Mechanics and Control

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See work with Crouch, Marsden, Murray, Krishnaprasad, Zenkov,

- **1. Examples and mathematical preliminaries.**
- **2. Geometric Mechanics**
- **3. Geometric Control**
- **4. Nonholonomic Mechanics**
- **5. Optimal Control and sub-Riemannian geometry.**

Nonholonomic Mechanics vs. Hamiltonian Mechanics

- **Energy Conservation**

Hamiltonian: Yes. Nonholonomic: Yes.

- **Momentum Conservation**

Hamiltonian: Yes, Noether's Theorem. Nonholonomic: No, Momentum Equation

- **Measure (volume) Preservation**

Hamiltonian: Yes. Nonholonomic: No, in general

- **Stability**

Hamiltonian: Never asymptotic. Nonholonomic: Can be asymptotic.

- **Key: Almost Poisson structure, Nonvariational**

Geometry and Kinematics of the Vertical Disk.

Configuration space: $Q = \mathbb{R}^2 \times S^1 \times S^1$, parameterized by coordinates $q = (x, y, \theta, \varphi)$, denoting the position of the contact point in the xy-plane, the rotation angle of the disk, and the orientation of the disk, respectively, as in figure 0.1.

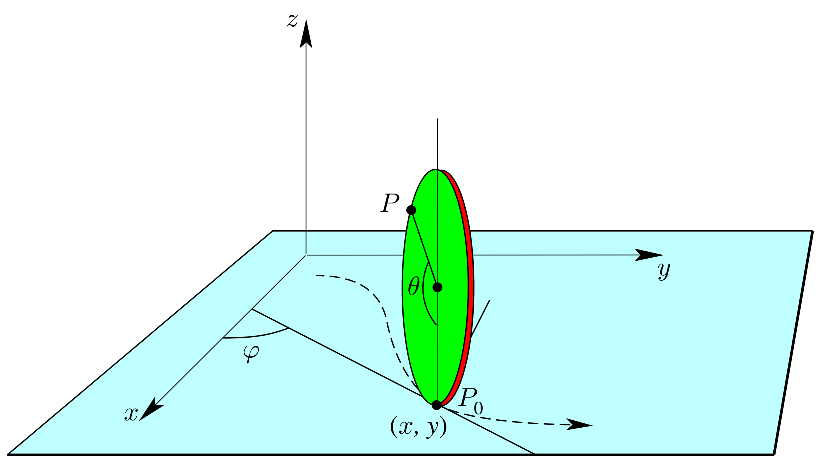

Figure 0.1: The geometry for the rolling disk.

The Lagrangian for the system: the kinetic energy

$$L(x, y, \theta, \phi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2.$$

If R is the radius of the disk, the nonholonomic constraints of rolling without slipping are

$$\begin{aligned} \dot{x} &= R(\cos\varphi)\dot{\theta} \\ \dot{y} &= R(\sin\varphi)\dot{\theta}, . \end{aligned}$$

Dynamics of the Controlled Disk. Consider the case where we have two controls, one that can steer the disk and another that determines the roll torque.

Lagrange d'Alembert equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = u_1 X_1 + u_2 X_2 + \lambda_1 W_1 + \lambda_2 W_2,$$

where

$$\frac{\partial L}{\partial \dot{q}} = (m\dot{x}, m\dot{y}, I\dot{\theta}, J\dot{\varphi})^T,$$

$$X_1 = (0,0,1,0)^T, X_2 = (0,0,0,1)^T,$$

and

$$W_1^T = (1, 0, -R\cos\varphi, 0), \quad W_2^T = (0, 1, -R\sin\varphi, 0)^T,$$

together with the constraint equations.

We may now eliminate the multipliers: gives the dynamic equations

$$\begin{aligned} (I + mR^2)\ddot{\theta} &= u_1 \\ J\ddot{\varphi} &= u_2\,, \end{aligned}$$

plus the constraints.

The ***free equations***, in which we set $u_1 = u_2 = 0$, are easily integrated.

• **Controlled case:** controls u_1, u_2. Call the variables θ and ϕ "base" or "controlled" variables and the variables x and y "fiber" variables. The distinction is that while θ and φ are controlled directly, the variables x and y are controlled indirectly via the constraints.

It is clear that the base variables are controllable in any sense we can imagine. One may ask whether the full system is controllable. Indeed it is: by virtue of the nonholonomic nature of the constraints.

The Kinematic Controlled Disk. In this case we imagine we have direct control over velocities rather than forces and, accordingly, we consider the most general first order system satisfying the constraints or lying in the "constraint distribution".

This system is

$$\dot{q} = u_1 \overline{X}_1 + u_2 \overline{X}_2$$

where $\overline{X}_1 = (\cos\varphi, \sin\varphi, 1, 0)^T$ and $\overline{X}_2 = (0, 0, 0, 1)^T$.

In fact, $\overline{X}_1$ and $\overline{X}_2$ comprise a maximal set of independent vector fields on Q satisfying the constraints.

•Nonholonomic Equations of Motion

See e.g. Bloch, Krishnaprsad, Marsden and Murray [1996] and Zenkov, Bloch and Marsden [1998], Bloch and Crouch [1995] and other references in these papers.

•The Lagrange-d'Alembert Principle

• Consider a system with a configuration space Q, local coordinates q^i and m nonintegrable constraints

$$\dot{s}^a + A^a_\alpha(r,s)\dot{r}^\alpha = 0$$

where $q = (r,s) \in \mathbb{R}^{n-p} \times \mathbb{R}^p$, which we write as $q^i = (r^\alpha, s^a)$, where $1 \leq \alpha \leq n-p$ and $1 \leq a \leq p$.

• Lagrangian $L(q^i, \dot{q}^i)$.

Equations of motion given by Lagrange-d'Alembert principle.

Definition 0.1 *The* ***Lagrange-d'Alembert equations of motion*** *for the system are those determined by*

$$\delta \int_a^b L(q^i, \dot{q}^i)\, dt = 0,$$

where we choose variations $\delta q(t)$ of the curve $q(t)$ that satisfy $\delta q(a) = \delta q(b) = 0$ and $\delta q(t)$ satisfies the constraints for each t where $a \leq t \leq b$.

• This principle is supplemented by the condition that the curve itself satisfies the constraints.

• Note that we take the variation *before* imposing the constraints; that is, we do not impose the constraints on the family of curves defining the variation.

• Equivalent to:

$$-\delta L = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) \delta q^i = 0$$

for all variations $\delta q^i = (\delta r^\alpha, \delta s^a)$ satisfying the constraints at each point of the underlying curve $q(t)$, *i.e.* such that $\delta s^a + A^a_\alpha \delta r^\alpha = 0$.

Substituting:

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial r^\alpha} \right) = A^a_\alpha \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{s}^a} - \frac{\partial L}{\partial s^a} \right)$$

for all $\alpha = 1, \ldots, n - p$.

Combined with the constraint equations

$$\dot{s}^a = -A^a_\alpha \dot{r}^\alpha$$

for all $a = 1, \ldots, p$, give the complete equations of motion of the system.

Useful way of reformulating equations (0.4) is to define a *constrained Lagrangian* by substituting the constraints (0.5) into the Lagrangian:

$$L_c(r^\alpha, s^a, \dot{r}^\alpha) := L(r^\alpha, s^a, \dot{r}^\alpha, -A^a_\alpha(r, s)\dot{r}^\alpha).$$

The equations of motion can be written in terms of the constrained Lagrangian in the following way, as a direct coordinate calculation shows:

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a} = -\frac{\partial L}{\partial \dot{s}^b} B^b_{\alpha\beta}\dot{r}^\beta,$$

where $B^b_{\alpha\beta}$ is defined by

$$B^b_{\alpha\beta} = \left(\frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha} + A^a_\alpha \frac{\partial A^b_\beta}{\partial s^a} - A^a_\beta \frac{\partial A^b_\alpha}{\partial s^a} \right).$$

• **The Nonholonomic and the Variational Systems.** Interesting to compare the dynamic equations, which can be shown to be consistent with Newton's second law $F = ma$ in the presense of reaction forces with the corresponding variational system. Long and distinguished history going back to the review article of Korteweg [1898].

• What is the difference in the two procedures?

Answer: with the dynamic Lagrange d'Alembert equations, we impose constraints only on the variations, whereas in the variational problem we impose the constraints on the velocity vectors of the class of allowable curves.

Show explicitly for penny that one really gets two *different* sets of equations.

• Variational system is obtained by using Lagrange multipliers with the Lagrangian rather than the equations:

$$\begin{aligned} L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}\dot{\varphi}^2 \\ &\quad + \mu_1(\dot{x} - R\dot{\theta}\cos\varphi) + \mu_2(\dot{x} - R\dot{\theta}\sin\varphi)\,, \end{aligned}$$

where now can relax the constraints and take variations over all curves.

• Variational equations with external forces therefore are

$$\begin{aligned} m\ddot{x} + \dot{\mu}_1 &= 0 \\ m\ddot{y} + \dot{\mu}_2 &= 0 \\ I\ddot{\theta} - R\frac{d}{dt}(\mu_1 \cos\varphi + \mu_2 \sin\varphi) &= u_1 \\ J\ddot{\varphi} + R\frac{\partial d}{\partial\varphi}(\mu_1\dot{\theta}\cos\varphi + \mu_2\dot{\theta}\sin\varphi) &= u_2\,. \end{aligned}$$

Substituting we obtain

$$\begin{aligned} (I + mR^2)\ddot{\theta} &= R\dot{\varphi}(-A\sin\varphi + B\cos\varphi) + u_1 \\ J\ddot{\varphi} &= R\dot{\theta}(A\sin\varphi - B\cos\varphi) + u_2\,. \end{aligned}$$

A,B, are consts.

- **The Falling Rolling Disk** More realistic disk allowed to fall over.

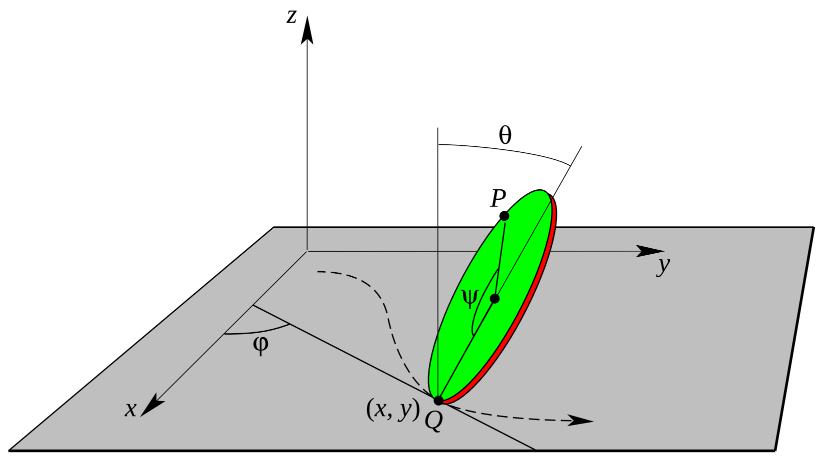

Figure 0.2: The geometry for the rolling disk.

This is a system which exhibits stability but not asymptotic stability.
Denote mass, the radius, and the moments of inertia of the disk by m, R, A, B.

$$L = \frac{m}{2}\left[(\xi - R(\dot\phi\sin\theta + \dot\psi))^2 + \eta^2\sin^2\theta + (\eta\cos\theta + R\dot\theta)^2\right]$$
$$+ \frac{1}{2}\left[A(\dot\theta^2 + \dot\phi^2\cos^2\theta) + B(\dot\phi\sin\theta + \dot\psi)^2\right] - mgR\cos\theta,$$

where $\xi = \dot x\cos\phi + \dot y\sin\phi + R\dot\psi$ and $\eta = -\dot x\sin\phi + \dot y\cos\phi$, while the constraints are given by

$$\dot x = -\dot\psi R\cos\phi, \qquad \dot y = -\dot\psi R\sin\phi.$$

Other systems:

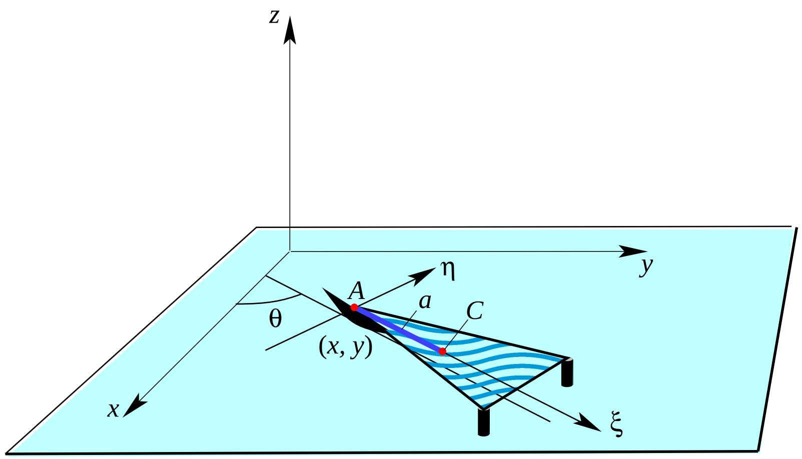

Figure 0.3: The Chaplygin sleigh is a rigid body moving on two sliding posts and one knife edge.

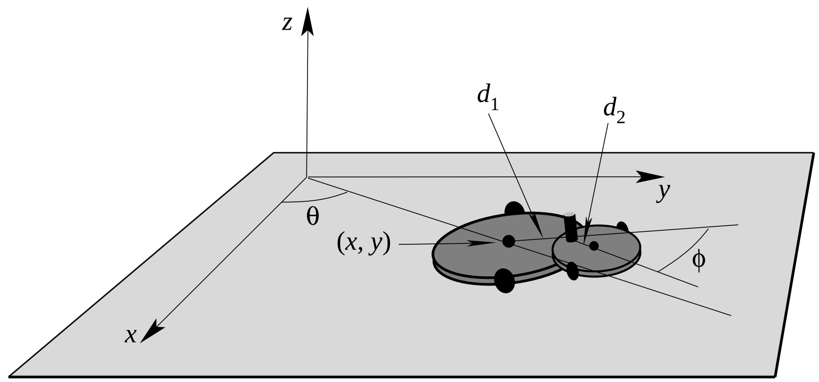

Figure 0.4: The geometry for the roller racer.

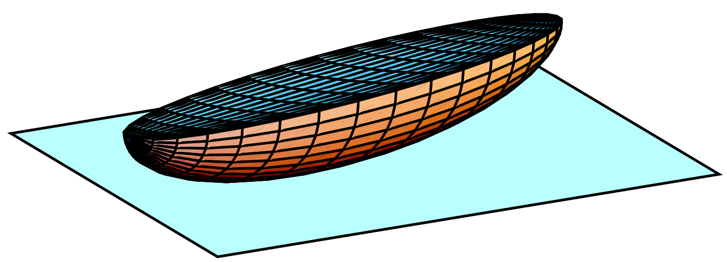

Figure 0.5: The rattleback.

•The Chaplygin Sleigh

• Perhaps the simplest mechanical system which illustrates the possible dissipative nature of energy preserving nonholonomic systems.

Compare the sleigh equations to the Toda lattice equations.

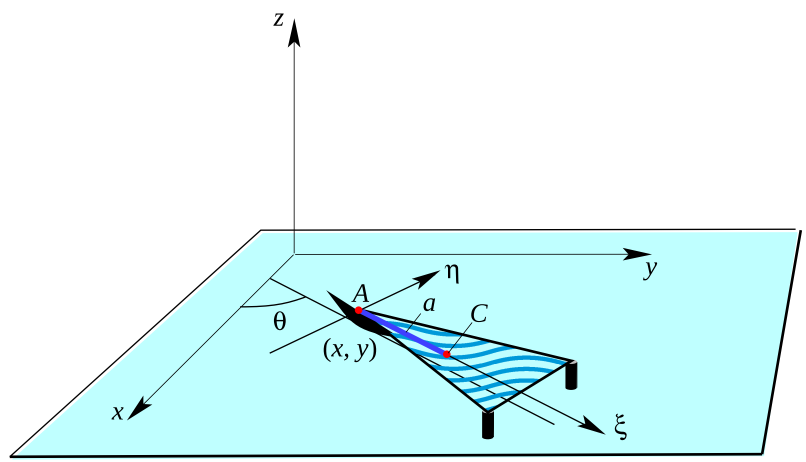

Figure 0.6: The Chaplygin sleigh is a rigid body moving on two sliding posts and one knife edge.

Equations:

$$\begin{aligned} \dot{v} &= a\omega^2 \\ \dot{\omega} &= -\frac{ma^2}{I + ma^2} v\omega \end{aligned}$$

Equations have a family of relative equilibria given by $(v, \omega) | v = \text{const}, \quad \omega = 0$.

Linearizing about any of these equilibria one finds one zero eigenvalue and one negative eigenvalue.

In fact the solution curves are ellipses in $v - \omega$ plane with the positive v-axis attracting all solutions.

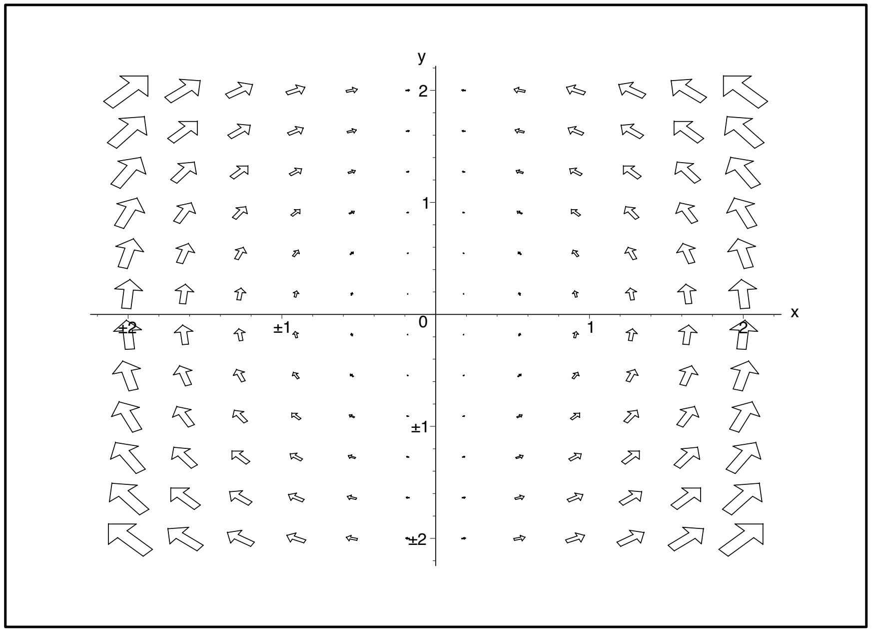

Figure 0.7: Chaplygin Sleigh/2d Toda phase portrait.

