An alternative definition for digital convexity

Context and objectives

Full convexity

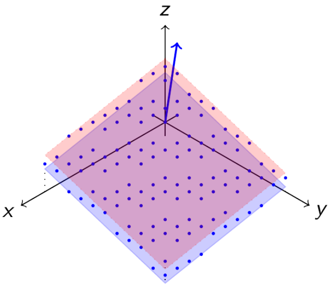
Some applications of full convexity

Fully convex enveloppe and polyhedral models

Some applications of full convexity

- full convexity of digital planes
- stability by intersection and local shape analysis
- tangency, tangential cover
- digital surface reconstruction
- shortest paths

Thick enough arithmetic planes are full convex

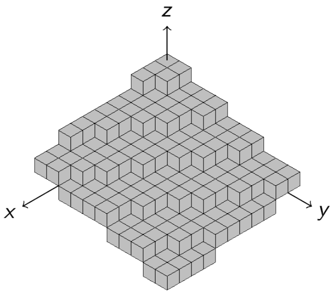


Arithmetic plane

- irreducible normal vector $N \in \mathbb{Z}^d$
- intercept $\mu \in \mathbb{Z}$
- positive thickness $\omega \in \mathbb{Z}, \omega > 0$

$$P(\mu, N, \omega) := \{x \in \mathbb{Z}^d, \mu \leqslant x \cdot N < \mu + \omega\}$$

Thick enough arithmetic planes are full convex



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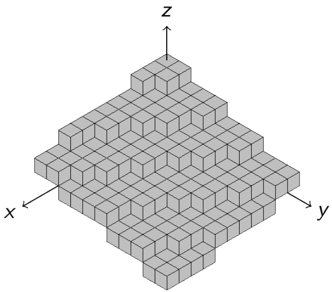

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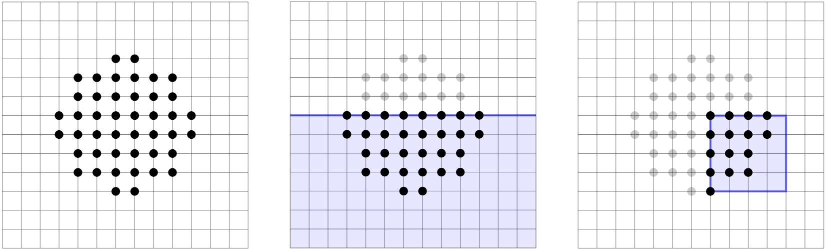
Theorem

Arithmetic planes are fully convex for thickness $\omega \geqslant \|N\|_\infty$.

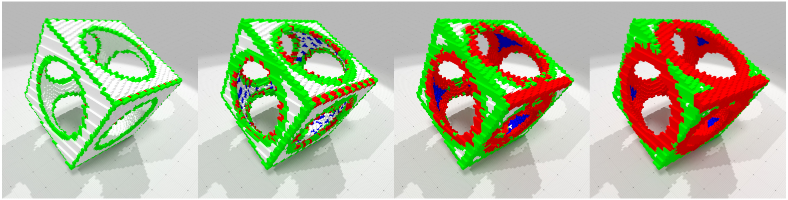
Stability by intersection

- A subset Y of $\mathbb{R}^d$ is said *stable* whenever Y is convex and, for any cell c of $\mathcal{C}^d$, $Y \cap c \neq \emptyset \Rightarrow \bar{c} \subset Y$.
- If $X \subset \mathbb{Z}^d$ is fully convex and $Y \subset \mathbb{R}^d$ is stable, then $X \cap Y$ is fully convex.
- Any intersection of stable sets is stable
- Any half-space of integer intercept and axis normal vector is stable.
- Any axis-aligned slice or any cubical neighborhood is stable



fully convex   still fully convex   yet fully convex

Local analysis of shape X



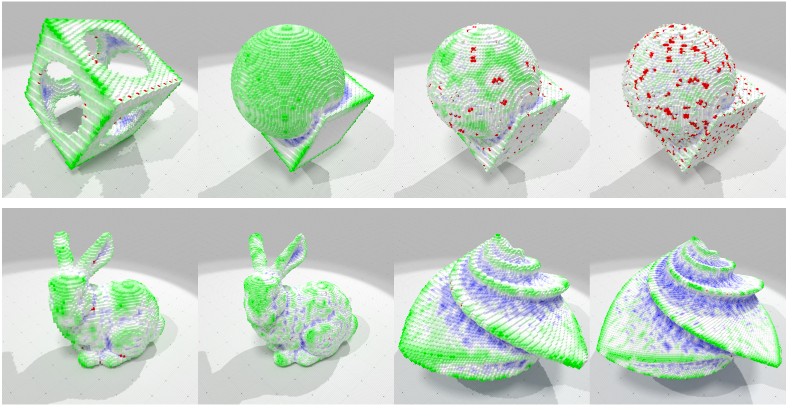
$k = 1$ $k = 2$ $k = 3$ $k = 4$

- Let $N_k(z)$ be the $(2k+1)^d$-neighborhood around z

$$X_k(z) := N_k(z) \cap X \qquad \bar{X}_k(z) := N_k(z) \cap (\mathbb{Z}^d \setminus X)$$

