Distance-mean functions and their geometric applications

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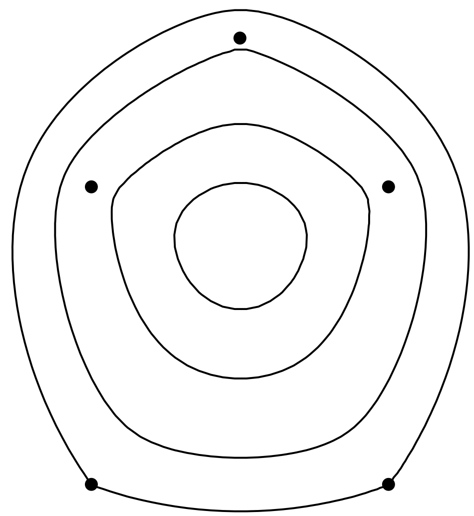
A distance-mean function measures the average distance of points from the elements of a given set of points (focal set) in the space. The level sets of a distance-mean function are called generalized conics.

Polyellipses (polyellipsoids): the level sets of a function measuring the arithmetic mean of distances from finitely many focal points (constant distance sum).



Polynomial lemniscates: the level sets of a function measuring the geometric mean of distances from finitely many focal points (constant distance product). In case of infinite focal points the average distance is typically given by integration over the focal set: the level sets (generalized conics) are Hausdorff limits of polyellipsoids (partitions, integral sums).

I. **Taxicab distance-mean functions** (reconstruction of planar bodies from the coordinate X-rays and some related results).

1. **A theoretical approach**. The general form of the functions we are interested in is

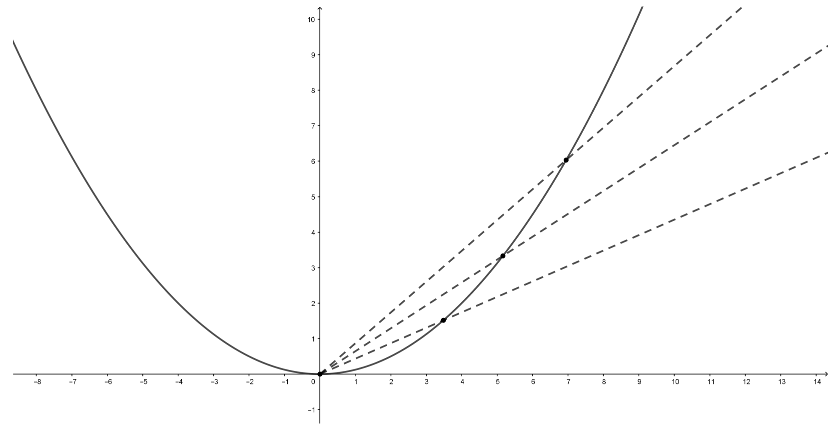
$$x \mapsto f_D(x) := \frac{1}{\mu(D)} \int_D u \circ d(x, y)\, d_\mu y, \tag{1}$$

where d measures the distance between the points, $D \subset \mathbb{R}^n$ is a compact subset with a finite positive measure with respect to μ, $u \colon \mathbb{R} \to \mathbb{R}$ is a strictly monote increasing convex function satisfying the initial condition $u(0) = 0$. Suppose that the distance function comes from a norm. According to the convexity of the integrand, the distance-mean function is a convex and, consequently, a continuous function. Using the increasing slope property of convex functions, we have that

$$\liminf_{t \to \infty} \frac{u(t)}{t} > 0. \tag{2}$$

Therefore the distance-mean function inherits a growth property of the form

$$\liminf_{\|x\| \to \infty} \frac{f_D(x)}{\|x\|} \geq \left(\liminf_{r \to \infty} \frac{u(r)}{r} \right) \mu(D) > 0. \tag{3}$$



The increasing slope property of convex functions

Theorem 1 *The sublevel sets of the distance-mean function are convex and compact.*

According to the convexity ($\Rightarrow$ continuity) of the distance-mean function it is enough to prove that the sublevel set $C_D := \{x \mid f_D(x) \leq c\} \subset \mathbb{R}^n$ is bounded: the existence of a sequence $x_n \in C_D$ such that $\lim_{n\to\infty} \|x_n\| = \infty$ implies that

$$\lim_{n\to\infty} \frac{f_D(x_n)}{\|x_n\|} \leq \lim_{n\to\infty} \frac{c}{\|x_n\|} = 0$$

which contradicts to the growth property (3).

Theorem 2 *The distance-mean function has a global minimizer.*

Weierstrass theorem: if all the level sets of a continuous function defined on a non-empty closed set in $\mathbb{R}^n$ are bounded, then it has a global minimizer.

Cs. Vincze and Á. Nagy, *On the theory of generalized conics with applications in geometric tomography*, J. of Approx. Theory 164 (2012), 371-390.

Cs. Vincze and Á Nagy, *On the average taxicab distance function and its applications*, Acta Appl. Math. 161, 201–220 (2019). https://doi.org/10.1007/s10440-018-0210-1.

2. **Bisection of bodies by coordinate hyperplanes** Let us choose $u(t) = t$ in formula (1) and suppose that distance measuring and integration are taken with respect to the taxicab distance

$$d_1(x, y) = \sum_{i=1}^{n} |x^i - y^i| \tag{4}$$

and the Lebesgue measure μ_n on $\mathbb{R}^n$, respectively.

