Generalized Microscopic Image Reconstruction Problems

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joint work with Amotz Bar-Noy, Zvi Lotker, David Peleg, and Dror Rawitz

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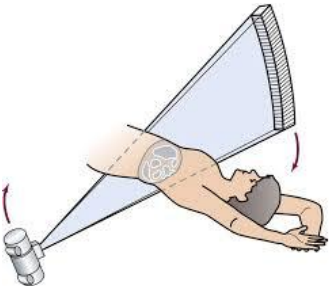
May 2-4, 2022, Politecnico di Milano

Medical Imaging

Medical Imaging

- measure the density of points in a space (2D or 3D)

Aggregate Measurements and Reconstruction



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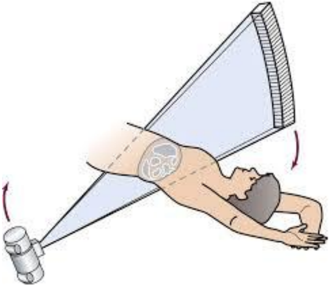

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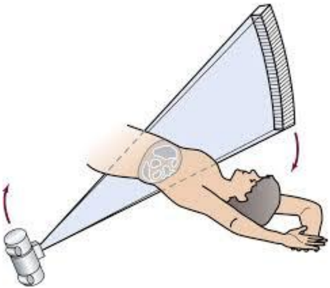

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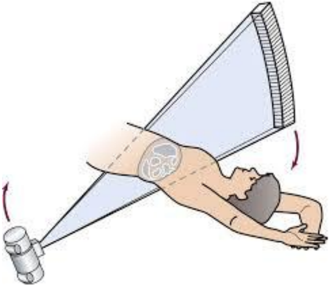

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- for a 2D-image, the points are arranged in a matrix

Discrete Tomography

- uncover the entries of a binary matrix

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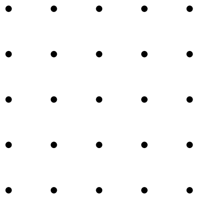

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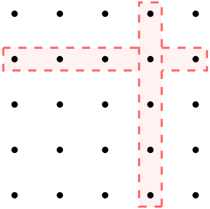

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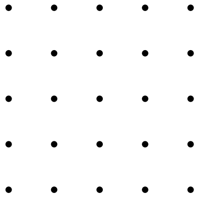

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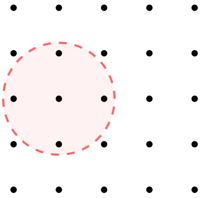

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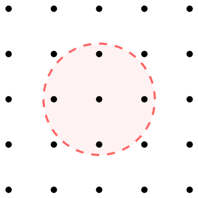

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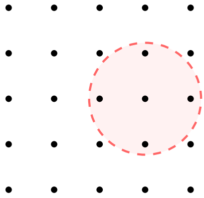

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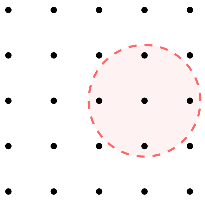

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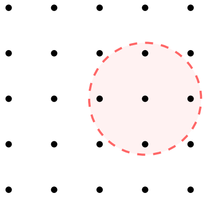

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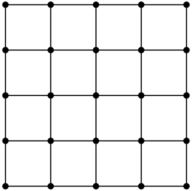

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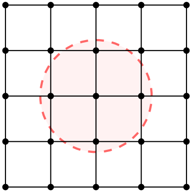

- regard the matrix as a grid graph
- scanning windows are centred at a vertex and capture its neighborhood

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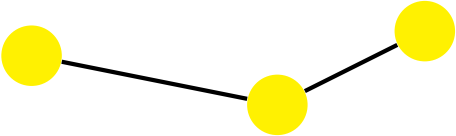

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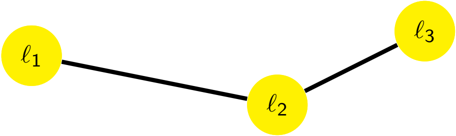

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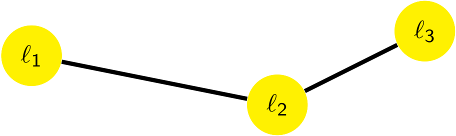

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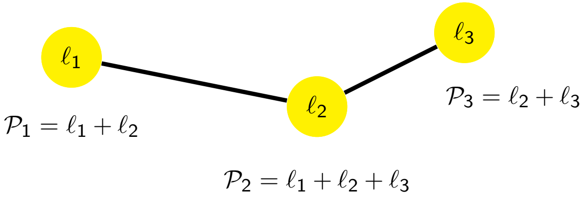