Normalizing, we have the equations

$$\begin{aligned} \dot{v} &= \omega^2 \\ \dot{\omega} &= -v\omega\,. \end{aligned}$$

Scaling time by a factor of two have: Chaplygin sleigh equations are equivalent to the two-dimensional Toda lattice equations except for the fact that there is no sign restriction on the variable ω. Hence can be written in Lax pair form and solved by the method of factorization.

•The Toda Lattice

Interacting particles on the line.

Non-periodic finite Toda lattice as analyzed by Moser [1974]:

$$H(x, y) = \frac{1}{2} \sum_{k=1}^{n} y_k^2 + \sum_{k-1}^{n-1} e^{(x_k - x_{k+1})} .$$

Hamiltonian equations:

$$\begin{aligned} \dot{x}_k &= \frac{\partial H}{\partial y_k} = y_k \\ \dot{y}_k &= -\frac{\partial H}{\partial x_k} = e^{x_{k-1} - x_k} - e^{x_k - x_{k-1}} , \end{aligned}$$

where assume $e^{x_0 - x_1} = e^{x_n - x_{n+1}} = 0$.

Flaschka:

$$a_k = \frac{1}{2} e^{(x_k - x_{k+1})/2} \quad b_k = -\frac{1}{2} y_k .$$

Get:

$$\begin{aligned} \dot{a}_k &= a_k(b_{k+1} - b_k) , \quad k = 1, \cdots , n-1 \\ \dot{b}_k &= 2(a_k^2 - a_{k-1}^2) , \quad k = 1, \cdots , n \end{aligned}$$

with the boundary conditions $a_0 = a_n = 0$ and where the $a_i > 0$.

Matrix form:

$$\frac{d}{dt}L = [B, L] = BL - LB,$$

If N is the matrix $\mathrm{diag}[1, 2, \cdots, n]$ the Toda flow can be written

$$\dot{L} = [L, [L, N]] \, .$$

Shows flow also gradient (on a level set of its integrals).

• Double bracket form of Brockett [1988] (see Bloch [1990], Bloch Brockett and Ratiu [1990, 1992]).

•The Two-dimensional Toda Lattice

In two-dimensional case matrices in the Lax pair are

$$L = \begin{pmatrix} b_1 & a_1 \\ a_1 & -b_1 \end{pmatrix} \; B = \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix} .$$

Equations of motion:

$$\begin{aligned} \dot{b_1} &= 2a_1^2 \\ \dot{a_1} &= -2a_1 b_1 \end{aligned}$$

For initial data $b_1 = 0$, $a_1 = c$, explicitly carrying out the factorization yields explicit solution

$$b_1(t) = -c\frac{\sinh 2ct}{\cosh 2ct} \, , \qquad a_1(t) = \frac{c}{\cosh 2ct}$$

Mathematical Preliminaries

Definition 0.1 *An n-dimensional differentiable manifold M is a set of points together with a finite or countably infinite set of subsets $U_\alpha \subset M$ and* 1*-to-*1 *mappings $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ such that:*

1. $\bigcup_\alpha U_\alpha = M$.
2. *For each nonempty intersection $U_\alpha \cap U_\beta$, $\phi_i(U_\alpha \cap U_\beta)$ is an open subset of $\mathbb{R}^n$, and the 1-to-1 and onto mapping $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is a smooth function.*
3. *The family $\{U_\alpha, \phi_\alpha\}$ is maximal with respect to conditions* 1 *and* 2.

Tangent Vectors to Manifolds. Two curves $t \mapsto c_1(t)$ and $t \mapsto c_2(t)$ in an n-manifold M are called ***equivalent at*** $x \in M$ if

$$c_1(0) = c_2(0) = x \quad \text{and} \quad (\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$$

in some chart φ, where the prime denotes the derivative with respect to the curve parameter. It is easy to check that this definition is chart independent. A tangent vector v to a manifold M at a point $x \in M$ is an equivalence class of curves at x. One proves that the set of tangent vectors to M at x forms a vector space. It is denoted by T_xM and is called the tangent space to M at $x \in M$. Given a curve $c(t)$, we denote by $c'(s)$ the tangent vector at $c(s)$ defined by the equivalence class of $t \mapsto c(s + t)$ at $t = 0$.

The tangent bundle of M, denoted by TM, is the differentiable manifold whose underlying set is the disjoint union of the tangent spaces to M at the points $x \in M$; that is,

$$TM = \bigcup_{x \in M} T_x M.$$

Differentiable Maps. Let E and F be vector spaces (for example, $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively), and let $f : U \subset E \to V \subset F$, where U and V are open sets, be of class C^{r+1}. We define the tangent map (the tangent map is sometimes denoted by f_*) of f to be the map $Tf : TU = U \times E \to TV = V \times F$ defined by

$$Tf(u, e) = (f(u), Df(u), e), \tag{0.10}$$

where $u \in U$ and $e \in E$. This notion from calculus may be generalized to the context of manifolds as follows. Let $f : M \to N$ be a map of a manifold M to a manifold N. We call f ***differentiable*** (or C^k) if in local coordinates on M and N it is expressed, or represented, by differentiable (or C^k) functions. The derivative of a differentiable map $f : M \to N$ at a point $x \in M$ is defined to be the linear map

$$T_x f : T_x M \to T_{f(x)} N$$

constructed in the following way. For $v \in T_x M$, choose a curve $c : \;]-\varepsilon, \varepsilon[\; \to M$ with $c(0) = x$, and velocity vector $dc/dt\,|_{t=0} = v$. Then $T_x f \cdot v$ is the velocity vector at $t = 0$ of the curve $f \circ c : \mathbb{R} \to N$; that is,

$$T_x f \cdot v = \frac{d}{dt} f(c(t))\bigg|_{t=0}.$$

Differential Forms

A **2-form** Ω on a manifold M is, for each point $x \in M$, a smooth skew-symmetric bilinear mapping $\Omega(x) : T_xM \times T_xM \to \mathbb{R}$. More generally, a **k-form** α (sometimes called a **differential form** of degree k) on a manifold M is a function $\alpha(x) : T_xM \times \cdots \times T_xM$ (there are k factors)$\to \mathbb{R}$ that assigns to each point $x \in M$ a smooth skew-symmetric k-multilinear map on the tangent space T_xM to M at x.

Without the skew-symmetry assumption, α would be referred to as a $(0, k)$-tensor.

Pull Back and Push Forward. Let $\varphi : M \to N$ be a C^∞ map from the manifold M to the manifold N and let α be a k-form on N. Define the **pull back** $\varphi^*\alpha$ of α by φ to be the k-form on M given by

$$(\varphi^*\alpha)_x(v_1, \ldots, v_k) = \alpha_{\varphi(x)}(T_x\varphi \cdot v_1, \ldots, T_x\varphi \cdot v_k). \tag{0.11}$$

If φ is a diffeomorphism, the **push forward** φ_* is defined by $\varphi_* = (\varphi^{-1})^*$.

Lie Groups

Definition 0.2 *A Lie group is a smooth manifold G that is a group and for which the group operations of multiplication, $(g, h) \mapsto gh$ for $g, h \in G$, and inversion, $g \mapsto g^{-1}$, are smooth.*

Definition 0.3 *A matrix Lie group is a set of invertible $n \times n$ matrices that is closed under matrix multiplication and that is a submanifold of $\mathbb{R}^{n \times n}$.*

- Tangent space at identity: Lie algebra.

Definition 0.4 *A fiber fiber bundle is a space Q for which the following are given: a space B called the* **base space***, a projection $\pi : Q \to B$ with fibers $\pi^{-1}(b), b \in B$, homeomorphic to a space F, a structure group G of homeomorphisms of F into itself, and a* **covering** *of B by open sets U_j, satisfying*

(i) *the bundle is locally trivial, i.e., $\pi^{-1}(U_j)$ is homeomorphic to the product space $U_j \times F$ and*

(ii) *if h_j is the map giving the homeomorphism on the fibers above the set U_j, for any $x \in U_j \cap U_k$ $h_j(h_k^{-1})$ is an element of the structure group G.*

If the fibers of the bundle are homeomorphic to the structure group, we call the bundle a **principal bundle**.

If the fibers of the bundle are homeomorphic to a vector space, we call the bundle a vector bundle.

Consider a bundle with projection map π and as usual let $T_q\pi$ denote its tangent map at any point. We call the kernel of $T_q\pi$ at any point the **vertical space** and denote it by V_q.

Definition 0.5 *An* **Ehresmann connection** A *is a vector-valued one-form on* Q *that satisfies:*

(i) A *is* **vertical valued***:* $A_q : T_qQ \to V_q$ *is a linear map for each point* $q \in Q$*.*

(ii) A *is a* **projection***:* $A(v_q) = v_q$ *for all* $v_q \in V_q$*.*

The key property of the connection is the following: If we denote by H_q or hor_q the kernel of A_q and call it the **horizontal space**, the tangent space to Q is the direct sum of the V_q and H_q; i.e., we can split the tangent space to Q into horizontal and vertical parts. For example, we can project a tangent vector onto its vertical part using the connection. Note that the vertical space at Q is tangent to the fiber over q.

Now define the fiber bundle coordinates $q^i = (r^\alpha, s^a)$ for the base and fiber. The coordinate representation of the projection π is just projection onto the factor r, and the connection A can be represented locally by a vector-valued differential form ω^a:

$$A = \omega^a \frac{\partial}{\partial s^a}, \quad \text{where} \quad \omega^a(q) = ds^a + A^a_\alpha(r, s) dr^\alpha.$$

We can see this as follows: Let

$$v_q = \sum_\beta \dot{r}^\beta \frac{\partial}{\partial r^\beta} + \sum_b \dot{s}^b \frac{\partial}{\partial s^b}$$

be an element of T_qQ. Then $\omega^a(v_q) = \dot{s}^a + A^a_\alpha \dot{r}^\alpha$ and

$$A(v_q) = (\dot{s}^a + A^a_\alpha \dot{r}^\alpha) \frac{\partial}{\partial s^a} \, .$$

This clearly demonstrates that A is a projection, since when A acts again only ds^a results in a nonzero term, and this has coefficient unity.

Given an Ehresmann connection A, a point $q \in Q$, and a vector $v_r \in T_r B$ tangent to the base at a point $r = \pi(q) \in B$, we can define the **horizontal lift** of v_r to be the unique vector v_r^h in H_q that projects to v_r under $T_q\pi$. If we have a vector $X_q \in T_q Q$, we shall also write its **horizontal part** as

$$\operatorname{hor} X_q = X_q - A(q) \cdot X_q.$$

In coordinates, the vertical projection is the map

$$(\dot{r}^\alpha, \dot{s}^a) \mapsto (0, \dot{s}^a + A^a_\alpha(r,s)\dot{r}^\alpha)\,,$$

while the horizontal projection is the map

$$(\dot{r}^\alpha, \dot{s}^a) \mapsto (\dot{r}^\alpha, -A^a_\alpha(r,s)\dot{r}^\alpha).$$

Next, we give the basic notion of curvature.

Definition 0.6 *The* **curvature** *of A is the vertical-vector-valued two-form B on Q defined by its action on two vector fields X and Y on Q by*

$$B(X,Y) = -A([\operatorname{hor} X, \operatorname{hor} Y]),$$

where the bracket on the right hand side is the Jacobi–Lie bracket of vector fields obtained by extending the stated vectors to vector fields.

The local expression for curvature is given by

$$B(X,Y)^a = B^a_{\alpha\beta} X^\alpha Y^\beta\,, \tag{0.12}$$

where the coefficients $B^a_{\alpha\beta}$ are given by

$$B^b_{\alpha\beta} = \left(\frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha} + A^a_\alpha \frac{\partial A^b_\beta}{\partial s^a} - A^a_\beta \frac{\partial A^b_\alpha}{\partial s^a} \right). \tag{0.13}$$

In the tangent bundle we can specify a linear connection by its action on vector fields, or by a map from vector fields (X, Y) to the vector field $\nabla_X Y$ that satisfies for smooth functions f and g and a vector fields X, Y, Z:

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$.

(ii) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$.

(iii) $\nabla_X(fY) = f\nabla_X Y + (\mathbf{d}f \cdot X)Y$,

where $\mathbf{d}f \cdot X$ is the directional derivative of f along X, or Lie derivative.

Given a basis of vector fields $\frac{\partial}{\partial r_j}$ we can represent ∇ by

$$\nabla_{\partial/\partial r_i} \frac{\partial}{\partial r_j} = \Gamma^k_{ij} \frac{\partial}{\partial r_k} \,. \tag{0.14}$$

For X, Y vector fields given locally by $X = X^i(\partial/\partial r_i),\ Y = Y^i(\partial/\partial r_i)$, (i) and (iii) imply

$$\nabla_X Y = \left(X^j \frac{\partial Y^i}{\partial r^j} + X^k Y^j \Gamma^i_{kj} \right) \frac{\partial}{\partial r^i}. \tag{0.15}$$

The geodesic equations (tangent vector to curve is always horizontal or curve is parallel transported) may be written

$$\nabla_{\dot r} \dot r = 0 \,. \tag{0.16}$$

We can see this directly by a simple computation, again using (i) and (iii):

$$\begin{aligned}
\nabla_{\dot r^i(\partial/\partial r_i)}\dot r^j \frac{\partial}{\partial r_j} &= \dot r^i \nabla_{\partial/\partial r_i}\dot r^j \frac{\partial}{\partial r_j} \\
&= \dot r^i \frac{\partial}{\partial r_i}\dot r^j \frac{\partial}{\partial r_j} + \dot r^i \dot r^j \Gamma^k_{ij}\frac{\partial}{\partial r_k} \\
&= (\ddot r^j + \Gamma^j_{ik}\dot r^i \dot r^k)\frac{\partial}{\partial r_j} \quad \text{(by the chain rule).}
\end{aligned}$$

Sometimes we will write

$$\nabla_{\dot r}\dot r = \frac{D^2 r}{dt^2}, \qquad \frac{DX}{dt} = \nabla_{\dot r(t)}X. \tag{0.17}$$

We define DX/dt to be the **covariant derivative**.

By (0.15), in local coordinates

$$\frac{DX}{dt} = \nabla_{\dot r}X = \left(\dot r^j \frac{\partial X^i}{\partial r^j} + \Gamma^i_{kj}X^k \dot r^j\right)\frac{\partial}{\partial r^i} = \left(\dot X^i + \Gamma^i_{kj}X^k \dot r^j\right)\frac{\partial}{\partial r^i}, \tag{0.18}$$

where $\dot r(t) = \dot r^i(\partial/\partial r^i)$. For $X = \dot r$ we of course recover the geodesic equations.

In the tangent bundle we can specify a linear connection by its action on vector fields, or by a map from vector fields (X, Y) to the vector field $\nabla_X Y$ that satisfies for smooth functions f and g and a vector fields X, Y, Z:

(i) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$.

(ii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$.

(iii) $\nabla_X(fY) = f\nabla_X Y + (\mathbf{d}f \cdot X)Y$,

where $\mathbf{d}f \cdot X$ is the directional derivative of f along X, or Lie derivative.

Given a basis of vector fields $\frac{\partial}{\partial r_j}$ we can represent ∇ by

$$\nabla_{\partial/\partial r_i} \frac{\partial}{\partial r_j} = \Gamma^k_{ij} \frac{\partial}{\partial r_k} \,. \tag{0.19}$$

For X, Y vector fields given locally by $X = X^i(\partial/\partial r_i),\ Y = Y^i(\partial/\partial r_i)$, (i) and (iii) imply

$$\nabla_X Y = \left(X^j \frac{\partial Y^i}{\partial r^j} + X^k Y^j \Gamma^i_{kj} \right) \frac{\partial}{\partial r^i}. \tag{0.20}$$

The geodesic equations above then may be written

$$\nabla_{\dot{r}} \dot{r} = 0 \,. \tag{0.21}$$

We can see this directly by a simple computation, again using (i) and (iii):

$$\begin{aligned}
\nabla_{\dot{r}^i(\partial/\partial r_i)} \dot{r}^j \frac{\partial}{\partial r_j} &= \dot{r}^i \nabla_{\partial/\partial r_i} \dot{r}^j \frac{\partial}{\partial r_j} \\
&= \dot{r}^i \frac{\partial}{\partial r_i} \dot{r}^j \frac{\partial}{\partial r_j} + \dot{r}^i \dot{r}^j \Gamma^k_{ij} \frac{\partial}{\partial r_k} \\
&= (\ddot{r}^j + \Gamma^j_{ik} \dot{r}^i \dot{r}^k) \frac{\partial}{\partial r_j} \quad \text{(by the chain rule).}
\end{aligned}$$

Sometimes we will write

$$\nabla_{\dot{r}} \dot{r} = \frac{D^2 r}{dt^2}, \qquad \frac{DX}{dt} = \nabla_{\dot{r}(t)} X. \tag{0.22}$$

We define DX/dt to be the **covariant derivative**.

By (0.15), in local coordinates

$$\frac{DX}{dt} = \nabla_{\dot{r}} X = \left(\dot{r}^j \frac{\partial X^i}{\partial r^j} + \Gamma^i_{kj} X^k \dot{r}^j \right) \frac{\partial}{\partial r^i} = \left(\dot{X}^i + \Gamma^i_{kj} X^k \dot{r}^j \right) \frac{\partial}{\partial r^i}, \tag{0.23}$$

where $\dot{r}(t) = \dot{r}^i(\partial/\partial r^i)$. For $X = \dot{r}$ we of course recover the geodesic equations.

Definition 0.7 *Let P be a manifold and let $\mathcal{F}(P)$ denote the set of smooth real-valued functions on P. Consider a bracket operation denoted by*

$$\{\,,\} : \mathcal{F}(P) \times \mathcal{F}(P) \to \mathcal{F}(P).$$

The pair $(P, \{\,,\})$ is called a **Poisson manifold** *if $\{\,,\}$ satisfies:*

(PB1)	**bilinearity**	*$\{f, g\}$ is bilinear in f and g.*
(PB2)	**anticommutativity**	*$\{f, g\} = -\{g, f\}$.*
(PB3)	**Jacobi's identity**	*$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$.*
(PB4)	**Leibniz's rule**	*$\{fg, h\} = f\{g, h\} + g\{f, h\}$.*

Notice that conditions (PB1)–(PB3) make $(\mathcal{F}(P), \{\,,\})$ into a Lie algebra.