Let K be a compact subset in $\mathbb{R}^n$, $\mu_n(K) = 1$ and consider the taxicab distance-mean function

$$f_K(x) = \int_K d_1(x,y)\,dy = \sum_{i=1}^{n} \int_K |x^i - y^i|\,dy.$$

Since the derivative of the integrand at x^i is ± 1 depending on $y^i < x^i$ or $x^i < y^i$, we can conclude that the value 1 occurs as many times as many points $y \in K$ is on the left hand side of x with respect to the i-th coordinate:

$$K \leq_i x^i := \{y \in K \mid y^i \leq x^i\}.$$

In a similar way, -1 occurs as many times as many points $y \in K$ is on the right hand side of x with respect to the i-th coordinate:

$$x^i \leq_i K := \{y \in K \mid x^i \leq y^i\}.$$

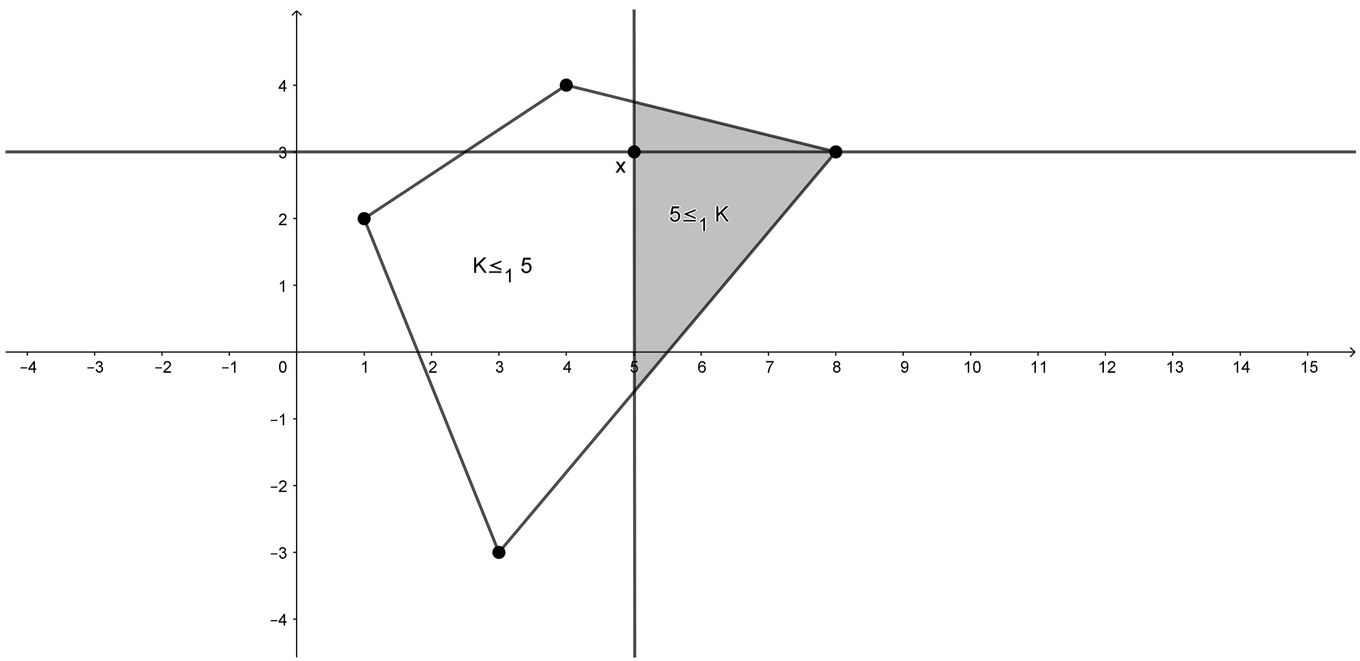
Since the set

$$K =_i x^i := \{y \in K \mid y^i = x^i\} \quad (i = 1, \ldots, n)$$

is of measure zero with respect μ_n we have that

$$D_i f_K(x) = \mu_n(K \leq_i x^i) - \mu_n(x^i \leq_i K) \quad (i = 1, \ldots, n). \tag{5}$$

Theorem 3 *The point $x \in \mathbb{R}^n$ is a minimizer of f_K if and only if each coordinate hyperplane at x divides K into two parts of equal measure.*



The partial derivative $D_1 f_K(x) > 0$, $x = (5, 3)$.

How to bisect a set of two parts of equal measure? Formula (5) shows that

$$|D_i f_K(x) - D_i f_K(y)| = 2\mu_n \left(\min\{x^i, y^i\} <_i K <_i \max\{x^i, y^i\}\right)$$

and the compactness of K implies that f_K has a Lipschitzian gradient.

Therefore the gradient descent method can be used to find the minimizer bisecting the measure of the integration domain K in the sense that each coordinate hyperplane passing through the minimizer divides the set into two parts of equal measure. Let us present the gradient descent method in terms of a stochastic algorithm: let P_k be a sequence of K-valued independent uniformly distributed random variables and consider the recursion

$$X_{k+1} = X_k - t_{k+1} Q_{k+1}, \tag{6}$$

where $X_0 := x_0 \in K$ is a starting point,

$$Q_{k+1} := \Big(\text{sgn }(X_k^1 - P_{k+1}^1), \ldots, \text{sgn }(X_k^n - P_{k+1}^n)\Big) \tag{7}$$

and the step size is a decreasing sequence of positive real numbers t_k satisfying conditions

$$\sum_{k=1}^{\infty} t_k = \infty \quad \text{and} \quad \sum_{k=1}^{\infty} t_k^2 < \infty. \tag{8}$$

