$\int_{\mathbb{R}^n} e^{-|\xi|}\frac{|\langle w, \xi\rangle|^p}{|\xi|^p}d\xi = \frac{1}{c_{np}}$$

is a constant independent of $w$. Denoting $A = (a_{ij})_{ij}$ the square root of $G$, with the change of variable $\xi = \eta A$ we have $\sum_s(\xi_s)^2 = \eta_k\eta_h g_{kh} = |\eta|_g^2$. Then the second member of (2.2) can be computed:

$$\int_{\mathbb{R}^n} e^{-|\eta|_g}\frac{|\langle v, \eta\rangle|^p}{|\eta|_g^p}\sqrt{g}d\eta = \int_{\mathbb{R}^n} e^{-|\xi|}\frac{|\langle vA^{-1}, \xi\rangle|^p}{|\xi|^p}d\xi$$

$$= \frac{1}{c_{np}} |vA^{-1}|^p = \frac{1}{c_{np}} |v_g|_g^p.$$

The first assertion is proved.

In order to prove the second one, we first note that

$$\delta_{ij} = c_{n2} \int_{\mathbb{R}^n} e^{-|\xi|} \frac{\xi_i \xi_j}{|\xi|^2} d\xi,$$

where $\delta_{ij}$ is the Kronecker delta. On the other hand, if we denote $A^{-1} = (a^{ij})_{ij}$,

$$\langle v_g, w_g \rangle_g = g^{hk} v_h w_k = a^{hi} \delta_{ij} a^{jk} v_h w_k = c_{n2} \int_{\mathbb{R}^n} e^{-|\xi|} \frac{v_h a^{hi} \xi_i \xi_j a^{jk} w_k}{|\xi|^2} d\xi$$

$$= c_{n2} \int_{\mathbb{R}^n} e^{-|\xi|} \frac{\langle \xi A^{-1}, v \rangle \langle \xi A^{-1}, w \rangle}{|\xi|^2} d\xi.$$

Then, with the same change of variable as before, $\eta = \xi A^{-1}$, we get the thesis. □

**3. Γ-convergence results: Proof of Theorem 1.1.** In this section we first recover formally the expression of the Euler–Lagrange functionals $F_\epsilon$; then we prove the Γ-convergence of the family $F_\epsilon$ to the Mumford and Shah functional

$$MS(u, \mathbb{R}^n) = \begin{cases} \alpha\, c_{np} \int_{\mathbb{R}^n} |\nabla_g u|_g^p \frac{h(x)}{\sqrt{g(x)}} dx + \beta\, c_{n1} \int_{S(u)} |\nu_g|_g \frac{h(x)}{\sqrt{g(x)}} dH^{n-1} & \text{if } u \in SBV, \\ +\infty & \text{otherwise.} \end{cases}$$

The proof of Theorem 1.1 is based on the slicing method, a general integral-geometric technique which allows us to represent the functional $F_\epsilon(u)$ in terms of its one-dimensional sections. In this way it is possible to reduce the dimension of the problem to one and recover the Γ-limit result through the study of the one-dimensional problem. The method we use is a combination of the techniques in [28] and [10], where similar convergence results are provided.

**3.1. An approximating family of discrete functionals.** Let us first formally write the expression of the Euler–Lagrange functional for (1.7), giving the definition of the space where the problem will be studied.

The equation is defined in terms of a metric $(g_{ij})_{ij}$ such that $g_{ij}$ are continuous functions on $R^n$ and that there are two positive constants $\lambda$ and $\Lambda$ such that

$$\lambda |\eta|^2 \leq g_{ij}(x) \eta^i \eta^j \leq \Lambda |\eta|^2 \quad \forall x, \eta \in \mathbb{R}^n. \tag{3.1}$$

Let us call $h : \mathbb{R}^n \to \mathbb{R}$ a continuous function such that

$$\lambda \leq h(x) \leq \Lambda \quad \forall x \in \mathbb{R}^n.$$

Let us recall here the assumptions required in Theorem 1.1. The function $\phi : \mathbb{R} \to \mathbb{R}$ is continuous, it is odd, $\phi > 0$ in $[0, \pi[$, and $\phi = 0$ on $[\pi, +\infty[$. If $\varphi : \mathbb{R} \to \mathbb{R}$ is a primitive function of $\phi$ null in 0, $\varphi$ is obviously of class $C^1([0, +\infty[)$ and constantly assumes a value $\beta$ in $[\pi, +\infty[$. Moreover, we require that (1.9) holds. A primitive $\varphi_\epsilon$ of the rescaled function $\phi_\epsilon$ defined in (1.4) is $\varphi_\epsilon(t) = \frac{1}{\epsilon} \varphi(\epsilon^{1/p} t)$.

We consider the following functional:

$$F_\epsilon : L^p \to \mathbb{R} \quad F_\epsilon(u) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-|\eta|_g}\varphi_{\epsilon|\eta|_g}\big(D^\epsilon_{g\eta}u(x)\big)h(x)dx\, d\eta, \tag{3.2}$$

where $D^\epsilon_{g\eta}u$ is defined in (1.6).

REMARK 3.1. *$F_\epsilon(u) < +\infty$ for every $u \in L^p$. Indeed, by the assumption* (1.9) *on $\varphi$ there exists $\delta > 0$ such that*

$$\varphi(z) \le c_1 z^p \quad \forall z \in [0,\delta] \tag{3.3}$$

*for a suitable constant $c_1$. Here and in what follows we will denote $c_i$ any constant depending only on the data of the problem. On the other hand, since $\phi$ is nonnegative, $\varphi$ is increasing and takes its maximum at $\pi$. It then follows that*

$$\varphi(z) \le \beta \le c_1 z^p \quad \forall z \ge \delta. \tag{3.4}$$

*Analogous inequalities hold for $\varphi_\epsilon$, with the same constant, so that*

$$F_\epsilon(u) \le c_\epsilon \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-|\eta|_g}\big|D^\epsilon_{g\eta}u(x)\big|^p dx\, d\eta \le c_{\epsilon p}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-|\eta|_g}\big|u(x)\big|^p dx\, d\eta,$$

*and this is finite if $u \in L^p(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$. In particular, due to* (3.3) *and* (3.4) *we also have the following: there exist positive constants $c_1, c_2$ such that*

$$c_1 \min\{\alpha z^p, \beta\} \le \varphi(z) \le c_2 \min\{\alpha z^p, \beta\}. \tag{3.5}$$

In order to recognize that $F_\epsilon$ is the Euler–Lagrange functional of the discrete phase equation, we will work in the following set of piecewise constant functions:

$$PC^p_\epsilon = \{u \in L^p(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z}) : u \text{ is constant on the cube } \epsilon z + [0,\epsilon]^n\ \forall z \in \mathbb{Z}^n\}.$$

PROPOSITION 3.1. *Let $\epsilon > 0$. Then we have the following:*

(i) *for every $u \in PC^p_\epsilon$ the gradient of $F_\epsilon$ in $u$ is given by*

$$(\nabla F_\epsilon(u))(x) = -\int_{\mathbb{R}^n} D^{-\epsilon}_\eta\left(he^{-|\eta|_g}\frac{|\eta|}{|\eta|_g}\phi_{\epsilon|\eta|_g}\big(D^\epsilon_{g\eta}u\big)\right)(x)d\eta,$$

*where we simply denote $D^\epsilon_\eta$ the difference quotient when the metric $g$ is the Euclidean metric;*

(ii) *$\nabla F_\epsilon$ is a Lipschitz continuous function on $PC^p_\epsilon$.*

