An alternative definition for digital convexity

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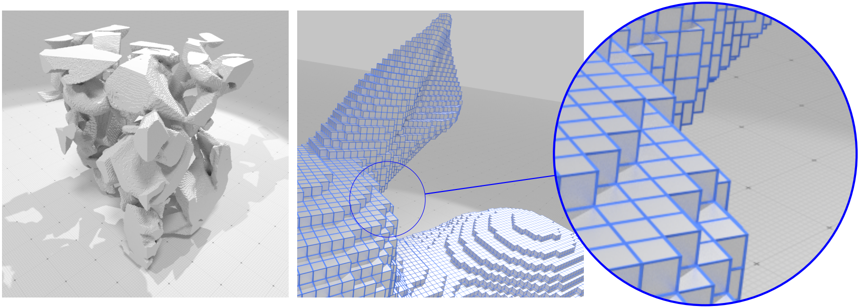
Why digital convexity ?



- no (infinitesimal) differential geometry for digital shapes
- convexity: a fundamental tool to analyze the geometry of shapes
- identifies convex/concave/flat/saddle regions
- gives locally its piecewise linear geometry
- facets give normal estimations

How well convexity remains meaningful in lattice spaces ?

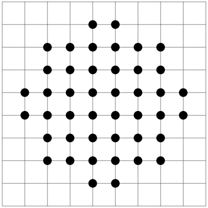
Some expectations when defining convexity in $\mathbb{Z}^d$:

- simple and elegant definition in arbitrary dimension
- straight lines, planes, half-spaces, balls, ..., are convex
- convex sets are connected, and even simply connected
- intersections of convex sets are convex
- deciding if a set $X \subset \mathbb{Z}^d$ is convex must be fast (polynomial time)
- convex hull leaves convex sets unchanged
- convex hull builds a convex set and is idempotent
- computing a convex hull must be fast
- convex hull is increasing, i.e. $A \subset B \Rightarrow \mathrm{Cvxh}\,(A) \subset \mathrm{Cvxh}\,(B)$

Natural digital convexity is not satisfactory

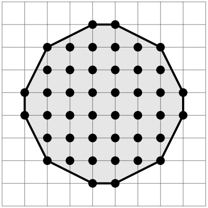
Definition (Natural digital convexity (or H-convexity))

$X \subset \mathbb{Z}^d$ is digitally convex iff $\mathrm{Cvxh}(X) \cap \mathbb{Z}^d = X$



X

$=$



$\mathrm{Cvxh}(X) \cap \mathbb{Z}^d$

$\Rightarrow$ convex

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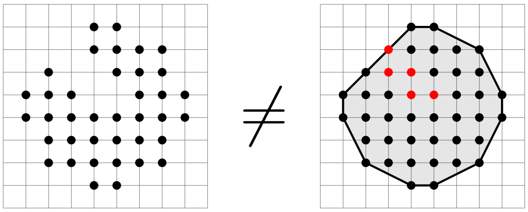

$X \neq \mathrm{Cvxh}(X) \cap \mathbb{Z}^d \Rightarrow$ not convex

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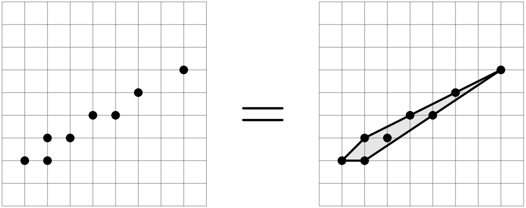

X $\qquad$ $\mathrm{Cvxh}\,(X) \cap \mathbb{Z}^d$

$\Rightarrow$ convex !

Natural digital convexity is not satisfactory

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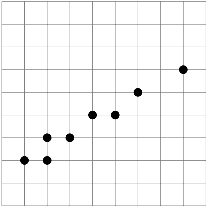
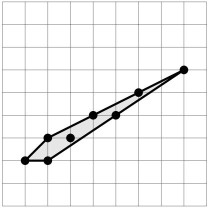


X

$=$



$\mathrm{Cvxh}\,(X) \cap \mathbb{Z}^d$

$\Rightarrow$ convex !

Digital convexity does not imply digital connectedness !

Summary

Properties of natural digital convex sets (H-convexity)

simple, generic	$+$	(indeed, $X = \mathrm{Cvxh}\,(X) \cap \mathbb{Z}^d$)
classical convex objects	$\approx$	(but weird sets are convex)
connectedness	$-$	(many convex sets are disconnected)
simple connectedness	$-$	(of course no)
intersection property	$+$	
fast convexity test	$+$	(quickhull+lattice enumeration)

Usual digital convexity adds connectedness

Definition (Usual digital convexity)

$X \subset \mathbb{Z}^d$ is digitally convex iff $\mathrm{Cvxh}(X) \cap \mathbb{Z}^d = X$ **and** X connected

- many more or less equivalent definitions **in 2D**: straight segment convexity, triangle convexity, . . . [Minsky, Papert 88], [Kim, Rosenfeld 82a], [Hübler, Klette, Voss89], . . .

