An Equity-Interest Rate Hybrid Model with Stochastic Volatility and the Interest Rate Smile

Lech A. Grzelak & Cornelis W. Oosterlee

Hybrid Models in Hybrid Markets

May 10, 2011
Politecnico di Milano - Dipartimento di Matematica
Via Bonardi 9 - Milano
Empirical evidence for non-zero correlation

Figure: Equity prices vs 10y Treasury Yield. Source: “Shifting Correlations” in Seeking Alpha.
The Objectives of the Research

To build an Equity-Interest Rate Hybrid model which:

⇒ generates a smile on the equity side;
⇒ includes stochastic interest rate with interest rate smile;
⇒ enables non-zero correlations between the underlying processes;
⇒ allows efficient calibration;
First, the Heston-Hull-White Hybrid model:

\[
\frac{dS}{S} = \ rdt + \sqrt{\sigma}dW^Q_x,
\]
\[
d\sigma = \ k(\bar{\sigma} - \sigma)dt + \gamma \sqrt{\sigma}dW^Q_\sigma,
\]
\[
dr = \ \lambda(\theta - r)dt + \eta dW^Q_r,
\]

with correlations: \( \rho_{x,\sigma} \neq 0, \rho_{x,r} \neq 0 \) and \( \rho_{\sigma,r} \neq 0 \).

With the Feynman-Kac theorem, for \( x = \log S \) the corresponding PDE is given by:

\[
r \phi = \ \phi_t + (r - 1/2\sigma) \phi_x + k(\bar{\sigma} - \sigma) \phi_\sigma + \lambda(\theta_t - r) \phi_r \\
+1/2\sigma \phi_{xx} + 1/2 \gamma^2 \sigma \phi_{\sigma\sigma} + 1/2 \eta^2 \phi_{rr} \\
+ \rho_{x,\sigma} \gamma \sigma \phi_{x,\sigma} + \rho_{x,r} \eta \sqrt{\sigma} \phi_{x,r} + \rho_{\sigma,r} \eta \gamma \sqrt{\sigma} \phi_{\sigma,r}.
\]

In the present form the model is not affine [Duffie et al. 2000].
By linearization of the non-affine terms in the covariance matrix we find an approximation:

\[
\begin{pmatrix}
\sigma & \rho_{x,\sigma} \gamma \sigma & \rho_{x,\eta} \sqrt{\sigma} \\
\gamma^2 \sigma & \rho_{\sigma,\eta} \gamma \sqrt{\sigma} & \rho_{\sigma,\eta} \sqrt{\sigma} \\
\end{pmatrix}
\approx
\begin{pmatrix}
\sigma & \rho_{x,\sigma} \gamma \sigma & \rho_{x,\eta} \Psi \\
\gamma^2 \sigma & \rho_{\sigma,\eta} \gamma \sqrt{\sigma} & \rho_{\sigma,\eta} \gamma \sqrt{\sigma} \\
\end{pmatrix}
\]

We linearize the non-affine term \( \sqrt{\sigma} \) by \( \Psi \):

\[
\Psi = \mathbb{E}(\sqrt{\sigma}) \quad \text{or} \quad \Psi = \mathcal{N}(\mathbb{E}(\sqrt{\sigma}), \text{Var}(\sqrt{\sigma}))
\]

The expectation for the CIR-type process is known analytically:

Affine approximation \( \Rightarrow \) efficient pricing!

The model with the modified covariance structure, \( \mathbf{C} \), constitutes the affine version of non-affine model.
Quality of the Approximations

\[ \Rightarrow \text{We set: } \kappa = 0.5, \gamma = 0.1, \lambda = 1, \eta = 0.01, \theta = 0.04 \text{ and } \rho_{x,\sigma} = -50\%, \rho_{x,r} = 60\%. \]