*Proof.* In order to prove (i) we calculate the Gâteaux derivative along a direction $v \in PC_\epsilon^{\frac{p}{p-1}}$:

$$\lim_{\delta\to 0}\frac{F_\epsilon(u+\delta v) - F_\epsilon(u)}{\delta}$$

$$= \lim_{\delta\to 0}\frac{1}{\delta}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x)\, e^{-|\eta|_g}\Big(\varphi_{\epsilon|\eta|_g}\big(D^\epsilon_{g\eta}u(x) + \delta D^\epsilon_{g\eta}v(x)\big) - \varphi_{\epsilon|\eta|_g}\big(D^\epsilon_{g\eta}u(x)\big)\Big)dx\, d\eta$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} he^{-|\eta|_g}\phi_{\epsilon|\eta|_g}\big(D^\epsilon_{g\eta}u\big)\frac{|\eta|}{|\eta|_g}D^\epsilon_\eta v\, dx\, d\eta$$

formally integrating by parts the difference quotient

$$= -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D^{-\epsilon}_\eta\Big(h(x)e^{-|\eta|_g}\frac{|\eta|}{|\eta|_g}\phi_{\epsilon|\eta|_g}\big(D^\epsilon_{g\eta}u\big)\Big)(x)v(x)\, dx\, d\eta.$$

Finally, $\nabla F_\epsilon$ is Lipschitz continuous because it is compositions of Lipschitz continuous functions. ∎

**3.2. The one-dimensional case.** Let us start with studying the simplest operator of the form (1.8) in $\mathbb{R}$:

$$\int_{\mathbb{R}} f(x)\varphi_{\epsilon|\eta|_g}\left(D^{\epsilon}_{g\eta}u(x)\right)dx, \tag{3.6}$$

where $\eta \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, is a continuous function such that

$$\lambda \le f(x) \le \Lambda \quad \forall x \in \mathbb{R},$$

with $\lambda$, $\Lambda$ positive constants. A metric in $\mathbb{R}$ is simply defined by a continuous function $b$ such that for every $\eta$,

$$|\eta|_{g(x)} = b(x)|\eta|.$$

Moreover, by simplicity in dimension 1 we will always assume that $\eta = 1$ so that the functional on an interval $I$ reduces to

$$\hat{F}_{\epsilon,1,f,b}(u, I) = \int_I f(x)\varphi_{\epsilon b(x)}\Big(\frac{D^{\epsilon}u(x)}{b(x)}\Big)dx, \tag{3.7}$$

where $D^{\epsilon} = D^{\epsilon}_1$ is the difference quotient with respect to the Euclidean metric.

We will give sufficient conditions for the $\Gamma$-convergence of the functional $\hat{F}_{\epsilon,1,f,b}(\cdot, I)$ to the Mumford and Shah functional

$$MS_{f,b}(u, I) = \begin{cases} \alpha \int_I f(x)\left(\frac{|u'(x)|}{b(x)}\right)^p dx + \beta \int_{I\cap S(u)} \frac{f(x)}{b(x)} dH^0(x) \\ \qquad\qquad\qquad \text{if} \in SBV(I), \\ \\ \qquad +\infty \quad \text{otherwise,} \end{cases} \tag{3.8}$$

where $\alpha$ is defined in (1.9) and $\beta = \varphi(\pi)$.

We recall the following regularity result, which is proved, for example, in Theorem 2.6 in [10].

THEOREM 3.2. *The functional $MS_{f,b}(u, I)$ is lower semicontinuous in $L^1_{loc}(I)$.*

In order to prove the $\Gamma$-convergence result in $L^1_{loc}(\mathbb{R}, \mathbb{R}/2\pi\mathbb{Z})$, we need an approximation lemma for sequences converging in $L^1_{loc}(\mathbb{R}, \mathbb{R}/2\pi\mathbb{Z})$; see [10].

LEMMA 3.3. *Let $u_\epsilon \to u$ in $L^1_{loc}(\mathbb{R}, \mathbb{R}/2\pi\mathbb{Z})$. We call $T^{\epsilon}_y v(x)$ a function whose values on the interval $[y+\epsilon(k, k+1)]$, $k \in \mathbb{Z}$, are between $u_\epsilon(y+k\epsilon)$ and $u_\epsilon(y+(k+1)\epsilon)$. Then, for almost every $y \in (0, \epsilon)$ and all choices of functions $T^{\epsilon}_y v(x)$, the family $T^{\epsilon}_y v(x)$ converges to $u$ in $L^1_{loc}(\mathbb{R})$.*

LEMMA 3.4. *Let us first assume that there are two positive constants $\tilde{\alpha}$ and $\tilde{\beta}$ such that*

$$\varphi(z) = \min\left\{\tilde{\alpha} z^p, \tilde{\beta}\right\}. \tag{3.9}$$

*Then for every $u \in L^1_{loc}(\mathbb{R}, \mathbb{R}/2\pi\mathbb{Z})$, for every sequence $u_j \to u$ in $L^1_{loc}(\mathbb{R}, \mathbb{R}/2\pi\mathbb{Z})$ there exists a sequence $\epsilon_j \to 0$ such that*

$$\lim_{j\to+\infty}\inf \hat{F}_{\epsilon_j,1,f,b}(u_j, \mathbb{R}) \ge MS_{f,b}(u, \mathbb{R}).$$

*Proof.* By simplicity of notation in the proof we will always denote $MS(u, I)$ instead of $MS_{f,b}(u, I)$, $\hat{F}_\epsilon(u, I)$ instead of $\hat{F}_{\epsilon,1,f,b}(u, I)$, and $D^\epsilon$ instead of $D^\epsilon_1$.

We will call $s \in \mathbb{R}$ such that $\varphi$ is constant in $[s, +\infty[$.

First we assume that $I$ is a bounded open interval of $\mathbb{R}$. Let $u_j \to u$ in $L^1(I, \mathbb{R}/2\pi\mathbb{Z})$ and show that

$$\liminf_j \hat{F}_{\epsilon_j}(u_j, I) \geq MS(u, I).$$

Let $\delta > 0$ be fixed. Arguing as in Braides [10, p. 82], we can assume that there exists a subsequence, always denoted $\epsilon_j$, and a sequence $(y_j)$, with $y_j \in (0, \epsilon_j)$ satisfying the thesis of Lemma 3.3 such that

$$\hat{F}_{\epsilon_j}(u, I) + \delta \geq \epsilon_j \sum_{k \in J_j} f(k\epsilon_j + y_j)\varphi_{\epsilon_j b}\Big(\frac{D^{\epsilon_j}u(k\epsilon_j + y_j)}{b(k\epsilon_j + y_j)}\Big),$$

where we have denoted

$$J_j = \{k \in \mathbb{Z} : ]\epsilon_j k + y_j, \epsilon_j(k+1) + y_j[ \subset I\}.$$

This is a particular version of the mean value theorem for integrals, where we have only one inequality, since we are not free to choose $y_j$ in an arbitrary way but only almost everywhere.

Since $J_j$ is finite, we can write

$$J_j = \{k_1^j, \dots, k_{N_j}^j\}$$

and denote

$$J_j^1 = \left\{ k \in J_j : \frac{|(u_j((k+1)\epsilon_j + y_j) - u_j(k\epsilon_j + y_j)) \, mod \, 2\pi|}{\epsilon_j b(k\epsilon_j + y_j)} \leq s\epsilon_j^{-1/p} \right\}, \quad J_j^2 = J_j \backslash J_j^1.$$

Then we define $v_j = T_y^{\epsilon_j} u_j$ as follows:

$$\begin{cases} \Big(\frac{t - y_j}{\epsilon_j} - k\Big)u_j(\epsilon_j(k+1) + y_j) + \Big((k+1) - \frac{t - y_j}{\epsilon_j}\Big)u_j(k\epsilon_j + y_j), & t \in \ y_j + \epsilon_j]k, k+1[, \ k \in J_j^1, \\ u_j(k\epsilon_j + y_j), & t \in \ y_j + \epsilon_j]k, k+1[, \ k \in J_j^2, \\ u_j(k_0^1\epsilon_j + y_j) & \text{if } t \leq y_j + k_1^j\epsilon_j, \\ u_j((k_{N_j}^k + 1)\epsilon_j + y_j) & \text{if } t \geq y_j + (k_{N_j}^j + 1)\epsilon_j. \end{cases}$$

The choice of $y_j$ is made, according to Lemma 3.3, in such a way that $v_j \to u$ in $L^1(I)$.

