The null label problem and the complexity of the reconstruction of $I_2(H)$

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Preliminary definitions

- A *graph* is a pair of sets $G = (V, E)$ where V is the set of vertices and $E \subseteq V \times V$ is the set of edges.

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- a hypergraph is *even* if every vertex has even degree .

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Definition (Null hypergraph)

An assignment of ± 1 to the (hyper)edges of a (hyper)graph is a *null label* if $d(v) = 0$, for all vertices $v \in V$. A (hyper)graph with a null labelling is said to be a *null (hyper)graph*.

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Hypergraph Null Labelling Problem: let H be a connected, even $k-$hypergraph. When can ± 1 be assigned to the hyperedges of H to produce a null-labelled $k-$hypergraph?

We study the problem in the case of $3-$hypergraphs.

Null label on graphs

The null label problem is completely solved for $2-$hypergraphs (i.e. graphs), thanks to the following Proposition.

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A graph G has a null labelling if and only if every connected component is an Eulerian graph with an even number of edges.

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Moving to 3−hypegraphs the situation becomes more complex. One of the first results use the notion of *intersection graph*.

Definition

Let $H = (V, E)$ be a 3−hypergraph. The *intersection graph* of H, denoted as $I(H)$, is a graph in which the nodes are the hyperedges of H and two hyperedges are adjacent if their intersection is non-empty.

Previous results

The following result holds.

Theorem

Let H be a connected, even $3-$hypergraph, in which every vertex has degree two. Then H has a null labelling if and only if $I(H)$ is bipartite.

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However, $I(H)$ is not useful in general.

Consider the following $3-$hypergraphs H_1 and H_2 on six vertices and whose hyperedges, arranged in matrix form, are:

$$H_1 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 6 \end{bmatrix} \qquad H_2 = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 4 \end{bmatrix}$$

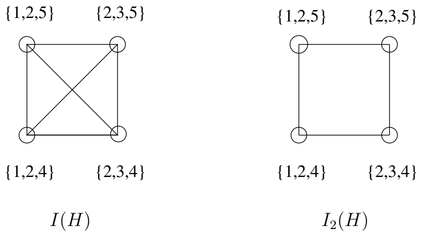
It is easy to check that the vector of labels $l = (1, -1, 1, -1)$ is a null label for H_2, while H_1 has no null labelling. However, H_1 and H_2 have the same intersection graph K_4, i.e. the complete graph on four vertices.

2−intersection graph

Relying on this fact, we decide to use the notion of *2-intersection graph*.

Definition

Let $H = (V, E)$ be a 3−hypergraph. The *2-intersection graph* $I_2(H)$ is a graph in which the nodes are the hyperedges of H and two hyperedges are adjacent if and only if they a share a common pair of vertices.



Hamiltonian cycle and null label

Idea: a connected, even graph G is Eulerian. An Euler tour in G corresponds to a Hamiltonian cycle in its line graph $L(G)$, and conversely. Furthermore, by alternate labelling this cycle we find a null labelling of G.

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Since the concept of $I_2(H)$ and $L(G)$ are very similar, we extend the previous idea and prove that, starting from an Hamiltonian cycle in $I_2(H)$ and alternate labelling its nodes (i.e. hyperedges of H), we can determine a null label of H.

Hamiltonian cycle and null label

Consider the 3-hypergraph $H = (V, E)$ on six vertices and $E = \{e_1, \dots, e_8\}$, where $e_1 = \{1, 2, 3\}, e_2 = \{1, 2, 4\}, e_3 = \{1, 2, 5\}, e_4 = \{1, 2, 6\}, e_5 = \{1, 3, 4\}, e_6 = \{1, 3, 5\}, e_7 = \{2, 3, 5\}, e_8 = \{2, 5, 6\}$.