Assuming $\mu_n(K) = 1$ we have the conditional probability

$$P(Q_{k+1} = (1, \ldots, 1) | X_k) = \mu_n \Big((K < X_k^1) \cap \ldots \cap (K < X_k^n) \Big) \tag{9}$$

because $Q_{k+1} = (1, \ldots, 1)$ means that X_k is greater than P_{k+1} with respect to the coordinatewise partial ordering $x \prec y \ \Leftrightarrow \ x^1 < y^1, \ldots, x^n < y^n$ and P_{k+1} is a uniformly distributed K-valued random variable In a similar way we have the conditional probability

$$P(Q_{k+1} = (1, -1, 1, \ldots, 1) | X_k) = \tag{10}$$

$$\mu_n \Big((K < X_k^1) \cap (X_k^2 < K) \cap (K < X_k^3) \cap \ldots \cap (K < X_k^n) \Big), \ldots$$

and so on. A direct computation shows that $\mathbb{E}(Q_{k+1} | X_k) = \text{grad } f_K(X_k)$.

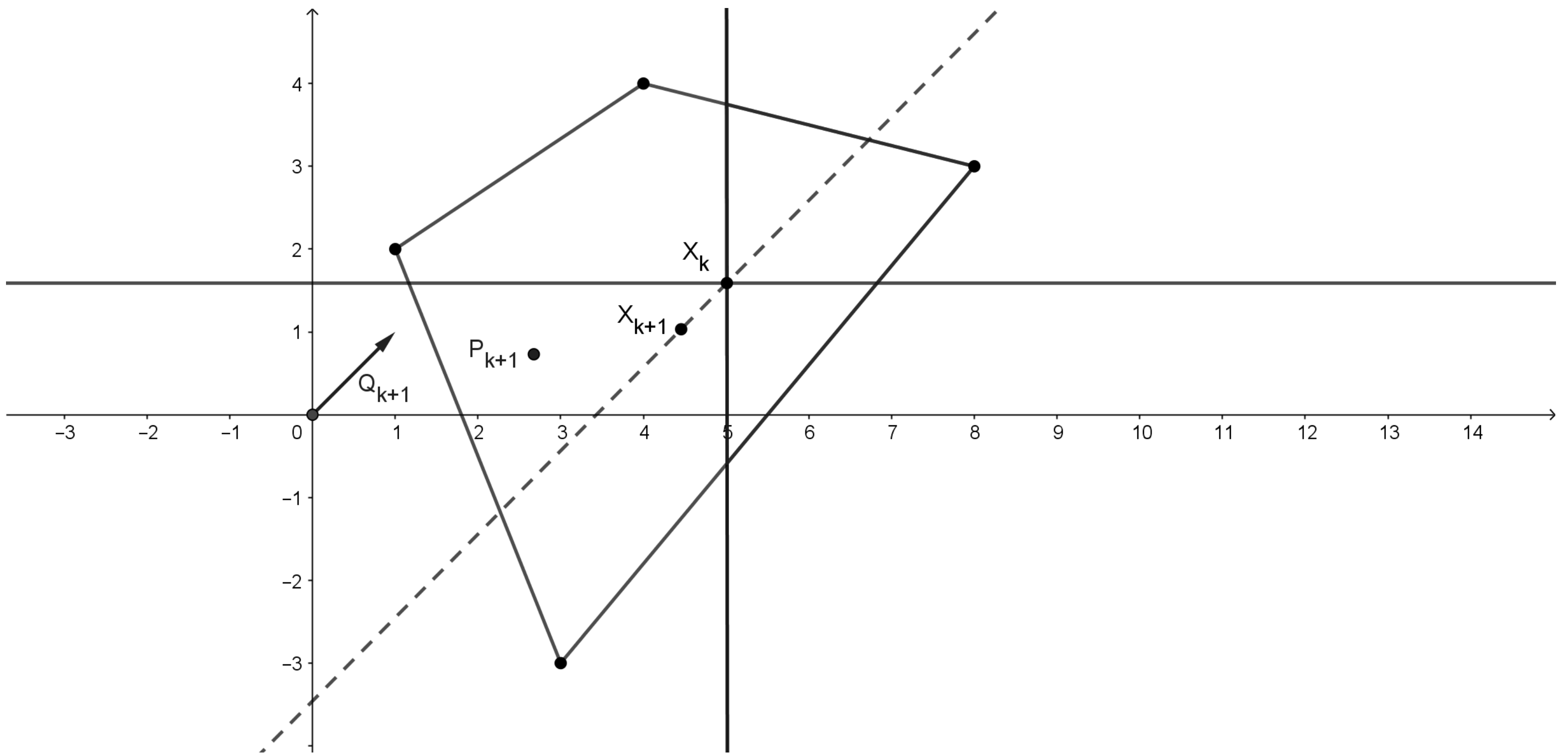
Theorem 4 *Let $K \subset \mathbb{R}^n$ be a connected compact body*[*]*. The sequence of the random variables X_k converges almost surely to the unique global minimizer x^* of the function f_K.*

Cs. Vincze and Á. Nagy, *On the theory of generalized conics with applications in geometric tomography*, J. of Approx. Theory 164 (2012), 371-390.

M. Barczy, Á Nagy, Cs. Noszály and Cs. Vincze, *A Robbins-Monro type algorithm for computing the global minimizer of generalized conic functions*, Optimization 64 (9) (2015), 1999-2020.

Cs. Vincze and Á Nagy, *On the average taxicab distance function and its applications*, Acta Appl. Math. 161, 201–220 (2019). https://doi.org/10.1007/s10440-018-0210-1.