- X is *j-convex at z* iff $X_j(z)$ is fully convex
- X is *j-concave at z* iff $\bar{X}_j(z)$ is fully convex
- X is *j-planar at z* iff it is j-convex and j-concave at z
- otherwise X is *j-atypical at z*

Local multiscale analysis of shape X



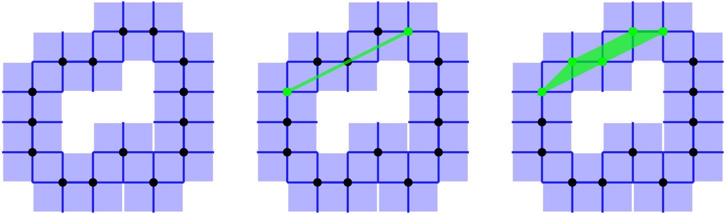
Lemma

- X $j+1$-convex at $z \Rightarrow X$ j-convex at z
- X $j+1$-concave at $z \Rightarrow X$ j-concave at z
- X $j+1$-planar at $z \Rightarrow X$ j-planar at z
- X j-atypical at $z \Rightarrow X$ $j+1$-atypical at z

Tangency

Definition

The digital set $A \subset X \subset \mathbb{Z}^d$ is said to be *k-tangent to X* for $0 \leqslant k \leqslant d$ whenever $\bar{\mathcal{C}}_k^d[\mathrm{Cvxh}\,(A)] \subset \bar{\mathcal{C}}_k^d[X]$. It is *tangent to X* if the relation holds for all such k.



X and $\bar{\mathcal{C}}^d[X]$ tangent tangent

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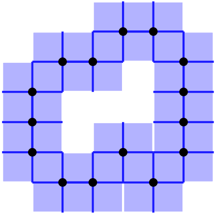

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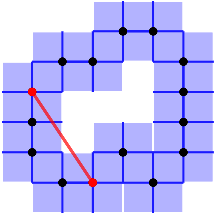
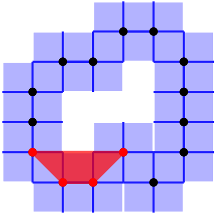
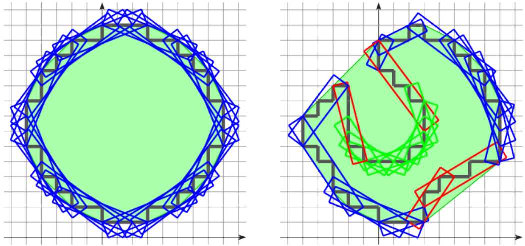


not tangent



not tangent

Side step: 2d tangential cover



Sequence of maximal digital straight segments = tangential cover of

[Feschet, Tougne 99]

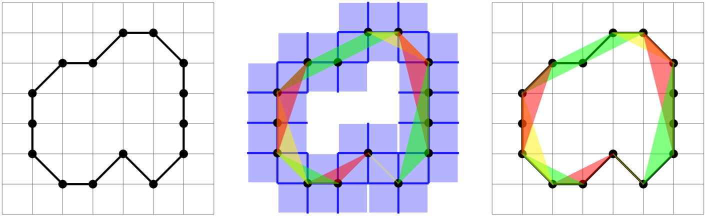
Theorem ([Debled-Rennesson,Reiter-Doerksen 04])

A 4- or 8-connected subset $S \subset \mathbb{Z}^2$ is digitally convex, iff the directions of its maximal digital straight segments are monotonous along $\mathrm{Bd}(S)$.

Tangential cover

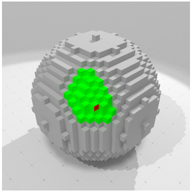
Theorem

In 2D, if C is a simple 2-connected digital contour, then the fully convex subsets of C that are maximal and tangent coincides with the classical tangential cover of [Feschet, Tougne 99].



Tangential cover in 3D ? dD ?

- can we define facets of X as inextensible connected pieces of arithmetic planes along X ?