Figure: Comparison of implied Black-Scholes volatilities from Monte Carlo (40,000 paths and 500 steps) and Fourier inversion.
⇒ The linearization method provides a high quality approximation;
⇒ The projection procedure can be simply extended to high dimensions;
⇒ The method is straightforward, and does not involve complex techniques;
⇒ Alternative methods for approximating the hybrid models are:
  ● Markovian projection based methods [Antonov-2008].
  ● Models with indirect correlation structure [Giese-2004, Andreasen-2006];
We now consider the Stochastic Volatility Libor Market Model [Andersen, Brotherton-Ratcliffe-2005], [Andersen, Andreasen-2000]. For $L_k := L(t, T_{k-1}, T_k)$ we define

$$L(t, T_{k-1}, T_k) \equiv \frac{1}{\tau_k} \left( \frac{P(t, T_{k-1})}{P(t, T_k)} - 1 \right), \text{ for } t < T_{k-1},$$

with the dynamics under their natural measure given by:

$$\begin{cases}
    dL_k = \sigma_k (\beta_k L_k + (1 - \beta_k) L_k(0)) \sqrt{V} dW^k_k, \\
    dV = \lambda (V(0) - V) dt + \eta \sqrt{V} dW^k_V,
\end{cases}$$

with $dW^k_i dW^k_j = \rho_{i,j} dt$, for $i \neq j$ and $dW^k_V dW^k_i = 0$.

Efficient calibration with Markovian Projection Method [Piterbarg-2005].
Fast pricing of European-style equity options:

\[
\Pi(t) = B(t) \mathbb{E}^Q \left( \frac{(S(T_N) - K)^+}{B(T_N)} \bigg| \mathcal{F}(t) \right), \text{ with } t < T_N,
\]

with \(K\) the strike, \(S(T_N)\) the stock price at time \(T_N\), filtration \(\mathcal{F}(t)\) and a numéraire \(B(T_N)\).

The money-savings account \(B(T_N)\) is assumed to be correlated with stock \(S(T_N)\).

We switch between the measures: From risk neutral \(Q\) to the \(T_N\)-forward \(Q^{T_N}\):

\[
\Pi(t) = P(t, T_N) \mathbb{E}^{T_N} \left( (F^{T_N}(T_N) - K)^+ \bigg| \mathcal{F}(t) \right), \text{ with } t < T_N,
\]

with \(F^{T_N}(t)\) the forward of the stock \(S(t)\), defined as:

\[
F^{T_N}(t) = \frac{S(t)}{P(t, T_N)}.
\]

The ZCB \(P(t, T_N)\) is not well-defined for all \(t\)!
⇒ Since $P(T_{k-1}, T_{k-1}) = 1$ we find for the ZCB $P(t, T_k)$:

$$P(t, T_k) = (1 + \tau_k L(t, T_{k-1}, T_k))^{-1}.$$  

⇒ For $t \neq T_{k-1}$ we use the interpolation from [Schlögl-2002]:

$$P(t, T_k) \approx (1 + (T_k - t)L(t, T_{k-1}, T_k))^{-1}, \text{ for } T_{k-1} \leq t \leq T_k.$$  

⇒ This ZCB interpolation is sufficient for calibration purposes but for pricing callable exotics more attention is needed [Piterbarg-2004, Davis et al.-2009, Beveridge & Joshi-2009].
Under the $T_N$-forward measure we have:

$\Rightarrow$ An equity part is driven by the Heston model:

$$
\frac{dS}{S} = (\ldots)dt + \sqrt{\xi}dW^N_x,
$$

$$
d\xi = \kappa(\bar{\xi} - \xi)dt + \gamma\sqrt{\xi}dW^N_{\xi}.
$$

$\Rightarrow$ The SV Libor Market Model under the $T_N$-measure is given by:

$$
dL_k = -\phi_k\sigma_kV \sum_{j=k+1}^{N} \frac{\tau_j\phi_j\sigma_j}{1 + \tau_jL_j} \rho_{k,j}dt + \sigma_k\phi_k\sqrt{V}dW^N_k,
$$

$$
dV = \lambda(V(0) - V)dt + \eta\sqrt{V}dW^N_V,
$$

with $\phi_k = \beta_kL_k + (1 - \beta_k)L_j(0)$.