With this notation the estimate of $\hat{F}_\epsilon$ becomes

$$\hat{F}_{\epsilon_j}(u, I) + \delta \geq \epsilon_j \sum_{k \in J_j} f(k\epsilon_j + y_j)\varphi_{\epsilon_j b}\Big(\frac{D^{\epsilon_j}u(k\epsilon_j + y_j)}{b(k\epsilon_j + y_j)}\Big)$$

$$= \tilde{\alpha} \sum_{k \in J_j^1} \epsilon_j f(k\epsilon_j + y_j)\Big|\frac{D^{\epsilon_j}u(k\epsilon_j + y_j)}{b(k\epsilon_j + y_j)}\Big|^p + \tilde{\beta} \sum_{k \in J_j^2} \frac{f(k\epsilon_j + y_j)}{b(k\epsilon_j + y_j)}$$

$$= \tilde{\alpha} \int_I f(x)\left|\frac{v_j'(x)}{b(x)}\right|^p dx + \tilde{\beta} \sum_{x \in S(v_j) \cap I} \frac{f(x)}{b(x)}.$$

The sequence $v_j$ converges to $u$ by its choice. On the other hand, the operator $MS$ is lower semicontinuous so that

$$\liminf_{j\to+\infty} \hat{F}_{\epsilon_j}(u_j, I) \geq MS(u, I) - \delta.$$

The arbitrariness of $\delta > 0$ gives the thesis in the case where $I$ is a bounded open interval. The result is still valid for $\mathbb{R}$ approximating from the interior by bounded and open interval $I$.

In order to deal with the general case, we recall the following theorem about supremum of family of positive measures, which can be found in [10].

PROPOSITION 3.5. *Let $\Omega$ be an open set and $A(\Omega)$ be the family of its open subsets. Let $\mu_1 : A(\Omega) \to [0, +\infty[$ be an open set function, superadditive on open sets with disjoint compact closures. Let $\mu$ be a positive measure, let $\psi_i$ be positive Borel functions such that $\mu_1(A) \geq \int_A \psi_i d\mu$ for all $A \in A(\Omega)$, and let $\psi(x) = \sup \psi_i(x)$. Then $\mu_1(A) \geq \int_A \psi d\mu$ for all $A \in A(\Omega)$.* □

THEOREM 3.6. *Let $\phi$ and $\varphi$ satisfy the assumptions stated in Theorem 1.1. Then for every $u \in L^1_{loc}(\mathbb{R}, \mathbb{R}/2\pi\mathbb{Z})$, for every sequence $u_j \to u$ in $L^1_{loc}(\mathbb{R}, \mathbb{R}/2\pi\mathbb{Z})$ there exists a sequence $\epsilon_j \to 0$ such that*

$$\liminf_{j\to+\infty} \hat{F}_{\epsilon_j,1,f,b}(u_j, \mathbb{R}) \geq MS_{f,b}(u, \mathbb{R}).$$

*Proof.* Let $a_i$ and $b_i$ be sequences of positive real numbers such that $\sup_i a_i = \alpha$, $\sup_i b_i = \beta$, and

$$\varphi_i(z) = \min\left\{a_i z^p b_i\right\} \leq \varphi(z) \quad \forall\, t \geq 0 \tag{3.10}$$

by Remark 3.1. Note that we do not require any monotonicity property on $a_i$ and $b_i$ so that their existence is ensured. From Lemma 3.4 we have

$$\liminf_{j\to+\infty} \hat{F}_{\epsilon_j,1,f,b}(u_j, \mathbb{R}) \geq a_i \int_I f(x) \left(\frac{|u'(x)|}{b(x)}\right)^p dx + b_i \int_{I\cap S(u)} \frac{f(x)}{b(x)} dH^0(x)$$

for every $i$. In order to apply Proposition 3.5 we set

$$\mu = \mathcal{L}_1 + \sum_{x\in S(u)} \delta_x,$$

where $\mathcal{L}_1$ is the Lebesgue measure and $\delta_x$ is the Dirac measure. We also set

$$\psi_i(x) = \begin{cases} a_i f(x) \left(\frac{|u'(x)|}{b(x)}\right)^p & \text{on } I\backslash S(u), \\ \\ b_i \frac{f(x)}{b(x)} & \text{on } S(u) \end{cases}$$

so that

$$\psi(x) = \sup \psi_i(x) = \begin{cases} \alpha f(x) \left(\frac{|u'(x)|}{b(x)}\right)^p & \text{on } I\backslash S(u), \\ \\ \beta \frac{f(x)}{b(x)} & \text{on } S(u). \end{cases}$$

By Proposition 3.5 we deduce

$$\liminf_{j\to+\infty} \hat{F}_{\epsilon_j,1,f,b}(u_j,\mathbb{R}) \geq \alpha \int_I f(x)\left(\frac{|u'(x)|}{b(x)}\right)^p dx + \beta \int_{I\cap S(u)} \frac{f(x)}{b(x)} dH^0(x).$$

This is the thesis. ◻

The opposite inequality is simpler. We start with a simple remark.

REMARK 3.2. *Let $I$ be a real interval, not necessarily bounded, and let $u \in BV(\mathbb{R}, \mathbb{R}/2\pi\mathbb{Z})$ such that $MS_{f,b}(u, I) < +\infty$. Then there exists a constant $c_1$ independent of $\epsilon$ such that for every*

$$A \subset \{x \in I : [x, x+\epsilon] \cap S(u) \neq \emptyset\}, \tag{3.11}$$

$$\int_A |D^\epsilon u(x)|^p dx \leq c_1 \int_{\tilde{A}_\epsilon} |u'(x)|^p dx, \tag{3.12}$$

*where $\tilde{A}_\epsilon = \cup_{x\in A}[x, x+\epsilon]$.*

Indeed,

$$\int_A |D^\epsilon u(x)|^p dx = \int_A \left| \int_0^1 u'(x+\epsilon s) ds \right|^p dx \leq c_1 \int_A \int_0^1 |u'(x+\epsilon s)|^p ds dx$$

(with the change of variable $y = x + \epsilon s$)

$$\leq c_1 \int_0^1 \int_{\tilde{A}_\epsilon} |u'(x)|^p dx ds = c_1 \int_{\tilde{A}_\epsilon} |u'(x)|^p dx.$$

THEOREM 3.7. *Let $\phi$ and $\varphi$ satisfy the assumptions stated in Theorem 1.1. Then for every $u \in L^1_{loc}(\mathbb{R}, \mathbb{R}/2\pi\mathbb{Z})$,*

$$\limsup_{\epsilon\to 0^+} \hat{F}_{\epsilon,1,f,b}(u, \mathbb{R}) \leq MS_{f,b}(u, \mathbb{R}).$$

*Proof.* Let us fix $\delta > 0$. We can obviously assume that $MS_{f,b}(u, \mathbb{R}) < +\infty$, which implies that $u$ has only a finite number of jumps. Then there exists $M > 0$ such that $u$ has no jumps in $I\backslash[-M, M]$. Since $u' \in L^p$, by the previous remark we can also assume that $M$ is chosen in such a way that for every $\epsilon$,

$$\int_{I\backslash[-M,M]} (|D^\epsilon u|^p + |u'|^p) dx \leq \delta. \tag{3.13}$$

From the previous remark it also follows that there exists $\sigma > 0$ independent of $\epsilon$ such that, for every $\epsilon$, for every $A$ satisfying (3.11), $A \subset [-M, M]$ and with Lebesgue measure $|A| < \sigma$, the following estimate holds:

$$\int_A (|D^\epsilon u|^p + |u'|^p) dx \leq \delta. \tag{3.14}$$

In particular, if we call

$$I_{k\epsilon} = \{x \in [-M, M] : [x, x+\epsilon] \cap S(u) \neq \emptyset, |D_\epsilon u(x)| > k\},$$

then, always for the previous remark,

$$|I_{k\epsilon}| \le \frac{1}{k^p}\int_{I_{k\epsilon}} |D^\epsilon u|^p dx \le \frac{1}{k^p}\int_{I\setminus S(u)} |u'(x)|^p dx \to 0$$

as $k \to +\infty$, uniformly in $\epsilon$. Then by (3.14) we can fix $k > 0$ such that for every $\epsilon$,

$$\int_{I_{k\epsilon}} |D^\epsilon u|^p dx \le \delta. \tag{3.15}$$

Let us denote $\{x_1, \dots, x_s\}$ the set of jumps of $u$, and let us call

$$J = \{x \in [-M, M] : [x, x+\epsilon] \cap S(u) = \Omega, |D_\epsilon u(x)| \le k\}, \quad I_S = \bigcup_j [x_j - \epsilon, x_j].$$