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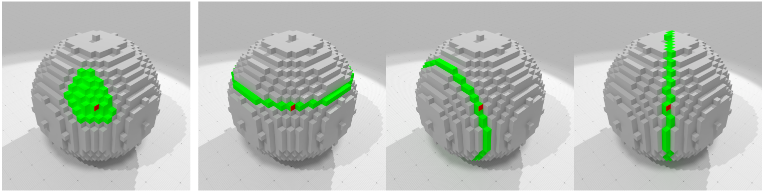

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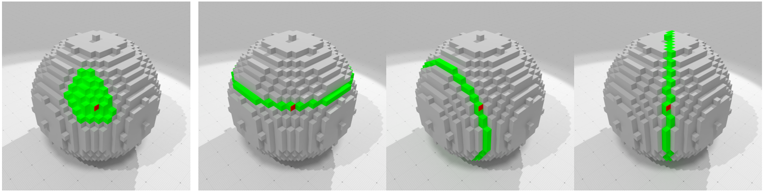

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- greedy methods to isolate meaningful ones:
  [Klette, Sun, Coeurjolly, Sivignon, Kenmochi, Provot, Debled-Rennesson, Charrier, L., ...]

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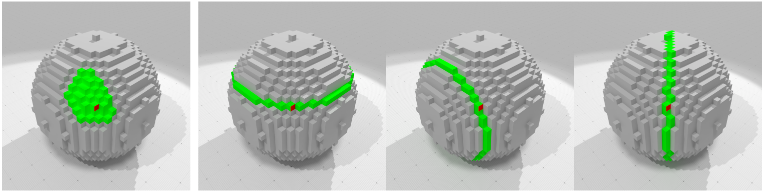

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Tangency extends to dD!

Tangent subsets in our sense are indeed tangent to X since their convex hull must lie close to X.

Piecewise linear reversible reconstruction in dD

Let $\mathrm{Del}\,(X)$ be the Delaunay complex of X.

Definition

The *tangent Delaunay complex* $\mathrm{Del}_{\mathrm{T}}\,(X)$ to X is the complex made of the cells τ of $\mathrm{Del}\,(X)$ such that the vertices of τ are tangent to X.

- its boundary is the convex hull when X is fully convex,

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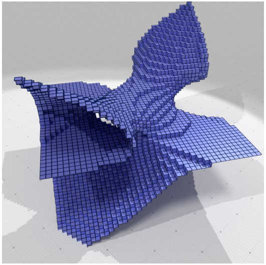
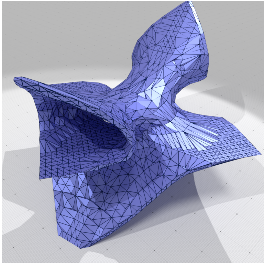
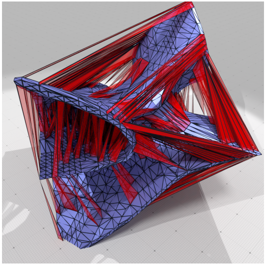


Input digital shape X



Reconstruction $\mathrm{Del}_{\mathrm{T}}\,(X)$

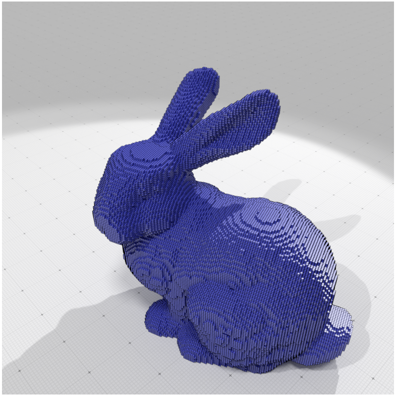


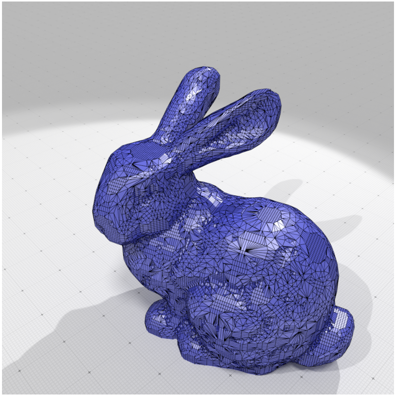
Bad simplices of $\mathrm{Del}\,(X)$

Theorem

The body of $\mathrm{Del}_{\mathrm{T}}\,(X)$ *is at Hausdorff* L_∞*-distance* 1 *to* X*.* $\mathrm{Del}_{\mathrm{T}}\,(X)$ *is a reversible polyhedrization, i.e.* $\|\mathrm{Del}_{\mathrm{T}}\,(X)\| \cap \mathbb{Z}^d = X$.