If $(P, \{\,,\})$ is a Poisson manifold, then one can show that because of (PB1) and (PB4), there is a tensor B on P assigning to each $z \in P$ a linear map $B(z) : T_z^*P \to T_zP$ such that

$$\{f, g\}(z) = \langle B(z) \cdot \mathbf{d}f(z), \mathbf{d}g(z) \rangle. \tag{0.24}$$

Here $\langle\,,\rangle$ denotes the natural pairing between vectors and covectors. Because of (PB2), $B(z)$ is antisymmetric. Letting z^I, $I = 1, \ldots, M$, denote coordinates on P, (0.24) becomes

$$\{f, g\} = B^{IJ} \frac{\partial f}{\partial z^I} \frac{\partial g}{\partial z^J}. \tag{0.25}$$

Definition 0.8 *Let P be a manifold and Ω a 2-form on P. The pair (P, Ω) is called a* **symplectic manifold** *if Ω satisfies*

(S1) $\mathbf{d}\Omega = 0$ *(i.e., Ω is closed) and*

(S2) Ω *is nondegenerate.*

Definition 0.9 *Let (P, Ω) be a symplectic manifold and let $f \in \mathcal{F}(P)$. Let X_f be the unique vector field on P satisfying*

$$\Omega_z(X_f(z), v) = \mathbf{d}f(z) \cdot v \quad \textit{for all} \quad v \in T_z P. \tag{0.26}$$

We call X_f the **Hamiltonian vector field** *of f.* **Hamilton's equations** *are the differential equations on P given by*

$$\dot{z} = X_f(z). \tag{0.27}$$

If (P, Ω) is a symplectic manifold, define the **Poisson bracket operation** $\{\cdot, \cdot\} : \mathcal{F}(P) \times \mathcal{F}(P) \to \mathcal{F}(P)$ *by*

$$\{f, g\} = \Omega(X_f, X_g). \tag{0.28}$$

Let G be a Lie group and $\mathfrak{g} = T_eG$ its Lie algebra with $[\,,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ the associated Lie bracket.

Proposition 1 *The dual space $\mathfrak{g}^*$ is a Poisson manifold with either of the two brackets*

$$\{f, k\}_\pm(\mu) = \pm \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu}\right]\right\rangle . \tag{0.29}$$

Here $\mathfrak{g}$ is identified with $\mathfrak{g}^{**}$ in the sense that $\delta f/\delta\mu \in \mathfrak{g}$ is defined by $\langle \nu, \delta f/\delta\mu\rangle = \mathbf{D}f(\mu)\cdot\nu$ for $\nu \in \mathfrak{g}^*$, where $\mathbf{D}$ denotes the derivative. Assuming that $\mathfrak{g}$ is finite-dimensional and choosing coordinates $(\xi^1, \ldots, \xi^m)$ on $\mathfrak{g}$ and corresponding dual coordinates $(\mu_1, \ldots, \mu_m)$ on $\mathfrak{g}^*$, the ***Lie–Poisson bracket*** (0.29) is

$$\{f, k\}_\pm(\mu) = \pm \mu_a C^a_{bc} \frac{\partial f}{\partial \mu_b}\frac{\partial k}{\partial \mu_c}; \tag{0.30}$$

here C^a_{bc} are the **structure constants** of $\mathfrak{g}$ defined by $[e_a, e_b] = C^c_{ab} e_c$, where $(e_1, \ldots, e_m)$ is the coordinate basis of $\mathfrak{g}$ and where for $\xi \in \mathfrak{g}$ we write $\xi = \xi^a e_a$, and for $\mu \in \mathfrak{g}^*, \mu = \mu_a e^a$, where $(e^1, \ldots, e^m)$ is the dual basis.

Definition 0.10 *A* **finite-dimensional nonlinear control system** *on a smooth n-manifold M is a differential equation of the form*

$$\dot{x} = f(x, u), \tag{0.31}$$

where $x \in M$, $u(t)$ is a time-dependent map from the nonnegative reals $\mathbb{R}^+$ to a constraint set $\Omega \subset \mathbb{R}^m$, and f is taken to be C^∞ (smooth) or C^ω (analytic) from $M \times \mathbb{R}^m$ into TM such that for each fixed u, f is a vector field on M. The map u is assumed to be piecewise smooth or piecewise analytic. Such a map u is said to be ***admissible****. The manifold M is said to be the* **state space** *or the* **phase space** *of the system.*

Affine control system:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \,, \tag{0.32}$$

Definition 0.11 *The system (0.32) is said to be* **controllable** *if for any two points* x_0 *and* x_f *in* M *there exists an admissible control* $u(t)$ *defined on some time interval* $[0, T]$ *such that the system (0.32) with initial condition* x_0 *reaches the point* x_f *in time* T*.*

To define accessibility we first need the notion of a reachable set. This notion will depend on the choice of a positive time T. The reachable set from a given point at time T will be defined to be, essentially, the set of points that may be reached by the system by traveling on trajectories from the initial point in a time at most T. In particular, if $q \in M$ is of the form $x(t)$ for some trajectory with initial condition $x(0) = p$ and for some t with $0 \leq t \leq T$, then q will be said to be reachable from p in time T. More precisely:

Definition 0.12 *Given* $x_0 \in M$ *we define* $R(x_0, t)$ *to be the set of all* $x \in M$ *for which there exists an admissible control* u *such that there is a trajectory of the system with* $x(0) = x_0$*,* $x(t) = x$*. The* **reachable set** *from* x_0 *at time* T *is defined to be*

$$R_T(x_0) = \bigcup_{0 \leq t \leq T} R(x_0, t)\,. \tag{0.33}$$

Definition 0.13 *The* **accessibility algebra** $\mathcal{C}$ *of the system is the smallest Lie algebra of vector fields on* M *that contains the vector fields* f *and* $g_1, \ldots, g_m$*.*

Note that the accessibility algebra is just the span of all possible Lie brackets of f and the g_i.

Definition 0.14 *We define the* **accessibility distribution** C *of the system to be the* *distribution generated by the vector fields in $\mathcal{C}$; i.e., $C(x)$ is the span of the vector fields X in $\mathcal{C}$ at x.*

Definition 0.15 *The system on M is said to be* **accessible** *from $p \in M$ if for every $T > 0$, $R_T(p)$ contains a nonempty open set.*

Roughly speaking, this means that there is *some* point q (not necessarily even close to a desired objective point) that is reachable from p in time no more than T and that points close to q are also reachable from p in time no more than T.

Accessibility, while relatively easy to prove, is far from proving controllability.

Theorem 0.1 *Consider the system and assume that the vector fields are C^∞. If $\dim C(x_0) = n$ (i.e., the accessibility algebra spans the tangent space to M at x_0), then for any $T > 0$, the set $R_T(x_0)$ has a nonempty interior; i.e., the system has the accessibility property from x_0.*

Note that while this spanning condition is an intuitively reasonable condition, the resulting theorem is quite weak, since it is far from implying controllability. The problem is that one cannot move "backward" along the drift vector field f. If f is absent, this is a strong condition; see below.

In certain special cases the accessibility rank condition *does* imply controllability, however. (We assume here that all vector fields are real analytic; the nonanalytic case can present difficulties.

Theorem 0.2 *Suppose the system is analytic. If $\dim C(x) = n$ everywhere on M and either*

1. $f = 0$, *or*
2. f *is divergence-free and M is compact and Riemannian,*

then (0.32) *is controllable.*

The idea behind this result is that one cannot move "backward" along the drift directions, and hence a spanning condition involving the drift vector field does not guarantee controllability. A particular case of item 2 above is that in which f is Hamiltonian. This ensures a drift "backward" eventually.

e.g. Heisenberg example Recall from Chapter 1 the Heisenberg system

$$\begin{aligned} \dot{x} &= u, \\ \dot{y} &= v, \\ \dot{z} &= vx - uy, \end{aligned} \tag{0.34}$$

which may be written as

$$\dot{q} = u_1 g_1 + u_2 g_2, \tag{0.35}$$

Another case of interest where accessibility implies controllability is a linear system of the form

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i = Ax + Bu, \tag{0.36}$$

where $x \in \mathbb{R}^n$, and $A \in \mathbb{R}^n \times \mathbb{R}^n$ and $B \in \mathbb{R}^n \times \mathbb{R}^m$ are constant matrices, b_i being the columns of B.

The Lie bracket of the drift vector field Ax with b_i is readily checked to be the constant vector field $-Ab_i$. Bracketing the latter field with Ax and so on tells us that $\mathcal{C}$ is spanned by $Ax, b_i, Ab_i, \ldots, A^{n-1}b_i$, $i = 1, \ldots, m$. Thus, the accessibility rank condition at the origin is equivalent to the classical controllability rank condition

$$\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n\,. \tag{0.37}$$

In fact, the following theorem holds.

Theorem 0.3 *The system is controllable if and only if the controllability rank condition holds.*

• **The Lagrange-d'Alembert-Poincaré equations.** Natural symmetries in the system:

Can rewrite the equation of motions in terms of a reduced constrained Lagrangian l_c.

Theorem 0.2 *The following* ***nonholonomic Lagrange-d'Alembert-Poincaré equations*** *hold for each* $1 \leq \alpha \leq \sigma$ *and* $1 \leq b \leq m$*:*

$$\begin{aligned}\frac{d}{dt}\frac{\partial l_c}{\partial \dot{r}^\alpha} - \frac{\partial l_c}{\partial r^\alpha} &= -\frac{\partial I^{cd}}{\partial r^\alpha} p_c p_d - \mathcal{D}^c_{b\alpha} I^{bd} p_c p_d \\ &\quad - \mathcal{B}^c_{\alpha\beta} p_c \dot{r}^\beta - \mathcal{D}_{\beta\alpha b} I^{bc} p_c \dot{r}^\beta - \mathcal{K}_{\alpha\beta\gamma} \dot{r}^\beta \dot{r}^\gamma, \\ \frac{d}{dt} p_b &= C^c_{ab} I^{ad} p_c p_d + \mathcal{D}^c_{b\alpha} p_c \dot{r}^\alpha + \mathcal{D}_{\alpha\beta b} \dot{r}^\alpha \dot{r}^\beta.\end{aligned}$$

Here $l_c(r^\alpha, \dot{r}^\alpha, p_a)$ *is the constrained* ***reduced*** *Lagrangian, i.e. the Lagrangian in the body frame;* r^α*,* $1 \leq \alpha \leq \sigma$*, are coordinates in the shape space; i.e. coordinates of system degrees of freedom* p_a*,* $1 \leq a \leq m$*, are components of the momentum map in the body representation.*

• The key to the qualitative behavior of this system are the terms on the right hand side of the momentum equation.

• Case of interest:the matrix $C^c_{ab} I^{ad}$ is skew. see Zenkov, Bloch and Marsden [1998] and divides into two cases: the term quadratic in $\dot{r}$ is present or not. If it vanishes, there are many cases where one does not obtain asymptotic stability, for example the rolling penny problem. When it is present asymptotically stable dynamics can occur as in the rattleback top.

• Key case: is the Euler-Poincaré-Suslov equations, where there are no internal or shape degrees of freedom, *i.e.* no coordinates r^α. Again, asymptotic behavior may occur in some of the variables.

Whether the nonholonomic systems exhibit asymptotic behavior or not it is striking that we have

Proposition 0.3 *The nonholonomic equations in the case that l_c is quadratic in p and $\dot{r}$, are time reversible.*

Proof. The equations are invariant under the discrete $\mathbb{Z}_2$ symmetry $(t \to -t, p \to -p, \dot{r} \to -\dot{r})$.

$\blacksquare$

In this setting it is easy to check that energy is always preserved.

•Almost Poisson Systems

Recall:

Definition 0.4 *An almost Poisson manifold is a pair* $(M, \{,\})$ *where* M *is a smooth manifold and (i)*$\{,\}$ *defines an almost Lie algebra structure on the* C^∞ *functions on* M*, i.e. the bracket satisfies all conditions for a Lie algebra except that the Jacobi identity is not satisfied and (ii)* $\{,\}$ *is a derivation in each factor.*

If in addition Jacobi satisfied, Poisson manifold.

An almost Poisson structure on M will be Poisson if its *Jacobiator*, defined by

$$J(f,g,h) = \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\}g\}$$

vanishes.

• One can define an *almost Poisson* vector field on M by

$$\dot{z}_i = \pi_{ij}(z)\frac{\partial H}{\partial z_j}\,.$$

.

• "Hamiltonian" Formulation of Nonholonomic Systems

Nonholonomic systems are almost Poisson.

Start on the Lagrangian side with a configuration space Q and a Lagrangian L (possibly of the form kinetic energy minus potential energy, *i.e.*,

$$L(q, \dot{q}) = \frac{1}{2}\langle\langle \dot{q}, \dot{q} \rangle\rangle - V(q),$$

As above, our nonholonomic constraints are given by a distribution $\mathcal{D} \subset TQ$. We also let $\mathcal{D}^0 \subset T^*Q$ denote the annihilator of this distribution. Using a basis ω^a of the annihilator $\mathcal{D}^o$, we can write the constraints as

$$\omega^a(\dot{q}) = 0,$$

where $a = 1, \ldots, k$.

Recall that the cotangent bundle T^*Q is equipped with a canonical Poisson bracket and is expressed in the canonical coordinates (q, p) as

$$\{F, G\}(q, p) = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} = \left(\frac{\partial F}{\partial q}^T, \frac{\partial F}{\partial p}^T\right) J \begin{pmatrix} \frac{\partial G}{\partial q} \\ \frac{\partial G}{\partial p} \end{pmatrix}.$$

Here J is the canonical Poisson tensor

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

A constrained t phase space $\mathcal{M} = \mathbb{F}L(\mathcal{D}) \subset T^*Q$ is defined so that the constraints on the Hamiltonian side are given by $p \in \mathcal{M}$. In local coordinates,

$$\mathcal{M} = \left\{ (q, p) \in T^*Q \;\middle|\; \omega_i^a \frac{\partial H}{\partial p_i} = 0 \right\}.$$

Let $\{X_\alpha\}$ be a local basis for the constraint distribution $\mathcal{D}$ and let $\{\omega^a\}$ be a local basis for the annihilator $\mathcal{D}^0$. Let $\{\omega_a\}$ span the complementary subspace to $\mathcal{D}$ such that $\langle \omega^a, \omega_b \rangle = \delta_b^a$, where δ_b^a is the usual Kronecker delta. Here $a = 1, \ldots, k$ and $\alpha = 1, \ldots, n - k$. Define a coordinate transformation $(q, p) \mapsto (q, \tilde{p}_\alpha, \tilde{p}_a)$ by

$$\tilde{p}_\alpha = X_\alpha^i p_i, \qquad \tilde{p}_a = \omega_a^i p_i.$$

In the new (generally not canonical) coordinates $(q, \tilde{p}_\alpha, \tilde{p}_a)$, the Poisson tensor becomes

$$\tilde{J}(q, \tilde{p}) = \begin{pmatrix} \{q^i, q^j\} & \{q^i, \tilde{p}_j\} \\ \{\tilde{p}_i, q^j\} & \{\tilde{p}_i, \tilde{p}_j\} \end{pmatrix}.$$

Use $(q, \tilde{p}_\alpha)$ as induced local coordinates for $\mathcal{M}$. It is easy to show that

$$\begin{aligned} \frac{\partial \tilde{H}}{\partial q^j}(q, \tilde{p}_\alpha, \tilde{p}_a) &= \frac{\partial H_{\mathcal{M}}}{\partial q^j}(q, \tilde{p}_\alpha), \\ \frac{\partial \tilde{H}}{\partial \tilde{p}_\beta}(q, \tilde{p}_\alpha, \tilde{p}_a) &= \frac{\partial H_{\mathcal{M}}}{\partial \tilde{p}_\beta}(q, \tilde{p}_\alpha), \end{aligned}$$

where $H_{\mathcal{M}}$ is the constrained Hamiltonian on $\mathcal{M}$ expressed in the induced coordinates. We can also truncate the Poisson tensor $\tilde{J}$ by leaving out its last k columns and last k rows and then describe the constrained dynamics on $\mathcal{M}$ expressed in the induced coordinates $(q^i, \tilde{p}_\alpha)$ as follows:

$$\begin{pmatrix} \dot{q}^i \\ \dot{\tilde{p}}_\alpha \end{pmatrix} = J_{\mathcal{M}}(q, \tilde{p}_\alpha) \begin{pmatrix} \frac{\partial H_{\mathcal{M}}}{\partial q^j}(q, \tilde{p}_\alpha) \\ \frac{\partial H_{\mathcal{M}}}{\partial \tilde{p}_\beta}(q, \tilde{p}_\alpha) \end{pmatrix}, \qquad \begin{pmatrix} q^i \\ \tilde{p}_\alpha \end{pmatrix} \in \mathcal{M}.$$

Here $J_{\mathcal{M}}$ is the $(2n-k) \times (2n-k)$ truncated matrix of $\tilde{J}$ restricted to $\mathcal{M}$ and is expressed in the induced coordinates.

The matrix $J_{\mathcal{M}}$ defines a bracket $\{\cdot\,,\cdot\}_{\mathcal{M}}$ on the constraint submanifold $\mathcal{M}$ as follows:

$$\{F_{\mathcal{M}}, G_{\mathcal{M}}\}_{\mathcal{M}}(q, \tilde{p}_\alpha) := \left(\frac{\partial F_{\mathcal{M}}^T}{\partial q^i}\ \frac{\partial F_{\mathcal{M}}^T}{\partial \tilde{p}_\alpha}\right) J_{\mathcal{M}}(q^i, \tilde{p}_\alpha) \begin{pmatrix} \frac{\partial G_{\mathcal{M}}}{\partial q^j} \\ \frac{\partial G_{\mathcal{M}}}{\partial \tilde{p}_\beta} \end{pmatrix},$$

for any two smooth functions $F_{\mathcal{M}}, G_{\mathcal{M}}$ on the constraint submanifold $\mathcal{M}$. Clearly, this bracket satisfies the first two defining properties of a Poisson bracket, namely, skew symmetry and the Leibniz rule, and one can show that it satisfies the Jacobi identity if and only if the constraints are holonomic. Furthermore, the constrained Hamiltonian $H_{\mathcal{M}}$ is an integral of motion for the constrained dynamics on $\mathcal{M}$ due to the skew symmetry of the bracket.

Following e.g. van der Schaft and Maschke [1994] and Koon and Marsden [1997] we can write the nonholonomic equations of motion as follows:

$$\begin{pmatrix} \dot{s}^a \\ \dot{r}^\alpha \\ \dot{\tilde{p}}_\alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 & -A^a_\beta \\ 0 & 0 & \delta^\alpha_\beta \\ (A^b_\alpha)^T & -\delta^\beta_\alpha & -p_c B^c_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \frac{\partial H_{\mathcal{M}}}{\partial s^b} \\ \frac{\partial H_{\mathcal{M}}}{\partial r^\beta} \\ \frac{\partial H_{\mathcal{M}}}{\partial \tilde{p}_\beta} \end{pmatrix}$$

Jacobiator of the Poisson tensor vanishes precisely when the curvature of the nonholonomic constraint distribution is zero or the constraints are holonomic.

• **The Momentum Equation** Simple constained physical systems that have symmetries do not have associated conservation laws.