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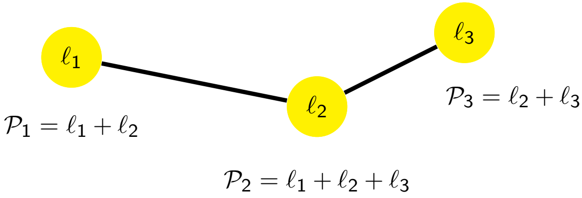

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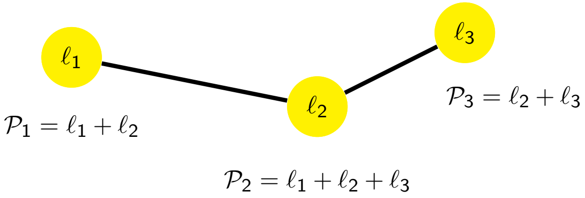

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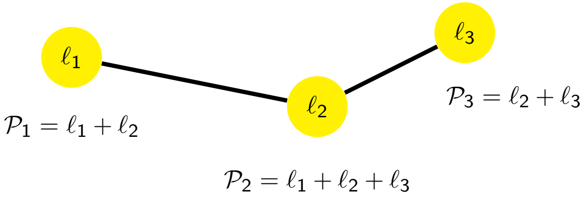

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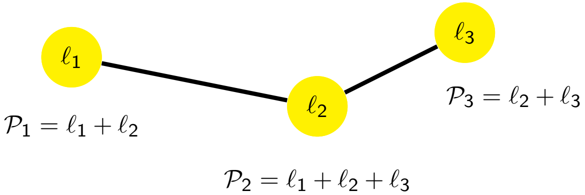

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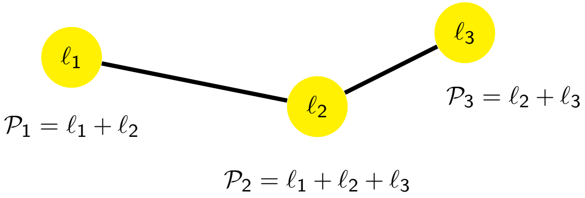

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Minimum Surgical Probing
Given graph G and probe vector $\vec{\mathcal{P}}$, uncover the labels $\vec{\ell}$ using a minimum number of surgical probes.

Talk Outline

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- otherwise, the system has degrees of freedom which we remove by surgical probes

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- exclusive variant with open neighborhoods using A_G

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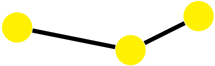

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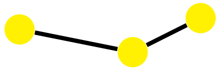

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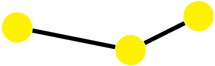

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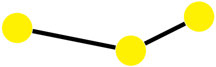

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Let graph G have (inclusion wise) maximal sets of co-duplicate vertices $D_1, \ldots D_k$. Solving Minimum Surgical Probing for G requires at least $\sum_{i=1}^{k}(|D_i| - 1)$ surgical probes.

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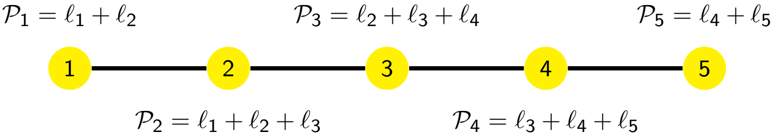

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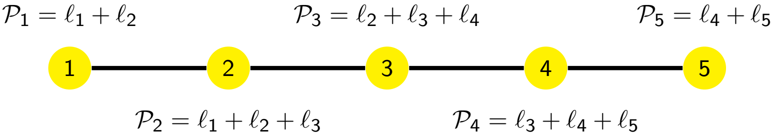

- linear-time algorithm to uncover $\vec{\ell}$ for a path graph

Graph Products

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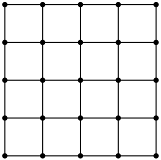

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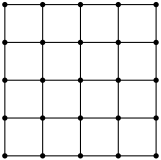

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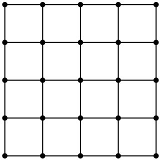

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- if $G = G_1 \times G_2$, then A_G has eigenvalue $\lambda + \mu$

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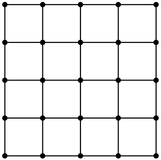

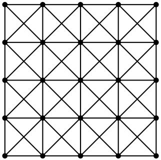

- λ and μ are eigenvalues of A_{G_1} and A_{G_2}, respectively
- if $G = G_1 \times G_2$, then A_G has eigenvalue $\lambda + \mu$