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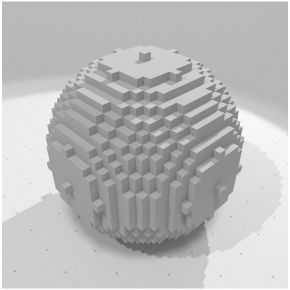
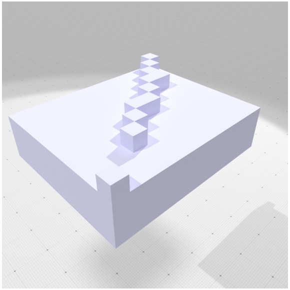
- **none extends well to 3D or more**



convex



convex !

Pros and cons

properties	H-convexity	H-convexity $+$ connectedness
simple, generic	$+$	$-$
classical convex objects	$\approx$	$\approx$
connectedness	$-$	$\approx$ (slices unconnected)
simple connectedness	$-$	$-$ (unclear)
intersection property	$+$	$-$
fast convexity test	$+$	$+$

Proposal: full convexity

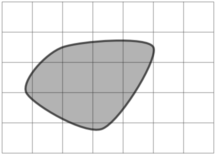
properties	H-convexity	H-convexity + connect.	Full convexity
simple, generic	$+$	$-$	$+$
classical convex objects	$\approx$	$\approx$	$+$
connectedness	$-$	$\approx$	$+$
simple connectedness	$-$	$-$	$+$
intersection property	$+$	$-$	$-$ (but...)
fast convexity test	$+$	$+$	$+$

An alternative definition for digital convexity

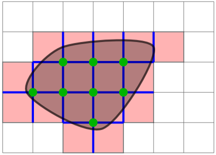
Cubical grid, intersection complex

- cubical grid complex $\mathcal{C}^d$
  - $\mathcal{C}_0^d$ vertices or 0-cells $= \mathbb{Z}^d$
  - $\mathcal{C}_1^d$ edges or 1-cells $=$ open unit segment joining 0-cells
  - $\mathcal{C}_2^d$ faces or 2-cells $=$ open unit square joining 1-cells
  - $\ldots$
- *intersection complex* of $Y \subset \mathbb{R}^d$

$$\bar{\mathcal{C}}_k^d[Y] := \{c \in \mathcal{C}_k^d, \bar{c} \cap Y \neq \emptyset\}$$



Y



cells $\bar{\mathcal{C}}_0^d[Y]$, $\bar{\mathcal{C}}_1^d[Y]$, $\bar{\mathcal{C}}_2^d[Y]$

Full convexity

Definition (Full convexity)

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathcal{C}}_k^d[X] = \bar{\mathcal{C}}_k^d[\mathrm{Cvxh}\,(X)]. \tag{1}$$

Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.

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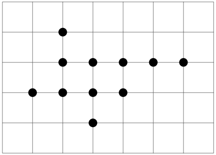
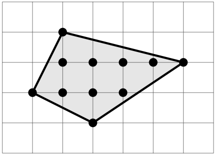


$\bar{\mathcal{C}}_0^d[X]$

$=$



$\bar{\mathcal{C}}_0^d[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex

Full convexity

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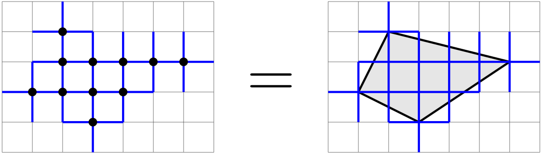
A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathcal{C}}^d_k[X] = \bar{\mathcal{C}}^d_k[\mathrm{Cvxh}\,(X)]. \quad (1)$$

Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.



$\bar{\mathcal{C}}^d_1[X] = \bar{\mathcal{C}}^d_1[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex, and 1-convex

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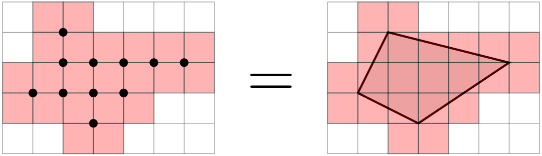

$\bar{\mathcal{C}}_2^d[X]$ $\qquad$ $\bar{\mathcal{C}}_2^d[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex, and 1-convex, and 2-convex, hence fully convex.