• Simplest situation: case of cyclic variables. Recall that the equations of motion have the form

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a} = -\frac{\partial L}{\partial \dot{s}^b} B^b_{\alpha\beta}\dot{r}^\beta.$$

If this has a cyclic variable, say r^1, this would mean that all the quantities L_c, L, $B^b_{\alpha\beta}$ would be independent of r^1. This is equivalent to saying that there is a translational symmetry in the r^1 direction.

Suppose also that the s variables are also cyclic. Then the above equation for the momentum $p_1 = \partial L_c / \partial \dot{r}^1$ becomes

$$\frac{d}{dt} p_1 = -\frac{\partial L}{\partial \dot{s}^b} B^b_{1\beta}\dot{r}^\beta.$$

Fails to be a conservation law in general. Note that the right hand side is linear in $\dot{r}$ (the first term is linear in p_r) and the equation does not depend on r^1 itself.

• Special case of the momentum equation.

General Momentum Equation Assume there is a Lie group G that acts freely and properly on the configuration space Q. The Lie algebra of G is denoted by $\mathfrak{g}$. A Lagrangian system is called *G-invariant* if its Lagrangian L is invariant under the induced action of G on TQ.

Recall the definition of the *momentum map* for an unconstrained Lagrangian system with symmetry:

The ***momentum map*** $J : TQ \to \mathfrak{g}^*$ is the bundle map taking TQ to the bundle $(\mathfrak{g}^Q)^*$ whose fiber over the point q is the dual Lie algebra $\mathfrak{g}^*$ that is defined by

$$\langle J(v_q), \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q \rangle := \frac{\partial L}{\partial \dot{q}^i} (\xi_Q)^i,$$

where $\xi \in \mathfrak{g}$, $v_q \in TQ$, and $\xi_Q \in TQ$ is the generator associated with the Lie algebra element ξ.

A nonholonomic system is called G-invariant if both the Lagrangian L and the constraint distribution $\mathcal{D}$ are invariant under the induced action of G on TQ. Let $\mathcal{D}_q$ enote the the fiber of the constraint distribution $\mathcal{D}$ at $q \in Q$.

Definition:

The ***nonholonomic momentum map*** J^{nhc} is defined as a collection of the components of the ordinary momentum map J that are ***consistent with the constraints***, i.e., the Lie algebra elements ξ in () are now chosen from the subspace $\mathfrak{g}^q$ of Lie algebra elements in $\mathfrak{g}$ whose infinitesimal generators evaluated at q lie in the intersection $\mathcal{D}_q \cap T_q(\mathrm{Orb}(q))$.

Thus, the nonholonomic momentum is a dynamic variable. The momentum dynamics is specified in the following theorem:

Theorem:

Assume that the Lagrangian is invariant under the group action and that ξ^q is a section of the bundle $\mathfrak{g}^{\mathcal{D}}$. Then any solution of the Lagrange–d'Alembert equations for a nonholonomic system must satisfy, in addition to the given kinematic constraints, the ***momentum equation***

$$\frac{d}{dt}\langle J^{\mathrm{nhc}}, (\xi^{q(t)})\rangle = \frac{\partial L}{\partial \dot{q}^i}\left[\frac{d}{dt}(\xi^{q(t)})\right]_Q^i . \tag{0.41}$$

A Lie algebra element ξ is said to act *horizontally* if $\xi_Q(q) \in \mathcal{D}_q$.

Corollary

If ξ is a horizontal symmetry, then the following conservation law holds:

$$\frac{d}{dt}\langle J^{\mathrm{nhc}}, (\xi)\rangle = 0. \tag{0.42}$$

Symmetries Symmetries play an important role in our analysis. Suppose we are given a nonholonomic system with Lagrangian $L : TQ \to \mathbb{R}$, and a (nonintegrable) constraint distribution $\mathcal{D}$. We can then look for a group G that acts freely and properly on the configuration space Q. It induces an action on the tangent space TQ and so it makes sense to ask that the Lagrangian L be *invariant*. Also, one can ask that the distribution be *invariant* in the sense that the action by a group element $g \in G$ maps the distribution $\mathcal{D}_q$ at the point $q \in Q$ to the distribution $\mathcal{D}_{gq}$ at the point gq. If these properties hold, we say that G is a *symmetry group*. The manifold Q/G is called the *shape space* of the system and the configuration space has the structure of a principal fiber bundle $\pi : Q \to Q/G$.

Geometry of Nonholonomic Systems with Symmetry

The group orbit through a point q, an (immersed) submanifold, is denoted

$$\mathrm{Orb}(q) := \{gq \mid g \in G\}.$$

Let $\mathfrak{g}$ denote the Lie algebra of the Lie group G. For an element $\xi \in \mathfrak{g}$, we denote by ξ_Q the vector field on Q arising from the corresponding infinitesimal generator of the group action, so these are also the tangent spaces to the group orbits. Define, for each $q \in Q$, the vector subspace $\mathfrak{g}^q$ to be the set of Lie algebra elements in $\mathfrak{g}$ whose infinitesimal generators evaluated at q lie in both $\mathcal{D}_q$ and $T_q(\mathrm{Orb}(q))$:

$$\mathfrak{g}^q := \{\xi \in \mathfrak{g} \mid \xi_Q(q) \in \mathcal{D}_q \cap T_q(\mathrm{Orb}(q))\}.$$

The corresponding bundle over Q whose fiber at the point q is given by $\mathfrak{g}^q$, is denoted by $\mathfrak{g}^{\mathcal{D}}$.

Reduced dynamics. Assuming that the Lagrangian and the constraint distribution are G-invariant, we can form the *reduced velocity phase space* TQ/G and the *reduced constraint space* $\mathcal{D}/G$. The Lagrangian L induces well defined functions, the *reduced Lagrangian*

$$l : TQ/G \to \mathbb{R}$$

and the *constrained reduced Lagrangian*

$$l_c : \mathcal{D}/G \to \mathbb{R},$$

satisfying $L = l \circ \pi_{TQ}$ and $L|_{\mathcal{D}} = l_c \circ \pi_{\mathcal{D}}$ where $\pi_{TQ} : TQ \to TQ/G$ and $\pi_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}/G$ are the projections. By general considerations, the Lagrange-d'Alembert equations induce well defined *reduced equations* on $\mathcal{D}/G$. That is, the vector field on the manifold $\mathcal{D}$ determined by the Lagrange-d'Alembert equations (including the constraints) is G-invariant, and so defines a reduced vector field on the quotient manifold $\mathcal{D}/G$. Call these equations the *Lagrange-d'Alembert-Poincaré equations.*

Let a local trivialization be chosen on the principle bundle $\pi : Q \to Q/G$, with a local representation having components denoted (r, g). Let r, an element of shape space Q/G, have coordinates denoted r^α, and let g be group variables for the fiber, G. In such a representation, the action of G is the left action of G on the second factor. The coordinates (r, g) induce the coordinates $(r, \dot{r}, \xi)$ on TQ/G, where $\xi = g^{-1}\dot{g}$. The Lagrangian L is invariant under the left action of G and so it depends on g and $\dot{g}$ only through the combination $\xi = g^{-1}\dot{g}$. Thus the reduced Lagrangian l is given by

$$l(r, \dot{r}, \xi) = L(r, g, \dot{r}, \dot{g}).$$

Therefore the full system of equations of motion consists of the following two groups:

1. The Lagrange-d'Alembert-Poincaré equation on $\mathcal{D}/G$ (see theorem 0.2).
2. The *reconstruction equation*
$$\dot{g} = g\xi.$$

The nonholonomic momentum in body representation. Choose a q-dependent basis $e_A(q)$ for the Lie algebra such that the first m elements span the subspace $\mathfrak{g}^q$ in the following way. First, one chooses, for each r, such a basis at the identity element $g = \mathrm{Id}$, say

$$e_1(r), e_2(r), \ldots, e_m(r), e_{m+1}(r), \ldots, e_k(r).$$

Now define the *body fixed basis* by

$$e_A(r, g) = \mathrm{Ad}_g\, e_A(r).$$

Then the first m elements will indeed span the subspace $\mathfrak{g}^q$ since the distribution is invariant. We denote the structure constants of the Lie algebra relative to this basis by C_{AB}^C.

To avoid confusion, we make the following index conventions:

1. The first batch of indices range from 1 to m corresponding to the symmetry directions along constraint space. These indices will be denoted $a, b, c, \ldots$.

2. The second batch of indices range from $m+1$ to k corresponding to the symmetry directions not aligned with the constraints. Indices for this range will be denoted by $a', b', c', \ldots$.

3. The indices $A, B, C, \ldots$ on the Lie algebra $\mathfrak{g}$ range from 1 to k.

4. The indices $\alpha, \beta, \ldots$ on the shape variables r range from 1 to σ. Thus, σ is the dimension of the shape space Q/G and so $\sigma = n - k$.

The summation convention for all of these indices will be understood.

Assume that the Lagrangian has the form of kinetic minus potential energy, and that the constraints and the orbit directions span the entire tangent space to the configuration space:

$$\mathcal{D}_q + T_q(\mathrm{Orb}(q)) = T_q Q.$$

Then it is possible to introduce a new Lie algebra variable Ω called the *body angular velocity* such that:

1. $\Omega = \mathcal{A}\dot{r} + \xi$, where the Lie algebra valued form $\mathcal{A} = \mathcal{A}_\alpha^A e_A(r) dr^\alpha$ is called the *nonholonomic connection* (see Bloch *et al.* [1996] for details).

2. The constraints are given by $\Omega \in \mathrm{span}\{e_1(r), \ldots, e_m(r)\}$ or $\Omega^{m+1} = \cdots = \Omega^k = 0$.

3. The reduced Lagrangian in the variables $(r, \dot{r}, \Omega)$ becomes

$$l(r^\alpha, \dot{r}^\alpha, \Omega^A) = \frac{1}{2} g_{\alpha\beta} \dot{r}^\alpha \dot{r}^\beta + \frac{1}{2} \mathbb{I}_{AB} \Omega^A \Omega^B + \lambda_{a'\alpha} \dot{r}^\alpha \Omega^{a'} - U(r). \tag{0.43}$$

Here $g_{\alpha\beta}$ are coefficients of the kinetic energy metric induced on the manifold Q/G, $\mathbb{I}_{AC}$ are components of the *locked inertia tensor* defined by

$$\langle \mathbb{I}(r)\xi, \eta \rangle = \langle\!\langle \xi_Q, \eta_Q \rangle\!\rangle, \qquad \xi, \eta \in \mathfrak{g},$$

where $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$ is the kinetic energy metric. The coefficients $\lambda_{a'\alpha}$ are defined by

$$\lambda_{a'\alpha} = \frac{\partial^2 l}{\partial \xi^{a'} \partial r^\alpha} - \frac{\partial^2 l}{\partial \xi^{a'} \partial \xi^B} \mathcal{A}_\alpha^B.$$

The constrained reduced Lagrangian becomes especially simple in the variables $(r, \dot{r}, \Omega)$:

$$l_c = \frac{1}{2} g_{\alpha\beta} \dot{r}^\alpha \dot{r}^\beta + \frac{1}{2} \mathbb{I}_{ab} \Omega^a \Omega^b - U. \tag{0.44}$$

We remark that this choice of Ω block-diagonalizes the kinetic energy metric, *i.e.*, eliminates the terms proportional to $\Omega^a \dot{r}^\alpha$ in (0.44).

The *nonholonomic momentum in body representation* is defined by

$$p_a = \frac{\partial l}{\partial \Omega^a} = \frac{\partial l_c}{\partial \Omega^a}, \qquad a = 1, \ldots, m.$$

Notice that the nonholonomic momentum may be viewed as a collection of components of the ordinary momentum map along the constraint directions.

The Lagrange-d'Alembert-Poincaré equations. As in Bloch *et al.* [1996], the reduced equations of motion are given by the next theorem.

Theorem 0.5 *The following* ***reduced nonholonomic Lagrange-d'Alembert-Poincaré equations*** *hold for each* $1 \leq \alpha \leq \sigma$ *and* $1 \leq b \leq m$*:*

$$\begin{aligned} \frac{d}{dt}\frac{\partial l_c}{\partial \dot{r}^\alpha} - \frac{\partial l_c}{\partial r^\alpha} &= -\mathcal{D}^c_{b\alpha} I^{bd} p_c p_d - \mathcal{K}_{\alpha\beta\gamma} \dot{r}^\beta \dot{r}^\gamma \\ &\quad - (\mathcal{B}^c_{\alpha\beta} - I_{c'a'} \mathbb{I}^{a'c} \mathcal{B}^{c'}_{\alpha\beta} + \mathcal{D}_{b\beta\alpha} I^{bc}) p_c \dot{r}^\beta, \qquad (0.45) \\ \frac{d}{dt} p_a &= (C^c_{ba} - C^{c'}_{ba} I_{c'a'} \mathbb{I}^{a'c}) I^{bd} p_c p_d + \mathcal{D}^c_{a\alpha} p_c \dot{r}^\alpha + \mathcal{D}_{a\alpha\beta} \dot{r}^\alpha \dot{r}^\beta. \qquad (0.46) \end{aligned}$$

Here and below $l_c(r^\alpha, \dot{r}^\alpha, \Omega^a)$ is the constrained Lagrangian, and I^{bd} and $I_{a'c'}$ are the inverse of the tensors $\mathbb{I}|_{\mathfrak{g}^q}$ and $\mathbb{I}^{-1}|_{(\mathfrak{g}^q)^*}$, respectively. We stress that in general $I^{bd} \neq \mathbb{I}^{bd}$ and $I_{a'c'} \neq \mathbb{I}_{a'c'}$.

The coefficients $\mathcal{B}^C_{\alpha\beta}$, $\mathcal{D}^c_{b\alpha}$, $\mathcal{D}_{b\alpha\beta}$, $\mathcal{K}_{\alpha\beta\gamma}$ are given by the formulae

$$\begin{aligned}
\mathcal{B}^C_{\alpha\beta} &= \frac{\partial \mathcal{A}^C_\alpha}{\partial r^\beta} - \frac{\partial \mathcal{A}^C_\beta}{\partial r^\alpha} - C^C_{BA}\mathcal{A}^A_\alpha \mathcal{A}^B_\beta + \gamma^C_{A\beta}\mathcal{A}^A_\alpha - \gamma^C_{A\alpha}\mathcal{A}^A_\beta, \\
\mathcal{D}^c_{b\alpha} &= -(C^c_{Ab} - C^{c'}_{Ab} I_{c'a'}\mathbb{I}^{a'c})\mathcal{A}^A_\alpha + C^{c'}_{ab}\lambda_{c'\alpha} I^{ac} + \gamma^c_{b\alpha} - \gamma^{c'}_{b\alpha} I_{c'a'}\mathbb{I}^{a'c}, \\
\mathcal{D}_{b\alpha\beta} &= \lambda_{c'\beta}(\gamma^{c'}_{b\alpha} - C^{c'}_{Ab}\mathcal{A}^A_\alpha), \\
\mathcal{K}_{\alpha\beta\gamma} &= \lambda_{c'\gamma}\mathcal{B}^{c'}_{\alpha\beta},
\end{aligned} \tag{0.47}$$

and the coefficients $\gamma^C_{b\alpha}$ are defined by

$$\frac{\partial e_b}{\partial r^\alpha} = \gamma^C_{b\alpha} e_C.$$

Equations (0.45) and (0.46) generalize the equations of motion in the orthogonal body frame (see Bloch *et al.* [1996]). Here we no longer assume that the body frame is orthogonal.

•Euler-Poincaré-Suslov Equations

Important special case of the reduced nonholonomic equations.

•Example: Euler-Poincaré-Suslov Problem on $SO(3)$ In this case the problem can be formulated as the standard Euler equations

$$I\dot{\omega} = I\omega \times \omega$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ are the system angular velocities in a frame where the inertia matrix is of the form $I = \mathrm{diag}(I_1, I_2, I_3)$ and the system is subject to the constraint

$$a \cdot \omega = 0$$

where $a = (a_1, a_2, a_3)$.

The nonholonomic equations of motion are then given by

$$I\dot{\omega} = I\omega \times \omega + \lambda a$$

subject to the constraint. Solve for λ:

$$\lambda = -\frac{I^{-1}a \cdot (I\omega \times \omega)}{I^{-1}a \cdot a}\,.$$

If a is an eigenvector of the moment of inertia tensor flow is measure preserving.

More generally:

Invariant Measures of the Euler-Poincaré-Suslov Equations An important special case of the reduced nonholonomic equations is the case when there is no shape space at all. In this case the system is characterized by the Lagrangian $L = \frac{1}{2}\mathbb{I}_{AB}\Omega^A\Omega^B$ and the left-invariant constraint

$$\langle a, \Omega\rangle = a_A\Omega^A = 0. \tag{0.52}$$

Here $a = a_A e^A \in \mathfrak{g}^*$ and $\Omega = \Omega^A e_A$, where e_A, $A = 1, \ldots, k$, is a basis for $\mathfrak{g}$ and e^A is its dual basis. Multiple constraints may be imposed as well. The two classical examples of such systems are the *Chaplygin Sleigh* and the *Suslov problem.* These problems were introduced by Chaplygin in 1895 and Suslov in 1902, respectively.

We can consider the problem of when such systems exhibit asymptotic behavior. Following Kozlov [1988] it is convenient to consider the unconstrained case first. In the absence of constraints the dynamics is governed by the basic Euler-Poincaré equations

$$\dot{p}_B = C^C_{AB}\mathbb{I}^{AD}p_C p_D = C^C_{AB}p_C\Omega^A \tag{0.53}$$

where $p_B = \mathbb{I}_{AB}\Omega^B$ are the components of the momentum $p \in \mathfrak{g}^*$. One considers the question of whether the (unconstrained) equations (0.53) have an absolutely continuous integral invariant $fd^k\Omega$ with summable density $\mathcal{M}$. If $\mathcal{M}$ is a positive function of class C^1 one calls the integral invariant an invariant measure. Kozlov [1988] shows

Theorem 0.6 *The Euler-Poincaré equations have an invariant measure if and only if the group G is unimodular.*

A group is said to be unimodular if it has a bilaterally invariant measure. A criterion for unimodularity is $C^C_{AC} = 0$ (using the Einstein summation convention). Now we know (Liouville's theorem) that the flow of a vector differential equation $\dot{x} = f(x)$ is phase volume preserving if and only if $\operatorname{div} f = 0$. In this case the divergence of the right hand side of equation (0.53) is $C^C_{AC}\mathbb{I}^{AD}p_D = 0$. The statement of the theorem now follows from the following theorem of Kozlov [1998]: *A flow due to a homogeneous vector field in $\mathbb{R}^n$ is measure-preserving if and only if this flow preserves the standard volume in $\mathbb{R}^n$.*

Now, turning to the case where we have the constraint (0.52) we obtain the *Euler-Poincaré-Suslov equations*

$$\dot{p}_B = C^C_{AB}\mathbb{I}^{AD}p_C p_D + \lambda a_B = C^C_{AB}p_C\Omega^A + \lambda a_B \tag{0.54}$$

together with the constraint (0.52). Here λ is the Lagrange multiplier. This defines a system on the subspace of the dual Lie algebra defined by the constraint. Since the constraint is assumed to be nonholonomic, this subspace is not a subalgebra. One can then formulate a condition for the existence of an invariant measure of the Euler-Poincaré-Suslov equations.