Piecewise linear reversible reconstruction in dD





Path

Definition (path)

Let $\gamma = (x_i)_{i=0,\dots,n}$, $n \geqslant 0$, be a sequence of points in some digital set X. The sequence γ is a *path from point a to point b in* X, if and only if, $x_0 = a$, $x_n = b$, and every two consecutive points of γ are co-tangent in X.

Its *embedding* $\overline{\overline{\gamma}}$ is the embedding of the straight segments joining consecutive points.

Lemma

$\overline{\overline{\gamma}} \subset \left\|\bar{\mathcal{C}}^d[\overline{\overline{\gamma}}]\right\| \subset \left\|\bar{\mathcal{C}}^d[X]\right\|$. *Hence the L_∞-distance of any point of $\overline{\overline{\gamma}}$ to X is smaller than 1.*

Shortest path

Definition (path length; shortest path)

The *length of* γ is $\mathrm{length}(\gamma) := \sum_{i=0}^{n-1} \|x_{i+1} - x_i\|$. The path γ from a to b is a *shortest path* from a to b if there exists no other path from a to b with a smaller length.

Definition (digital distance)

The *digital distance* d_X *in* $X \subset \mathbb{Z}^d$ is

$$\forall x, y \in X, \mathrm{d}_X(x, y) := \begin{cases} +\infty \text{ if } \mathcal{P}_X(x, y) \text{is empty,} \\ \inf_{\gamma \in \mathcal{P}_X(x,y)} \mathrm{length}(\gamma) \text{ otherwise.} \end{cases}$$

where $\mathcal{P}_X(x, y)$ is the set of path from x to y.

Lemma

If X is d-connected then the infimum above is a minimum, i.e. $\mathrm{d}_X(a, b) = \mathrm{length}(\gamma)$ with γ any shortest path of $\mathcal{P}_X(a, b)$.

Shortest paths and metric space

Theorem

If $X \subset \mathbb{Z}^d$ is d-connected and non-empty, then (X, d_X) is a metric space.

Theorem

If X is a fully convex set, then for any pair of points $x, y \in X$, (x, y) is the shortest path between x and y and $\mathrm{d}_X = \|x - y\|$.

Computing shortest paths

Lemma

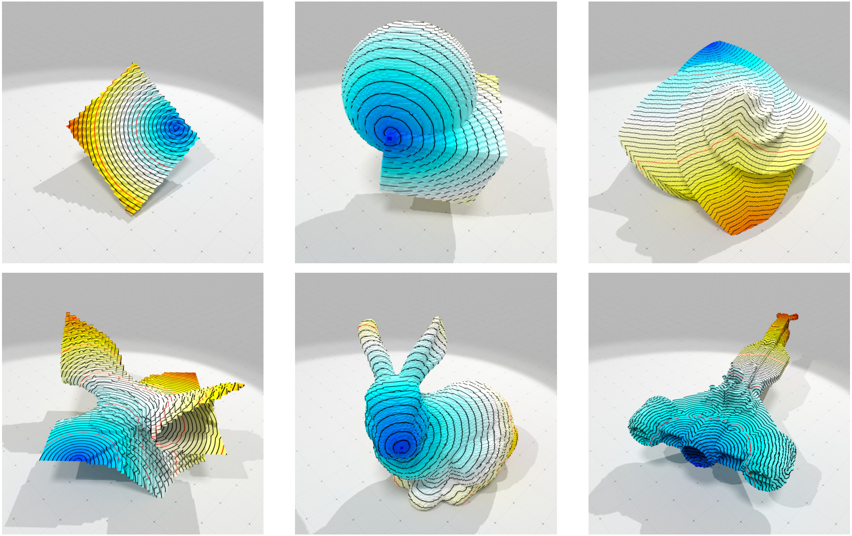
If γ is a path in X between a and b, then there exists a d-connected path P of points in X between a and b such that $P \subset \mathrm{Cl}\left(\left\|\bar{\mathcal{C}}^d[\overline{\overline{\gamma}}]\right\|\right)$ (it stays close *to γ), and P visits the digital points of γ in the same order.*

- we can find shortest paths by visiting direct neighbors.