By (3.15) and (3.13) and the fact that $\varphi_\epsilon(z) \le c_2 z^p$ for every $z \in \mathbb{R}$, with $c_2$ independent of $\epsilon$, the discrete functional can be estimated as

$$\hat{F}_{\epsilon,1,f,b}(u, I) = 2c_2\delta + \sum_j \int_{x_j-\epsilon}^{x_j} f(x)\varphi_{\epsilon b}\Big(\frac{D^\epsilon u(x)}{b(x)}\Big)dx \tag{3.16}$$

$$+ \int_J f(x)\varphi_{\epsilon b}\Big(\frac{D^\epsilon u(x)}{b(x)}\Big)dx.$$

Each of the integrals in the first sum can be estimated using the definition of $\varphi_\epsilon$ and the fact that $\max \varphi = \beta$:

$$\int_{x_j-\epsilon}^{x_j} f(x)\varphi_{\epsilon b}\Big(\frac{D^\epsilon u(x)}{b(x)}\Big)dx \le \frac{\beta}{\epsilon}\int_{x_j-\epsilon}^{x_j} \frac{f(x)}{b(x)}dx \to \beta\frac{f(x_j)}{b(x_j)} \tag{3.17}$$

as $\epsilon$ tends to 0.

In the last integral of (3.16) we use the fact that $D^\epsilon u(x)$ takes values in the compact set $[-k, k]$ and punctually tends to $u'$, while $\varphi_{\epsilon b(x)}(z) \to \alpha\frac{|z|^p}{b^p(x)}$ uniformly if $(x, z)$ belong to a compact set. Hence

$$\varphi_{\epsilon b}\Big(\frac{D^\epsilon u(x)}{b(x)}\Big) \to \alpha\,\frac{|u'(x)|^p}{b^p(x)}$$

almost everywhere. Using again the fact that $D^\epsilon u(x)$ is bounded by $k$ we can apply Lebesgue's dominate convergence theorem on the bounded set $[-M, M]\setminus S(u)$ and obtain

$$\int_J f(x)\varphi_{\epsilon b}\Big(\frac{D^\epsilon u(x)}{b(x)}\Big)dx \to \alpha\int_{[-M,M]\setminus S(u)} \frac{|u'(x)|^p}{b^p(x)}dx. \tag{3.18}$$

Putting together (3.16), (3.17), and (3.18) we obtain

$$\limsup_{\epsilon\to 0^+} \hat{F}_{\epsilon,1,f,b}(u, \mathbb{R}) \le 2c_2\delta + MS_{f,b}(u, \mathbb{R}),$$

and this implies the thesis, since $\delta$ is arbitrary. $\square$

Finally, from Lemma 3.4, Theorem 3.6, and (3.5) we have the following corollary.

COROLLARY 3.8. *Let $\phi$ and $\varphi$ satisfy the assumptions stated in Theorem* 1.1. *Then*

$$\Gamma - \lim_{\epsilon\to 0} \hat{F}_{\epsilon,1,f,b}(u,\mathbb{R}) = MS_{f,b}(u,\mathbb{R}) \quad in \quad L^1_{loc}(\mathbb{R},\mathbb{R}/2\pi\mathbb{Z}),$$

$$\lim_{\epsilon\to 0}\hat{F}_{\epsilon,1,f,b}(u,\mathbb{R}) = MS_{f,b}(u,\mathbb{R}) \quad for \quad every \quad u\in SBV(\mathbb{R},\mathbb{R}/2\pi\mathbb{Z}),$$

*and*

$$\hat{F}_{\epsilon,1,f,b}(u,\mathbb{R}) \le C\ MS_{f,b}(u,\mathbb{R}) \quad for \quad every \quad u\in L^1(\mathbb{R},\mathbb{R}/2\pi\mathbb{Z}),$$

*with $C$ a positive constant.* □

**3.3. The $n$-dimensional case.** In this section we will deduce the general $n$-dimensional case from the one-dimensional result, using the slicing method already used in the nonperiodic, isotropic case by Braides [10].

This procedure is formally similar to a standard reduction in the integral so that we fix $\eta\in\mathbb{R}^n\setminus\{0\}$ and denote $\langle\eta\rangle^\perp = \{z\in\mathbb{R}^n : \langle\eta,z\rangle = 0\}$ the orthogonal space to $\eta$ with respect to the Euclidean metrics. For every $y\in\langle\eta\rangle^\perp$ consider the function $u_{\eta y}$ defined by

$$u_{\eta y}(t) = u\Big(y + t\frac{\eta}{|\eta|}\Big), \quad t\in\mathbb{R}.$$

With these notations the operator $F_\epsilon$ defined in (3.2) becomes

$$F_\epsilon(u,\mathbb{R}^n) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x)e^{-|\eta|_g}\varphi_{\epsilon|\eta|_g}\Big(D^\epsilon_{g\eta}u(x)\Big)dxd\eta \tag{3.19}$$

$$= \int_{\mathbb{R}^n}\int_{\langle\eta\rangle^\perp} \hat{F}_{\epsilon,\eta,f,b}(u_{\eta y},\mathbb{R})dyd\eta,$$

where

$$f(t) = \Big(he^{-|\eta|_g}\frac{|\eta|}{|\eta|_g}\Big)_{\eta y}(t), \qquad b(t) = \frac{|\eta|_g(y+t\frac{\eta}{|\eta|})}{|\eta|}, \tag{3.20}$$

and

$$\hat{F}_{\epsilon,\eta,f,b}(u_{\eta y}(t),\mathbb{R}) = \int_{\mathbb{R}} f(t)\varphi_{\epsilon\eta}\Big(\frac{D^\epsilon_{|\eta|}u_{\eta y}(t)}{b(t)}\Big)dt.$$

In this way the functional $F_\epsilon$ is represented in terms of one-dimensional sections.

Also the functional $MS$, defined in (1.10), can be represented in terms of its sections, and the function $u$ belongs to $BV$ if and only if its sections $u_{\eta y}$ belong to $BV(\mathbb{R})$.

THEOREM 3.9. (i) *Let $u\in SBV(\mathbb{R}^n,\mathbb{R}/2\pi\mathbb{Z})$. Then for all $\eta\in\mathbb{R}^n$ we have $u_{\eta y}\in SBV(\mathbb{R},\mathbb{R}/2\pi\mathbb{Z})$ for almost everywhere $y\in\langle\eta\rangle^\perp$ and, moreover,*

$$u'_{\eta y}(t) = \Big\langle \nabla u\Big(y+t\frac{\eta}{|\eta|}\Big), \frac{\eta}{|\eta|}\Big\rangle \quad for\ a.e.\ t\in\mathbb{R},$$

$$S(u_{\eta y}) = \left\{t \in \mathbb{R} : y+t\frac{\eta}{|\eta|} \in S(u)\right\}, \quad u^+_{\eta y}(t) = u^+\Big(y+t\frac{\eta}{|\eta|}\Big), \quad u^-_{\eta y}(t) = u^-\Big(y+t\frac{\eta}{|\eta|}\Big),$$

*where* $u^+$ *and* $u^-$ *are defined at the end of Definition* 2.1.

(ii) *Let* $u \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$*, and let* $MS_{f,b}$ *be the operator defined in* (3.8)*. If*

$$\int_{\langle\eta\rangle^\perp} MS_{f,b}(u_{\eta y}, \mathbb{R})dy < +\infty$$

*for every* $\eta \in B$*,* $B$ *a basis of vector space* $\mathbb{R}^n$*, then* $u \in SBV(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$*.*

We refer to Ambrosio [1] for the proof.

Applying the previous result we get the expression of our Mumford and Shah functional.