The related 2-intersection graph $I_2(H)$ in Fig. 1 has the Hamiltonian cycle $C_1 = (v_{e_1}, v_{e_3}, v_{e_2}, v_{e_4}, v_{e_8}, v_{e_7}, v_{e_6}, v_{e_5}, v_{e_1})$

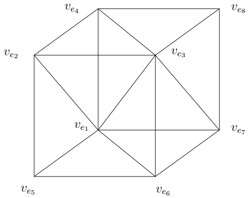

Figure: The 2-intersection graph of the 3-hypergraph considered in the Example.

Hamiltonian cycle and null label

It is easy to check that alternately labelling ± 1 the vertices of C_1, starting with $+1$, we obtain the null labelling $l_1 = (1, 1, -1, -1, -1, 1, -1, 1)$ on the eight hyperedges of H such that $l_1(i)$ is the label of e_i, with $1 \leq i \leq 8$.

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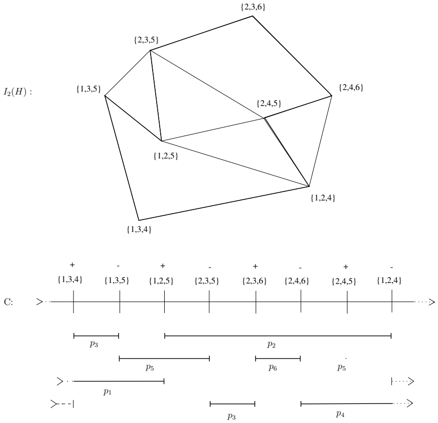
Unfortunately, not every Hamiltonian cycle provides a null labelling. A second Hamiltonian cycle $C_2 = (v_{e_1}, v_{e_2}, v_{e_3}, v_{e_4}, v_{e_8}, v_{e_7}, v_{e_6}, v_{e_5}, v_{e_1})$ exists such that the alternating labelling $l_2 = (1, -1, 1, -1, -1, 1, -1, 1)$ is not null on H, as $d(v_4) = -2$ and $d(v_5) = +2$.

Notation

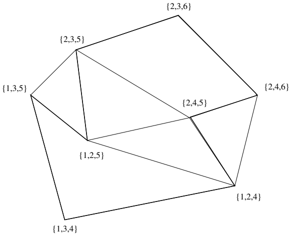
Suppose that v_{e_i} and v_{e_j} are two consecutive vertices of the alternate labelled Hamiltonian cycle C of $I_2(H)$, with $e_i = \{u, x, y\}$ and $e_j = \{v, x, y\}$. We see that $v \notin e_i$. There may be several consecutive vertices of C that contain v. Denote by $p_v = (v_{e_{j_1}}, \ldots, v_{e_{j_k}})$ the longest sub-path of C starting in v_{e_j} such that every vertex of p_v contains v.



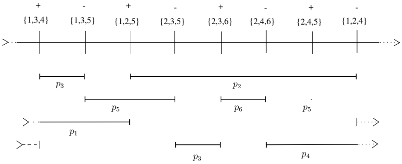
Notation

Let $l(p_v)$ denote the labels of the vertices of p_v, and let $\sigma(l(p_v))$ denote the sum of the elements of $l(p_v)$. Moreover, let $|p_v|$ denote the length $k-1$ of p_v.

$I_2(H)$:

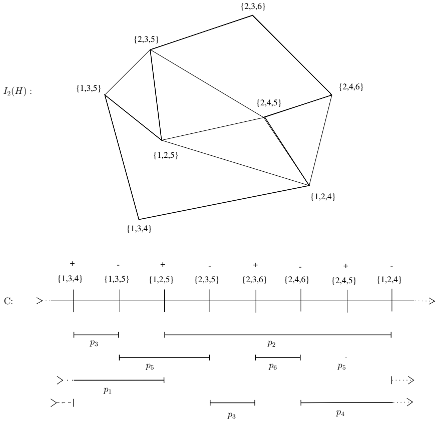


C:



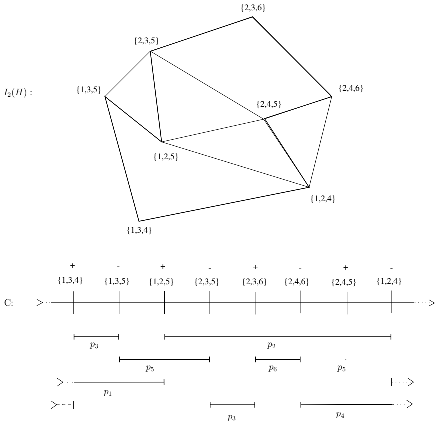
Notation

Moreover, we define the *distance* between two paths p_u and p_v as the distance along C between the last point of p_u and the first point of p_v. Finally, given a path p_u, $next(p_u)$ is the path beginning in the first vertex following the last vertex of p_u.



Notation

Note: in general, C may contain several different sub-paths of the form p_v, for each vertex v; we indicate them by $p_v^1, \ldots, p_v^n$.



Subpaths' properties

Property

Given an alternating labelling ± 1 on the vertices of a Hamiltonian cycle C of $I_2(H)$. For each sub-path $p_v = (v_{e_{j_1}}, \ldots, v_{e_{j_k}})$, the following holds:

- if p_v has odd length, then $\sigma(l(p_v)) = 0$, so that the labels of the hyperedges $e_{j_1}, \ldots, e_{j_k}$ containing v sum to zero in H. In this case the first and the last vertex of p_v have different labels;

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- if p_v has even length, then $\sigma(l(p_v)) \neq 0$ and the sum of the labels of the hyperedges $e_{j_1}, \ldots, e_{j_k}$ containing v contribute $+1$ or -1 to the signed degree of v. In this case, the extremal vertices of p_v have the same label.

Subpaths' properties

Lemma

Let H be an even 3-hypergraph and $I_2(H)$ its 2-intersection graph. If $I_2(H)$ has a Hamiltonian cycle C, an alternating ± 1 labelling $l(C)$ defines a null label of H if and only if, for each $v \in V$:

i) each subpath p_v has odd length; OR

ii) the number of subpaths of v having even length is even and the sum of their labels is zero.

Algorithm

Let C be an Hamiltonian cycle not satisfying the conditions of Lemma 1. We define an algorithm to obtain a null label from C. It relies on the *Switch*() operator defined as follows:

Definition

Given two sub-paths $p_u = (v_{e_{i_1}}, \ldots, v_{e_{i_k}})$ and $p_v = (v_{e_{j_1}}, \ldots, v_{e_{j_{k'}}})$, where $p_v = next(p_u)$, and $e_{i_k} \neq e_{j_1}$, the operator $Switch(p_u, p_v)$ produces a new labelling $l'(C)$ by changing the signs of e_{i_k} and e_{j_1}: $l'(e_{i_k}) = -l(e_{i_k})$ and $l'(e_{j_1}) = -l(e_{j_1})$ and keeping the remaining labels of $l(C)$ unchanged.

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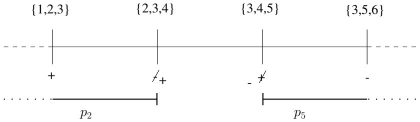
This is an example of $\mathit{Switch}(p_2, p_5)$ between the two consecutive paths p_2 and p_5, i.e., such that $p_5 = \mathit{next}(p_2)$.



Algorithm

We will start with an alternating labelling $l(C)$ and gradually change it using $Switch()$.

Property

Let H be a even 3-hypergraph, C a Hamiltonian cycle of $I_2(H)$ and l a ± 1 labelling of C. Consider a sub-path p_u of C whose last element v_{e_i} with label $+1$, and the sub-path $p_v = next(p_u)$ whose first element v_{e_j} with label -1. The operator $Switch(p_u, p_v)$ modifies l into l' so that $d_{l'}(u) = d_l(u) - 2$, $d_{l'}(v) = d_l(v) + 2$ and all the remaining signed degrees are left unchanged.