Theorem 0.7 *Equations* (0.54) *have an invariant measure if and only if*

$$K\operatorname{ad}^*_{\mathbb{I}^{-1}a}a + T = \mu a, \qquad \mu \in \mathbb{R}, \tag{0.55}$$

where $K = 1/\langle a, \mathbb{I}^{-1}a\rangle$ and $T \in \mathfrak{g}^*$ is defined by $\langle T, \xi\rangle = \operatorname{Tr}(\operatorname{ad}_\xi)$.

This theorem was proved by Kozlov [1988] for compact algebras and for arbitrary algebras by Jovanović [1998]. In coordinates, condition (0.55) becomes

$$KC^C_{AB}\mathbb{I}^{AD}a_C a_D + C^C_{BC} = \mu a_B.$$

For a compact algebra (0.55) becomes

$$[\mathbb{I}^{-1}a, a] = \mu a, \qquad \mu \in \mathbb{R}, \tag{0.56}$$

where we identified $\mathfrak{g}^*$ with $\mathfrak{g}$.

The proof of theorem 0.7 reduces to the computation of the divergence of the vector field in (0.54).

In the compact case only constraint vectors a which commute with $\mathbb{I}^{-1}a$ allow the measure to be preserved. This means that a and $\mathbb{I}^{-1}a$ must lie in the same maximal commuting subalgebra. In particular, if a is an eigenstate of the inertia tensor, the reduced phase volume is preserved. When the maximal commuting subalgebra is one-dimensional this is a necessary condition. This is the case for groups such as $SO(3)$.

We thus have the following result which reflects a symmetry requirement on the constraints:

Theorem 0.8 *A compact Euler-Poincaré-Suslov system is measure preserving (i.e. does not exhibit asymptotic dynamics) if the constraint vectors a are eigenvectors of the inertia tensor, or if the constrained system is $\mathbb{Z}_2$ symmetric about each of its principal axes. If the maximal commuting subalgebra is one-dimensional this condition is necessary.*

Invariant Measures of Systems with Internal Degrees of Freedom In this section we extend the result of Kozlov [1988] and Jovanović [1998] to nonholonomic systems with nontrivial shape space. One can think of these systems as the Euler-Poincaré-Suslov systems with internal degrees of freedom. Recall that the constraints are of the form $\Omega^{m+1} = \cdots = \Omega^k = 0$. To simplify the exposition, we consider below systems with a single constraint. The results are valid for systems with multiple constraints as well.

Consider a nonholonomic system with the reduced Lagrangian $l(r, \dot{r}, \Omega)$ and a constraint $\langle a(r), \Omega \rangle = 0$. The subspace of the Lie algebra defined by the constraint at the configuration q is denoted here by $\mathfrak{g}^q$. The orientation of this subspace in $\mathfrak{g}$ depends on the shape configuration of the system, r. The dimension of $\mathfrak{g}^q$ however stays the same. As discussed in section , we choose a special moving frame in which $\mathfrak{g}^q$ is spanned by the vectors $e_1(r), \ldots, e_{k-1}(r)$. The equation of the constraint in this basis becomes $\Omega^k = 0$. Recall that the horizontal part of the kinetic energy metric is $g_{\alpha\beta}(r)$.

Theorem 0.9 *The system associated with the reduced Lagrangian $l(r, \dot{r}, \Omega)$ and the constraint $\langle a(r), \Omega \rangle = 0$ has an integral invariant with a C^1 density $\mathcal{M}(r)$ if and only if*

(i) $\left(C^a_{ba} - C^k_{ba} \dfrac{\mathbb{I}^{ka}}{\mathbb{I}^{kk}} \right) - g^{\alpha\delta} \mathcal{D}_{b\alpha\delta} = 0,$

(ii) the form $\left[\mathcal{D}^b_{b\beta} - g^{\alpha\delta} \lambda_{k\delta} \mathcal{B}^k_{\alpha\beta} \right] dr^\beta$ *is exact.*

Systems with One-Dimensional Shape Space. Assume that condition (i) of theorem 0.9 is satisfied. In this case the equation for the density of the invariant measure becomes

$$d(\ln \mathcal{M}) = d(\ln g) + \mathcal{D}_b^b dr. \tag{0.57}$$

The solution of this equation is globally defined if the shape space is either noncompact (and thus diffeomorphic to $\mathbb{R}$), or compact and the average of the function $\mathcal{D}_b^b$ equals zero.

Systems with Conserved Momentum. If the nonholonomic momentum is a constant of motion, then condition (i) of theorem 0.9 is trivially satisfied. Moreover, condition (ii) now asks that the form

$$g^{\alpha\delta}\lambda_{k\delta}\mathcal{B}^k_{\alpha\beta}r^{\beta} \tag{0.58}$$

is exact. The system thus preserves the measure with the density

$$\mathcal{M} = \det g \exp\left(-\int g^{\alpha\delta}\lambda_{k\delta}\mathcal{B}^k_{\alpha\beta}r^{\beta}\right).$$

Examples

The Routh Problem. This mechanical system consists of a uniform sphere rolling without slipping on the inner surface of a vertically oriented surface of revolution. He described the family of stationary periodic motions and obtained a necessary condition for stability of these motions. Routh noticed as well that integration of the equations of motion may be reduced to integration of a system of two linear differential equations with variable coefficients and considered a few cases when the equations of motion can be solved by quadratures. Modern references that treat this system are Hermans [1995] and Zenkov [1995].

This problem is $SO(2) \times SO(2)$-invariant, where the first copy of $SO(2)$ represents rotations about the axis of the surface of revolution while the second copy of $SO(2)$ represents rotations of the sphere about its radius through the contact point of the surface and the sphere.

Let r be the latitude of this contact point, a be the radius of the sphere, $c(r)+a$ be the reciprocal of the curvature of the meridian of the surface, and $b(r)$ be the distance from the axis of the surface to the sphere's center. The shape metric is $c^2(r)\dot{r}^2/2$ while the momentum equations are

$$\dot{p}_1 = \frac{c(r)\sin r}{b(r)} p_1 \dot{r} - \frac{2}{7} p_2 \dot{r}, \qquad \dot{p}_2 = \left(1 - \frac{c(r)\cos r}{b(r)}\right) p_1 \dot{r}.$$

The shape space is one-dimensional, the symmetry group $SO(2)\times SO(2)$ is commutative, and there are no terms proportional to $\dot{r}^2$ in the momentum equations. The trace term in (0.57) equals $c(r)\sin r/b(r)$, and thus the density of the invariant measure for the Routh problem is

$$\mathcal{M} = c^2(r) e^{\int \frac{c(r)\sin r}{b(r)}\, dr}. \tag{0.59}$$

The group action in this problem is singular: the intersection points of the surface of revolution and its axis have nontrivial isotropy subgroups. The shape coordinate r equals $\pm\pi/2$ at these points. As a result,

$$\lim_{r\to-\pi/2} \mathcal{M}(r) = \lim_{r\to\pi/2} \mathcal{M}(r) = \infty.$$

The Falling Disk. Consider a homogeneous disk rolling without sliding on a horizontal plane. This mechanical system is $SO(2) \times SE(2)$-invariant; the group $SO(2)$ represents the symmetry of the disk while the group $SE(2)$ represents the Euclidean symmetry of the overall system.

Classical references for the rolling disk are Vierkandt [1892], Korteweg [1899], and Appel [1900]. In particular, Vierkandt showed that on the reduced space $\mathcal{D}/SE(2)$—the constrained velocity phase space modulo the action of the Euclidean group $SE(2)$—most orbits of the system are periodic.

The shape of the system is specified by a single coordinate—the tilt of the disk denoted here by r. The momentum equations are

$$\begin{aligned}
\dot{p}_1 &= mR^2 \left(-\frac{\sin r}{A \cos r} p_1 + \left(\frac{\cos r}{mR^2 + B} + \frac{\sin^2 r}{A \cos r} \right) p_2 \right) \dot{r}, \\
\dot{p}_2 &= mR^2 \left(-\frac{1}{A \cos r} p_1 + \frac{\sin r}{A \cos r} p_2 \right) \dot{r}.
\end{aligned}$$

Hence, the trace terms $\mathcal{D}_b^b$ in (0.57) vanish, and the density of the invariant measure equals the component of the shape metric $g(r)$. The latter equals the moment of inertia of the disk with respect to the line through the rim of the disk and parallel to its diameter. Since the density of the measure is determined up to a constant factor, we conclude that the dynamics preserves the reduced phase space volume.

The 3D Chaplygin Sleigh with an Oscillating Mass. The three-dimensional Chaplygin sleigh is a free rigid body subject to the nonholonomic constraint $v^3 = 0$, where v^3 is the third component of the (linear) velocity relative to the body frame. The Lagrangian of this system is

$$\frac{1}{2}M\left[(v^1)^2 + (v^2)^2 + (v^3)^2\right] + \frac{1}{2}\left[I_1(\Omega^1)^2 + I_2(\Omega^2)^2 + I_3(\Omega^3)^2\right].$$

In this formula M is the mass of the body, I_j are the eigenvalues of its inertia tensor, and $(\Omega^1, \Omega^2, \Omega^3)$ and (v^1, v^2, v^3) are the angular and linear velocities relative to the body frame. The dynamics of this system is discussed in Neimark and Fufaev [1972].

We couple this system with an oscillator moving along the third coordinate axis of the body frame. The mass of this oscillator is m and the displacement from the origin is r. To keep the notation uniform with the general theory, we write the components of the linear velocity relative to the body frame as $(\Omega^4, \Omega^5, \Omega^6)$. The vector $(\Omega^1, \Omega^2, \Omega^3, \Omega^4, \Omega^5, \Omega^6)$ should be viewed as an element of the Lie algebra $se(3)$. The Lagrangian of this new system is

$$\begin{aligned} L = \frac{1}{2}\left[I_1(\Omega^1)^2 + I_2(\Omega^2)^2 + I_3(\Omega^3)^2\right] + \frac{M}{2}\left[(\Omega^4)^2 + (\Omega^5)^2 + (\Omega^6)^2\right] \\ + \frac{m}{2}\left[(\Omega^4 + \Omega^2 r)^2 + (\Omega^5 - \Omega^1 r)^2 + (\Omega^6 + \dot{r})^2\right] - U(r). \end{aligned} \tag{0.60}$$

The configuration space is $\mathbb{R} \times SE(3)$, and the system is invariant under the left action of $SE(3)$ on the second factor. We have not specified the potential energy as its choice does not affect the existence of the invariant measure. The shape space is just the first factor of $\mathbb{R} \times SE(3)$ and is one dimensional, and thus the above theory is applicable. To show the existence of the invariant measure, we note the following:

1. The constrained Lagrangian does not contain terms that simultaneously depend on $\dot{r}$ and p_a. The constraint is $\Omega^6 = 0$. Therefore, all the coefficients of the nonholonomic connection as well as its curvature form vanish. This implies that the terms $\mathcal{D}_{a\alpha\beta}$ and $\mathcal{K}_{\alpha\beta\gamma}$ vanish too. The differential form from condition (ii) of theorem 0.9 is therefore trivial.

2. The moving frame is r-independent. Therefore all of the coefficients $\gamma_{A\alpha}^{B}$ are trivial. Condition (i) of theorem 0.9 is satisfied since the group $SE(3)$ is unimodular and e_6 is the eigenvector of the inertia tensor.

3. The shape metric is r-independent.

The system's dynamics preserves the volume in the reduced phase space.

This can be verified by a straightforward computation of the divergence of the vector field that defines the equations of motion:

$$\begin{aligned}
\ddot{r} &= -\frac{\partial U_a}{\partial r}, \\
\dot{p}_1 &= -\Omega^2 p_3 + \Omega^3 p_2 - m\Omega^5 \dot{r}, \\
\dot{p}_2 &= -\Omega^3 p_1 + \Omega^1 p_3 + m\Omega^4 \dot{r}, \\
\dot{p}_3 &= -\Omega^1 p_2 + \Omega^2 p_1 - \Omega^4 p_5 + \Omega^5 p_4, \\
\dot{p}_4 &= \Omega^3 p_5 - m\Omega^2 \dot{r}, \\
\dot{p}_5 &= -\Omega^3 p_4 + m\Omega^1 \dot{r}.
\end{aligned}$$

Chaplygin Sphere. This system consists of a sphere rolling without slipping on a horizontal plane. The center of mass of this sphere is at the geometric center, but the principal moments of inertia are distinct. Chaplygin [1903] proved integrability of this problem. Modern references for the Chaplygin sphere include Kozlov [1985] and Schneider [2002].

One may view this system as a nonholonomic version of the Euler top. The configuration space is diffeomorphic to $SO(3) \times \mathbb{R}^2$. We choose the Euler angles (θ, ϕ, ψ) and the Cartesian coordinates (x, y) as the configuration parameters of the Chaplygin sphere. The Lagrangian and constraints written in these coordinates become

$$\begin{aligned} L = \frac{I_1}{2}(\dot\theta \cos\psi + \dot\phi \sin\psi \sin\theta)^2 + \frac{I_2}{2}(-\dot\theta \sin\psi + \dot\phi \cos\psi \sin\theta)^2 \\ + \frac{I_3}{2}(\dot\psi + \dot\phi \cos\theta)^2 + \frac{M}{2}(\dot x^2 + \dot y^2) \end{aligned}$$

and

$$\dot x - \dot\theta \sin\phi + \dot\psi \cos\phi \sin\theta = 0, \qquad \dot y + \dot\theta \cos\phi + \dot\psi \sin\phi \sin\theta = 0,$$

respectively.

This system is $SE(2)$ invariant. The action by the group element (α, a, b) on the configuration space is given by

$$(\theta, \psi, \phi, x, y) \mapsto (\theta, \psi, \phi + \alpha, x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b).$$

The shape space for the Chaplygin sphere is diffeomorphic to the two-dimensional sphere. The nonholonomic momentum map has just one component and is moreover preserved. Straightforward computations show that the form (0.58) is exact. The conditions for measure existence are therefore satisfied. The density of the invariant measure is computed in *overdetermined coordinates* in Chaplygin [1903] (see also Kozlov [1985]).

The invariant manifolds of the Chaplygin sphere are two-dimensional tori. The phase flow on these tori is measure preserving and thus there are angle variables (x, y) on each torus in which the flow equations become

$$\dot{x} = \frac{\lambda}{\mathcal{M}(x,y)}, \qquad \dot{y} = \frac{\mu}{\mathcal{M}(x,y)}.$$

See Kolmogorov [1953] and Kozlov [1985] for details. In general, these equations cannot be rewritten as

$$\dot{x} = \lambda, \qquad \dot{y} = \mu.$$

The flow however becomes quasi-periodic after a time substitution $dt = \mathcal{M}(x,y)d\tau$ (see Kozlov [1985] for details). This example thus shows that the flow on the nonholonomic invariant tori can be more complicated than a Hamiltonian flow.

It follows that adding a symmetry preserving potential to the Lagrangian of the Chaplygin sphere leaves the new system measure preserving with the same measure density. This was pointed out by Kozlov for a specific potential (see Kozlov [1985] for details).

•**The General Suslov Rigid Body Problem** We discuss this problem just briefly here. For more details see Federov and Koslov [1995]. A different non-asymptotic form is analyzed in Zenkov and Bloch [1999].

The equations of motion are those of an n-dimensional rigid body with skew-symmetric angular velocity matrix Ω with entries Ω_{ij} and symmetric moment of inertia matrix $I = I_{ij}$.

One then introduces the constraints $\Omega_{ij} = 0, i, j \geq 2$.

$$\begin{aligned}
(I_{11} + I_{22})\,\dot{\Omega}_{12} &= I_{12}\left(\Omega_{13}^2 + \Omega_{14}^2 + \cdots + \Omega_{1n}^2\right) \\
&\quad - \left(I_{13}\Omega_{13} + I_{14}\Omega_{14} + \cdots + I_{1n}\Omega_{1n}\right)\Omega_{12} \\
(I_{11} + I_{33})\,\dot{\Omega}_{13} &= I_{13}\left(\Omega_{12}^2 + \Omega_{14}^2 + \cdots + \Omega_{1n}^2\right) \\
&\quad - \left(I_{12}\Omega_{12} + I_{14}\Omega_{14} + \cdots + I_{1n}\Omega_{1n}\right)\Omega_{13} \\
\cdots\cdots &\quad \cdots\cdots \\
(I_{11} + I_{nn})\,\dot{\Omega}_{1n} &= I_{1n}\left(\Omega_{12}^2 + \Omega_{13}^2 + \cdots + \Omega_{1n-1}^2\right) \\
&\quad - \left(I_{12}\Omega_{12} + I_{13}\Omega_{13} + \cdots + I_{1n-1}\Omega_{1n-1}\right)\Omega_{1n}\,.
\end{aligned}$$

This system has the energy integral

$$H = \frac{1}{2}\left((I_{11} + I_{22})\Omega_{12}^2 + (I_{11} + I_{33})\Omega_{13}^2 + \cdots + (I_{11} + I_{nn})\Omega_{nn}^2\right)\,.$$

.

Defining the momenta $M_{1j} = (I_{11} + I_{jj})\Omega_{1j}$ by the Legendre transform, we can write the system as one of almost Poisson form $\dot{M} = J(M)\nabla H(M)$.

This system exhibits asymptotic behavior as indicated by the fact that the function

$$F = (I_{11} + I_{22})I_{12}\Omega_{12} + (I_{11} + I_{33})I_{13}\Omega_{13} + \cdots + (I_{11} + I_{nn})I_{1n}\Omega_{1n}$$

satisfies

$$\dot{F} = \sum_{i<j}^{n} (I_{1i}\Omega_{1j} - I_{1j}\Omega_1)^2$$

along the flow and is positive everywhere except at points of the line $\{\Omega_{12} = I_{12}\mu, \cdots, \Omega_{1n} = I_{1n}\mu\}$, $\mu \in \mathbb{R}$.

Thus motion occurs on the energy ellipsoid (a generalization of the Toda/Chaplygin ellipse) and asymptotes to a point on the line intersecting the ellipsoid.

- **The Lyapunov-Malkin Theorem**

Can be used to show asymptotic stability in a large class of nonholonomic systems, for example the roller racer and rattleback top. See Zenkov, Bloch and Marsden [1998] and Bloch. Baillieul, Crouch and Marsden [2002] for further details.

