A Geometric Approach to Coalition Resilient Outcomes in Social Graphs modeled by the Max k-Cut Game

Giulia Palma

Meeting on Tomography and Applications Discrete Tomography, Neuroscience and Image Reconstruction 16th Edition

04th May 2022

What the talk is about

- This talk provides a formal proof of the conjecture stating that optimal colorings in max k-cut games over unweighted and undirected graphs do not allow the presence of any strongly divergent coalition. Specifically, we formally prove that do not exist any subsets of nodes able to increase their own payoffs simultaneously.

What the talk is about

- This talk provides a formal proof of the conjecture stating that optimal colorings in max k-cut games over unweighted and undirected graphs do not allow the presence of any strongly divergent coalition. Specifically, we formally prove that do not exist any subsets of nodes able to increase their own payoffs simultaneously.
- The result is obtained by splitting the nodes of the graph into three subsets: the coalition itself, the coalition boundary and the nodes without relationship with the coalition.

What the talk is about

- Moreover, we prove an intermediate result highlighting the fact that any payoff improvement will correspondingly imply a larger payoff reduction and we propose a novel approach based on discrete geometry and algorithms on graphs to study the properties of the adjacency matrix of the graph.

Introduction, The problem

- The **max k-cut problem** consists in assigning colors to the vertices of a graph with the aim of ensure the highest heterogeneity of colors in the graph, that is, by partitioning the vertices of the graph, in such a way that each of them has the largest possible number of nodes having a different color from its own.

Introduction, The problem

- The **max k-cut problem** consists in assigning colors to the vertices of a graph with the aim of ensure the highest heterogeneity of colors in the graph, that is, by partitioning the vertices of the graph, in such a way that each of them has the largest possible number of nodes having a different color from its own.
- This problem is particularly interesting not only from a theoretical point of view, but also from the **applicative perspective**. Indeed, it is linked to significant real-life applications with selfish agents and, moreover, it is related to fundamental classes of games.

Introduction, Applications

In many contexts, individuals may find themselves wanting to choose from multiple options set, the least popular, so as not to overlap with the choices of others in order to have a global maximum reward society.

Introduction, Applications

In many contexts, individuals may find themselves wanting to choose from multiple options set, the least popular, so as not to overlap with the choices of others in order to have a global maximum reward society. Some examples are:

- drilling companies that have to choose a land in which to dig by minimizing the number of competitors digging in the same terrain;

Introduction, Applications

In many contexts, individuals may find themselves wanting to choose from multiple options set, the least popular, so as not to overlap with the choices of others in order to have a global maximum reward society. Some examples are:

- drilling companies that have to choose a land in which to dig by minimizing the number of competitors digging in the same terrain;
- the choice of frequencies on which to operate by several telematic operators;

Introduction, Applications

Some examples are:

- a company that has to decide which products to develop in order to minimize the redundancy between these;

Introduction, Applications

Some examples are:

- a company that has to decide which products to develop in order to minimize the redundancy between these;
- biodiversity in the environmental field, where to have a greater number of animal species e plants in the same ecosystem guarantees better resistance to perturbations.

Introduction, State of the art

- A **strong equilibrium** corresponds to assign colorings in which no coalition, assuming the actions of its complements as given, can cooperatively deviate in a way that benefits all of its members, in other words each player of the coalition strictly improves its utility.

Introduction, State of the art

- A **strong equilibrium** corresponds to assign colorings in which no coalition, assuming the actions of its complements as given, can cooperatively deviate in a way that benefits all of its members, in other words each player of the coalition strictly improves its utility.
- The most important existing result has been provided by Carosi et al., who showed that on undirected unweighted graphs, optimal colorings are **5-Strong Equilibria** (5-SE), i.e. colorings in which no coalition of at most 5 vertices can profitably deviate.

Introduction, Our results

- In this talk we consider a **max k-cut game** played by n individuals or players. The individuals are assumed to be arranged on an undirected and unweighted graph; specifically, nodes of the graph represent the individuals, while the edges describe the connections among them. The strategy space of each player is composed by a set of k available colors (i.e. $\{1, \ldots, k\}$).