$\Rightarrow$ We assume non-zero correlation between asset $S(t)$ and Libor rates $L_j(t)$.
⇒ The forward $F^{T_N}$ is a martingale under the $T_N$-forward measure:

$$
\frac{dF^{T_N}(t)}{P(t, T_N)} = \frac{1}{dS(t)} - \frac{S(t)}{P^2(t, T_N)}dP(t, T_N).
$$

⇒ Dynamics for $S(t)$ are known (the Heston model), for ZCB $P(t, T_N)$ we find:

$$
\frac{1}{P(t, T_N)} = (1 + (T_m(t) - t)L_m(t)(T_{m(t)-1})) \prod_{j=m(t)+1}^{N} (1 + \tau_j L(t, T_{j-1}, T_j)).
$$

interpolation

rolling

with $m(t) = \min\{k : t \leq T_k\}$.
⇒ For the ZCB $P(t, T_N)$ we are only interested in diffusion coefficients:

$$\frac{dP(t, T_N)}{P(t, T_N)} = (...)dt - \sqrt{V} \sum_{j=m(t)+1}^{N} \frac{\tau_j \sigma_j \phi_j}{1 + \tau_j L_j} dW_j^N.$$

⇒ The forward $F_{TN}^T(t)$ dynamics are now given by:

$$\frac{dF_{TN}^T}{F_{TN}^T} = \sqrt{\xi} dW_x^N_{\text{asset}} + \sqrt{V} \sum_{j=m(t)+1}^{N} \frac{\tau_j \sigma_j \phi_j}{1 + \tau_j L_j} dW_j^N_{\text{interest rate}}.$$

⇒ The model is not affine!
We freeze the Libor rates [Glasserman, Zhao-1999], [Hull, White-1996], [Jäckel, Rebonato-2000], i.e.:

\[ L_j(t) \approx L_j(0) \implies \phi_j(t) \approx L_j(0). \]

Now, the linearized dynamics are given by:

\[
\frac{dF^T_N}{F^T_N} \approx \sqrt{\xi} dW^N_x + \sqrt{V} \sum_{j=m(t)+1}^{N} \frac{\tau_j \sigma_j L_j(0)}{1 + \tau_j L_j(0)} dW^N_j.
\]

The model does not depend on the Libor processes! It is fully described by the volatility structure.
⇒ The model is now given by:

\[
dF^T_N / F^T_N \approx \sqrt{\xi} dW^N_x + \sqrt{V} \Sigma^T dW^N,
\]

\[
d\xi = \kappa(\bar{\xi} - \xi) dt + \gamma \sqrt{\xi} dW^N_\xi,
\]

\[
dV = \lambda(V(0) - V) dt + \eta \sqrt{V} dW^N_V,
\]

with appropriate column vectors \( \Sigma \) and \( dW^N \).

⇒ Under the log-transform, \( x = \log F^T_N \), we find:

\[
dx \approx -\frac{1}{2} \left( \sqrt{\xi} dW^N_x + \sqrt{V} \Sigma^T dW^N \right)^2 + \sqrt{\xi} dW^N_x + \sqrt{V} \Sigma^T dW^N.
\]

⇒ Since \( dW^N_x \) is correlated with \( dW^N \) cross terms are still not affine!
We set: \( A = m(t) + 1, \ldots, N \) and \( \psi_j = \frac{\tau_j \sigma_j L_j(0)}{1+\tau_j L_j(0)} \).

The dynamics for \( x = \log F^{TN} \) are given by:

\[
dx \approx -\frac{1}{2} \left( \xi + A_1(t) V + 2\sqrt{V} \sqrt{\xi} A_2(t) \right) dt + \sqrt{\xi} dW^N_x + \sqrt{V} \Sigma^T dW^N.
\]

\( A_1(t) \) and \( A_2(t) \) are deterministic piecewise constant functions!

The drift and covariance matrix include the non-affine term \( \sqrt{V} \sqrt{\xi} \), we linearize it by:

\[
\sqrt{\xi} \sqrt{V} \approx \mathbb{E}(\sqrt{\xi} \sqrt{V})
\]

\[
\mathbb{E}(\sqrt{\xi}) \mathbb{E}(\sqrt{V}) =: \vartheta(t).
\]
⇒ With Feynman-Kac theorem we find the corresponding PDE:

\[
0 = \phi_t + 1/2 (\xi + A_1 V + 2A_2 \vartheta(t)) (\phi_{xx} - \phi_x) \\
+ \kappa(\bar{\xi} - \xi) \phi_x + \lambda(V(0) - V) \phi_V + 1/2 \eta^2 V \phi_V, V \\
+ 1/2 \gamma^2 \xi \phi_{\xi,\xi} + \rho_x, \xi \gamma \xi \phi_{x,\xi},
\]

subject to \( \phi(u, X(T), 0) = \exp(iux(T_N)) \).