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Proof.

Without loss of generality, assume that $e_i = \{u, x, y\}$ and $e_j = \{v, x, y\}$. It is immediate that the change of the opposite labels of e_i and e_j keeps the signed degrees of x and y, while it subtracts 2 from u and adds 2 to v. A symmetric result holds. □

Algorithm

The algorithm $Balance()$ modifies a labelling $l(C)$ of a Hamiltonian cycle C of $I_2(H)$ in order to change the signed degree of two input vertices u and v of H, if possible, otherwise it gives failure.

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$Balance()$ uses consecutive iterations of $Switch()$ and it can be summarized in the following steps:

1. choose a subpath p_u^i (suppose $d_l(u) = 2$ and $d_l(v) = -2$);
2. if $|p_u^i|$ is even, suppose $p_t = next(p_u)$. Apply $Switch(p_u, p_t)$. Now $d_l'(u) = 0$ and and $d_l'(t) = d_l(t) + 2$. Set $t = u$ and repeat step 2)
3. if $|p_u^i|$ is odd, we know that there exists another subpath p_u^j such that $\sigma(p_u^j) = 1$ (otherwise is not possible that $d_l(u) = 2$). Choose p_u^j and apply step 2).

The procedure stops when we (eventually) obtain $d(u) = d(v) = 0$

Correctness of *Balance*

First, we proved that $Balance()$ computes a null labelling starting from the alternating labelling $l(C)$ in the easiest case of having only two signed degrees u and v different from zero, in particular $+2$ and -2, respectively.

Lemma

Let H be a 3-hypergraph, C a Hamiltonian cycle of $I_2(H)$, and l an alternating labelling of C. If u and v are the only nodes of H with signed degree different from zero, in particular $d_l(u) = +2$ and $d_l(v) = -2$, then $Balance(u, v, l(C))$ returns a null labelling $l'(C)$ of H.

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The key of the proof relies on the fact that $Switch()$ **always** changes $+$ with a $-$.

Correctness of *Balance*

The previous Lemma can be generalized thanks to two final results.

Lemma

Let $H = (V, E)$ be a 3-hypergraph, C a Hamiltonian cycle of $I_2(H)$ and l an alternating labelling of C. If v_1 and v_2 are the only nodes of H with signed degree different from zero with respect to l, say $d_l(u) = +2k$ and $d_l(v) = -2k$, where $k \geq 1$, then H admits a null labelling.

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Theorem

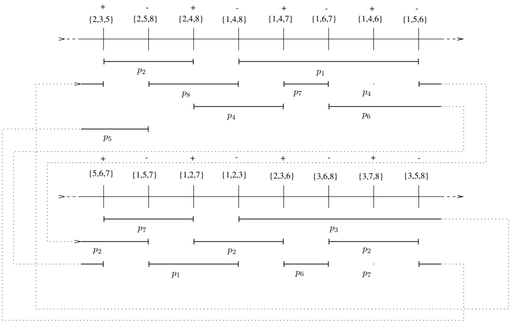
Let H be a 3-hypergraph. If the 2-intersection graph $I_2(H)$ is Hamiltonian, then H admits a null labelling.

The proof of these last results is simply based on a multiple iteration of *Balance*.

Example

Consider the following 3−hypergraph $H = (V, E)$ with $V = \{1, \ldots, 8\}$ and E = {{2,3,5}, {2,5,8}, {2,4,8}, {1,4,8}, {1,4,7}, {1,6,7}, {1,4,6}, {1,5,6}, {5,6,7}, {1,5,7}, {1,2,7}, {1,2,3}, {2,3,6}, {3,6,8}, {3,7,8}, {3,5,8}}

This figure shows a Hamiltonian cycle C of $I_2(H)$, and one of its alternating labellings $l(C)$.