[*]A nonemty compact set is called a body if it is the closure of its interior.



$$\text{The step } X_k \mapsto X_{k+1} = X_k - t_{k+1}Q_{k+1}.$$

The lines parallel to the coordinate axis at X_k divide the plane into four quadrants. The value of P_{k+1} is the position of the highest probability, i.e. it is in the quadrant containing the part of K of the highest measure. The gradient of f_K at X_k is pointed in the first quadrant represented by the value $(1, 1)$ of the stochastic vector Q_{k+1}.

Therefore the step of the highest probability is taken into the opposite direction.

3. **Applications in geometric tomography** The function f_K is strongly related to the parallel X-rays as follows: by the Cavaliéri principle, the formula

$$D_i f_K(x) = \mu_n(K \leq_i x^i) - \mu_n(x^i \leq_i K) \quad (i = 1, \ldots, n)$$

of the first partial derivatives implies that

$$D_i D_i f_K(x) =_{\text{a.e.}} 2X_i K(x^i) \quad (i = 1, \ldots, n), \tag{11}$$

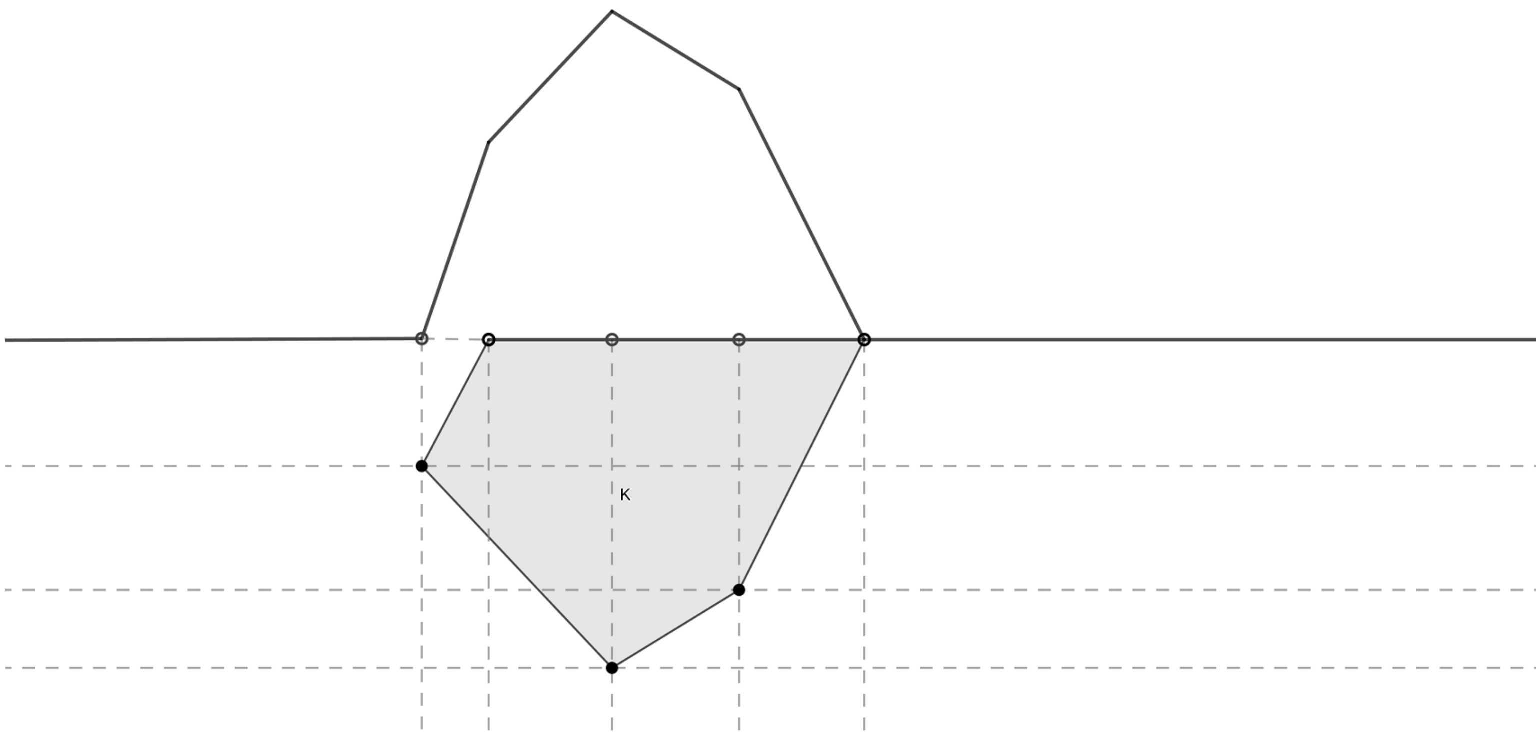
where $X_i K(x^i) := \mu_{n-1}(x^i =_i K)$ is the $(n-1)$-dimensional Lebesgue measure of the set