THEOREM 3.10. *For every function* $u \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$ *we have that*

$$\int_{\mathbb{R}^n}\left(\int_{\langle\eta\rangle^\perp} MS_{f,b}(u_{\eta y}, \mathbb{R})dy\right)d\eta = MS(u, \mathbb{R}^n).$$

*Proof.* We can assume that $u \in SBV(\mathbb{R}^n)$. In this case by Theorem 3.9 we have that

$$\int_{\langle\eta\rangle^\perp} MS_{f,b}(u_{\eta y}, \mathbb{R})dy$$

$$= \alpha \int_{\langle\eta\rangle^\perp}\int_{\mathbb{R}} f(t)\Big(\frac{|u'_{\eta y}(t)|}{b(t)}\Big)^p dtdy + \beta\int_{\langle\eta\rangle^\perp}\int_{S(u_{\eta y})}\frac{f(t)}{b(t)}dH^0(t)dy$$

by Theorem 3.9 and by definition (3.20)

$$= \alpha\int_{\mathbb{R}^n} h(x)e^{-|\eta|_g}\left|\Big\langle\nabla u(x), \frac{\eta}{|\eta|_g}\Big\rangle\right|^p dx + \beta\int_{S(u)} h(x)e^{-|\eta|_g}\left|\Big\langle\nu, \frac{\eta}{|\eta|_g}\Big\rangle\right| dH^{n-1}(x),$$

where the equality follows from [10] for the second integral. Integrating in $\eta$ the preceding equality we get

$$\int_{\mathbb{R}^n}\int_{\langle\eta\rangle^\perp} MS_{f,b}(u_{\eta y}, \mathbb{R})dyd\eta$$

$$= \alpha\int_{\mathbb{R}^n} h(x)\left(\int_{\mathbb{R}^n} e^{-|\eta|_g}\left|\Big\langle\nabla u(x), \frac{\eta}{|\eta|_g}\Big\rangle\right|^p d\eta\right)dx$$

$$+\beta\int_{S(u)} h(x)\left(\int_{\mathbb{R}^n} e^{-|\eta|_g}\left|\Big\langle\nu, \frac{\eta}{|\eta|_g}\Big\rangle\right| d\eta\right)dH^{n-1}(x)$$

(by Proposition 2.5)

$$= \alpha c_{np}\int_{\mathbb{R}^n}\frac{h(x)}{\sqrt{g(x)}}|\nabla_g u(x)|_g^p dx + \beta\, c_{n1}\int_{S(u)}\frac{h(x)}{\sqrt{g(x)}}|\nu_g|_g dH^{n-1}(x). \qquad \square$$

**Proof of Theorem 1.1.** We sketch the proof which follows from the convergence Corollary 3.8 and the representation of the limit functional provided in Theorem 3.10. Let $u, u_j \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$, $u_j \to u$ in $L^1_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$, let $\epsilon_j \to 0$, and let us prove that

$$\lim\inf_{j\to+\infty} F_{\epsilon_j}(u_j) \geq MS(u, \mathbb{R}^n).$$

Indeed, by (3.19) and the Fatou lemma

$$\lim\inf_{j\to+\infty} F_{\epsilon_j}(u_j) \geq \int_{\mathbb{R}^n}\int_{\langle\eta\rangle^\perp} \lim\inf_{j\to+\infty} \hat{F}_{\epsilon_j,\eta,f,b}((u_j)_{\eta y}, \mathbb{R})dyd\eta$$

(by Corollary 3.8)

$$\geq \int_{\mathbb{R}^n}\int_{\langle\eta\rangle^\perp} MS_{f,b}((u_j)_{\eta y}, \mathbb{R})dyd\eta = MS(u, \mathbb{R}^n)$$

by Theorem 3.10. Finally, the dominated convergence asserted in Corollary 3.8 ensures that $MS(u) = \lim_\epsilon F_\epsilon(u)$ for every $u$, and this proves the second requirement in the definition of $\Gamma$-convergence. ∎

**4. Existence of a minimum for the Mumford and Shah functional.** We will give here an approximation result of the minimization problem for the Riemannian Mumford and Shah functional. It is based on the existence of the minimum for every $F_\epsilon$, on the $\Gamma$-convergence property, and on a suitable compactness result.

**4.1. An embedding theorem.** In this section we will prove an embedding theorem which extends the classical compactness result in the space $BV$. Indeed, due to the particular expression of the functional $F_\epsilon$, we will deal with family $(u_\epsilon)$ of functions such that the quantity

$$N(u_\epsilon) = \int_\Omega |u_\epsilon|dx + \int_{\mathbb{R}^n} e^{-|\eta|_g} \int_\Omega |D^\epsilon_\eta u_\epsilon(x)|dx\, d\eta \tag{4.1}$$

is bounded if $\Omega$ is bounded.

THEOREM 4.1. *Let $(u_\epsilon)$ be a family of functions in $L^1_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$ such that for every bounded set $\Omega$, $N(u_\epsilon)$ is bounded. Then there exists a sequence $\epsilon_j$ convergent to 0 and a function $u$ in $BV_{loc}$ such that $u_{\epsilon_j}$ converges to $u$ in $L^1_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$.*

*Proof.* Let us choose a nonnegative radially symmetric cut off function $\eta$ of class $C^\infty_0(\mathbb{R}^n)$, supported in the unitary sphere, and with integral 1. For every $\epsilon > 0$ we set

$$u^\epsilon_\epsilon(x) = \int_{\mathbb{R}^n} \eta(\xi)u_\epsilon(x+\epsilon\xi)d\xi.$$

Then we have for $\Omega$ bounded

$$\int_\Omega |u^\epsilon_\epsilon(x)|dx \leq \int_\Omega \left(\int_{\mathbb{R}^n} \eta(\xi)u_\epsilon(x+\epsilon\xi)d\xi\right)dx < c_1,$$

since $\eta$ is bounded in $L^\infty$ and $(u_\epsilon)$ in $L^1_{loc}$. By definition the gradient of $(u^\epsilon_\epsilon)$ is

$$\nabla u^\epsilon_\epsilon(x) = \frac{1}{\epsilon}\int_{\mathbb{R}^n} \nabla\eta(\xi)u_\epsilon(x+\epsilon\xi)d\xi$$

since $\eta$ is radially symmetric

$$= \int_{\mathbb{R}^n} \nabla\eta(\xi)\Big(\frac{u_\epsilon(x+\epsilon\xi) - u_\epsilon(x)}{\epsilon}\Big)d\xi.$$

Then for any bounded $\Omega$

$$\int_\Omega |\nabla u_\epsilon^\epsilon(x)|dx \leq \int_{|\xi|\leq 1}\int_\Omega |D_\xi^\epsilon u_\epsilon(x)|dxd\xi < c_2$$

by the assumption on $N_\epsilon$. By the standard compactness theorem in $BV_{loc}$ it follows that $(u_\epsilon^\epsilon)$ has a subsequence $(u_{\epsilon_j}^{\epsilon_j})$ converging in $L^1_{loc}$ to a $BV_{loc}$ function $u$. On the other side,

$$(u_{\epsilon_j} - u_{\epsilon_j}^{\epsilon_j})(x) = \int_{\mathbb{R}^n} \eta(\xi)\Big(u_{\epsilon_j}(x+\epsilon_j\xi) - u_{\epsilon_j}(x)\Big)d\xi \leq \epsilon_j \int_{|\xi|\leq 1}\Big|D_\xi^{\epsilon_j} u_{\epsilon_j}(x)\Big|d\xi.$$

Integrating over $\Omega$ we get

$$\int_\Omega |u_{\epsilon_j} - u_{\epsilon_j}^{\epsilon_j}|(x)dx \leq \epsilon_j \int_{|\xi|\leq 1}\int_\Omega \Big|D_\xi^{\epsilon_j} u_{\epsilon_j}(x)\Big|dxd\xi \leq c_3\epsilon_j.$$

It immediately follows that $u_{\epsilon_j}$ has the same limit as $u_{\epsilon_j}^{\epsilon_j}$ in $L^1_{loc}$. ◻

**4.2. A compactness result.** Let us now prove a compactness result for a family $(u_\epsilon)$ of functions such that $F_\epsilon(u_\epsilon)$ is bounded. Since the argument of the function $\varphi_\epsilon$ in the expression of $F_\epsilon$ is the difference quotient and the functions we are interested in have a different behavior when the argument is small or big, we will also denote the following: $D_{g\xi}^{\epsilon,+}u_\epsilon(x) = D_{g\xi}^\epsilon u_\epsilon(x)$ if $|D_{g\xi}^\epsilon u_\epsilon(x)| > \pi(\epsilon|\xi|)^{-\frac{1}{p}}$ and $D_{g\xi}^{\epsilon,+}u_\epsilon(x) = 0$ otherwise, and we will call

$$I_{\epsilon\xi}^+ = \{x \in \mathbb{R}^n | D_{g\xi}^{\epsilon,+}u_\epsilon(x) \neq 0\}. \tag{4.2}$$

This notation will be useful when studying the limit for $\epsilon \to 0$, since the term $D_{g\xi}^{\epsilon,-}u_\epsilon(x)$ will recover the gradient of $u$, while $D_{g\xi}^{\epsilon,+}u_\epsilon(x)$ will describe the jump set of the function.