Introduction, Our results

- In this talk we consider a **max k-cut game** played by n individuals or players. The individuals are assumed to be arranged on an undirected and unweighted graph; specifically, nodes of the graph represent the individuals, while the edges describe the connections among them. The strategy space of each player is composed by a set of k available colors (i.e. $\{1, \ldots, k\}$).
- Given a strategy profile or a coloring, the **utility** (or payoff) of a player g is the sum of the weights of edges $\{g, v\}$ incident to g, such that the color chosen by g is different from the one chosen by v.

Main approaches

1. Set as a goal for each individual to share the chosen strategy with the fewer individuals, it is natural to model this type of problem through **Game Theory**, which allows to combine the strategies of individuals in relation to the strategies of others.
2. Instead of formulating a basic approach based only on the notions and tools of game theory, we can use a novel approach based on **Discrete Geometry and algorithms on graphs** to study the properties of the adjacency matrix of the graph and obtain significant information on the coalition and its boundary.

Main notions and notations

The population

- $\mathcal{V} = \{1, \ldots, N\}$ is the set of nodes, with $N \geq 2$.

Main notions and notations

The population

- $\mathcal{V} = \{1, \ldots, N\}$ is the set of nodes, with $N \geq 2$.
- $\mathrm{A} = \{a_{v,w}\} \in \{0,1\}^{N \times N}$ is the undirected adjacency matrix ($\mathrm{A} = \mathrm{A}^\top$)

Main notions and notations

The population

- $\mathcal{V} = \{1, \dots, N\}$ is the set of nodes, with $N \geq 2$.
- $\mathrm{A} = \{a_{v,w}\} \in \{0,1\}^{N \times N}$ is the undirected adjacency matrix ($\mathrm{A} = \mathrm{A}^\top$)
- $\delta_v = \sum_{w \in \mathcal{V}} a_{v,w}$ is the degree of node v

Main notions and notations

The population

- $\mathcal{V} = \{1, \dots, N\}$ is the set of nodes, with $N \geq 2$.
- $\mathrm{A} = \{a_{v,w}\} \in \{0,1\}^{N \times N}$ is the undirected adjacency matrix ($\mathrm{A} = \mathrm{A}^\top$)
- $\delta_v = \sum_{w \in \mathcal{V}} a_{v,w}$ is the degree of node v
- $\delta_v(S) = \sum_{w \in S} a_{v,w}$, with $S \subset \mathcal{V}$

Main notions and notations

The population

- $\mathcal{V} = \{1, \dots, N\}$ is the set of nodes, with $N \geq 2$.
- $\mathrm{A} = \{a_{v,w}\} \in \{0,1\}^{N \times N}$ is the undirected adjacency matrix ($\mathrm{A} = \mathrm{A}^\top$)
- $\delta_v = \sum_{w \in \mathcal{V}} a_{v,w}$ is the degree of node v
- $\delta_v(S) = \sum_{w \in S} a_{v,w}$, with $S \subset \mathcal{V}$
- Given $S_1 \subseteq S_2$, then $\delta_v(S_2 \setminus S_1) = \delta_v(S_2) - \delta_v(S_1)$

Main notions and notations

The colorings

- $\mathcal{K} = \{1, \dots, M\}$ is the **set of colors**, with $M \geq 2$.

Main notions and notations

The colorings

- $\mathcal{K} = \{1, \ldots, M\}$ is the **set of colors**, with $M \geq 2$.
- Given a set $S \subseteq \mathcal{V}$, the set of colors in S is $\mathcal{K}(S) \subseteq \mathcal{K}$.

Main notions and notations

The colorings

- $\mathcal{K} = \{1, \dots, M\}$ is the **set of colors**, with $M \geq 2$.
- Given a set $S \subseteq \mathcal{V}$, the set of colors in S is $\mathcal{K}(S) \subseteq \mathcal{K}$.
- A **coloring** $\sigma \in \mathcal{K}^N$ is an assignment of colors to each node of the graph.