Computing shortest path; Dijkstra algorithm (variant)

```
Procedure ShortestPath( In X, In a, Out A, Out D )
In X : subset of ℤ^d ;                    // Any non-empty subset of ℤ^d
In a : Point ;                            // source point in X
Out A : map<Point,Point> ;                // ancestor in the geodesic
Out D : map<Point,Real> ;                 // distance to a
Type Node = tuple<Point,Point,Real>
Var V : set<Point> ;                      // visited points
Var Q : priority_queue<Node>
begin
    foreach p ∈ X do D[p] ← +∞;
    Q.push((a, a, 0.0)) ;                 // Starting point
    while ¬Q.empty() do
        (q, r, d) ← Q.pop(); ;            // pop top node
        if d > D[q] then continue ;
        A[q] ← r, D[q] ← d, V.insert(q);
        N ← CotangentPoints(X, q, V);
        foreach p ∈ N do
            d′ ← D[q] + ‖p − q‖;
            if d′ < D[p] then d′ ← D[p], Q.push((p, q, d′)); ;
        end
    end
end
```

Results



An alternative definition for digital convexity

With Fabien Feschet
Université d'Auvergne

What about digital convex hull ?

- digital convex hull $\mathrm{Cvxh}_{\mathbb{Z}^d}(A) := \mathrm{Cvxh}(A) \cap \mathbb{Z}^d$

properties	H-convexity	H-convexity + connect.
$\mathrm{Cvxh}_{\mathbb{Z}^d}(A)$ convex	$+$	$-$
$\mathrm{Cvxh}_{\mathbb{Z}^d}(A) = A$ (for A cvx)	$+$	$+$
idempotence	$+$	$+$
fast computation	$+$	$+$
increasing	$+$	$+$

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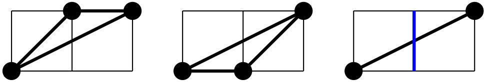
idempotence	+	+
fast computation	+	+
increasing	+	+

How can we build fully convex sets from arbitrary $A \subset \mathbb{Z}^d$?

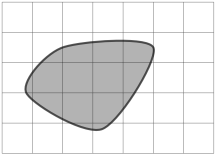
Fully convex hull through intersections ?

- half-spaces are fully convex
- can we intersect support half-spaces to get fully convex hull ?
- intersections of fully convex sets are not fully convex in general

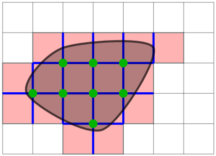


A $\quad$ B $\quad$ $A \cap B$

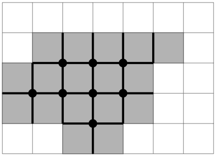
Local operators $\mathrm{Star}(\cdot)$, $\mathrm{Skeleton}(\cdot)$, $\mathrm{Extrema}(\cdot)$



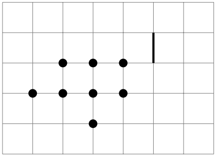
Y



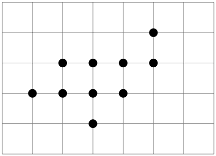
$\mathrm{Star}(Y) = \bar{\mathcal{C}}_0^d[Y] \cup \bar{\mathcal{C}}_1^d[Y] \cup \bar{\mathcal{C}}_2^d[Y]$



K



$K' = \mathrm{Skeleton}(K)$

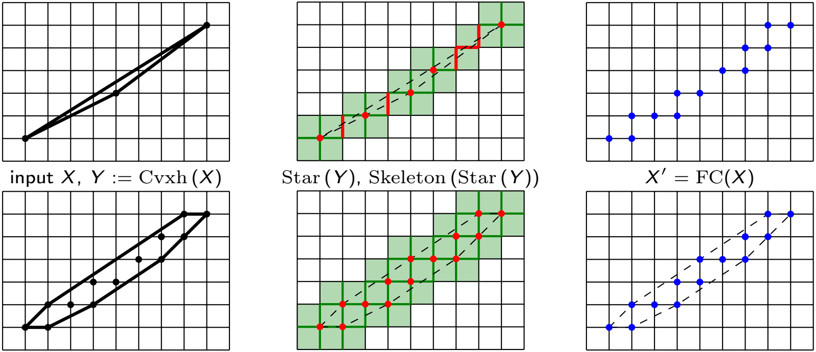


$\mathrm{Extrema}(K')$

- For any $Y \subset \mathbb{R}^d$, let $\mathrm{Star}(Y) := \bar{\mathcal{C}}^d[Y]$ (coincides with the usual star of combinatorial topology)
- For any complex $K \subset \mathcal{C}^d$, let $\mathrm{Skeleton}(K) := \bigcap_{K' \subset K \subset \mathrm{Star}(K')} K'$
- For any complex $K \subset \mathcal{C}^d$, let $\mathrm{Extrema}(K) := \mathrm{Cl}(K) \cap \mathbb{Z}^d$