THEOREM 4.2. *Let $(u_\epsilon)$ be a family of functions in $L^1_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$ such that $F_\epsilon(u_\epsilon)$ is bounded; then $N_\epsilon(u_\epsilon)$ is bounded.*

*Proof.* Let us call $c_1$ a constant such that

$$\varphi_\epsilon(z) \geq c_1 z^p \tag{4.3}$$

for all $z$ such that $\epsilon^{1/p}z \leq \pi$. Note that $c_1$ is independent of $\epsilon$. Let us now fix an open set $\Omega \subset\subset \mathbb{R}^n$ and estimate separately the integral on $I_{\epsilon\xi}^+$ and the complement set. Since $u_\epsilon$ takes values in $[-\pi, \pi]$, we have

$$\int_{\mathbb{R}^n} e^{-|\xi|_g}\int_{I_{\epsilon\xi}^+\cap\Omega} |D_{g\xi}^\epsilon u_\epsilon(x)|dxd\xi \leq c_2\int_{\mathbb{R}^n} e^{-|\xi|_g}\int_{I_{\epsilon\xi}^+\cap\Omega}\frac{1}{\epsilon|\xi|_g}dxd\xi \tag{4.4}$$

(since $\varphi$ takes constantly the value $\beta$ in $[\pi, +\infty]$)

$$\leq \frac{c_2}{\beta}\int_{\mathbb{R}^n} e^{-|\xi|_g}\int_{I_{\epsilon\xi}^+\cap\Omega}\varphi_{\epsilon|\xi|}\big(D_{g\xi}^\epsilon u_\epsilon\big)dxd\xi \leq c_3 F_\epsilon(u_\epsilon).$$

By condition (4.3) and the assumption (3.1) on $(g_{ij})_{ij}$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus I^+_{\epsilon\xi}} e^{-|\xi|}\left|D^\epsilon_{g\xi}u_\epsilon(x)\right|^p dxd\xi \le c_4\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus I^+_{\epsilon\xi}} e^{-|\xi|_g}\varphi_{\epsilon|\xi|}\left(D^\epsilon_{g\xi}u_\epsilon(x)\right)dxd\xi$$

$$\le c_4 F_\epsilon(u_\epsilon).$$

Consequently,

$$\int_{\mathbb{R}^n} e^{-|\xi|_g}\int_\Omega |D^\epsilon_{g\xi}u_\epsilon(x)|dxd\xi = \int_{\mathbb{R}^n} e^{-|\xi|_g}\left(\int_{\Omega\setminus I^+_{\epsilon\xi}} + \int_{\Omega\cap I^+_{\epsilon\xi}}\right)|D^\epsilon_{g\xi}u(x)|dxd\xi$$

by (4.4) and Hölder inequality

$$\le c_3F_\epsilon(u_\epsilon) + \int_{\mathbb{R}^n} e^{-|\xi|_g}\left(\int_{\Omega\setminus I^+_{\epsilon\xi}}|D^\epsilon_{g\xi}u_\epsilon(x)|^p dx + c_5|\Omega|\right)d\xi$$

$$\le (c_3+1)\,F_\epsilon(u_\epsilon) + c_6|\Omega|$$

for suitable constants $c_i$. Here $|\cdot|$ indicate the Lebesgue measure in $\mathbb{R}^n$.

Then lemma is proved. □

**4.3. Approximation of the minima for the Riemannian Mumford and Shah functional.** Let us first modify the functional $F_\epsilon$ in such a way that its minimum is a $BV$ function.

Lemma 4.3. *Let $g \in L^q(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$ and for every $\epsilon > 0$ let us denote*

$$(4.5)\quad G_\epsilon(u) = \begin{cases} F_\epsilon(u) + \int_{\mathbb{R}^n}|u-g|^q dx & \text{if } u \in BV(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z}), |Du|(\mathbb{R}^n) \le \frac{1}{\epsilon}, \\ +\infty & \text{otherwise.}\end{cases}$$

*Then the family $G_\epsilon(u)$ $\Gamma$-converges as $\epsilon \to 0$ in $L^1_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$ to the functional*

$$G_0(u) = MS(u, \mathbb{R}^n) + \int_{\mathbb{R}^n}|u-g|^q dx.$$

*Proof.* The lim inf-inequality follows from the $\Gamma$-convergence of $(F_\epsilon)$. The lim sup follows from the pointwise convergence of $(F_\epsilon)$ if $u \in SBV$ and by a truncation argument for all $u$. □

**Proof of Theorem 1.2.** Since the functional $G_\epsilon$ is lower semicontinuous in $L^1_{loc}(\mathbb{R}^n)$ and the set

$$\{u \in BV(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z}) : |Du|(\mathbb{R}^n) \le 1/\epsilon\}$$

is compact in $L^1_{loc}(\mathbb{R}^n)$, the existence of minimizers for $G_\epsilon$ follows from the direct method of the calculus of variations.

We then prove that all the minimizers belong to the same compact set $K$. Let $(u_\epsilon)$ be a family of minimizers. Since

$$G_\epsilon(u_\epsilon) \le G_\epsilon(0) \le |g|^q_{L^q(\mathbb{R}^n)},$$

we can apply Theorems 4.1 and 4.2 and deduce that the family $(u_\epsilon)$ is relatively compact in $L^1_{loc}$ and has a limit in $BV$.

Finally, by the general property of $\Gamma$-convergence stated in Theorem 2.3, any limit point of $(u_\epsilon)$ is a minimizer for the problem (1.12) and $m_\epsilon \to m_0$ as $\epsilon \to 0$. □

**5. The evolution problem.** In this chapter we fix a function $u_0 \in BV$, approximate it by a piecewise constant function, and for every $\epsilon$ study the solution $(u_\epsilon)$ of problem (1.13) in section 1. Then we establish the properties of the limit of this family as $\epsilon \to 0$.

We now define the space

$$X = \{u \in SBV_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi Z) : MS(u, \mathbb{R}^n) < +\infty\}.$$

Let $u_0 \in X$ be an initial datum for the parabolic problem. Since the functional $F_\epsilon$ is defined on piecewise constant functions, we consider an approximation of $u_0$ in the space $PC^p_\epsilon(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$ defined in section 2.

PROPOSITION 5.1. *If $u_0 \in X$, there exists a family $(u_{0\epsilon}) \in PC^p_\epsilon(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z})$ such that*

$$u_{0\epsilon} \to u_0 \quad in \quad L^p_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z}),$$

$$\lim_{\epsilon \to 0} F_\epsilon(u_{0\epsilon}) = MS(u_0, \mathbb{R}^n),$$

*and*

$$\sup_{\epsilon>0}\{F_\epsilon(u_{0\epsilon})\} < +\infty$$

(*see* [29, *p.* 167] *for the proof*).

Then we consider the evolution problem in (1.13). By the standard Cauchy–Lipschitz existence result (cf. [31]), we have the following theorem.

THEOREM 5.2. *For every $\epsilon > 0$ the initial value problem* (1.13) *has a unique solution $u_\epsilon \in C^1([0, +\infty[, PC^p_\epsilon)$ which depends continuously on the initial datum.*

Let us now study the limit of the family $(u_\epsilon)$.