Main notions and notations

The payoff

- $\mu_v(\sigma) = \sum_{\substack{w \in \mathcal{V} \\ \sigma_w \neq \sigma_v}} a_{v,w}$ is the **payoff of node** v;

Main notions and notations

The payoff

- $\mu_v(\sigma) = \sum\limits_{\substack{w \in \mathcal{V} \\ \sigma_w \neq \sigma_v}} a_{v,w}$ is the **payoff of node** v;
- $\mu_v(S, \sigma) = \sum\limits_{\substack{w \in S \\ \sigma_w \neq \sigma_v}} a_{v,w}$ is the payoff of node v gained with players in $S \subseteq \mathcal{V}$;
- $\mu(S, \sigma) = \sum\limits_{v \in S} \mu_v(\sigma)$ is the payoff of the set $S \subseteq \mathcal{V}$. Notice that $\mu(\mathcal{V}, \sigma)$ is the payoff of the whole population $\mathcal{V}$ which is using the coloring σ.

Main notions and notations

The payoff

- $\Delta\mu_v(\gamma, \sigma) = \mu_v(\gamma) - \mu_v(\sigma)$ is the **payoff difference** for v when the coloring of the graph is changed from σ to γ;

Main notions and notations

The payoff

- $\Delta\mu_v(\gamma,\sigma) = \mu_v(\gamma) - \mu_v(\sigma)$ is the **payoff difference** for v when the coloring of the graph is changed from σ to γ;
- $\Delta\mu(S,\gamma,\sigma) = \sum_{v\in S} \Delta\mu_v(\gamma,\sigma) = \mu(S,\gamma) - \mu(S,\sigma)$ is the **variation of payoff of players in** $S \subseteq \mathcal{V}$ when the coloring changes from σ to γ. Notice that $\Delta\mu(\mathcal{V},\gamma,\sigma)$ is the global variation of payoff the coloring changes from σ to γ.

Main notions and notations

Definition (Deviating coalition)

Given two colorings σ and γ and a coalition C, we say that C **deviates** from σ to γ if and only if $\sigma_v = \gamma_v \ \forall v \notin C$ and $\sigma_v \neq \gamma_v \ \forall v \in C$.

Main notions and notations

Definition (Deviating coalition)

Given two colorings σ and γ and a coalition C, we say that C **deviates** from σ to γ if and only if $\sigma_v = \gamma_v \ \forall v \notin C$ and $\sigma_v \neq \gamma_v \ \forall v \in C$.

Definition (Strong deviation)

Given two colorings σ and γ and a coalition C, we say that C **strongly deviates** from σ to γ if and only if C deviates from σ to γ and

$$\Delta\mu_v(\gamma, \sigma) \geq 1 \ \forall v \in C.$$

Main notions and notations

Definition (Optimal coloring)

A coloring σ is **optimal** if and only if $\mu(\mathcal{V}, \sigma)$ is maximum, or equivalently

$$\Delta\mu(\mathcal{V}, \gamma, \sigma) \leq 0 \ \forall \gamma \in \mathcal{K}^N.$$

The equal sign holds if and only if γ is also an optimal coloring.

Main notions and notations

Remark

If σ is an optimal coloring, then:

- there are nodes such that $\mu_v(\sigma) = \delta_v$. In this case, v is not connected to any player w such that $\sigma_v = \sigma_w$;

Main notions and notations

Remark

If σ is an optimal coloring, then:

- there are nodes such that $\mu_v(\sigma) = \delta_v$. In this case, v is not connected to any player w such that $\sigma_v = \sigma_w$;
- there are nodes such that $\mu_v(\sigma) < \delta_v$. In this case, v is connected to at least one player w such that $\sigma_v = \sigma_w$.

Main notions and notations

Remark

No node such that $\mu_v(\sigma) = \delta_v$ can belong to a strongly deviating coalition.

Main notions and notations

Remark

No node such that $\mu_v(\sigma) = \delta_v$ can belong to a strongly deviating coalition.