Operator $\mathrm{FC}(\cdot)$ and fully convex enveloppe $\mathrm{FC}^*(\cdot)$

- Iterative method for computing a fully convex enveloppe
- Let $\mathrm{FC}(X) := \mathrm{Extrema}\left(\mathrm{Skeleton}\left(\mathrm{Star}\left(\mathrm{Cvxh}\left(X\right)\right)\right)\right)$
- Iterative composition $\mathrm{FC}^n(X) := \underbrace{\mathrm{FC} \circ \cdots \circ \mathrm{FC}}_{n \text{ times}}(X)$
- *Fully convex envelope* of X is $\mathrm{FC}^*(X) := \lim_{n\to\infty} \mathrm{FC}^n(X)$.



input X, $Y := \mathrm{Cvxh}\left(X\right)$ — $\mathrm{Star}\left(Y\right)$, $\mathrm{Skeleton}\left(\mathrm{Star}\left(Y\right)\right)$ — $X' = \mathrm{FC}(X)$

input X', $Y' := \mathrm{Cvxh}\left(X'\right)$ — $\mathrm{Star}\left(Y'\right)$, $\mathrm{Skeleton}\left(\mathrm{Star}\left(Y'\right)\right)$ — $X'' = \mathrm{FC}(X') = \mathrm{FC}^2(X)$

The fully convex enveloppe is well defined

Lemma

For any $X \subset \mathbb{Z}^d$, $X \subset \mathrm{FC}(X)$.

Lemma

For any finite $X \subset \mathbb{Z}^d$, X *and* $\mathrm{FC}(X)$ *have the same bounding box.*

Theorem

For any finite digital set $X \subset \mathbb{Z}^d$, *there exists a finite* n *such that* $\mathrm{FC}^n(X) = \mathrm{FC}^{n+1}(X)$, *hence* $\mathrm{FC}^*(X)$ *exists and is equal to* $\mathrm{FC}^n(X)$.

Consistency and idempotence of fully convex enveloppe

The fully convex enveloppe acts as a fully convex hull operator

Lemma

If $X \subset \mathbb{Z}^d$ is fully convex, then $\mathrm{FC}(X) = X$. So $\mathrm{FC}^*(X) = X$.

Lemma

If $X \subset \mathbb{Z}^d$ is not fully convex, then $X \subsetneq \mathrm{FC}(X)$

Theorem

$X \subset \mathbb{Z}^d$ is fully convex if and only if $X = FC(X)$.

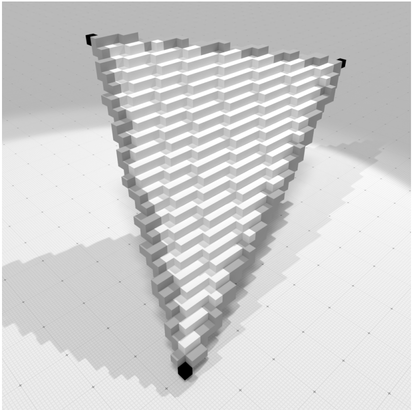
Theorem

For any finite $X \subset \mathbb{Z}^d$, $\mathrm{FC}^*(X)$ is fully convex.

Theorem

Computation of $\mathrm{FC}(\cdot)$ is bounded by $O\left(n^{\lfloor \frac{d}{2} \rfloor}\right)$, with $n = \#(X)$.

A 3D digital triangle



vertices $A = (8, 4, 18)$, $B = (-22, -2, 4)$, $C = (18, -20, -8)$ (black),

edges $\mathrm{FC}^*(\{A, B\})$, $\mathrm{FC}^*(\{A, C\})$, $\mathrm{FC}^*(\{B, C\})$ (grey+black)

triangle $\mathrm{FC}^*(\{A, B, C\})$ (white+grey+black)

Is the fully convex enveloppe a hull operator ?

properties	fully convex enveloppe
$\mathrm{FC}^*(A)$ convex	$+$
$\mathrm{FC}^*(A) = A$ (for A fully cvx)	$+$
idempotence	$+$
fast computation	$\approx$ (# iterations)
increasing	$-$

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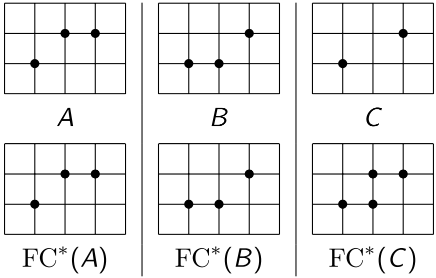

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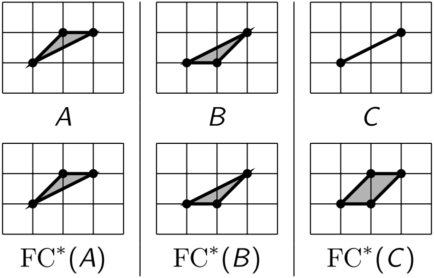