LEMMA 5.3. *Let $\Omega$ be a compact set in $\mathbb{R}^n$, and let $(u_\epsilon)$ be the family of solutions of the initial value problem found in Theorem* 5.2. *There exists a sequence $(\epsilon_k)$ convergent to* 0 *such that $(u_{\epsilon_k})$ is relatively compact in $C([0, +\infty[; L^p_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z}))$ and has a limit $u \in C([0, +\infty[; L^p_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z}))$ such that $u(t) \in BV(\Omega)$ for every $t \in [0, +\infty[$.*

*Proof.* We first note that the function $t \to F_\epsilon(u_\epsilon(t))$ is nonincreasing. Indeed,

$$\frac{d}{dt}F_\epsilon(u_\epsilon(t)) = \langle \nabla F_\epsilon(u_\epsilon(t)), u'_\epsilon(t)\rangle_{L^2(\mathbb{R}^n)} \tag{5.1}$$

$$= -||u'_\epsilon(t)||_{L^2(\mathbb{R}^n)} = -||\nabla F_\epsilon(u_\epsilon(t))||_{L^2(\mathbb{R}^n)}.$$

This implies that

$$F_\epsilon(u_\epsilon(t)) \le F_\epsilon(u_{0\epsilon}) \le \sup_{\epsilon>0} F_\epsilon(u_{0\epsilon}) < +\infty$$

by the choice of the family $(u_{0\epsilon})$ in Proposition 5.1. By Theorems 4.1 and 4.2, this implies that the family $(u_\epsilon(t))$ is relatively compact in $L^1_{loc}(\mathbb{R}^n)$, and for every $t$ the limit $u(t)$ belongs to $BV$.

We have to prove the continuity of this limit. Since the functions $(u_\epsilon)$ take values in $[-\pi, \pi]$, the compactness in $L^1_{loc}$ implies compactness in $L^p_{loc}$ for every $p$. Moreover,

$$||u_\epsilon(t_1) - u_\epsilon(t_2)||_{L^2(\mathbb{R}^n)} \le \int_{t_1}^{t_2} ||u'_\epsilon(t)||_{L^2(\mathbb{R}^n)} dt \le \left(\int_{t_1}^{t_2} ||u'_\epsilon(t)||^2_{L^2(\mathbb{R}^n)} dt\right)^{\frac{1}{2}} |t_1 - t_2|^{\frac{1}{2}}$$

$$\leq (F_\epsilon(u_{0\epsilon}(t)))^{\frac{1}{2}}\,|t_1-t_2|^{\frac{1}{2}} \leq c\,|t_1-t_2|^{\frac{1}{2}}$$

for any $\epsilon > 0$. Letting $\epsilon$ go to 0, we obtain the continuity of $u$. □

REMARK 5.1. *Let us note that if*

$$\frac{\phi(z)}{z} \to 2\alpha \quad \textit{as } t \to 0, \tag{5.2}$$

*then condition* (1.9) *is satisfied with* $p = 2$*, and all the previous results hold true. Moreover, if* $\phi$ *is of class* $C^2$*, there exists a constant* $c$ *such that*

$$|\phi_{\epsilon|\xi|}(z) - \alpha z| \leq c\sqrt{\epsilon}(\varphi_{\epsilon|\xi|}(z) + |\xi|^2) \text{ when } |z| \leq \pi(\epsilon|\xi|)^{-\frac{1}{2}}. \tag{5.3}$$

Let us prove the following theorem, where we will assume $p = 2$.

THEOREM 5.4. *Assume as before that* $\phi$ *is continuous, it is odd,* $\phi > 0$ *in* $[0, \pi[$, $\phi = 0$ *on* $[\pi, +\infty[$*, and assume that assumptions* (5.2) *and* (5.3) *are satisfied. If* $u_\epsilon$ *is the solution of problem* (1.13) *and* $u$ *its limit, then*

$$\partial_t u_\epsilon \to 2\alpha\, c_{n2} \operatorname{div}\left(\frac{g^{ij}}{\sqrt{g}}\partial_j u\right) \textit{ weakly in } L^2_{loc}([0, +\infty[\times\mathbb{R}^n, \mathbb{R}).$$

*Proof.* Let us fix a bounded set $\Omega$. By assumption we have

$$\int_0^T \int_{\mathbb{R}^n} \int_{\Omega\cap I^+_{\epsilon\eta}} \phi_{\epsilon|\eta|_g}\big(D^\epsilon_{g\eta} u_\epsilon(x)\big) dx d\eta dt = 0,$$

where $I^+_{\epsilon\eta}$ is defined in (4.2). If $U \subset\subset \mathbb{R}^n$ is bounded, by (5.3)

$$\int_0^T \int_U \int_{\Omega\setminus I^+_{\epsilon\eta}} \Big|\phi_{\epsilon|\eta|_g}\big(D^\epsilon_{g\eta} u_\epsilon(x)\big) - \alpha D^\epsilon_{g\eta} u_\epsilon(x)\Big| dx d\eta dt \leq \sqrt{\epsilon}(F_\epsilon(u_\epsilon) + c_1) \to 0$$

as $\epsilon \to 0$.

This means that

$$\phi_{\epsilon|\eta|_g}\big(D^\epsilon_{g\eta} u_\epsilon(x)\big) - \alpha D^\epsilon_{g\eta} u_\epsilon(x) \to 0 \text{ in } L^1_{loc}([0,T] \times \mathbb{R}^n \times \Omega) \quad \text{as} \quad \epsilon \to 0. \tag{5.4}$$

On the other side, by Lemma 3.6 in [29]

$$D^\epsilon_\eta u_\epsilon(x) \to \left\langle \nabla u, \frac{\eta}{|\eta|}\right\rangle \text{weakly} * \text{in } L^1_{loc}([0,T] \times \mathbb{R}^n \times \Omega)$$

so that

$$\phi_{\epsilon|\eta|_g}\big(D^\epsilon_{g\eta} u_\epsilon(x)\big) \to \left\langle \nabla u, \frac{\eta}{|\eta|}\right\rangle \text{weakly} * \text{in } L^1_{loc}([0,T] \times \mathbb{R}^n \times \Omega). \tag{5.5}$$

Now let $\Phi \in C^\infty_0(]0, +\infty[\times\mathbb{R}^n)$. Since $u_\epsilon$ is a solution of the evolution equation, we have

$$\int_0^\infty \int_{\mathbb{R}^n} u_\epsilon \frac{\partial \Phi}{\partial t} dx dt$$

$$= \int_0^{+\infty} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} h\, e^{-|\eta|_g} \phi_{\epsilon|\eta|_g}\big(D^\epsilon_{g\eta} u_\epsilon(x)\big) D^\epsilon_{g\eta}\Phi(x,t) d\eta\right) dx dt$$

by (5.5) and the uniform convergence of $D^\epsilon_{g\eta}\Phi$ to $\langle \nabla\Phi, \frac{\eta}{|\eta|_g}\rangle$ as $\epsilon \to 0$

$$\to \alpha \int_0^{+\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h\, e^{-|\eta|_g} \left\langle \nabla u, \frac{\eta}{|\eta|_g} \right\rangle \left\langle \nabla \Phi, \frac{\eta}{|\eta|_g} \right\rangle d\eta dx dt$$

by Proposition 2.5

$$= \alpha\, c_{n2} \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{g^{hk}(x)}{\sqrt{g}} \partial_h u \partial_k \Phi dx dt.$$

On the other side, $\partial_t u_\epsilon$ is bounded in $L^2_{loc}([0, +\infty[\times\mathbb{R}^n, \mathbb{R})$, and the thesis is proved. ∎

COROLLARY 5.5. *Under the assumptions of Theorem* 5.4 *the function* $u$ *belongs to* $C([0,+\infty[; L^2_{loc}(\mathbb{R}^n, \mathbb{R}/2\pi\mathbb{Z}))$ *and satisfies the following:* $u(0) = u_0$, $MS(u(t), \mathbb{R}^n) \leq MS(u_0, \mathbb{R}^n)$, *for every* $t \geq 0$. *Moreover, the function* $u = u(x,t)$ *is a distributional solution in* $]0, +\infty[\times\mathbb{R}^n$ *of the equation*

$$\frac{\partial u}{\partial t} = 2\alpha\, c_{n2} D \left( \frac{g^{ij}}{\sqrt{g}} \nabla u \right),$$

*where* $D$ *is the distributional* $x$*-derivative, out of the jump set of* $u$.