- Strong product of two path graphs is the King's graph

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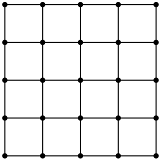

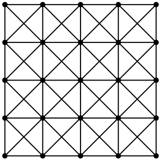

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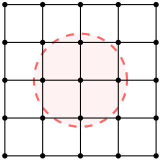

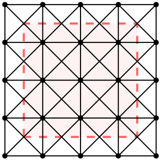

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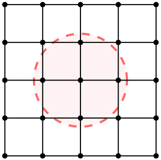

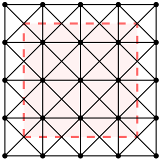

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- Non-Complete Extended p-sum (NEPS) operation

Graph Products

- for cycle graphs we have expressions for eigenvalues

Graph Products

- for cycle graphs we have expressions for eigenvalues
- $\mathcal{I}^a_b(n) = 1$ if and only if $n \bmod a \equiv b$

Graph	# surgical probes	at most
Grid	$\mathcal{I}^3_2(n_1)\mathcal{I}^2_1(n_2) + \mathcal{I}^2_1(n_1)\mathcal{I}^3_2(n_2) + 2\mathcal{I}^5_4(n_1)\mathcal{I}^5_4(n_2)$	4
Path	$\mathcal{I}^3_2(n)$	1
Tube	$2\mathcal{I}^2_1(n_1)\mathcal{I}^3_0(n_2) + \mathcal{I}^3_2(n_1)\mathcal{I}^2_0(n_2) + 2\mathcal{I}^3_2(n_1)\mathcal{I}^4_0(n_2) + 4\mathcal{I}^5_4(n_1)\mathcal{I}^5_0(n_2)$	9
Cycle	$2\mathcal{I}^3_0(n)$	2
Torus	$4\mathcal{I}^3_0(n_1)\mathcal{I}^4_0(n_2) + 4\mathcal{I}^4_0(n_1)\mathcal{I}^3_0(n_2) + 2\mathcal{I}^2_0(n_1)\mathcal{I}^6_0(n_2) + 2\mathcal{I}^6_0(n_1)\mathcal{I}^2_0(n_2) + 8\mathcal{I}^5_0(n_1)\mathcal{I}^5_0(n_2)$	20

Graph Products

- for cycle graphs we have expressions for eigenvalues
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Path	$\mathcal{I}_2^3(n)$	1
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- constant number of surgical probes (engineering)

Graph Products

Graph	# surgical probes	at most
King's graph	$\mathcal{I}_2^3(n_1)n_2 + \mathcal{I}_2^3(n_2)n_1 - \mathcal{I}_2^3(n_1)\mathcal{I}_2^3(n_2)$	$n_1 + n_2 - 1$
Rook graph	0	0
Hypercube	$\mathcal{I}_1^2(d)\binom{d}{(d-1)/2}$	$\binom{d}{(d-1)/2}$
Sudoku graph	$\Theta(n^2)$	$\Theta(n^2)$

Graph Products

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Graph Products

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- Rook graph, Cartesian product of two complete graphs
- Hypercube, *d times* Cartesian product of P_2
- Sudoku graph, NEPS of four complete graphs
- similar results for exclusive variant

Talk Outline

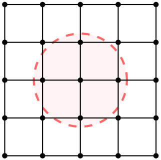
- MINIMUM SURGICAL PROBING
- Solving MINIMUM SURGICAL PROBING
- Spectral Graph Theory and Graph Products
- Weighted MINIMUM SURGICAL PROBING
- Bipartite Graphs and Trees

Weighted Probing

Weighted Probing

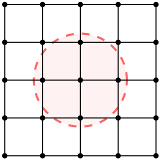
- measure the temperature of a surface

Weighted Probing



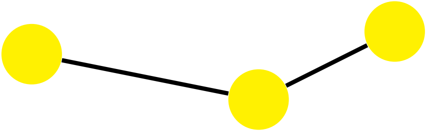
- measure the temperature of a surface
- cover the surface with a grid and measure each point

Weighted Probing



- measure the temperature of a surface
- cover the surface with a grid and measure each point
- centre point has a higher contribution than neighbors

Weighted Minimum Surgical Probing



- consider a simple graph $G = (V, E)$

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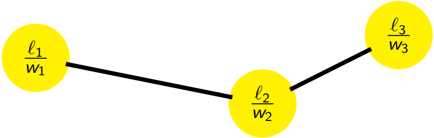

- consider a simple graph $G = (V, E)$
- each vertex $v \in V$ has a label $\ell_v \in \mathbb{R}$ and a weight $w_v \in \mathbb{R}$