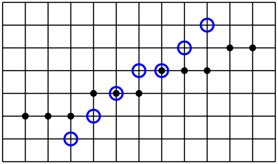
A relative fully convex enveloppe

- For $X \subset Y$, let $\mathrm{FC}_{|Y}(X) := \mathrm{FC}(X) \cap Y$
- $\mathrm{FC}^n_{|Y}(X) := \mathrm{FC}_{|Y} \circ \cdots \circ \mathrm{FC}_{|Y}(X)$, composed n times
- *Fully convex envelope of X relative to Y* is $\mathrm{FC}^*_{|Y}(X) := \lim_{n\to\infty} \mathrm{FC}^n_{|Y}(X)$
- we have $\mathrm{FC}^*(X) = \mathrm{FC}^*_{|\mathbb{Z}^d}(X)$

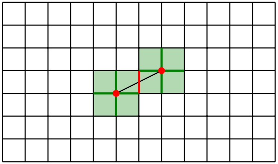
Theorem

Let $X \subset \mathbb{Z}^d$ and $Y \subset \mathbb{Z}^d$ fully convex.
Then $\mathrm{FC}^*_{|Y}(X \cap Y)$ is fully convex and is included in Y.

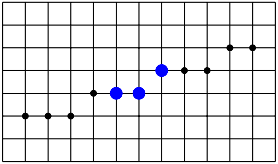
Intersections of fully convex sets



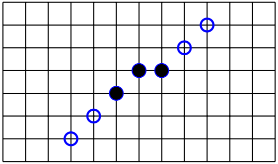
X, Y



$\mathrm{Skeleton}(\mathrm{Star}(\mathrm{Cvxh}\,(X \cap Y)))$



$\mathrm{FC}^*_{|Y}\,(X \cap Y)$



$\mathrm{FC}^*_{|X}\,(X \cap Y)$

Polyhedral models (here 3D)

- polyhedron $\mathcal{P}$ made of k-cells (facets, edges, vertices), with incidence relations
- use relative full convexity to define facets that are pieces of arithmetic planes
- $T \subset \mathbb{Z}^3$ made of coplanar points, $P_1(T)$ is the median standard plane (resp. $P_\infty(T)$ the median naive plane) defined by T.

Definition (standard digital polyhedron)

$\mathcal{P}_1^*$ is the collection of digital cells subsets of $\mathbb{Z}^d$:

- if σ is a facet of $\mathcal{P}$ with vertices $V(\sigma)$, then σ_1^* is a cell of $\mathcal{P}_1^*$ with $\sigma_{\mathbf{1}}^* := \mathrm{FC}^*_{|P_{\mathbf{1}}(V(\sigma))}(V(\sigma))$.
- if τ is an edge, then it has as many geometric realizations as incident facets σ: $(\tau,\sigma)_1^* := \mathrm{FC}^*_{|\sigma_{\mathbf{1}}^*}(V(\tau))$.
- vertices are simply digital points.

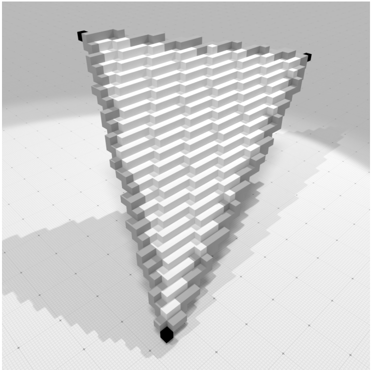
Definition (naive digital polyhedron)

$\mathcal{P}_\infty^*$ defined similarly by replacing 1 with ∞ above.

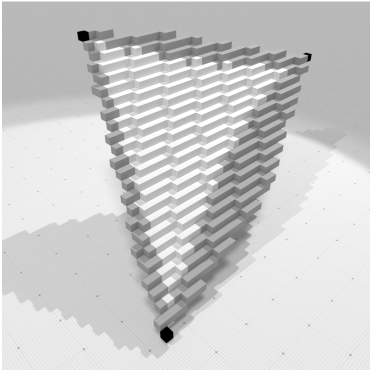
Standard and naive 3D triangle

Theorem

All digital cells are fully convex.