$$x^i =_i K := \{y \in K \mid y^i = x^i\}. \tag{12}$$

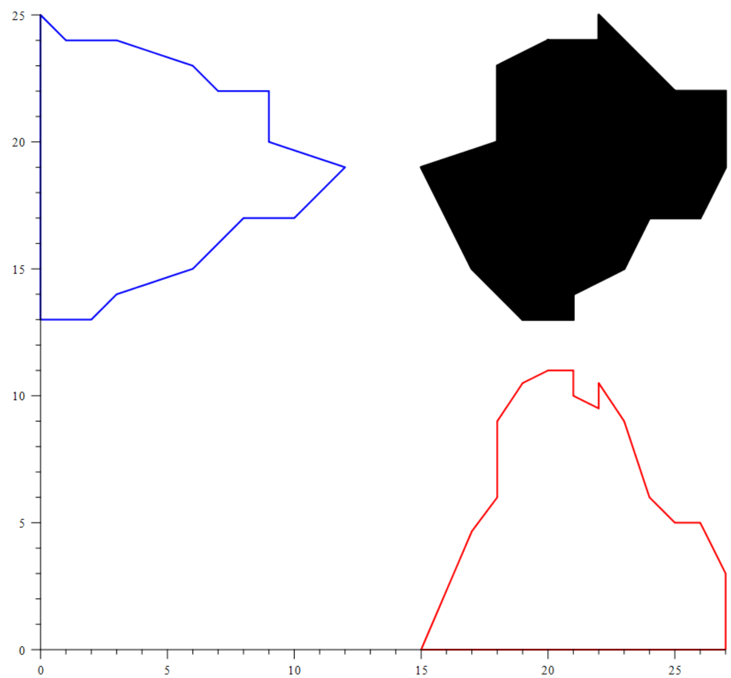
The functions

$$X_i K(t) := \mu_{n-1}(t =_i K) \quad (t \in \mathbb{R} \text{ and } i = 1, \ldots, n)$$

are called the coordinate X-rays of K (X-rays parallel to the coordinate hyperplanes).



The case of compact convex planar bodies (X-rays measure the length of the chords parallel to the coordinate axis):



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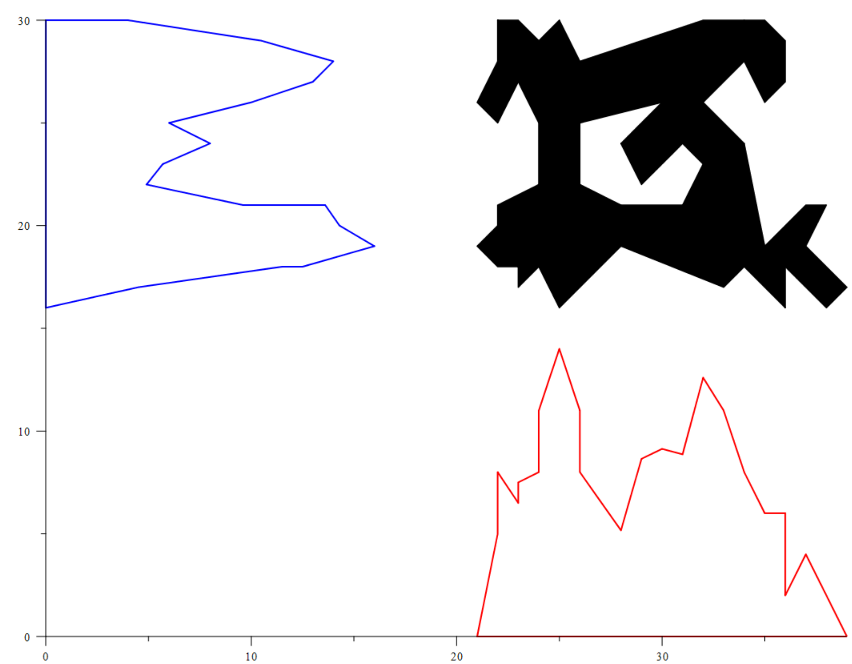

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In terms of the coordinate X-rays

$$f_K(x) = \int_K d_1(x,y)\,dy = \sum_{i=1}^{n} \int_K |x^i - y^i|\,dy = \sum_{i=1}^{n} \int\limits_{-\infty}^{\infty} |x^i - t| X_i K(t)\,dt. \quad (13)$$

Theorem 5 $f_K = f_L$ *iff the coordinate X-rays of* K *and* L *coincide almost everywhere.*

Let $K \subset \mathbb{R}^n$ be a compact set and consider the function

$$f_K^p \colon \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto f_K^p(x) := \int_K d_p(x,y)\,dy, \quad (14)$$

where d_p is the distance function coming from the p-norm

$$d_p(x,y) = \sqrt[p]{(x^1 - y^1)^p + \ldots + (x^n - y^n)^p} \quad (p \geq 1).$$

The sublevel sets of type

$$C := \{x \in \mathbb{R}^n \mid f_K^p(x) \leq c\}$$

are called generalized p-conic bodies. (They are convex compact subsets in the space.)

Theorem 6 *Let C be a generalized p-conic and suppose that C^* is a compact set of the same measure as C. If the distance-mean functions f_C^p and $f_{C^*}^p$ associated to C and C^* coincide then $C \approx C^*$, i.e. C is equal to C^* except on a set of measure zero.*

Let C be defined by the inequality $f_K^p(x) \leq c$. Observe that the decompositions $C = (C \setminus C^*) \cup (C \cap C^*)$ and $C^* = (C^* \setminus C) \cup (C^* \cap C)$ imply that $\mu_n(C \setminus C^*) = \mu_n(C^* \setminus C)$ because of $\mu_n(C^*) = \mu_n(C)$. On the other hand, the double integral form

$$\int_C f_K^p(x)\, dx = \int_C \int_K d_p(x,y)\, dy\, dx \tag{15}$$

shows that $\int_C f_K^p = \int_{C^*} f_K^p$ because of $f_C^p = f_{C^*}^p$. Therefore

$$\int_{C\setminus C^*} f_K^p = \int_C f_K^p - \int_{C\cap C^*} f_K^p = \int_{C^*} f_K^p - \int_{C\cap C^*} f_K^p = \int_{C^*\setminus C} f_K^p. \tag{16}$$

The sublevel rate c is working as an upper bound for f_K^p on $C \setminus C^*$ but it is a (strict) lower bound for f_K^p on $C^* \setminus C$ and we have that $\mu_n(C \setminus C^*) = \mu_n(C^* \setminus C) = 0$.