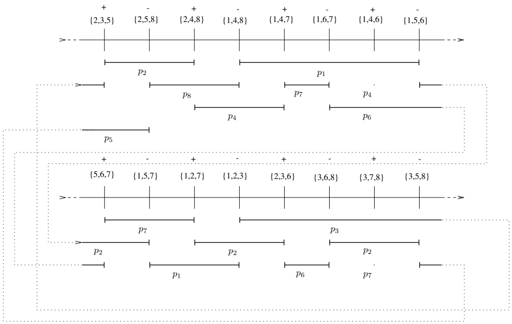


Example

The chosen labelling is not a null labelling of H. The vector of the signed degrees of the vertices of H is

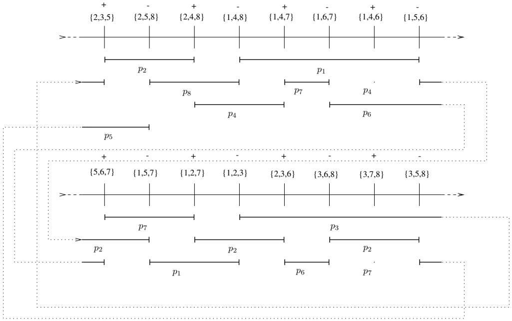
$$d = (-2, 2, 0, 2, -2, 0, 2, -2).$$

Let us perform a sequence of runs of $Balance()$ to compute a null labelling of H starting from $l(C)$.



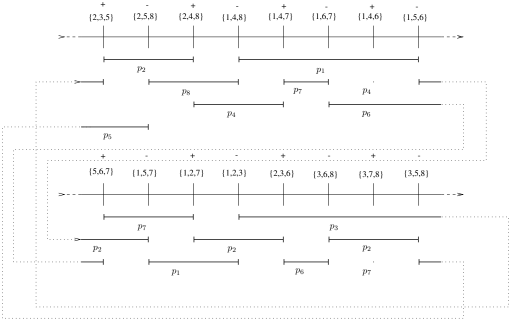
Example

Let us start, as an example, the run $Balance(2, v, l(C))$ in the p_2 sub-path having $\{2, 3, 5\}$ as first element. It calls $Switch(p_2, p_1)$, with $p_1 = next(p_2)$ and $|p_1|$ even. Since $d_l(1) = -2$, we perform the choice $v = 1$, and the switchings of $\{2, 4, 8\}$ and $\{1, 4, 8\}$ leading to the labelling $l^1(C)$ such that $d_{l^1}(1) = d_{l^1}(2) = 0$, leaving the remaining labels unchanged.



Example

Choose the vertex 7 such that $d_{l^1}(7) = +2$ and run $Balance(7, v, l^1(C))$ with the starting p_7 sub-path whose first element is $\{5,6,7\}$. The sub-path $p_3 = next(p_7)$ has odd length so the labels of $\{1,2,7\}$ and $\{1,2,3\}$ are switched and we obtain $d(7) = 0$ and $d(3) = +2$. Now $p_8 = next(p_3)$ and the labels of $\{2,3,5\}$ and $\{2,5,8\}$ are switched obtaining $d(3) = d(8) = 0$. Since $|p_8|$ is even, the run $Balance(7, v, l^1(C))$ ends setting $v = 8$. A new labelling $l^2(C)$ is returned as output.