Definition

Given a coloring $\gamma \in \mathcal{K}^N$, a color $a \in \mathcal{K}$ and a set $S \subseteq \mathcal{V}$, we define:

$$S_a(\gamma) = \{v \in S : \gamma_v = a\}.$$

Main notions and notations

Definition

Given a coloring $\gamma \in \mathcal{K}^N$, a color $a \in \mathcal{K}$, a set $S \subseteq \mathcal{V}$ and a node $v \in \mathcal{V}$, we define:

$$S_{a,v}(\gamma) = \{w \in S_a(\gamma) : a_{v,w} = 1\},$$

and

Main notions and notations

Definition

Given a coloring $\gamma \in \mathcal{K}^N$, a color $a \in \mathcal{K}$, a set $S \subseteq \mathcal{V}$ and a node $v \in \mathcal{V}$, we define:

$$S_{a,v}(\gamma) = \{w \in S_a(\gamma) : a_{v,w} = 1\},$$

and

$$\delta_v(S_a, \gamma) = |S_{a,v}(\gamma)|.$$

Our results, The idea of our approach

Aim: Prove that, given an optimal system, there is no group of individuals who autonomously come to an agreement, forming a coalition, to try to earn more at the expense of the overall system.

Our results, The idea of our approach

To prove the impossibility of building such a coalition, we basically base on two facts:

Our results, The idea of our approach

To prove the impossibility of building such a coalition, we basically base on two facts:

1. **Fact 1:** Given an optimal strategy setup, each individual will be connected in the graph to at least one individual of every other strategy available.

Our results, The idea of our approach

To prove the impossibility of building such a coalition, we basically base on two facts:

1. **Fact 1:** Given an optimal strategy setup, each individual will be connected in the graph to at least one individual of every other strategy available.
2. **Fact 2:** Given an optimal strategy setup, for each individual the number of individuals with his own strategy connected to him is less than number of individuals with different strategies from their own connected to him.

Our results, The idea of our approach

To prove the impossibility of building such a coalition, we basically base on two facts:

1. **Fact 1:** Given an optimal strategy setup, each individual will be connected in the graph to at least one individual of every other strategy available.
2. **Fact 2:** Given an optimal strategy setup, for each individual the number of individuals with his own strategy connected to him is less than number of individuals with different strategies from their own connected to him.

These facts make us understand that optimality corresponds in a natural way the diversification of strategies.

Our results

Let σ be an optimal coloring. We define:

Our results

Let σ be an optimal coloring. We define:

- $C^\sigma = \{v \in \mathcal{V} : \mu_v(\sigma) < \delta_v\}$, the set of the nodes **candidate to belong to a strong deviation**;

Our results

Let σ be an optimal coloring. We define:

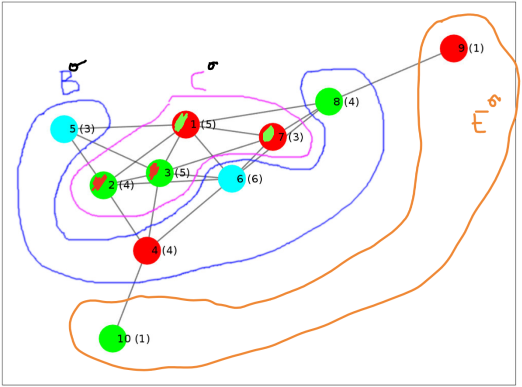
- $C^\sigma = \{v \in \mathcal{V} : \mu_v(\sigma) < \delta_v\}$, the set of the nodes **candidate to belong to a strong deviation**;
- $B^\sigma = \{v \in \mathcal{V} : \mu_v(\sigma) = \delta_v \wedge \exists w \in C^\sigma : a_{v,w} = 1\}$, the **boundary set of** C^σ, i.e. it contains all nodes not in C^σ which are connected to some node in C^σ;

Our results

Let σ be an optimal coloring. We define:

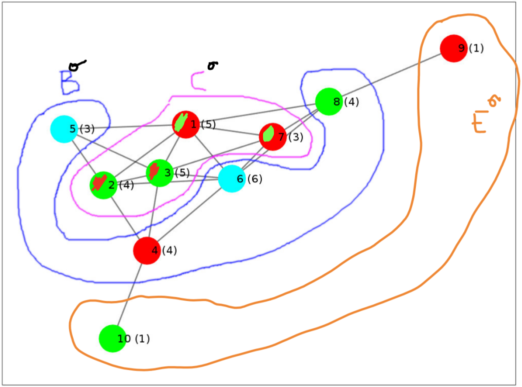
- $C^\sigma = \{v \in \mathcal{V} : \mu_v(\sigma) < \delta_v\}$, the set of the nodes **candidate to belong to a strong deviation**;
- $B^\sigma = \{v \in \mathcal{V} : \mu_v(\sigma) = \delta_v \wedge \exists w \in C^\sigma : a_{v,w} = 1\}$, the **boundary set of** C^σ, i.e. it contains all nodes not in C^σ which are connected to some node in C^σ;
- $E^\sigma = \{v \in \mathcal{V} : \mu_v(\sigma) = \delta_v \wedge a_{v,w} = 0 \ \forall w \in C^\sigma\} = \mathcal{V} \setminus (C^\sigma \cup B^\sigma)$, the **external set of** C^σ, i.e. the set of the nodes which are not connected to C^σ.

An example



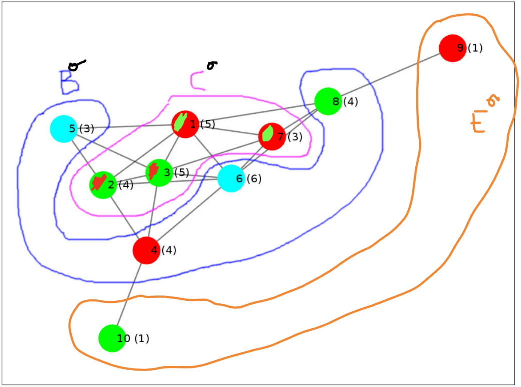
Consider a graph $G = (V, E)$ as depicted in the figure, referred to an optimal coloring σ. Note that vertex v_1 belongs to the set C^{σ}, since its degree 5 is less than its profit 4.

An example



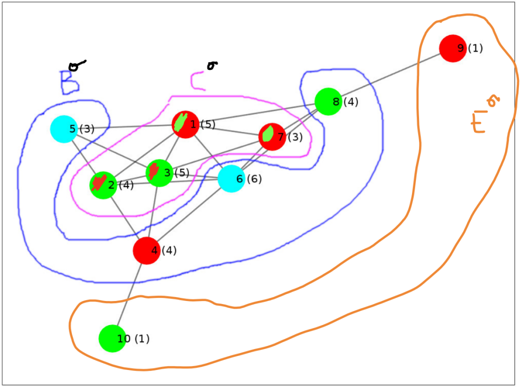
Similarly we reason for the vertices v_2, v_3 and v_7.

An example



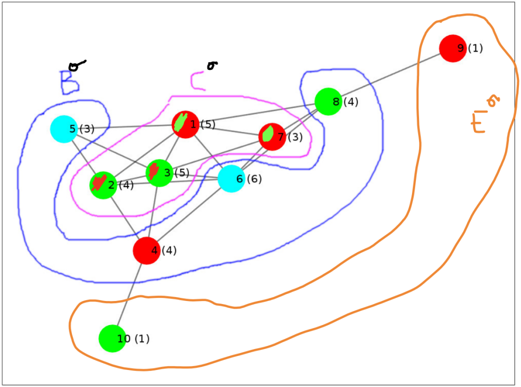
Instead, vertex v_5 belongs to the set B^{σ}, since its profit is exactly equal to its degree, i.e. 3. and v_5 is the neighbor of at least one vertex in C^{σ}, e.g. v_1.

An example



Similarly we reason for the vertices v_4 and v_8.

An example



Similarly we reason for the vertices v_4 and v_8. Finally, v_9 belongs to E^σ as its profit is exactly equal to its degree, i.e. 1 and has no neighbors in C^σ.

Our results

Remark

Let $\sigma \in \mathcal{K}^N$ be an optimal coloring.

Our results

Remark

Let $\sigma \in \mathcal{K}^N$ be an optimal coloring. Then:

$$\delta_v(B^{\sigma}_{\sigma_v}(\sigma)) = 0 \ \forall v \in C^{\sigma}. \quad (1)$$

Our results

Remark

Let $\sigma \in \mathcal{K}^N$ be an optimal coloring. Then:

$$\delta_v(B^{\sigma}_{\sigma_v}(\sigma)) = 0 \ \forall v \in C^{\sigma}. \tag{1}$$

Indeed, suppose that for a player $v \in C^{\sigma}$, there exists a $w \in B^{\sigma}_{\sigma_v}(\sigma)$ connected to v.