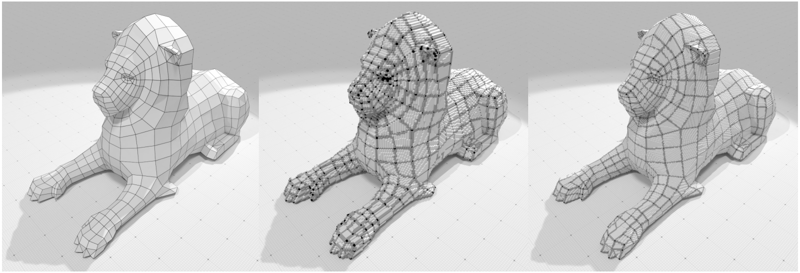
standard triangle $\mathcal{T}_1^*$
985 points



naive triangle $\mathcal{T}_\infty^*$
567 points

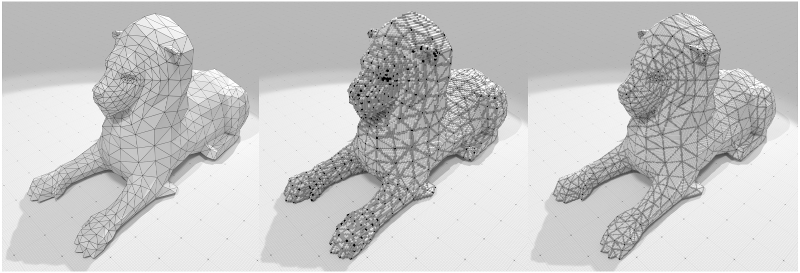
Polyhedron $\mathcal{T}$ with vertices $A = (8, 4, 18)$, $B = (-22, -2, 4)$, $C = (18, -20, -8)$, edges $\{(A, B), (A, C), (B, C)\}$ and one facet $\{(A, B, C)\}$.

Generic/standard/naive digital polyhedron



Quad-mesh $\mathcal{Q}$, non planar faces | $\sharp\mathcal{Q}^* = 81044$ | $\sharp\mathcal{Q}^* = 373225$

Generic/standard/naive digital polyhedron

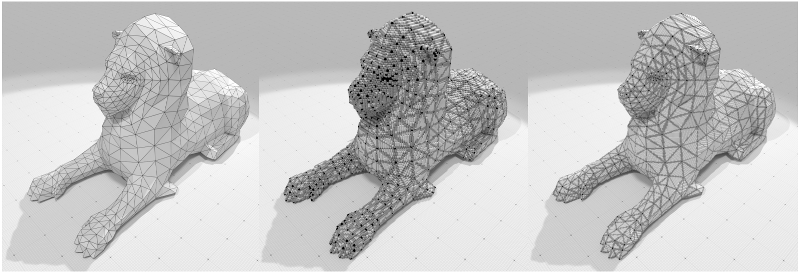


Tri-mesh $\mathcal{T}$, planar faces

$\sharp \mathcal{T}_1^* = 68603$

$\sharp \mathcal{T}_1^* = 275931$

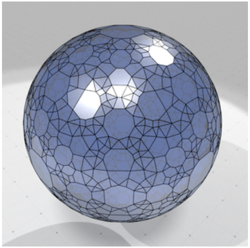
Generic/standard/naive digital polyhedron



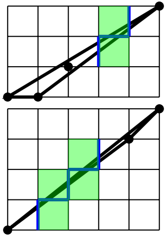
Tri-mesh $\mathcal{T}$, planar faces | $\sharp\mathcal{T}^*_\infty = 46639$ | $\sharp\mathcal{T}^*_\infty = 182451$

Full convexity packages in DGtal

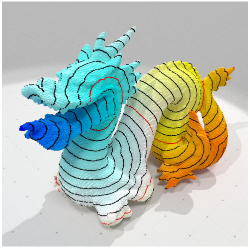
dgtal.org



*d*D convex hull and Delaunay triangulation



Full convexity tests in *d*D



Local shape analysis, geodesic shortest paths

- most of full convexity and applications implemented in DGtal
- open source library
- efficient generic C++

Conclusion

Pros of full convexity

- natural definition in arbitrary dimension that uses $\mathbb{Z}^d \subset \mathcal{C}^d$
- guarantees connectedness and simple connectedness
- morphological characterization that allows simple convexity check
- thick enough arithmetic planes are fully convex
- entails a consistent definition of tangency
- simple tight and reversible polyhedrization
- local shape analysis, shortest paths
- fully convex enveloppe, digital polyhedra

Cons of full convexity

- $(2^d - 1)$ times slower to check convexity

Future works

Theoretical side

- intersection property and increasingness of enveloppe still under study
- convergence of maximal tangent planes for normal estimation ?
- lattice cells enumeration falls into Ehrhart theory: are polynomials specific ?

Algorithmic side

- fast lattice point enumeration within polytopes
- fast computation of maximal tangent planes (link with plane probing)

Explore its natural applications

- other polyhedrization algorithms using tangency