Full convexity

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A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

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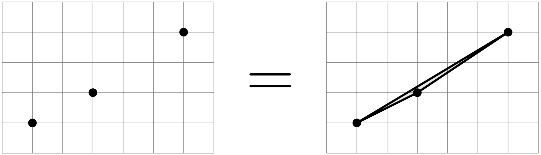
Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.



$\bar{\mathcal{C}}^d_0[X]$ $\quad$ $\bar{\mathcal{C}}^d_0[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex

Full convexity

Definition (Full convexity)

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$$\bar{\mathcal{C}}^d_k[X] = \bar{\mathcal{C}}^d_k[\mathrm{Cvxh}\,(X)]. \tag{1}$$

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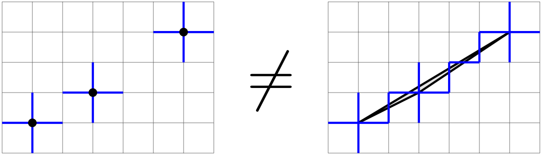

$\bar{\mathcal{C}}^d_1[X]$ $\bar{\mathcal{C}}^d_1[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex, but neither 1-convex

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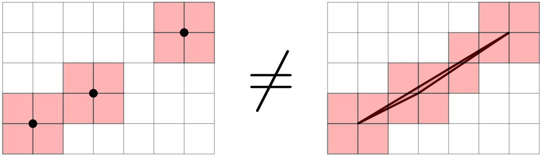

$\bar{\mathcal{C}}_2^d[X]$ $\quad\neq\quad$ $\bar{\mathcal{C}}_2^d[\mathrm{Cvxh}\,(X)]$

X is digitally 0-convex, but neither 1-convex, nor 2-convex.

Full convexity

Definition (Full convexity)

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

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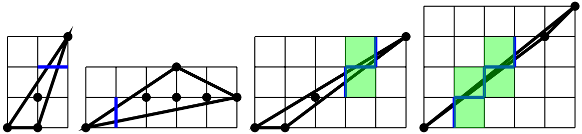
Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.

Full convexity eliminates too thin digital convex sets in arbitrary dimension.



Elementary properties

Lemma

Digital 0-convexity is classical digital convexity (H-convexity).

Lemma

A finite non-empty subset $X \subset \mathbb{Z}^d$ *is* digitally k-convex *for* $0 \leqslant k \leqslant d$ *iff* $\#\left(\bar{\mathcal{C}}^d_k[X]\right) \geqslant \#\left(\bar{\mathcal{C}}^d_k[\mathrm{Cvxh}\,(X)]\right)$.

Lemma

If $Z \subset \mathbb{Z}^d$ *is digitally* k*-convex for* $0 \leqslant k < d$*, it is also digitally* d*-convex, hence fully convex.*

Proof.

Use Jordan-Brouwer surface separation theorem. □

Elementary properties

Lemma

Digital 0-convexity is classical digital convexity (H-convexity).

Lemma

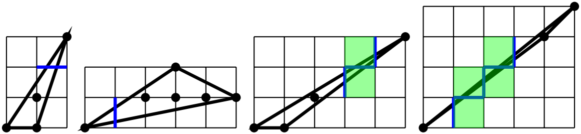
A finite non-empty subset $X \subset \mathbb{Z}^d$ is digitally k-convex *for $0 \leqslant k \leqslant d$ iff* $\#\left(\bar{\mathcal{C}}^d_k[X]\right) \geqslant \#\left(\bar{\mathcal{C}}^d_k[\mathrm{Cvxh}\,(X)]\right)$.

Lemma

If $Z \subset \mathbb{Z}^d$ is digitally k-convex for $0 \leqslant k < d$, it is also digitally d-convex, hence fully convex.

Proof.

Use Jordan-Brouwer surface separation theorem. □



Digital connectedness

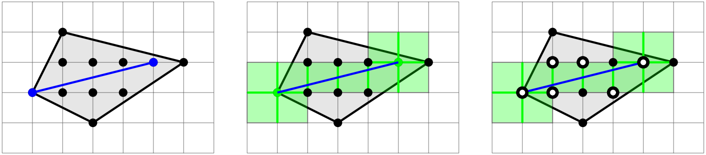
Theorem

If the digital set $X \subset \mathbb{Z}^d$ is fully convex, then X is d-connected.

Proof.