Take:

$$
\begin{aligned}
\dot{x} &= Ax + X(x, y), \\
\dot{y} &= Y(x, y).
\end{aligned}
$$

Theorem 0.10 *Consider this system of equations. If $X(0, y) = 0$, $Y(0, y) = 0$, and all the eigenvalues of the matrix A have negative real parts, then the system has n local integrals in the neighborhood of $x = 0$, $y = 0$.*

Theorem 0.11 (Lyapunov-Malkin) *Consider the above system of differential equations where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, A is an $m \times m$-matrix, and $X(x, y)$, $Y(x, y)$ represent nonlinear terms. If all eigenvalues of the matrix A have negative real parts, and $X(x, y)$, $Y(x, y)$ vanish when $x = 0$, then the solution $x = 0$, $y = c$ of the system is stable with respect to x, y, and asymptotically stable with respect to x. If a solution $x(t)$, $y(t)$ is close enough to the solution $x = 0$, $y = c$, then*

$$
\lim_{t \to \infty} x(t) = 0, \qquad \lim_{t \to \infty} y(t) = c.
$$

•Example

$$\dot{x} = -x + xy$$
$$\dot{y} = xy$$

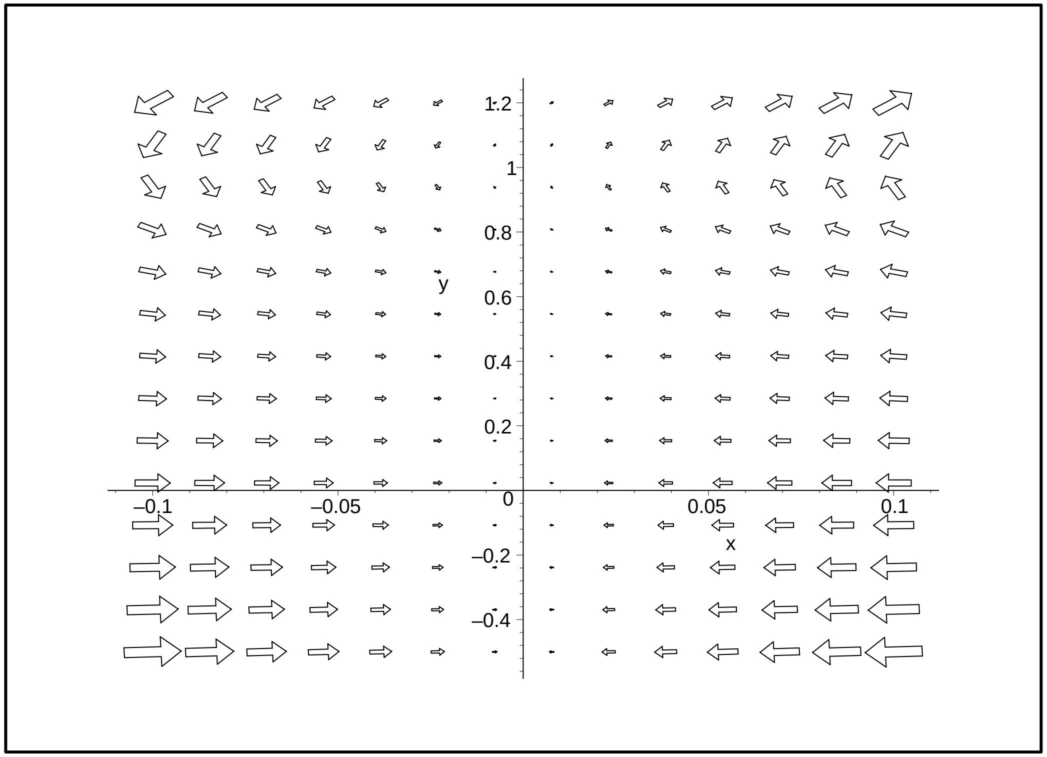

Figure 0.8: Phase Portrait for Lyapunov Malkin example.

As can be seen from the phase portrait and simulation below, exhibits asymptotic stability for small values of x and y.

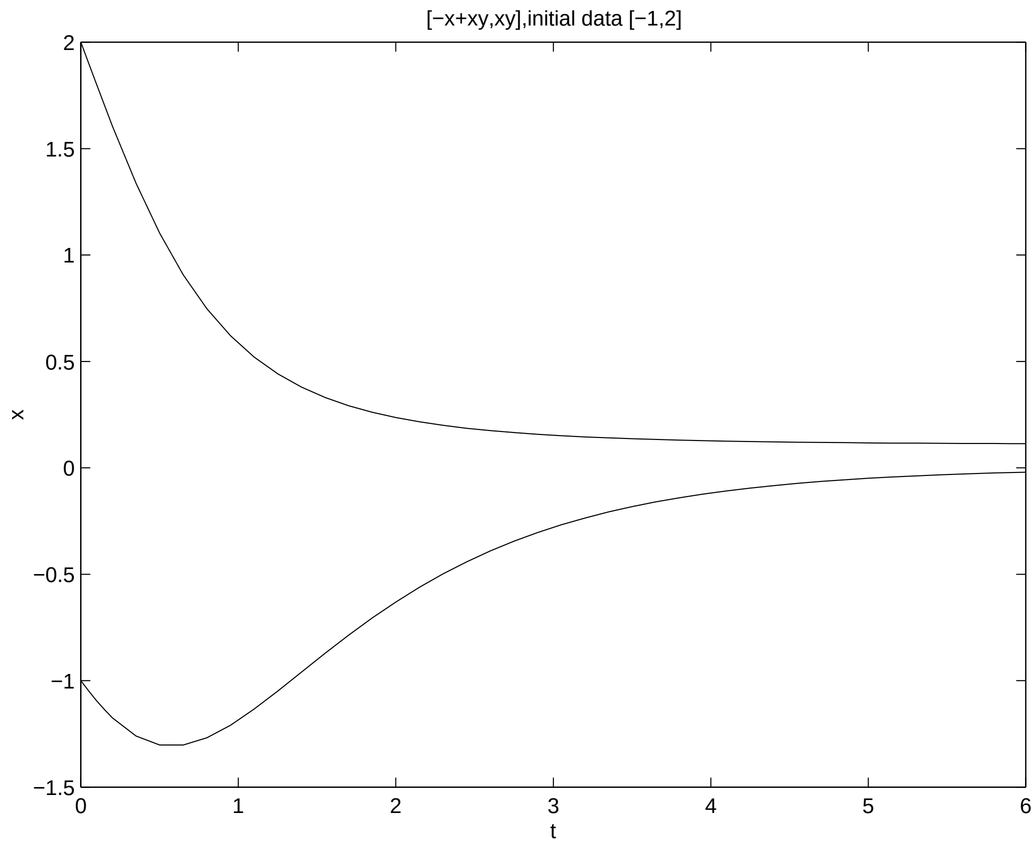

Figure 0.9: Flow for Lyapunov-Malkin example.

The Constrained Routhian. This function is defined by analogy with the usual Routhian by

$$R(r^\alpha, \dot r^\alpha, p_a) = l_c(r^\alpha, \dot r^\alpha, I^{ab}p_b) - I^{ab}p_a p_b,$$

and in terms of it, the reduced equations of motion become

$$\begin{aligned}
\frac{d}{dt}\frac{\partial R}{\partial \dot r^\alpha} - \frac{\partial R}{\partial r^\alpha} &= -\mathcal{D}^c_{b\alpha} I^{bd} p_c p_d - \mathcal{B}^c_{\alpha\beta} p_c \dot r^\beta \\
&\quad - \mathcal{D}_{\beta\alpha b} I^{bc} p_c \dot r^\beta - \mathcal{K}_{\alpha\beta\gamma} \dot r^\beta \dot r^\gamma, \\
\frac{d}{dt} p_b &= C^c_{ab} I^{ad} p_c p_d + \mathcal{D}^c_{b\alpha} p_c \dot r^\alpha + \mathcal{D}_{\alpha\beta b} \dot r^\alpha \dot r^\beta.
\end{aligned}$$

we define the function E by

$$E = \frac{1}{2} g_{\alpha\beta} \dot r^\alpha \dot r^\beta + U(r^\alpha, p_a),$$

which represents the reduced constrained energy in the coordinates r^α, $\dot r^\alpha$, p_a, where $U(r^\alpha, p_a)$ is the ***amended potential*** defined by

$$U(r^\alpha, p_a) = \frac{1}{2} I^{ab} p_a p_b + V(r^\alpha),$$

and $V(r^\alpha)$ is the potential energy of the system.

Can show: that the reduced constrained energy is conserved along the flow.

Stability of Nonholonomic Systems

Skew Symmetry Assumption.

We assume that the tensor $C^c_{ab} I^{ad}$ is skew-symmetric in c, d. Under this assumption, the terms quadratic in p in the momentum equation vanish, and the equations of motion become

$$\begin{aligned} \frac{d}{dt}\frac{\partial R}{\partial \dot{r}^\alpha} - \frac{\partial R}{\partial r^\alpha} &= -\mathcal{D}^c_{b\alpha} I^{bd} p_c p_d - \mathcal{B}^c_{\alpha\beta} p_c \dot{r}^\beta \\ &\quad - \mathcal{D}_{\beta\alpha b} I^{bc} p_c \dot{r}^\beta - \mathcal{K}_{\alpha\beta\gamma} \dot{r}^\beta \dot{r}^\gamma, \\ \frac{d}{dt} p_b &= \mathcal{D}^c_{b\alpha} p_c \dot{r}^\alpha + \mathcal{D}_{\alpha\beta b} \dot{r}^\alpha \dot{r}^\beta. \end{aligned}$$

Convenient to consider 3 cases:

Pure Transport Case Terms quadratic in $\dot{r}$ are not present in the momentum equation, so it is in the form of a transport equation—i.e., the momentum equation is an equation of parallel transport and the equation itself defines the relevant connection.

Under certain integrability conditions transport equation defines invariant surfaces, which allow us to use a type of energy-momentum method for stability analysis in a similar fashion to the manner in which the holonomic case uses the level surfaces defined by the momentum map. Key difference: here additional invariant surfaces do not arise from conservation of momentum. One gets stable, but not asymptotically stable, relative equilibria. Examples include: rolling disk, a body of revolution rolling on a horizontal plane.

Integrable Transport Case Terms quadratic in $\dot{r}$ are present in the momentum equation and thus it is not a pure transport equation. However, in this case, we assume that the transport part is integrable. In this case relative equilibria may be asymptotically stable. Can find a generalization of the energy-momentum method which gives conditions for asymptotic stability. An example is the roller racer.

Nonintegrable Transport Case Again, terms quadratic in $\dot{r}$ are present in the momentum equation and thus it is not a pure transport equation. However, the transport part is not integrable. Able to demonstrate asymptotic stability using the Lyapunov-Malkin Theorem and to relate it to an energy-momentum type analysis under certain eigenvalue hypotheses. Example: is the rattleback top. Another example is a nonhomogeneous sphere with a center of mass lying off the planes spanned by the principal axis body frame.

• **The Pure Transport Case.** Here we assume that

H1 $\mathcal{D}_{\alpha\beta b}$ are skew-symmetric in α, β. Under this assumption, the momentum equation can be written as the vanishing of the connection one form defined by $dp_b - \mathcal{D}^c_{b\alpha} p_c dr^\alpha$.

H2 The curvature of the preceding connection form is zero.

A nontrivial example of this case is that of Routh's problem of a sphere rolling in a surface of revolution. See Zenkov [1995].

Under the above two assumptions, the distribution defined by the momentum equation is integrable, and so we get invariant surfaces, which makes further reduction possible. Under the assumptions H1 and H2 made so far, the equations of motion become

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{r}^\alpha} - \frac{\partial R}{\partial r^\alpha} = -\mathcal{D}^c_{b\alpha} I^{bd} p_c p_d - \mathcal{B}^c_{\alpha\beta} p_c \dot{r}^\beta - \mathcal{K}_{\alpha\beta\gamma} \dot{r}^\beta \dot{r}^\gamma,$$
$$\frac{d}{dt} p_b = \mathcal{D}^c_{b\alpha} p_c \dot{r}^\alpha.$$

A ***relative equilibrium*** is a point $(r, \dot{r}, p) = (r_0, 0, p_0)$ which is a fixed point for the dynamics determined by these equations. Under assumption H1 the point (r_0, p_0) is seen to be a critical point of the amended potential.

Because of our zero curvature assumption H2, the solutions of the momentum equation lie on surfaces of the form $p_a = P_a(r^\alpha, k_b)$, $a, b = 1, \ldots, m$, where k_b are constants labeling these surfaces.

Using the functions $p_a = P_a(r^\alpha, k_b)$ we introduce the ***reduced amended potential*** $U_k(r^\alpha) = U(r^\alpha, P_a(r^\alpha, k_b))$. We think of the function $U_k(r^\alpha)$ as being the restriction of the function U to the invariant manifold

$$Q_k = \{(r^\alpha, p_a) \mid p_a = P_a(r^\alpha, k_b)\}.$$

Theorem 0.4 *Let assumptions H1 and H2 hold and let (r_0, p_0), where $p_0 = P(r_0, k^0)$, be a relative equilibrium. If the reduced amended potential $U_{k^0}(r)$ has a nondegenerate minimum at r_0, then this equilibrium is Lyapunov stable.*

Theorem 0.5 (Nonholonomic energy-momentum) *Under assumptions H1 and H2, the point $q_e = (r_0^\alpha, p_a^0)$ is a relative equilibrium if and only if there is a $\xi \in \mathfrak{g}^{q_e}$ such that q_e is a critical point of the* ***augmented energy*** *$E_\xi : \mathcal{D}/G \to \mathbb{R}$ (i.e., E_ξ is a function of $(r, \dot{r}, p)$), defined by*

$$E_\xi = E - \langle p - P(r, k), \xi \rangle.$$

This equilibrium is stable if $\delta^2 E_\xi$ restricted to $T_{q_e} Q_k$ is positive definite (here δ denotes differentiation with respect to all variables except ξ).

- Example. Falling disc.

Momentum equations:

$$\frac{dp_1}{dt} = mR^2 \cos\theta \left(-\frac{\sin\theta}{A\cos^2\theta} p_1 + \left(\frac{1}{mR^2 + B} + \frac{\sin^2\theta}{A\cos^2\theta} \right) p_2 \right) \dot{\theta},$$
$$\frac{dp_2}{dt} = mR^2 \cos\theta \left(-\frac{1}{A\cos^2\theta} p_1 + \frac{\sin\theta}{A\cos^2\theta} p_2 \right) \dot{\theta}.$$

The right hand sides of these eqns do not have terms quadratic in the shape variable θ. The distribution, defined by by the equations is integrable and defines two integrals of the form

$p_1 = P_1(\theta, k_1, k_2)$, $p_2 = P_2(\theta, k_1, k_2)$. It is known that these integrals may be written down explicitly in terms of the hypergeometric function. (Goes back to Appel [1900], Chaplgin [1897] Korteweg [1899].

Matching and Controlled Lagrangians

Lagrangian Matching. Consider a mechanical system specified by the Lagrangian $L = K - V$. The kinetic energy K is given by the Riemannian metric g_{ij} on the configuration manifold Q. The potential energy $V(q)$ has a critical point at q_0. Assuming that the equilibrium q_0 is unstable, we would like to find the feedback control inputs that stabilize this equilibrium. This problem becomes interesting and nontrivial if the system is *underactuated, i.e.,* the number of the control inputs is smaller than $\dim Q$.

Denote the unactuated and actuated variables by $x = (x^1, \ldots, x^m)$ and $y = (y^1, \ldots, y^n)$, respectively. The controlled dynamics is governed by the equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y} + u,$$

where $u = (u_1, \ldots, u_n)$ represents the control inputs.

According to the method of controlled Lagrangians, one introduces a new function $\widetilde{L} = \widetilde{K} - \widetilde{V}$ and considers the system

$$\frac{d}{dt}\frac{\partial \widetilde{L}}{\partial \dot{x}} = \frac{\partial \widetilde{L}}{\partial x}, \qquad \frac{d}{dt}\frac{\partial \widetilde{L}}{\partial \dot{y}} = \frac{\partial \widetilde{L}}{\partial y}.$$

One then requires that the vector fields defined by the two sets of equations are identical. This determines the feedback control inputs u. If in addition $\widetilde{K} + \widetilde{V}$ has a minimum (maximum) at $(q_0, 0)$, the equilibrium q_0 of the closed loop system is neutrally stable.

Nonholonomic Reduced Equations. Assume also: The curvature of the *nonholonomic connection* zero, the controls affect some of the shape variables,t he *momentum equation* is in the form of a parallel transport equation.

The *Routhian* of the system equals

$$\mathcal{R}(r, \dot{r}, p) = \frac{1}{2} g_{\alpha\beta}(r) \dot{r}^{\alpha} \dot{r}^{\beta} - U(r, p),$$

where the first term represents the *shape metric* and the second term, called the *amended potential*, is defined by

$$U(r, p) = \frac{1}{2} I^{ab}(r) p_a p_b + V(r).$$

The reduced equations of a system satisfying the assumptions 1–3 become

$$\begin{aligned}
\frac{d}{dt} \frac{\partial \mathcal{R}}{\partial \dot{r}^{\alpha'}} &= \nabla_{\alpha'} \mathcal{R}, \\
\frac{d}{dt} \frac{\partial \mathcal{R}}{\partial \dot{r}^{\alpha''}} &= \nabla_{\alpha''} \mathcal{R} + u_{\alpha''}, \\
\dot{p}_a &= \mathcal{D}^b_{a\alpha} p_b \dot{r}^{\alpha}.
\end{aligned}$$

In the above, $r^{\alpha'}$ and $r^{\alpha''}$ are the unactuated and actuated shape variables, respectively, and $u_{\alpha''}$ are the control inputs. The operators ∇_α are defined by

$$\nabla_\alpha = \frac{\partial}{\partial r^\alpha} + \mathcal{D}^b_{a\alpha} p_b \frac{\partial}{\partial p_a}.$$

The equilibria of these equations represent the steady state motions of the original system.

Require also as a part of the controller design, that the actuated variables $r^{\alpha''}$ are cyclic.

Elimination of the Momentum Variables. Since the momentum equation is in the form of a parallel transport equation, it defines a distribution

$$dp_a = \mathcal{D}^c_{a\alpha'} p_c dr^{\alpha'}.$$

We assume in this paper that the curvature of this distribution vanishes (hence the name *flat* in the title of the paper). This defines the global invariant manifolds Q_c:

$$p_a = \mathcal{P}_a(r^{\alpha'}, c_b), \qquad c_b = \text{const.} \tag{0.70}$$

Stabilization of the Unicycle with Rotor Dynamical model of a homogeneous disk on a horizontal plane with a rotor. The rotor is free to rotate in the plane orthogonal to the disk. The rod connecting the centers of the disk and rotor keeps the direction of the radius of the disk through the contact point with the plane (*i.e.*, the appropriate controller has already been implemented).