*Proof.* It is a consequence of the results we have proved on the function $u$ in the previous theorems. ∎

**6. A numerical example.** We consider here a simple numerical example showing how the phase equation (1.5) is able to segment an object by reaching phase locking in semantically homogeneous areas of an image and by decoupling phases between object and background. We will consider the figure completion of the well-known square of Kanizsa (Figure 6.1). In this example we consider an image $(x_1, x_2) \to I(x_1, x_2)$ as a real positive function defined in a rectangular domain $\Omega \subset \mathbb{R}^2$. Following [37], we suppose that the image induces a local change of the connectivity $e$ in proximity of its discontinuities in such a way that hypercolumns appear decoupled across the boundaries of a figure. We choose a simple edge indicator as the connectivity function, namely

$$s(x_1, x_2) = \frac{1}{1 + (|\nabla G_\sigma(x_1, x_2) \star I(x_1, x_2)|/c)^2}, \tag{6.1}$$

where

$$G_\sigma(x_1, x_2) = \frac{\exp(-(|(x_1, x_2)|/\sigma)^2)}{\sigma\sqrt{\pi}}, \tag{6.2}$$

and $\star$ denotes the convolution. The denominator is the gradient magnitude of a smoothed version of the initial image. Thus, the value of $s$ is closer to 1 in flat areas ($|\nabla I| \to 0$) and closer to 0 in areas with large changes in image intensity, i.e., the local edge features. The minimal size of the details that are detected is related to the size of the kernel, which acts like a scale parameter. By viewing $s$ as a potential function, we note that its minima denote the position of edges, as depicted in Figure 6.1.

The edge indicator $s$ also induces a metric $g\delta_{ij}$, where $g = \frac{1}{s^2}$ and $\delta_{ij}$ is the Kronecker function. Since this metric is conformal, we get

$$|\eta|_g = g|\eta|, \quad D^\epsilon_{g\eta} u = \frac{1}{\sqrt{g}} D^\epsilon_\eta u,$$

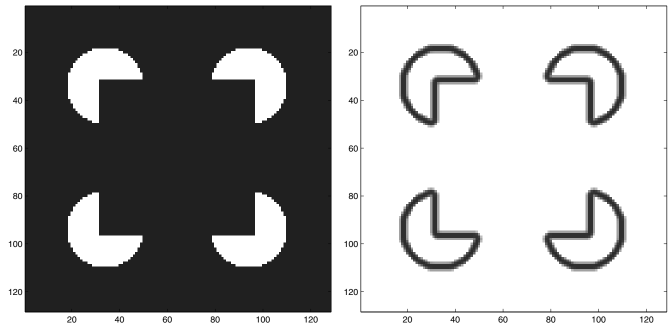


FIG. 6.1. *The Kanizsa square (left) with the connectivity map g (right).*

and, in order to study a curvature equation, we will choose

$$h = g.$$

The phase equation (1.7) becomes

$$\partial_t u(t) = -\int_{\mathbb{R}^n} D_\eta^{-\epsilon}\left(e^{-\sqrt{g}|\eta|}\phi_{\epsilon|\eta|_g}\left(D^\epsilon_{g\eta}u\right)\sqrt{g}\right)d\eta$$

(using the definition of difference quotient)

$$= -\int_{\mathbb{R}^n}\left(\frac{e^{-\sqrt{g}|\eta|}}{(\epsilon|\eta|)^{3/2}}\phi\Big(\frac{u(x)-u(x-\epsilon\eta)}{\sqrt{\epsilon|\eta|g}}\Big) - \frac{e^{\sqrt{g}|\eta|}}{(\epsilon|\eta|)^{3/2}}\phi\Big(\frac{u(x+\epsilon\eta)-u(x)}{\sqrt{\epsilon|\eta|g}}\Big)\right)d\eta$$

since $\phi$ is odd

$$= 2\int_{\mathbb{R}^n}\frac{e^{-\sqrt{g}|\eta|}}{(\epsilon|\eta|)^{3/2}}\phi\Big(\frac{u(x+\epsilon\eta)-u(x)}{\sqrt{\epsilon|\eta|g}}\Big)d\eta.$$

We note that the exponential kernel $e^{-\sqrt{g}|\eta|}$ can be substituted by a compactly supported kernel $\chi = \chi(\sqrt{g}|\eta|)$. The new equation and the corresponding functional $F_\epsilon$ satisfy the same convergence results as before. We will assume that $\chi$ is the indicatrix function of the square $[-1,1]^2$ so that in the numerical simulations the integral will be approximated with the sum on the vectors

$$\eta = (i,j), \quad i,j \in \{0,1,-1\}.$$

According to the introduction, the function $\phi$ will be the sin function, extended with zero, outside of the interval $[-\pi,\pi]$. To perform numerical simulations the phase equation has been approximated by forward differences in time:

$$u^{n+1}_{l,m} = u^n_{l,m} + 2\Delta t \sum_{(i,j)\in\{0,-1,1\}} \frac{1}{\epsilon^{3/2}(i^2+j^2)^{3/4}} \sin\Big(\frac{u^n(l+i,m+j)-u^n(l,m)}{\sqrt{\epsilon(i^2+j^2)^{1/2}g(l+i/2,m+j/2)}}\Big),$$

where $\epsilon = 0.03$ is the space increment and $\Delta t = 0.01$ is the time discretization. As in [37], the initial condition is given by a function $u_0 = \mathcal{D}$ that is proportional to

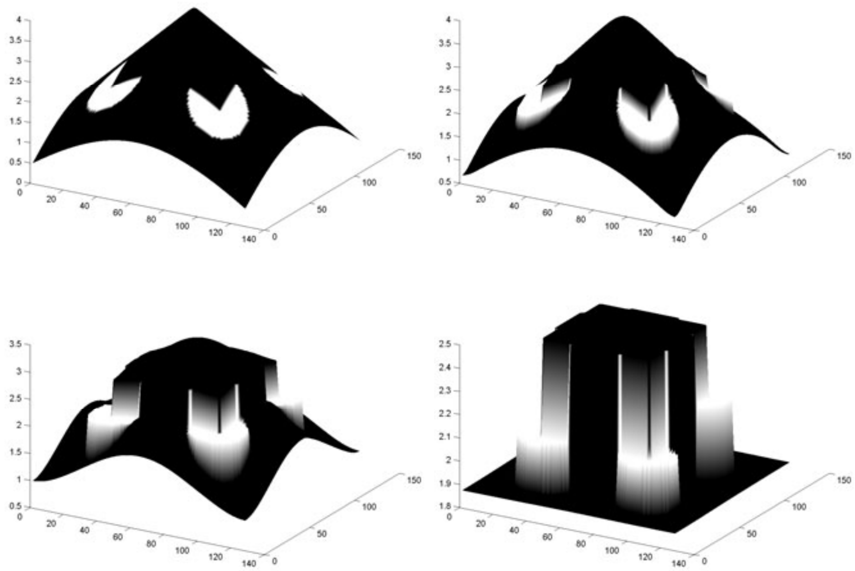

FIG. 6.2. *Evolution of the phase equation towards the phase locking solution segmenting the Kanizsa square.*

the distance from a point internal to the object. We impose Neumann boundary conditions.

During the flow, the surface evolves towards the piecewise constant solution by continuation and closing of the boundary fragments and the filling in of the homogeneous regions (Figure 6.2). In regions of the image where edge information exists, the level sets of the surface get attracted to the edges and accumulate. Consequently, the spatial gradient increases, and the surface begins to develop a discontinuity. In the regions of the image corresponding to subjective contours (i.e., contours that are perceived without any existing discontinuity in the image) discontinuities of $u$ are propagated from existing edge fragments (Figure 6.2).

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