- for $x, y \in X$, segment $[x, y]$ intersects cells $c_0, c_1, \ldots, c_m$,
- full convexity $\Rightarrow$ each c_i touches at least one corner $z_i \in X$,
- each c_i is a face of c_{i+1} or inversely,
- implies z_i and z_{i+1} shares a unit cube, hence d-connected

$\square$



$[x, y]$ intersected cells c_i points z_i

Simple connectedness

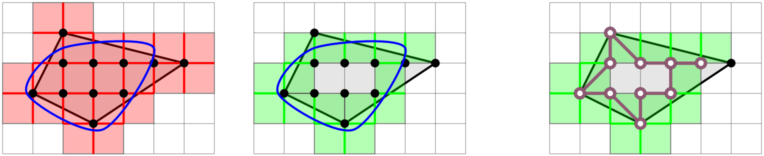
Theorem

If the digital set $X \subset \mathbb{Z}^d$ is fully convex, then the body of its intersection complex is **simply** *connected.*

Proof.

- let $\mathcal{A} := \{x(t), t \in [0,1]\}$ be a closed curve in $\left\|\bar{\mathcal{C}}^d[X]\right\|$
- sequence of intersected cells $c_i \in \bar{\mathcal{C}}^d[X]$
- sequence of associated corners $z_i \in X$
- homotopy between $\mathcal{A}$ and path $z_0 - z_1 - \cdots - z_n - z_0$
- path $z_0 - z_1 - \cdots - z_n - z_0$ subset of $\mathrm{Cvxh}\,(X) \Rightarrow$ contractible

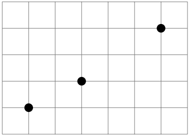
$\square$



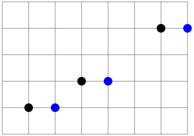
$\mathcal{A}$ | intersected cells (c_i) | path $z_0 - z_1 - \cdots - z_n - z_0$

Discrete Minkowski sum U_α

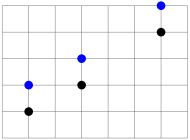
- let $X \subset \mathbb{Z}^d$, denote $\mathrm{e}_i(X)$ the translation of X with axis vector e_i
- let $I^d := \{1, \ldots, d\}$ be the set of possible directions
- let $U_\emptyset(X) := X$, and, for $\alpha \subset I^d$ and $i \in \alpha$, recursively $U_\alpha(X) := U_{\alpha\setminus i}(X) \cup \mathrm{e}_i(U_{\alpha\setminus i}(X))$.



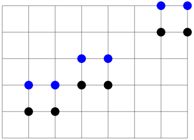
$U_\emptyset(X) = X$



$U_{\{1\}}(X) = U_\emptyset(X) \cup \mathrm{e}_1(U_\emptyset(X))$



$U_{\{2\}}(X) = U_\emptyset(X) \cup \mathrm{e}_2(U_\emptyset(X))$



$U_{\{1,2\}}(X) = U_{\{1\}}(X) \cup \mathrm{e}_2(U_{\{1\}}(X))$

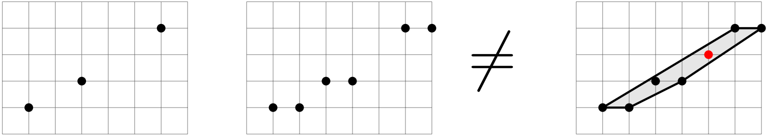
A morphological characterization

Theorem

A non empty subset $X \subset \mathbb{Z}^d$ is digitally k-convex for $0 \leqslant k \leqslant d$ iff

$$\forall \alpha \in I_k^d, U_\alpha(X) = \mathrm{Cvxh}\left(U_\alpha(X)\right) \cap \mathbb{Z}^d. \qquad (2)$$

It is thus fully convex if the previous relations holds for all $k, 0 \leqslant k \leqslant d$.



X $\quad$ $U_{\{1\}}(X)$ $\quad$ $\mathrm{Cvxh}\left(U_{\{1\}}(X)\right) \cap \mathbb{Z}^d$

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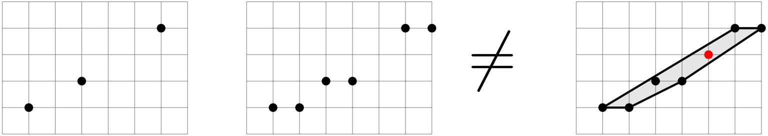

X $\quad$ $U_{\{1\}}(X)$ $\quad$ $\neq$ $\quad$ $\mathrm{Cvxh}\left(U_{\{1\}}(X)\right) \cap \mathbb{Z}^d$

Algorithm in arbitrary dimension

$U_\alpha(X)$ easily computed while convex hull algorithms exist in arbitrary dimension. Slowest part is lattice point enumeration in convex hull.