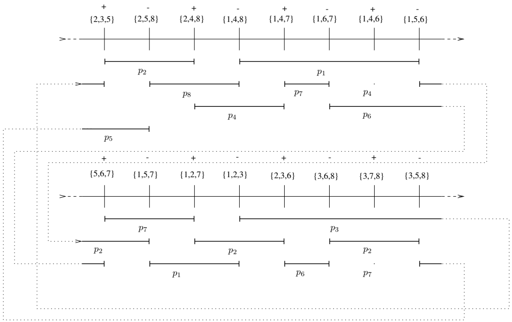


Example

Only the vertices 4 and 5 are left. A last run of $Balance(4, 5, l^2(C))$ is performed. Taking the p_4 subpath containing only $\{1, 4, 6\}$, we have $p_5 = next(p_4)$ with $|p_5|$ even. Therefore, switching the sign of $\{1, 4, 6\}$ and $\{1, 5, 6\}$ we obtain a new labelling l^3 such that $d_{l^3}(4) = d_{l^3}(5) = 0$ and $Balance(4, 5, l^2(C))$ ends. Therefore, the labelling

$$l^3 = (-1, 1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, 1, -1)$$

is a null labelling of H.



The importance of the $2-$intersection graph in the null labelling problem lead to the study of the following problem.

Problem: given a graph G, it is possible to decide in polynomial time the existence of a 3-hypergraph H such that $G = I_2(H)$?

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Problem: given a graph G, it is possible to decide in polynomial time the existence of a 3-hypergraph H such that $G = I_2(H)$?

In general, we say that a graph G has the $2-$*intersection property* if it is the $2-$intersection graph of some $3-$hypergraph H.

Properties of $I_2(H)$

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However, this Property does not characterize the cliques of $I_2(H)$ since there may appear triangles (K_3 cliques) whose edges do not share a common label. More precisely, the configurations $v_1 = \{x, y, z\}$, $v_2 = \{x, y, t\}$, and $v_3 = \{x, z, t\}$ and $v_1 = \{x, y, z\}$, $v_2 = \{x, y, t\}$, and $v_3 = \{x, y, k\}$ are both triangle in $I_2(H)$.

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Property

The edges of a 3-hypergraph H sharing the same couple of vertices originate a clique in the 2-intersection graph $I_2(H)$.

However, this Property does not characterize the cliques of $I_2(H)$ since there may appear triangles (K_3 cliques) whose edges do not share a common label. More precisely, the configurations $v_1 = \{x, y, z\}$, $v_2 = \{x, y, t\}$, and $v_3 = \{x, z, t\}$ and $v_1 = \{x, y, z\}$, $v_2 = \{x, y, t\}$, and $v_3 = \{x, y, k\}$ are both triangle in $I_2(H)$.

In order to distinguish them, we indicate the first and the second ones with T and K triangles, respectively.

A similar situation holds for K_4 cliques, since they can arise from T or K triangles. Maintaining the introduced notation, we indicate square configuration with S, and we differentiate it from the K square, i.e., the 4-clique whose edges have a common label.

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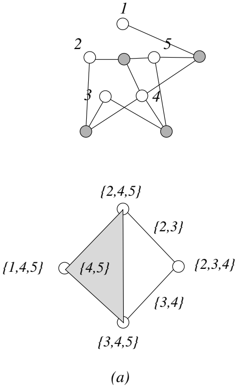

Figure: (a) example of adjacent T and K triangles. (b) example of S square.

Property

The biggest clique that admit a label without a common couple is a S square.

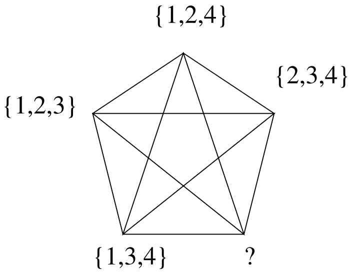

Figure: Labelling a K_5 clique without a common couple lead to an unsatisfactory labeling.

Lemma

If G is has the $2-$intersection property, then every node of G must belong to at most 3 maximal cliques or 5 maximal cliques two of which being non-adjacent T triangles.

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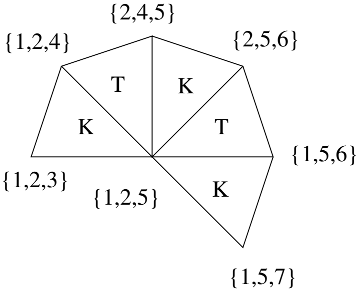

Figure: Reconstruction of the configuration in which a point belongs to 5 triangles.