The configuration space for this system is $Q = S^1 \times S^1 \times S^1 \times SE(2)$, which we parameterize with coordinates $(\theta, \chi, \psi, \phi, x, y)$.

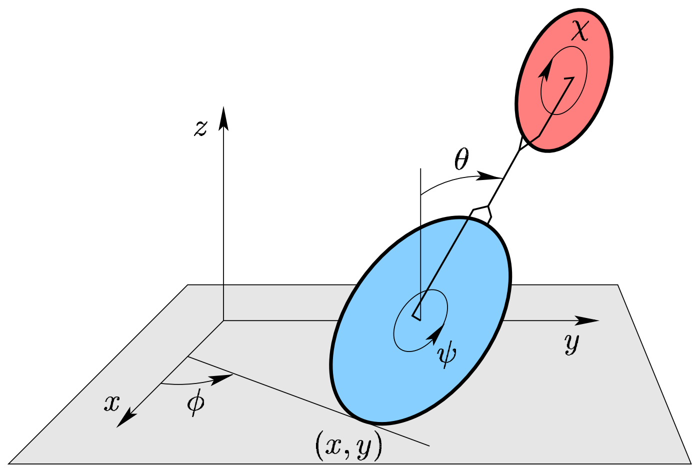

Figure 0.10: Unicycle with rotor.

The reduced Lagrangian for the unicycle with rotor is

$$L_c = \frac{1}{2}\big(\alpha\dot\theta^2 + 2\beta\dot\theta\dot\chi + \beta\dot\chi^2 \\ + I_{11}(\theta)\dot\phi^2 + 2I_{12}\dot\phi\dot\psi + I_{22}\dot\psi^2\big) - V(\theta),$$

The slow vertical steady state motions of this system are represented by the relative equilibria

$$\theta = 0, \qquad \dot\chi = 0, \qquad p_1 = 0, \qquad p_2 = p_2^0.$$

Momentum Reduction and Stabilization. The momentum equations define an integrable distribution. The dynamics on the invariant manifolds Q_c is governed by the equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}_c}{\partial \dot\theta} = \frac{\partial \mathcal{L}_c}{\partial \theta}, \qquad \frac{d}{dt}\frac{\partial \mathcal{L}_c}{\partial \dot\chi} = u_c,$$

where

$$\mathcal{L}_c = \frac{1}{2}(\alpha\dot\theta^2 + 2\beta\dot\theta\dot\chi + \beta\dot\chi^2) - U_c(\theta),$$

and

$$U_c(\theta) = \frac{1}{2}I^{ab}(\theta)\mathcal{P}_a(\theta, c), \mathcal{P}_b(\theta, c) + V(\theta)$$

is the amended potential for the unicycle with rotor restricted to the invariant manifolds. Observe that the components of the shape metric for the unicycle with rotor are constants. We thus construct the controlled Lagrangians of the form

$$
\begin{aligned}
\widetilde{\mathcal{L}}_c = \frac{1}{2}(\alpha\dot{\theta}^2 + 2\beta\dot{\theta}(\dot{\chi} + k\dot{\theta}) + \beta(\dot{\chi} + k\dot{\theta})^2) \\
+ \frac{\sigma}{2}(k\dot{\theta})^2 - U_c(\theta),
\end{aligned}
$$

The steady state motions under consideration become stable if one chooses

$$
k > \frac{\alpha - \beta}{\beta^2}.
$$

• **Nonoholonomic Control Systems on Riemannian Manifolds**

First, consider the holonomic or unconstrained case:

Let $(Q, \langle , \rangle)$ be an n-dimensional Riemannian manifold, with metric $g(\quad , \quad) = \langle \quad , \quad \rangle$. Denote the norm of a tangent vector X at the point p by $\|X_p\| = \langle X_p, X_p \rangle^{\frac{1}{2}}$. The geodesic flow on Q is then given by

$$\frac{D^2 q}{dt^2} = 0, \tag{0.72}$$

where $\frac{Dq}{dt}$ denotes the covariant derivative. This flow minimizes the integral $\int_0^1 \left\| \frac{Dq}{dt} \right\|^2 dt$ along parametrized paths.

We define a **controlled holonomic system** to be a system of the form

$$\frac{D^2 q}{dt^2} = \sum_{i=1}^{N} u_i X_i, \tag{0.73}$$

where $\{X_i\}$ is an arbitrary set of control vector fields, the u_i are functions of time, and $N \leq n$. (Note that here we do not consider systems evolving under the influence of a potential, but the analysis is easily extended to include a potential.) Such systems are sometimes now called **affine connection control systems**.

We now consider the formulation of controlled nonholonomic systems in this Riemannian setting.

Classical nonholonomic systems are obtained from Lagrange–d'Alembert's principle, as discussed before.

The equations are

$$\frac{D^2 q}{dt^2} = \sum_{k=1}^{m} \lambda_i W_i, \tag{0.74}$$

subject to

$$\omega_k \left(\frac{Dq}{dt} \right) = \left\langle W_k, \frac{Dq}{dt} \right\rangle = 0, \qquad 1 \leq k \leq m,$$

where $\omega_k(X) = \langle W_k, X \rangle$ and the λ_i are Lagrange multipliers. The constraints are given by the 1-forms ω_k, $1 \leq k \leq m$, which define a (smooth) distribution H on Q.

We now define a **controlled nonholonomic mechanical system** to be a system of the form

$$\frac{D^2 q}{dt^2} = \sum_{i=1}^{m} \lambda_i W_i + \sum_{i=1}^{N} u_i X_i \tag{0.75}$$

subject to

$$\left\langle W_k, \frac{Dq}{dt} \right\rangle = 0 \qquad 1 \leq k \leq m\,, \tag{0.76}$$

where the $u_i(t)$ are controls and the X_i are arbitrary smooth (control) vector fields.

• **Variational Nonholonomic Problems**

Variational nonholonomic problems, on the other hand, are equivalent to the classical Lagrange problem of minimizing a functional over a class of curves with fixed extreme points and satisfying a given set of equalities.

More precisely, we have the following : Let Q be a smooth manifold and TQ its tangent bundle with coordinates $(q^i, \dot{q}^i)$. Let $L : TQ \to \mathbb{R}$ be a given smooth Lagrangian and let $\Phi : TQ \to \mathbb{R}^{n-m}$ be a given smooth function.

Definition 0.16 *The* **Lagrange problem** *is given by*

$$\min_{q(\cdot)} \int_0^T L(q, \dot{q}) dt \tag{0.77}$$

subject to the fixed endpoint conditions $q(0) = 0$, $q(T) = q_T$, *and subject to the constraints*

$$\Phi(q, \dot{q}) = 0.$$

The falling cat problem is an abstraction of the problem of how a falling cat should optimally (in some sense) move its body parts so that it achieves a 180° reorientation during its fall.

In this case we begin with a Riemannian manifold Q (the configuration space of the problem) with a free and proper isometric action of a Lie group G on Q (the group SO(3) for the falling cat). Let $\mathcal{A}$ denote the mechanical connection; that is, it is the principal connection whose horizontal space is the metric orthogonal to the group orbits. The quotient space $Q/G = X$, the shape space, inherits a Riemannian metric from that on Q. Given a curve $c(t)$ in Q, we shall denote the corresponding curve in the shape space X by $r(t)$.

The problem under consideration is as follows:

Falling Cat problem: *Fixing two points $q_1, q_2 \in Q$, among all curves $q(t) \in Q$, $0 \leq t \leq 1$, such that $q(0) = q_0, q(1) = q_1$, and $\dot{q}(t) \in \mathrm{hor}_{q(t)}$ (horizontal with respect to the mechanical connection $\mathcal{A}$), find the curve or curves $q(t)$ such that the energy of the shape space curve, namely,*

$$\frac{1}{2}\int_0^1 \|\dot{r}\|^2 dt,$$

is minimized.

Local Solution. We can proceed to solve the Lagrange problem locally by forming the modified Lagrangian

$$\Lambda(q, \dot{q}, \lambda) = L(q, \dot{q}) + \lambda \cdot \Phi(q, \dot{q}), \tag{0.78}$$

with $\lambda \in \mathbb{R}^{n-m}$. The Euler–Lagrange equations then take the form

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\Lambda(q, \dot{q}, \lambda) - \frac{\partial}{\partial q}\Lambda(q, \dot{q}, \lambda) = 0, \tag{0.79}$$

$$\Phi(q, \dot{q}) = 0. \tag{0.80}$$

The case we are particularly interested in is the case of of classical (linear in the velocity) nonholonomic constraints:

$$\omega_i(q, \dot{q}) = \sum_{k=1}^{n} a_{ik}(q)\dot{q}^k = 0, \qquad i = 1, \ldots, n-m. \tag{0.81}$$

In the case that these constraints are integrable (equivalent to functions of q only) and L is physical, i.e., it is a holonomic mechanical system, this system will represent physical dynamics. In the nonholonomic case, these equations will not be physical; one needs the Lagrange–d'Alembert principle, as we have seen in Chapters 1, 3, and 5. The following theorem gives the differential equations for the Lagrange problem.

Theorem 0.6 *A solution of the Lagrange problem* Definition 0.77 *with constraints of the* *form (0.81) satisfies the following equations:*

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}L - \frac{\partial}{\partial q_i}L + \sum_{j=1}^{n-m}\left(\frac{d}{dt}\lambda_j\right)a_{ji} + \sum_{j=1}^{n-m}\lambda_j\left(\dot{a}_{ji} - \sum_{k=1}^{n}\frac{\partial a_{jk}}{\partial q_i}\dot{q}_k\right) = 0 \tag{0.82}$$

with the constraints

$$\sum_{k=1}^{n} a_{ik}\dot{q}^k = 0. \tag{0.83}$$

Contrast these equations of motion with the nonholonomic equations of motion with Lagrange multipliers obtained in Chapters 1 and 5 from the Lagrange–d'Alembert principle:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}L - \frac{\partial}{\partial q_i}L = \sum_{j=1}^{n-m}\lambda_j a_{ji}\,. \tag{0.84}$$

Observe that if we (formally) set $\lambda_j = 0$ and $\dot{\lambda}_j = \lambda_j$ in the variational nonholonomic equations, we recover the nonholonomic equations of motion. It is precisely the omission of the $\dot{\lambda}_j$ term that destroys the variational nature of the nonholonomic equations.

A General Formulation of Optimal Control Problems. We state a typical optimal control problem,

$$\min_{u(\cdot)} \int_0^T g(x, u)dt, \tag{0.85}$$

subject to the following conditions:

(i) a differential equation constraint $\dot{x} = f(x, u)$, and a state space constraint $x \in M$, and a constraint on the controls $u \in \Omega \subset \mathbb{R}^k$;

(ii) the endpoint conditions: $x(0) = x_0$ and $x(T) = x_T$,

where f and $g \geq 0$ are smooth, Ω is a closed subset of $\mathbb{R}^k$, and M is a smooth manifold of dimension n that is the state space of the system. The integrand g is sometimes referred to as the **cost function**.

The Pontryagin Maximum Principle

To state necessary conditions dictated by the Pontryagin maximum principle, we introduce a parametrized Hamiltonian function on T^*M:

$$\hat{H}(x, p, u) = \langle p, f(x, u) \rangle - p_0 g(x, u), \tag{0.86}$$

where $p_0 \geq 0$ is a fixed positive constant, and $p \in T^*M$. Note that p_0 is the multiplier of the cost function and that $\hat{H}$ is linear in p.

We denote by $t \mapsto u^*(t)$ a curve that satisfies the following relationship along a trajectory $t \mapsto (x(t), p(t))$ in T^*M:

$$H(x(t), p(t), u^*(t)) = \max_{u \in \Omega} \hat{H}(x(t), p(t), u). \tag{0.87}$$

Then if u^* is defined by equation (0.87), we can define H^* by

$$H^*(x(t), p(t), t) = \max_{u \in \Omega} \hat{H}(x(t), p(t), u). \tag{0.88}$$

The time-varying Hamiltonian function H^* defines a time-varying Hamiltonian vector field X_{H^*} on T^*M with respect to the canonical symplectic structure on T^*M.

One statement of Pontryagin's maximum principle gives necessary conditions for extremals of the problem (0.85) as follows: An **extremal trajectory** $t \mapsto x(t)$ of the problem (0.85) is the projection onto M of a trajectory of the flow of the vector field X_{H^*} that satisfies the boundary condition (0.85) (ii), and for which $t \mapsto (p(t), p_0)$ is not identically zero on $[0, T]$.

The extremal is called **normal** when $p_0 \neq 0$ (in which case we may set $p_0 = 1$ by normalizing the Hamiltonian function). When $p_0 = 0$ we call the extremal **abnormal**, corresponding to the case where the extremal is determined by constraints alone.

If the extremal control function u^* is not determined by the system (0.87) along the extremal trajectory, then the extremal is said to be ***singular***, in which case further (higher-order) necessary conditions are needed to determine u^*.

Consider here nonsingular case.

We also suppose that the data are sufficiently regular that u^* is determined uniquely from the condition

$$0 \equiv \frac{\partial \hat{H}}{\partial u}(x(t), p(t), u^*(t)), \quad t \in [0, T]. \tag{0.89}$$

(Since u^* maximizes the function $\hat{H}$, its partial derivative in u evaluated at u^* must vanish.)

It follows from the implicit function theorem that there exists a function k such that $u^*(t) = k(x(t), p(t))$. We then set

$$H(x, p) \triangleq \hat{H}(x, p, k(x, p)). \tag{0.90}$$

Thus along an extremal,

$$H(x(t), p(t)) = H^*(x(t), p(t), t). \tag{0.91}$$

We briefly motivate our statement of the Pontryagin maximum principle in the presence of regularity conditions alluded to above: In particular, we assume that $\Omega = \mathbb{R}^m$ and that $u^*(t)$ is uniquely determined by the condition (0.89). Treating the optimal control problem (0.85) as a variational problem with constraints, we augment the cost function and constraints (in the form of the constraining state differential equation) by multipliers $p_0 \in \mathbb{R}^+$ and $p \in T^*_M$. We obtain necessary conditions in the form

$$\delta \int_0^T \left(p\left(f(x,u) - \dot{x}\right) - p_0 g(x,u)\right) dt = 0, \tag{0.92}$$

where the variations are taken over pairs (x, u) satisfying the constraints $\dot{x} = f(x, u)$ and the boundary conditions $x(0) = x_0$, $x(T) = x_T$.

We may restate the condition (0.92) as

$$\delta \int_0^T \left(\hat{H}(x,p,u) - p\dot{x}\right) dt = 0. \tag{0.93}$$

Under the assumed regularity we may eliminate the variation with respect to u, and from (0.91) the necessary condition becomes

$$\delta \int_0^T \left(\hat{H}(x,p) - p\dot{x}\right) dt = 0. \tag{0.94}$$

This is, of course, just Hamilton's principle for the Hamiltonian H, which yields necessary conditions in terms of the usual Hamiltonian equations. Now from (0.89) and (0.91) the Hamiltonian equations for H may be replaced by the Hamiltonian equations for H^*, resulting in the statement of the maximum principle above. Note that whereas $\hat{H}$ and H^* are affine in p, H is in general *not* affine in p.

The main point of the Pontryagin maximum principle is that the result stated above is true under far less severe regularity conditions and in particular where Ω is a proper subset of $\mathbb{R}^n$.

Kinematic Sub-Riemannian Optimal Control Problems

We consider control systems of the form

$$\dot{x} = \sum_{i=1}^{m} X_i u_i, \quad x \in M, \quad u \in \Omega \subset \mathbb{R}^m, \tag{0.95}$$

where Ω contains an open subset that contains the origin, M is a smooth manifold of dimension n, and each of the vector fields in the collection $F := \{X_1, \ldots, X_k\}$ is complete.

We assume that the system satisfies the accessibility rank condition and is thus controllable, since there is no drift term. Then we can pose the optimal control problem

$$\min_{u(\cdot)} \int_0^T \frac{1}{2} \sum_{i=1}^{m} u_i^2(t) dt \tag{0.96}$$

subject to the dynamics (0.95) and the endpoint conditions $x(0) = x_0$ and $x(T) = x_T$.

To view this as a constrained variational problem we make some additional regularity assumptions. These are not necessary, but even when they hold, they produce a very rich class of problems.

Assumption.

(i) The system defined by (0.95) satisfies the accessibility rank condition.

(ii) The dimension of D_F is constant on M and equal to k. (Thus the vector fields $X_1, \ldots, X_k$ are everywhere independent.)

(iii) There exist exactly $n-k=m$ one-forms on M $\omega_1,\ldots,\omega_m$ such that the codistribution

$$D_F^{\perp}(x)=\{\bar{\omega}\in T_x^*M;\quad \bar{\omega}D_F(x)=0\}$$

is spanned by $\omega_1,\ldots,\omega_m$ everywhere. (This condition implies that M is parallelizable.)

Since D_F has constant dimension on M, we may define a norm on each subspace $D_F(x)$; if $X\in D_F(x)$ and $X=\sum_{i=1}^k \alpha_i X_i(x)$, then we define

$$|X|:=\sum_{i=1}^{k}\alpha_i^2.$$

This norm defines an inner product on $D_F(x)$, denoted by $\langle\cdot,\cdot\rangle_x$, which can be extended to a metric on M. The optimal control problem (0.96) is now equivalent to the following constrained variational problem when the assumptions (i), (ii), (iii) hold:

$$\min_{x(\cdot)}\frac{1}{2}\int_0^T\langle\dot{x},\dot{x}\rangle_x dt \tag{0.97}$$

subject to the condition that $x(\cdot)$ is a piecewise C^1 curve in M such that $x(0)=x_0$, $x(T)=x_T$, and $\omega_i(x)(\dot{x})=0$, $1\leq i\leq m$. This problem is often referred to as the **sub-Riemannian geodesic problem**, to distinguish it from the Riemannian geodesic problem, in which the constraints are absent.

The singular nature of the sub-Riemannian geodesic problem is manifested in many ways, such as the existence of distinct abnormal extremals and the singular nature of the sub-Riemannian geodesic ball, as first investigated by Brockett. If we define a metric on M by setting

$$\begin{aligned} d(x_0, x_T) &= \min_{x(\cdot)} \int_0^T |\dot{x}| dt, \dot{x} \in D_F(x), x(0) = x_0, x(T) = x_T, \\ B_\varepsilon^F(x_0) &= \{\bar{x} \in M; d(\bar{x}, x_0) \leq \varepsilon\}, \end{aligned}$$

then the sub-Riemannian geodesic ball $S_\varepsilon^F(x_0)$ is simply the boundary of $B_\varepsilon^F(x_0)$.