Corollary 1 *Generalized 1-conic bodies are determined by their X-rays parallel to the coordinate hyperplanes among compact sets.*

The coordinate X-rays determine both the measure and the taxicab distance-mean function $f_K = f_K^1$ of the sets.

Example. Circles are determined by their X-rays in the coordinate directions among compact bodies in the plane. They are level sets of the taxicab distance-mean function f_B associated to the circumscribed square: if $B :=$ conv $\{(0,0),(1,0),(1,1),(0,1)\}$; then we have that

$$f_B(x) = (x^1 - (1/2))^2 + (x^2 - (1/2))^2 + (1/2)$$

for any interior point $(x^1, x^2) \in B$

Cs. Vincze and Á. Nagy, *On the theory of generalized conics with applications in geometric tomography*, J. of Approx. Theory 164 (2012), 371-390.

Cs. Vincze and Á Nagy, *On the average taxicab distance function and its applications*, Acta Appl. Math. 161, 201–220 (2019). https://doi.org/10.1007/s10440-018-0210-1.

Concluding remarks. The taxicab distance-mean function f_K accumulates the coordinate X-ray information. Instead of the X-rays we can investigate a convex function independently of the convexity of the integration domain. Technics and results based on f_K are typically working in higher dimensional spaces as well.

4. **Reconstruction of planar sets by their coordinate X-rays**. The problem is also motivated by Gardner's unicity problem: Characterize those convex bodies that can be determined by two X-rays.

R. J. Gardner, Geometric Tomography, 2nd ed. Cambridge University Press, New York (2006).

Let K be a compact subset in the plane. The coordinate X-rays of K provide us to construct an axis-parallel bounding box containing K. Since f_K is also given by the coordinate X-rays the reconstruction is based on the best approximation of f_K by the distance-mean functions of a special class of sets. They are constituted by subrectangles of the bounding box under a given resolution: $f_{L_n} \to f_K$.

Let K^* be the limit set of a convergent subsequence in L_n. For a convergent reconstruction process we have to find special classes of sets to provide the continuity of the mapping $L \mapsto f_L$. The continuity implies that the distance-mean functions of K^* and K coincide. So do the coordinate X-rays (almost everywhere).

Cs. Vincze and Á. Nagy, *On the theory of generalized conics with applications in geometric tomography*, J. of Approx. Theory 164 (2012), 371-390.

Cs. Vincze and Á Nagy, *Reconstruction of hv-convex sets by their coordinate X-ray functions*, J. Math. Imaging and Vis. Volume 49 (3) (2014), pp. 569–582.

Cs. Vincze and Á. Nagy, *Generalized conic functions of hv-convex planar sets: continuity properties and X-rays*, Aequat. Math. 89 (4) (2015), pp. 1015-1030. Arxiv:1303.4412.

Let K be a compact subset in the plane. The outer parallel body K_ε is the union of closed Euclidean disks centered at the points of K with radius $\varepsilon > 0$. The Hausdorff distance between compact subsets K and L is given by the formula

$$H(K, L) := \inf\{\varepsilon > 0 \mid K \subset L_\varepsilon \text{ and } L \subset K_\varepsilon\}.$$

Theorem 7 *If $L_n \to K$ with respect to the Hausdorff metric then*

$$\limsup_{n\to\infty} f_{L_n}(x) \leq f_K(x).$$

The Hausdorff convergence $L_n \to K$ is called regular iff L_n tends to K in measure: $\lim_{n\to\infty} \mu_2(L_n) = \mu_2(K)$. Under the hypothesis of the Hausdorff convergence the regularity is equivalent to the convergence in symmetric difference: $\lim_{n\to\infty} \mu_2(L_n \triangle K) = 0$.

Theorem 8 *If the Hausdorff convergence $L_n \to K$ is regular then*

$$\lim_{n\to\infty} f_{L_n}(x) = f_K(x)$$

and the convergence $f_{L_n} \to f_K$ is uniform over any compact subset in $\mathbb{R}^2$.

X-regularity of the Hausdorff convergence $L_n \to K$ means the convergence of $I_n := \cap_{n=i}^{\infty} L_i$ to K in measure: $\lim_{n\to\infty} \mu_2(I_n) = \mu_2(K)$.

Theorem 9 *If the Hausdorff convergence $L_n \to K$ is X-regular then it is regular and the coordinate X-rays tend to the coordinate X-rays of the limit set almost everywhere.*

Example 1. If each L_n is obtained from a compact set L via finitely many Steiner symmetrizations and Euclidean isometries then the Hausdorff convergence $L_n \to K$ is regular.

G. Bianchi, A. Burchard, P. Gronchi and A. Volcic, *Convergence in Shape of Steiner Symmetrization*, Indiana University Math. Journal, Vol. 61, No. 4. (2012), 1695-1709.

Example 2. Any outer Hausdorff approximation $K \subset L_n \to K$ is X-regular.

Cs. Vincze and Á. Nagy, *On the theory of generalized conics with applications in geometric tomography*, J. of Approx. Theory **164** (2012), 371-390.

Example 3. If L_n is a sequence of compact connected hv-convex sets tending to the limit K with respect to the Hausdorff metric then the convergence is regular.