Complexity of the problem

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Given an instance C of MAX-3-SAT, we construct a graph G_C so that there is a solution of the MAX-3-SAT instance if and only if G_C is a 2-intersection graph.

First results

Property

If two triangles intersecting in one vertex have the 2-intersection property, and there are no edges joining the triangles, then they are not both T-triangles.

We will call two triangles sharing one vertex, with no edges between them, a *ribbon* configuration.

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Figure: Two possible labels of a ribbon configuration. In each of them at most one triangle is a T-triangle.

First results

Property

Let T_1 and T_2 be two triangular cliques. Suppose there are just two edges joining a vertex of T_1 to a vertex of T_2, and that these edges have no common endpoints. Then the obtained configuration has the 2-intersection property. Furthermore, T_1 and T_2 cannot both be T-triangles. It is possible for T_1 and T_2 to be both K-triangles.

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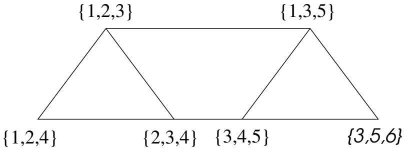
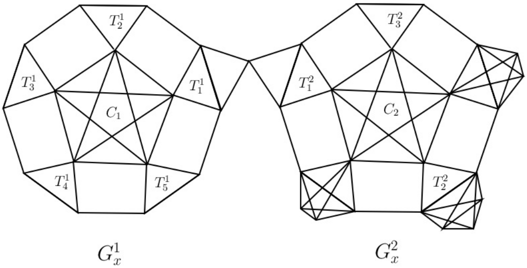


Figure: The configuration obtained by two triangles with joined by two edges. One possible labelling is shown.

Variables representation

Define a *variable gadget*, denoted G_x, to represent a variable x in U. The gadget is a 2-intersection graph obtained by the union of different configurations as defined in the following figure.



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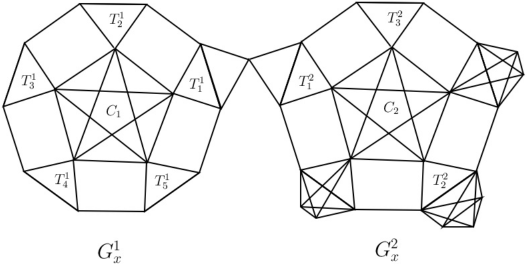


Property

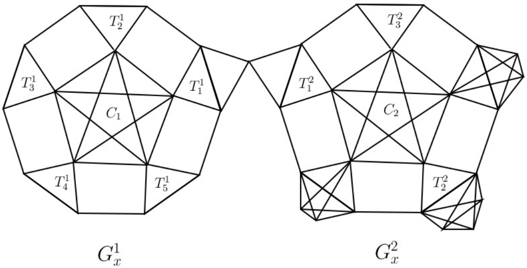
The variable gadget G_x is a 2-intersection graph.

Variables representation

Lemma

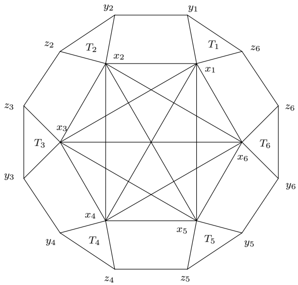
A labelling of the vertices of G_x allows only the following configurations for triangles T_2^1, T_5^1 and T_3^2:

$$i)\ \text{if } T_3^2 = T \text{ then } T_2^1 = T_5^1 = K;$$
$$ii)\ \text{if } T_5^1 = T \text{ (resp. } T_2^1 = T\text{) then } T_3^2 = K.$$



Clauses representation

The *clause gadget* G_c in Figure represents a single clause $c \in C$. It consists of a central clique K_6 whose vertices also belong to six different K_3 cliques, called *boundary triangles*.