Our results

Remark

Let $\sigma \in \mathcal{K}^N$ be an optimal coloring. Then:

$$\delta_v(B^{\sigma}_{\sigma_v}(\sigma)) = 0 \ \forall v \in C^{\sigma}. \tag{1}$$

Indeed, suppose that for a player $v \in C^{\sigma}$, there exists a $w \in B^{\sigma}_{\sigma_v}(\sigma)$ connected to v. But this means that $\mu_w(\sigma) < \delta_w$, since it is connected to v which has the same color.

Our results

Remark

Let $\sigma \in \mathcal{K}^N$ be an optimal coloring. Then:

$$\delta_v(B^{\sigma}_{\sigma_v}(\sigma)) = 0 \ \forall v \in C^{\sigma}. \tag{1}$$

Indeed, suppose that for a player $v \in C^{\sigma}$, there exists a $w \in B^{\sigma}_{\sigma_v}(\sigma)$ connected to v. But this means that $\mu_w(\sigma) < \delta_w$, since it is connected to v which has the same color. This is contradiction with the membership of w in the set B^{σ}.

Our results

Remark

Hence,

$$\delta_v(B^{\sigma}_{\sigma_v}(\sigma)) = 0 \ \forall v \in C^{\sigma}.$$

Our results

Remark

Hence,

$$\delta_v(B^\sigma_{\sigma_v}(\sigma)) = 0 \ \forall v \in C^\sigma.$$

Moreover, similarly, from the definition of C^σ, it follows that:

$$\delta_v(C^\sigma_{\sigma_v}(\sigma)) \geq 1. \tag{2}$$

Our results

Remark

Concerning the case $C^\sigma = \emptyset$, we can make the following topological observations.

- If a graph is a **star**, then two colors are enough to have $C^\sigma = \emptyset$. In fact, it is sufficient to color the central vertex of one color and the remaining nodes of the other.

Our results

Remark

Concerning the case $C^\sigma = \emptyset$, we can make the following topological observations.

- If a graph is a **star**, then two colors are enough to have $C^\sigma = \emptyset$. In fact, it is sufficient to color the central vertex of one color and the remaining nodes of the other.
- If a graph is **bipartite**, then two colors are enough to have $C^\sigma = \emptyset$. Indeed, the vertices of such a graph can be partitioned into two sets and it will be sufficient to color the nodes in one of the two sets of one color and the other ones in the other set with the remaining color.

Our results

Remark

- If a graph is **complete**, since each node is connected to all the others, the only possibility to have $C^\sigma = \emptyset$ is that the number of colors is greater than or equal to the number of nodes.

Our results

Remark

- If a graph is **complete**, since each node is connected to all the others, the only possibility to have $C^\sigma = \emptyset$ is that the number of colors is greater than or equal to the number of nodes.
- If a graph is such that **each node has a degree less than the number of colors** then there exists an optimal coloring such that $C^\sigma = \emptyset$.

Our results

Lemma

Let $\sigma \in \mathcal{K}^N$ be an optimal coloring.

Our results

Lemma

Let $\sigma \in \mathcal{K}^N$ be an optimal coloring. Then:

$$\forall v \in C^{\sigma}, \forall b \in \mathcal{K} \setminus \{\sigma_v\}, \delta_v(C_b^{\sigma}(\sigma)) + \delta_v(B_b^{\sigma}(\sigma)) \geq 1.$$

Our results

Lemma

Let $\sigma \in \mathcal{K}^N$ be an optimal coloring.

Our results

Lemma

Let $\sigma \in \mathcal{K}^N$ *be an optimal coloring.*

Then: $\forall v \in C^\sigma, \forall b \in \mathcal{K} \setminus \{\sigma_v\}, \delta_v(C^\sigma_{\sigma_v}(\sigma)) \leq \delta_v(C^\sigma_b(\sigma)) + \delta_v(B^\sigma_b(\sigma)).$

Our results

Lemma

Let $\sigma \in \mathcal{K}^N$ *be an optimal coloring.*

Then: $\forall v \in C^\sigma, \forall b \in \mathcal{K} \setminus \{\sigma_v\}, \delta_v(C^\sigma_{\sigma_v}(\sigma)) \leq \delta_v(C^\sigma_b(\sigma)) + \delta_v(B^\sigma_b(\sigma))$.