Cs. Vincze and Á. Nagy, *Generalized conic functions of hv-convex planar sets: continuity properties and X-rays*, Aequat. Math. 89 (4) (2015), pp. 1015-1030. Arxiv:1303.4412.

Example 4. The Hausdorff convergence of compact convex subsets L_n to K with non-empty interior is X-regular.

Cs. Vincze and Á Nagy, *On the average taxicab distance function and its applications*, Acta Appl. Math. 161, 201–220 (2019). https://doi.org/10.1007/s10440-018-0210-1.

In the sense of Example 4, the Hausdorff convergence in the class of compact convex sets (with nonempty) interior implies X-regularity and the reconstruction can be based on direct comparisons of X-rays: the coordinate X-rays converges to the coordinate X-rays of the limit set.

R. J. Gardner and M. Kiderlen, *A solution to Hammer's X-ray reconstruction problem*, Advances in Mathematics, 214 (2007) 323–343.

In the sense of Example 3, the Hausdorff convergence in the class of compact connected hv-convex sets implies the regularity and the reconstruction can be based on direct comparisons of the distance-mean functions: $\lim_{n\to\infty} f_{L_n}(x) = f_K(x)$ and the convergence $f_{L_n} \to f_K$ is uniform over any compact subset in $\mathbb{R}^2$.

Cs. Vincze and Á Nagy, *Reconstruction of hv-convex sets by their coordinate X-ray functions*, J. Math. Imaging and Vis. Volume 49 (3) (2014), pp. 569–582.

Theorem 10 *Consider the collection of compact connected hv-convex sets contained in the axis parallel bounding box $B \subset \mathbb{R}^2$ and let K be one of them; for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_B |f_L(x) - f_K(x)| \, dx < \delta$$

implies that $H(L, K^*) < \varepsilon$, where $f_K = f_{K^*}$, i.e. K and K^* have the same coordinate X-rays almost everywhere.

Such a continuity property allows us to reconstruct compact connected hv-convex planar sets by the coordinate X-rays as follows.

Input: $n \in \mathbb{N}$ and X_1K, X_2K.

STEP 1: Let B and the function f_K associated to K be given by the formulas

$$B = \text{supp } (X_1K) \times \text{supp } (X_2K), \tag{17}$$

$$f_K(x) = \int_{-\infty}^{\infty} \left| x^1 - t \right| X_1K(t) \, dt + \int_{-\infty}^{\infty} \left| x^2 - t \right| X_2K(t) \, dt. \tag{18}$$

STEP 2: Let $t_i^1 \in [a,b]$ and $t_i^2 \in [c,d]$ be equally spaced points with $t_0^1 = a$, $t_n^1 = b$ and $t_0^2 = d$, $t_n^2 = c$.

$$t_i^1 = a + i\frac{b-a}{n}, \quad t_i^2 = d - i\frac{d-c}{n} \quad (i = 0, \ldots, n)$$

STEP 3: $B_{ij}^n = [t_{i-1}^1, t_i^1] \times [t_j^2, t_{j-1}^2]$, where $i, j = 1, \ldots, n$.

STEP 4: The control grid $G^n(K) := \left\{ y_{ij} \in B_K \mid i, j = 1, \ldots, n \right\}$ consists of the centers of the subrectangles.

STEP 5: The feasible set $\mathcal{H}_n$ consists of compact connected hv-convex sets given by the union of some subrectangles such that

$$f_L(y_{ij}) \geq f_K(y_{ij}) \quad (i, j = 1, \ldots, n). \tag{19}$$

STEP 6: Choose $L_n \in \mathcal{H}_n$ that minimizes

$$\sum_{i,j=1}^{n} \frac{f_{L_n}(y_{ij}) - f_K(y_{ij})}{n^2}$$

Output: L_n.