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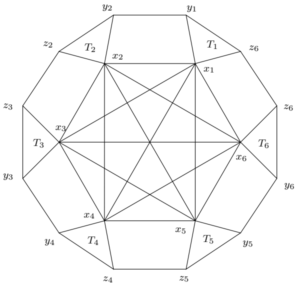

Lemma

A clause gadget G_c is a 2-intersection graph if and only if its boundary triangles do not contain either exactly three or exactly one T-triangles.

Note that by previous results G_c cannot have more than 3 $T-triangles$ as boundary triangles.

NP-Completeness

Let us consider the instance $C = \{c_1, \dots, c_n\}$ of MAX-3-SAT involving the variables in the set $U = \{x_1, \dots x_m\}$. Based on the gadgets already defined, we construct a graph G_C whose labels determine its 2-intersection property and express the desired valuations of C.

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We can assume that each variable must appear at most three times: one in a form and two in the opposite form. For each variable $x_i \in U$, we define a variable gadget G_{x_i}, and associate the triangle T_3^2 with the single occurrences of a literal x_i. The at-most-two remaining occurrences of the opposite literal are associated with the triangles T_2^1 and T_5^1.

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For each clause $c_j \in C$, we construct a clause gadget G_{c_j} and label its boundary triangles $T_1 \ldots T_6$. Finally, we connect variable gadgets and clause gadgets together as follows: for each clause c_j, with $1 \leq j \leq n$, we use a ribbon to the triangle T_{2i-1} of the clause gadget G_c, to the corresponding triangle associated with the i^{th} literal of c in the G_x gadget of its variable.

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Theorem

Given an instance C of MAX-3-SAT, the graph G_C has the 2-intersection property if and only if the instance C has a solution.

Sketch of the proof

Let us assume that there exists a valuation for the MAX-3-SAT instance C. Given a variable $x \in U$, for each literal with value *true* associated with x, we assign the triangles associated with it to be T-triangles, and we assign the triangles associated with its negation to be K-triangles. For each clause $c_j \in C$, in its clause gadget G_{c_j}, we assign the triangles associated with the literals having valuation *true* to be K-triangles, but T-triangles for the literals having valuation *false*. The previous Lemmas assure that G_C has the 2−intersection property.

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Suppose G_C has the 2-intersection property. For each clause gadget G_{c_j}, with $c_j \in C$, there exists at least one triangle among T_1, T_3 and T_5, say T' of type K. Previous properties assures that T' having type K leads to a T-triangle in the variable gadget G_x to which a literal, say l, is associated. We assign such a literal value *true*. The opposite literal $\bar{l}$ is then associated with *false*. The valuation defined is a solution of the MAX-3-SAT instance C.

Example

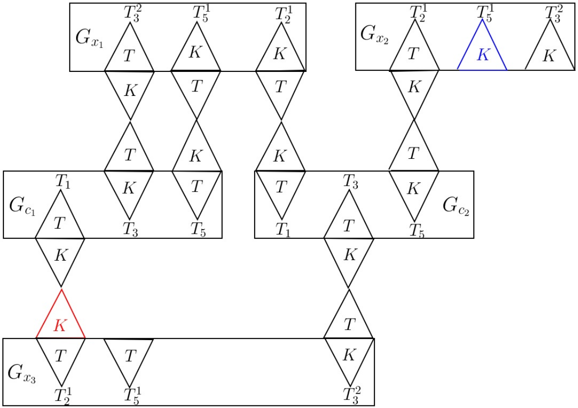

Figure: Example of the construction of the gadget for $C = \{c_1, c_2\}$, $c_1 = (x_3, x_1, \overline{x}_1)$, $c_2 = (\overline{x}_1, \overline{x_3}, x_2)$. One valuations obtained by the labelling of the corresponding G_C graph is $x_1 = true$, $x_2 = true$ and $x_3 = false$.

Thanks for your attention!