Formulation on Riemannian Manifolds

Let M be a Riemannian manifold of dimension n with metric denoted by $\langle \cdot, \cdot \rangle$. The corresponding Riemannian connection and covariant derivative will be denoted by ∇ and $D/\partial t$, respectively. Now assume that M is such that there exist smooth vector fields $X^1(q), \ldots, X^n(q)$ satisfying $\langle X^i(q), X^j(q) \rangle = \delta_{ij}$, an orthonormal frame for T_qM for all $q \in M$.

We now define the **kinematic control system** on M by

$$\frac{dq}{dt} = \sum_{i=1}^{m} u_i X^i(q), \quad m < n. \tag{0.98}$$

The optimal control problem is defined by

$$\min_u \int_0^T \frac{1}{2} \sum_{i=1}^{m} u_i^2(t) dt; \quad q(0) = q_0, \quad q(T) = q_T, \tag{0.99}$$

subject to (0.98).

This may be posed as a variational problem on M as follows: Define the constraints

$$\omega_k \left(\frac{dq}{dt} \right) = \left\langle X^k, \frac{dq}{dt} \right\rangle = 0, \quad m < k \leq n, \tag{0.100}$$

and let

$$Z_t = \sum_{k=m+1}^{n} \lambda_k(t) X^k, \tag{0.101}$$

where the λ_k are Lagrange multipliers. By the orthonormality of the X^i the optimal control problem then becomes

$$\min_q J(q) = \min_q \int_0^T \left(\frac{1}{2} \left\langle \frac{dq}{dt}, \frac{dq}{dt} \right\rangle + \left\langle Z_t, \frac{dq}{dt} \right\rangle \right) dt, \tag{0.102}$$

$$\left\langle Z_t, \frac{dq}{dt} \right\rangle = 0. \tag{0.103}$$

We now briefly derive necessary conditions for the regular extremals of this variational problem.

Firstly, we have to define the variations we are going to use: The tangent space to the space Ω of C^2 curves satisfying the boundary conditions of (0.99) is denoted by $T_q\Omega$. It is the space of C^1 vector fields $t \longrightarrow W_t$ along $q(t)$ satisfying $W_0 = 0 = W_T$. The curve $t \longrightarrow \frac{DWt}{\partial t}$ in TM is continuous. Exponentiating a vector field in $T_q\Omega$ we obtain a one-parameter variation of q:

$$\alpha : [0, T] \times (-\varepsilon, \varepsilon) \longrightarrow M, \tag{0.104}$$

$$\alpha_u(t) = \alpha(t, u) = \exp_{q(t)}(uW_t), \tag{0.105}$$

where exp is the exponential mapping (integral curve) on M. Note that

$$\alpha_u(0) = q(0) = q_0, \quad \alpha_u(T) = q(T) = q_T, \quad \alpha_0(t) = q(t),$$
$$\frac{\partial \alpha_0(t)}{\partial u} = W_t, \qquad 0 \leq t \leq T.$$

The necessary conditions for regular extremals are obtained from

$$\frac{d}{du} J(\alpha_u)|_{u=0} = 0, \tag{0.106}$$

where

$$J(\alpha_u) = \int_0^T \left(\frac{1}{2} \left\langle \frac{\partial \alpha_u}{\partial t}, \frac{\partial \alpha_u}{\partial t} \right\rangle + \left\langle Z_t(\alpha_u), \frac{\partial \alpha_u}{\partial t} \right\rangle \right) dt\,. \tag{0.107}$$

Now

$$\left.\frac{DJ(\alpha_u)}{du}\right|_{u=0} = \int_0^T \left(\left\langle \frac{dq}{dt}, \frac{DW_t}{\partial t} \right\rangle + \left\langle \nabla_{W_t} Z_t, \frac{dq}{dt} \right\rangle + \left\langle Z_t, \frac{D}{\partial t} W_t \right\rangle \right) dt$$
$$= \int_0^T \Bigg(- \left\langle \frac{D}{dt} V_t, W_t \right\rangle - \left\langle \frac{D}{\partial t} Z_t, W_t \right\rangle$$
$$- \langle \nabla_{Z_t} V_t, W_t \rangle + \langle [W_t, Z_t], V_t \rangle \Bigg) dt, \tag{0.108}$$

where

$$V_t = \frac{dq}{dt} = \sum_{i=1}^{m} v_i(t) X^i(q).$$

Necessary Conditions on a Compact Semisimple Lie Group.

Now let $M = G$, G a compact semisimple Lie group, with Lie algebra $\mathfrak{g}$, and let $\langle\langle\cdot,\cdot\rangle\rangle = -\frac{1}{2}\kappa(\cdot,\cdot)$, where κ is the Killing form on $\mathfrak{g}$.

Let J be a positive definite linear mapping $J : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$\langle\langle JX, Y\rangle\rangle = \langle\langle X, JY\rangle\rangle, \tag{0.109}$$

$$\langle\langle JX, X\rangle\rangle \geq 0 \quad (= 0 \text{ if and only if } X = 0). \tag{0.110}$$

Now we can define a right-invariant metric on G as follows: If $X, Y \in \mathfrak{g}$ and R_g is right translation on G by $g \in G$, then

$$X^r_g = X^r(g) = R_{g^*}X \quad \text{and} \quad Y^r_g = Y^r(g) = R_{g^*}Y$$

are corresponding right-invariant vector fields. Now

$$\langle X^r(g), Y^r(g)\rangle = \langle\langle X, JY\rangle\rangle \tag{0.111}$$

defines a right-invariant metric on G. Corresponding to the right-invariant metric $\langle\cdot,\cdot\rangle$ there is a unique Riemannian connection ∇ , and ∇ defines a bilinear form on $\mathfrak{g}$:

$$(X, Y) \to \nabla_X Y = \frac{1}{2}\{[X, Y] + J^{-1}[X, JY] + J^{-1}[Y, JX]\}, \quad X, Y \in \mathfrak{g}. \tag{0.112}$$

The expression for ∇ on right-invariant vector fields on G is

$$(\nabla_{X^r} Y^r)(g) = (\nabla_X Y)^r_g. \tag{0.113}$$

We now show how to reduce the variational problem to one in the Lie algebra: Choose an orthonormal basis e_i on $\mathfrak{g}$, $\langle\langle e_i, Je_j\rangle\rangle = \delta_{ij}$, and extend it to a right-invariant orthonormal frame on T_gG, $X^i(g) = R_{g^*}e_i \equiv X^{ir}(g)$.

Find the necessary conditions on $\mathfrak{g}$ are

$$\dot{V}_t + J^{-1}[V_t, JZ_t] + \dot{Z}_t + J^{-1}[V_t, JV_t] = 0 \tag{0.114}$$

with the constraint

$$\left\langle \frac{dg}{dt}, Z_t \right\rangle = \langle\langle V_t, JZ_t \rangle\rangle = 0\,. \tag{0.115}$$

The Case of Symmetric Space Structure

Suppose now that G/K is a **Riemannian symmetric space**, G as above, K a closed subgroup of G with Lie algebra $\mathfrak{k}$. Then $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ with $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$, $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, and $\langle\langle \mathfrak{k}, \mathfrak{p} \rangle\rangle = 0$. We now want to consider the necessary conditions (0.114) in this case. We shall see that they simplify in an intriguing fashion, giving us a singular case of the so-called generalized rigid body equations.

The generalized rigid body equations are a natural generalization of the classical 3-dimensional rigid body equations. We recall that the left-invariant generalized rigid body equations on $SO(n)$ may be written as

$$\begin{aligned} \dot{Q} &= Q\Omega, \\ \dot{M} &= [M, \Omega], \end{aligned} \tag{0.116}$$

where $Q \in \mathrm{SO}(n)$ denotes the configuration space variable (the attitude of the body), $\Omega = Q^{-1}\dot{Q} \in \mathfrak{so}(n)$ is the body angular velocity, and

$$M := J(\Omega) = \Lambda\Omega + \Omega\Lambda \in \mathfrak{so}(n)$$

is the body angular momentum. Here $J : \mathfrak{so}(n) \to \mathfrak{so}(n)$ is the symmetric (with respect to the inner product defined by the Killing form), positive definite, and hence invertible operator defined by

$$J(\Omega) = \Lambda\Omega + \Omega\Lambda,$$

where Λ is a diagonal matrix satisfying $\Lambda_i + \Lambda_j > 0$ for all $i \neq j$. For $n = 3$ the elements of Λ_i are related to the standard diagonal moment of inertia tensor I by $I_1 = \Lambda_2 + \Lambda_3$, $I_2 = \Lambda_3 + \Lambda_1$,

$I_3 = \Lambda_1 + \Lambda_2$.

Since $\dot{V}_t + J^{-1}[V_t, JZ_t] \in \mathfrak{p}$ and $\dot{Z}_t + J^{-1}[V_t, JV_t] \in \mathfrak{k}$, the necessary conditions (0.114) become

$$\begin{aligned} \dot{V}_t &= J^{-1}[JZ_t, V_t], \\ \dot{Z}_t &= J^{-1}[JV_t, V_t], \end{aligned} \tag{0.117}$$

or, if we define $P_t = JV_t$ and $Q_t = JZ_t$,

$$\begin{aligned} \dot{P}_t &= [Q_t, J^{-1}P_t], \\ \dot{Q}_t &= [P_t, J^{-1}P_t]. \end{aligned} \tag{0.118}$$

We will now show that equations (0.118) are Hamiltonian with respect to the Lie–Poisson structure on $\mathfrak{g}$.

Recall that for F, H functions on $\mathfrak{g}$, their $(-)$ Lie–Poisson bracket is given by

$$\{F, H\}(X) = -\langle\langle X, [\nabla F(X), \nabla H(X)]\rangle\rangle, \quad X \in \mathfrak{g}, \tag{0.119}$$

where $dF(X) \cdot Y = \langle\langle \nabla F(X), Y\rangle\rangle$.

For $H(X)$ a given Hamiltonian, we thus have the Lie–Poisson equations $\dot{F}(X) = \{F, H\}(X)$. Letting $F(X) = \langle\langle A, X\rangle\rangle$, $A \in \mathfrak{g}$, we obtain

$$\langle\langle A, \dot{X}\rangle\rangle = -\langle\langle X, [A, \nabla H(X)]\rangle\rangle = \langle\langle A, [X, \nabla H(X)]\rangle\rangle \tag{0.120}$$

and hence

$$\dot{X} = [X, \nabla H(X)]. \tag{0.121}$$

For $H(M) = \frac{1}{2}\langle\langle M, J^{-1}M\rangle\rangle$, $M \in \mathfrak{g}$, and J as in the previous subsection, we obtain the generalized rigid body equations

$$\dot{M} = [M, J^{-1}M]\,. \tag{0.122}$$

Now for $X = P + Q \in \mathfrak{p} \oplus \mathfrak{k}$, let $H(X) = H(P) = \frac{1}{2}\langle\langle P, J^{-1}P\rangle\rangle$, $P \in \mathfrak{p}$. Then $\nabla H(X) = J^{-1}P \in \mathfrak{p}$, and equations (0.121) become

$$(Q_t + P_t)^{\cdot} = [Q_t + P_t, J^{-1}P_t], \tag{0.123}$$

or

$$\begin{aligned} \dot{P}_t &= [Q_t, J^{-1}P_t], \\ \dot{Q}_t &= [P_t, J^{-1}P_t], \end{aligned} \tag{0.124}$$

which are precisely equations (0.118).

Thus equations (0.118) are Lie–Poisson with respect to the "singular" Hamiltonian $H(P)$. Summarizing then, we have the following result:

Theorem 0.7 *The optimal trajectories for the singular optimal control problem* (0.98), (0.99) *on a Riemannian symmetric space are given by equations* (0.118)*. These equations are Lie–Poisson with respect to a singular rigid body Hamiltonian on* $\mathfrak{g}$*.*

We see, therefore, that we can obtain the singular optimal trajectories by letting $J\mid_{\mathfrak{k}} \to \infty$ in the full rigid body Hamiltonian $H(X) = \frac{1}{2}\langle\langle X, J^{-1}X\rangle\rangle$, thus obtaining the singular Hamiltonian

$$H(P) = \frac{1}{2}\langle\langle P, J^{-1}P\rangle\rangle.$$

This observation also enables us to obtain the singular rigid body equations directly by a limiting process from the full rigid body equations. The key is the correct choice of angular velocity and momentum variables corresponding to the Lie algebra decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$.

In the notation of equation (0.118) we write an arbitrary element of $\mathfrak{g}$ as $M = JV + Q$, $JV \in \mathfrak{p}$, $Q \in \mathfrak{k}$. Then the generalized rigid body equations (0.122) become

$$\begin{aligned} J\dot{V}_t &= [Q_t, V_t] + [JV_t, J^{-1}Q_t], \\ \dot{Q}_t &= [Q_t, J^{-1}Q_t] + [JV_t, V_t] \,. \end{aligned} \tag{0.125}$$

Letting $J \mid_{\mathfrak{k}} \longrightarrow \infty$ we obtain

$$\begin{aligned} J\dot{V}_t &= [Q_t, V_t], \\ \dot{Q}_t &= [JV_t, V_t]. \end{aligned} \tag{0.126}$$

We note that this is a mixture between the Lagrangian and Hamiltonian pictures. While the variables in $\mathfrak{k}$ are momenta (and should really be viewed as lying in $\mathfrak{k}^*$), the variables in $\mathfrak{p}$ are velocities.

The necessary conditions above may also be derived directly from the maximum principle developed for Lie groups, yielding an invariant maximum principle. The Hamiltonian in the maximum principle of the system (0.118) is precisely $\frac{1}{2}\langle P_t, J^{-1}P_t\rangle$. This is just the sum of the Hamiltonians corresponding to each of the vector fields X_i.

Write $M \in \mathfrak{g}$ as $M = JZ + P$, $Z \in \mathfrak{k}$, $P \in \mathfrak{p}$, which we can do, since $J : \mathfrak{p} \longrightarrow \mathfrak{p}$ and

$J : \mathfrak{k} \rightarrow \mathfrak{k}$. Then the Hamiltonian (in the maximum principle) becomes

$$\begin{aligned} H(M) = \langle M, J^{-1}M \rangle &= \langle JZ + P, J^{-1}(JZ + P) \rangle \\ &= \langle JZ, Z \rangle + \langle P, J^{-1}P \rangle. \end{aligned} \tag{0.127}$$

Letting $J \mid_{\mathfrak{k}} \rightarrow \infty$ we see that the cost becomes infinite unless $Z = 0$, i.e., unless the constraints are satisfied.

Example We consider a simple but nontrivial example: the symmetric space $\mathrm{SO}(3)/\mathrm{SO}(2)$. In this case $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ becomes $\mathfrak{so}(3) = \mathfrak{so}(2) \oplus \mathbb{R}^2$ relative to a given choice of z-axis used to embed $\mathrm{SO}(2)$ into $\mathrm{SO}(3)$. We may thus represent matrices in $\mathfrak{so}(3)$ as

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{0.128}$$

with the lower 2×2 block in $\mathfrak{so}(2)$.

This example illustrates the importance of writing the optimal equations in the natural variables $M = JV + Q$ in order to understand the limiting process in equations (0.125) and (0.126).

We write here

$$M = \begin{bmatrix} 0 & -J_3\omega_3 & J_2\omega_2 \\ J_3\omega_3 & 0 & -m_1 \\ -J_2\omega_2 & m_1 & 0 \end{bmatrix}. \tag{0.129}$$

Here $Q_t \in \mathfrak{so}(2)$ has "momentum" variable m_1. Then the equations

$$(JV_t + Q_t)^{\cdot} = [JV_t + Q_t, V_t] \tag{0.130}$$

become, for $\mathfrak{g} = \mathfrak{so}(3)$,

$$\begin{aligned} \dot{m}_1 &= (J_2 - J_3)\omega_2\omega_3, \\ J_2\dot{\omega}_2 &= -m_1\omega_3, \\ J_3\dot{\omega}_3 &= m_1\omega_2. \end{aligned} \tag{0.131}$$

The full rigid body equations in these variables are

$$(JV_t + Q_t)^{\cdot} = [JV_t + Q_t, V_t + J^{-1}Q_t], \tag{0.132}$$

which for $\mathfrak{g} = \mathfrak{so}(3)$ are

$$\begin{aligned} \dot{m}_1 &= (J_2 - J_3)\omega_2\omega_3, \\ J_2\dot{\omega_2} &= \left(\frac{J_3}{J_1} - 1\right)\omega_3 m_1, \\ J_3\dot{\omega}_3 &= \left(1 - \frac{J_2}{J_1}\right) m_1\omega_2, \end{aligned} \tag{0.133}$$

which clearly approaches to (0.131) as $J_1 \to \infty$.

Note that if we write the rigid body equations in the usual form,

$$J(V_t + Z_t)^{\cdot} = \left[J(V_t + Z_t), V_t + Z_t\right], \tag{0.134}$$

we obtain, for $\mathfrak{g} = \mathfrak{so}(3)$,

$$\begin{aligned} J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3, \\ J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_1\omega_3, \\ J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_2\omega_1. \end{aligned} \tag{0.135}$$

In this formulation, where we do not distinguish between $\mathfrak{p}$ and $\mathfrak{k}$, the limiting process described above is not obvious. The same is true for the rigid body in the momentum representation.

We remark that this set of equations, despite its singular nature, is still integrable, for we still have two conserved quantities, the Hamiltonian $H(\omega) = J_2\omega_2^2 + J_3\omega_3^2 (= \frac{1}{2}\langle P, J^{-1}P\rangle)$ and the Casimir

$$C(\omega) = m_1^2 + J_2^2\omega_2^2 + J_3^2\omega_2^2.$$

(Recall that a Casimir function for a Poisson structure is a function that commutes with every other function under the Poisson bracket.)

It is interesting to consider the case $J = I$. Equations (0.118) then become

$$\begin{aligned} \dot{P}_t &= [Q_t, P_t], \\ \dot{Q}_t &= 0. \end{aligned} \tag{0.136}$$

Hence $Q_t = Q$ is constant.

Similarly, considering (0.117), we obtain

$$\dot{V}_t = [Z_t, V_t], \quad Z_t = Z, \tag{0.137}$$

Z a constant. This is, of course, solvable: $V_t = \mathrm{Ad}_{e^{Zt}} V_0$ and $u_i(t) = \langle\langle e_i, \mathrm{Ad}_{e^{Zt}} V_0\rangle\rangle$.

Consider again the case SO(3)/$SO(2)$. Since $V_t \in \mathbb{R}^2$ and $Z_t \in \mathfrak{so}(2)$, we may set

$$Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\phi \\ 0 & \phi & 0 \end{bmatrix}, \tag{0.138}$$

where ϕ is fixed. Then

$$e^{Zt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi t & -\sin\phi t \\ 0 & \sin\phi t & \cos\phi t \end{bmatrix}. \tag{0.139}$$

Hence the optimal evolution of V_t (or equivalently the optimal controls) is given by rotation.