⇒ The corresponding characteristic function is given by:

\[
\phi(u, X(t), \tau) = \exp(A(u, \tau) + iux(t) + B(u, \tau) \xi(t) + C(u, \tau) V(t)),
\]

with \( \tau = T_N - t \).

⇒ The ODEs for \( A(u, \tau), B(u, \tau), C(u, \tau) \) are of Heston-type and can be solved recursively [Andersen,Andreasen-2000].
We price an equity call option and investigate the accuracy of the approximation.

For equity we take:
\[ \kappa = 1.2, \quad \bar{\xi} = 0.1, \quad \gamma = 0.5, \quad S(0) = 1, \quad \xi(0) = 0.1. \]

For the interest rate model we take term structure:
\[ P(0, T) = \exp(-0.05T), \] with
\[ \beta_k = 0.5, \quad \sigma_k = 0.25, \quad \lambda = 1, \quad V(0) = 1, \quad \eta = 0.1. \]

The correlation structure is given by:

\[
\begin{pmatrix}
1 & \rho_{x,\xi} & \rho_{x,1} & \cdots & \rho_{x,N} \\
\rho_{x,\xi} & 1 & \rho_{x,1} & \cdots & \rho_{x,N} \\
\rho_{x,1} & \rho_{x,1} & 1 & \cdots & \rho_{x,N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{x,N} & \rho_{x,N} & \rho_{x,N} & \cdots & 1
\end{pmatrix}
= \begin{pmatrix}
1 & -30\% & 50\% & \cdots & 50\% \\
-30\% & 1 & 0 & \cdots & 0 \\
50\% & 0 & 1 & \cdots & 98\% \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
50\% & 0 & 98\% & \cdots & 1
\end{pmatrix}
\]
Figure: Comparison of implied Black-Scholes volatilities for the European equity option, obtained by Fourier inversion of approximation and by Monte Carlo simulation.
We investigate the effect of $\beta$ on equity implied vol. with Monte Carlo simulation of the full-scale model:

Figure: The effect of the interest rate skew, controlled by $\beta_k$, on the equity implied volatilities. The Monte Carlo simulation was performed with for maturity $T = 10$.

⇒ The prices of the European style options are rather insensitive to skew parameter $\beta$!
We consider an investor who is willing to take some risk in one asset class in order to obtain a participation in a different asset class.

An example of such hybrid product is *minimum of several assets* [Hunter-2005] with payoff defined as:

\[
\text{Payoff} = \max \left( 0, \min \left( C_n(T), k\% \times \frac{S(T)}{S(t)} \right) \right),
\]

where \( C_n(T) \) is an \( n \)-years CMS, and \( S(T) \) is a stock.

By taking \( T = \{1, 2, \ldots, 10\} \) and the payment date \( T_N = 5 \) we get:

\[
\frac{\Pi_H(t)}{P(t, T_5)} = \mathbb{E}^{T_5} \left[ \max \left( 0, \min \left( \frac{1 - P(T_5, T_{10})}{\sum_{k=6}^{10} P(T_5, T_k)}, k\% \times \frac{S(T_5)}{S(t)} \right) \right) \bigg| \mathcal{F}(t) \right].
\]
**Figure:** The value for a *minimum of several assets* hybrid product. The prices are obtained by Monte Carlo simulation with 20,000 paths and 20 intermediate points. Left: Influence of $\beta$; Right: Influence of $\rho_{x,L}$.
Now, we compare the results with Heston-Hull-White model
⇒ From calibration routine we have: \( \lambda = 0.0614, \eta = 0.0133, r_0 = 0.05 \) and \( \kappa = 0.65, \gamma = 0.469, \bar{\xi} = 0.090, \rho_{x,\xi} = -0.222 \) and \( \xi_0 = 0.114 \).
⇒ Calibration ensures that prices on the equities are the same, so the hybrid price differences can only result from the interest rate component!

Figure: LEFT: Hybrid prices obtained by two different hybrid models, H-LMM and HHW. The models were calibrated to the same data set., RIGHT: CMS rate for the SV LMM and the Hull-White models.
We have developed an efficient approximation method projecting non-affine models on affine versions;

The models with modified covariance structure are affine *by construction*;

We have presented an extension of the Heston model with stochastic interest rates:
- Short-rate processes;
- SV LMM;

The model can be easily generalized to FX and Inflation;
References