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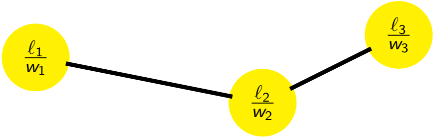

$$\mathcal{P}_2 = w_2 \cdot \ell_2 + \ell_1 + \ell_3$$

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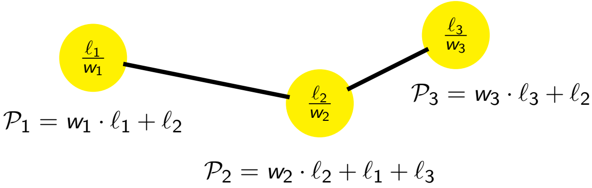

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- given graph G, weights $\vec{w}$ and probes $\vec{\mathcal{P}}$, uncover the labels $\vec{\ell}$

Weighted Minimum Surgical Probing

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WEIGHTED MINIMUM SURGICAL PROBING
Given graph G, weight vector $\vec{w}$, and probe vector $\vec{\mathcal{P}}$, uncover the labels $\vec{\ell}$ using a minimum number of surgical probes.

Weighted Minimum Surgical Probing

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Given graph G, weight vector $\vec{w}$, and probe vector $\vec{\mathcal{P}}$, uncover the labels $\vec{\ell}$ using a minimum number of surgical probes.

- $\bar{A}_G \triangleq A_G + \mathrm{diag}(\vec{w})$ adjacency matrix with $\vec{w}$ on the diagonal

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- number of surgical probes is equal to the nullity of $\bar{A}_G$

Weights and the Number of Surgical Probes

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Weights and the Number of Surgical Probes

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Boundary weight vector $\bar{w}_G$

- $\bar{w}_G$ at least one surgical probe is required
- $\bar{w}_G + \epsilon$ (or $\bar{w}_G - \epsilon$) no surgical probes are required

Boundary Weights and the Laplacian Matrix

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Let G be a connected graph. If $\vec{w} = -\vec{\delta}(G)$, then uncovering $\vec{\ell}$ requires one surgical probe.

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Then, matrix $L_G^- - \mathrm{diag}(\varepsilon)$ has full rank, i.e., uncovering $\vec{\ell}$ requires no surgical probes.

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Positive boundary weight

- bipartite graphs: $\vec{\delta}(G)$
- non-bipartite graphs: $\vec{\delta}(G) - \mu$ where μ is the smallest eigenvalue of G

Talk Outline

- Minimum Surgical Probing
- Solving Minimum Surgical Probing
- Spectral Graph Theory and Graph Products
- Weighted Minimum Surgical Probing
- Bipartite Graphs and Trees

Probing Bipartite Graphs and Trees

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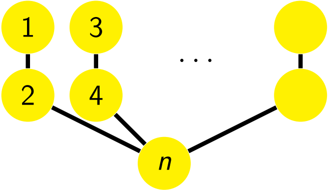
Probing Bipartite Graphs and Trees

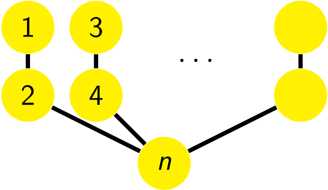
- for a connected, bipartite graph, $\vec{\ell}$ can be uncovered with $\left\lfloor \frac{n}{2} \right\rfloor - 1$ surgical probes
- there exist n-vertex trees, for odd n, that require $\left\lfloor \frac{n}{2} \right\rfloor - 1$ surgical probes



$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ \vdots & & & & & \\ 0 & 1 & 0 & 1 & \cdots & 1 \end{pmatrix}$$

Probing Bipartite Graphs and Trees

- for a connected, bipartite graph, $\vec{\ell}$ can be uncovered with $\lfloor \frac{n}{2} \rfloor - 1$ surgical probes
- there exist n-vertex trees, for odd n, that require $\lfloor \frac{n}{2} \rfloor - 1$ surgical probes



$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ \vdots & & & & & \\ 0 & 1 & 0 & 1 & \cdots & 1 \end{pmatrix}$$

- stars and perfect k-ary trees require no surgical probes

Talk Outline

- Minimum Surgical Probing
- Solving Minimum Surgical Probing
- Spectral Graph Theory and Graph Products
- Weighted Minimum Surgical Probing
- Bipartite Graphs and Trees

Open Questions

Open Questions

- probes with a distance d on a grid

Open Questions

- probes with a distance d on a grid
- surgical probes with costs

Open Questions

- probes with a distance d on a grid
- surgical probes with costs
- restrictions on the labels ℓ_v

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Thanks!