This Lemma asserts that for every player in C^σ, the number of nodes connected to it with the same color is always lower than the number of connected nodes with different colors. Therefore, since no color different from σ_v can provide higher payoff, v **cannot change unilaterally and with profit its own strategy**.

Our results

Remark

From the definitions of the sets C^σ, B^σ and E^σ and the previous lemmas follow directly the following observations on the degrees of the nodes in each of these sets, referring to an optimal coloring $\sigma \in \mathcal{K}^N$:

Our results

Remark

From the definitions of the sets C^σ, B^σ and E^σ and the previous lemmas follow directly the following observations on the degrees of the nodes in each of these sets, referring to an optimal coloring $\sigma \in \mathcal{K}^N$:

- $\forall v \in C^\sigma, \delta_v \geq |K|$. Indeed, we have shown that each node v belonging to C^σ must have a neighbor in B^σ for each of the colors different from its own; furthermore, v must have a neighbor in C^σ with its own color, otherwise it would have profit equal to the degree, against the definition of C^σ.

Our results

Remark

- $\forall v \in B^\sigma, \delta_v \geq M$, where M is the number of colors in C^σ. Indeed, from the definition of the set C^σ it is clear that each vertex v in B^σ must have at least one neighbor in C^σ for each color present in C^σ, otherwise a node in C^σ not connected to v could take the color of v and increase its profit.

Our results

Remark

- $\forall v \in B^{\sigma}, \delta_v \geq M$, where M is the number of colors in C^{σ}. Indeed, from the definition of the set C^{σ} it is clear that each vertex v in B^{σ} must have at least one neighbor in C^{σ} for each color present in C^{σ}, otherwise a node in C^{σ} not connected to v could take the color of v and increase its profit.
- $\forall v \in E^{\sigma}, \delta_v \geq 0$. Indeed, a vertex in E^{σ} could be an isolated point and in this case have zero degree.

Our results

Theorem

Let $\sigma \in \mathcal{K}^N$ be an optimal coloring. Then any set $C \subseteq C^\sigma$ such that $\sigma_v = c, \ \forall v \in C$ is not a strongly deviating coalition.

Our results

A first theorem which constitutes a first step in proving the conjecture for the multicolor case.

Our results

A first theorem which constitutes a first step in proving the conjecture for the multicolor case.

Theorem

Let γ be a deviation on C^σ.

Our results

A first theorem which constitutes a first step in proving the conjecture for the multicolor case.

Theorem

Let γ be a deviation on C^σ. Then, $\gamma_v \neq \sigma_v, \forall v \in C^\sigma, \gamma_v = \sigma_v, \forall v \notin C^\sigma$.

Our results

A first theorem which constitutes a first step in proving the conjecture for the multicolor case.

Theorem

Let γ be a deviation on C^σ. Then, $\gamma_v \neq \sigma_v, \forall v \in C^\sigma$, $\gamma_v = \sigma_v, \forall v \notin C^\sigma$.

If γ satisfies the property:

$$\sum_{v \in C^\sigma} \delta_v(C^\sigma_{\gamma_v}(\gamma)) = \sum_{v \in C^\sigma} \delta_v(C^\sigma_{\sigma_v}(\sigma)) \tag{3}$$

Our results

A first theorem which constitutes a first step in proving the conjecture for the multicolor case.

Theorem

Let γ be a deviation on C^σ. Then, $\gamma_v \neq \sigma_v$, $\forall v \in C^\sigma$, $\gamma_v = \sigma_v$, $\forall v \notin C^\sigma$.

If γ satisfies the property:

$$\sum_{v \in C^\sigma} \delta_v(C^\sigma_{\gamma_v}(\gamma)) = \sum_{v \in C^\sigma} \delta_v(C^\sigma_{\sigma_v}(\sigma)) \tag{3}$$

Then γ is not a strong deviation.