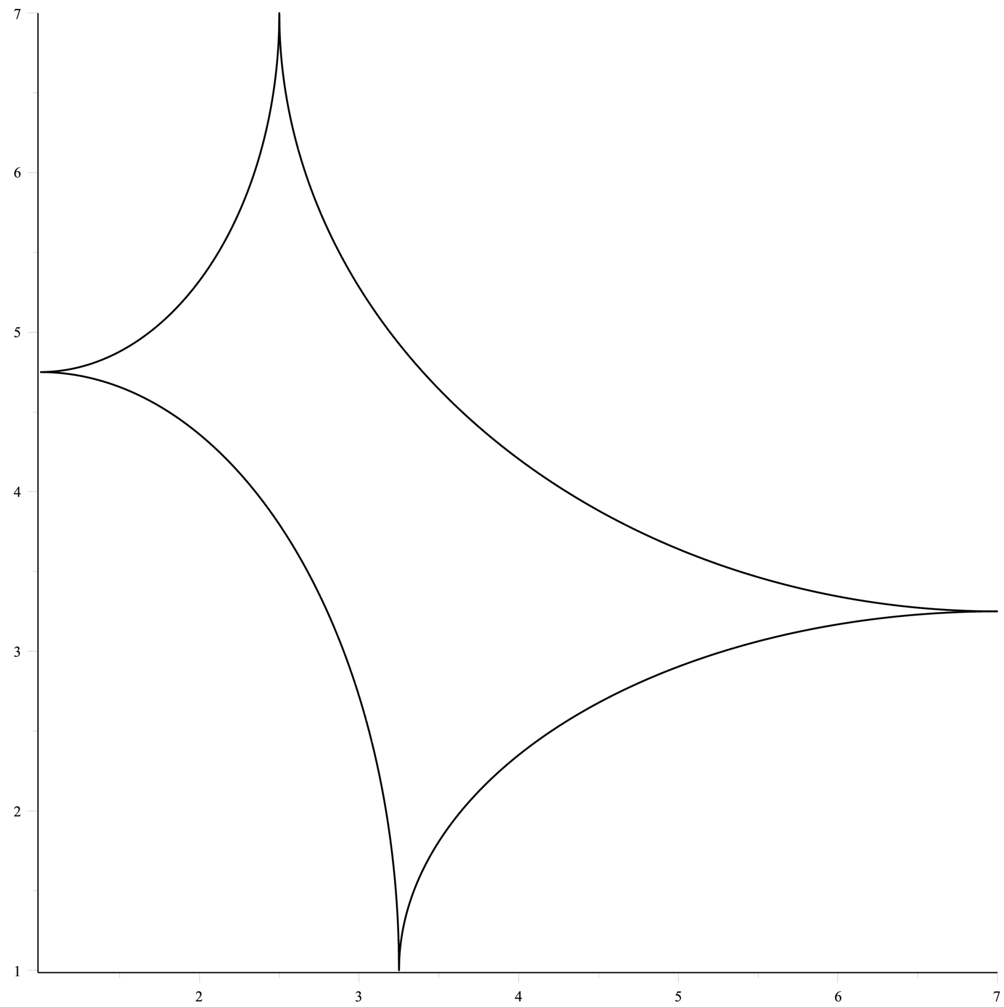
The procedure can be formulated in terms of a linear 0 - 1 programming because any element L in the feasible set can be represented as a $0-1$ interval matrix by the variables x_{kl}, where $x_{kl} = 1$ if $B^n_{kl} \subset L$ and $x_{kl} = 0$ otherwise $(k, l = 1, \ldots, n)$.

The applications of the greedy or the antigreedy algorithmic paradigms are also possible. They are based on deleting the subrectangle which causes the extremal (the greatest or the least) average descent of f_{L_n} at the control points. In general the antigreedy version increases the number of the possible outputs for making some voting processes more effective. The algorithm is adapted to finitely many and/or noisy measurements of the coordinate X-rays as well.

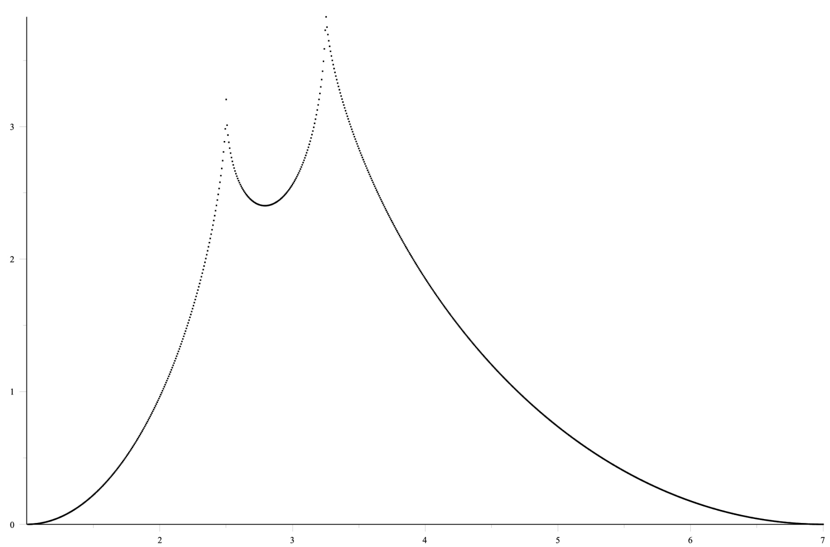
Cs. Vincze and Á Nagy, *Reconstruction of hv-convex sets by their coordinate X-ray functions*, J. Math. Imaging and Vis. Volume 49 (3) (2014), pp. 569–582.

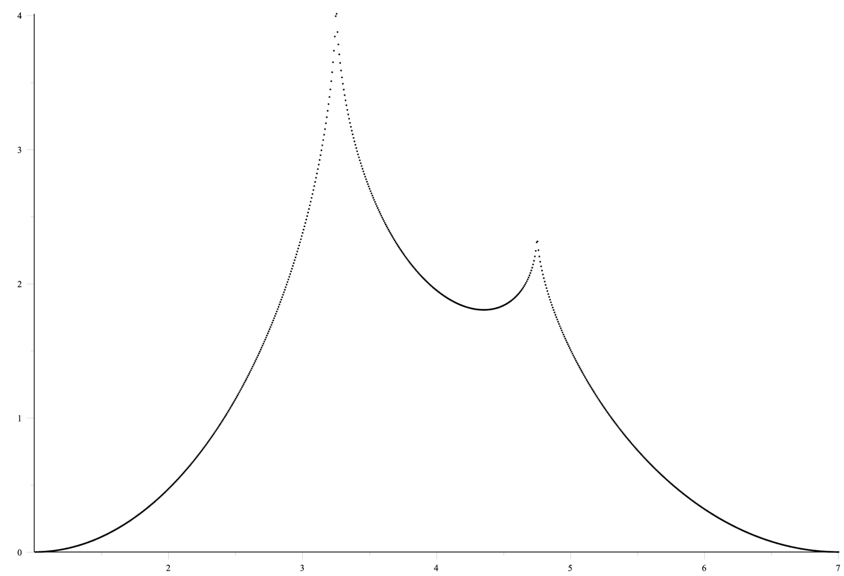
Cs. Vincze and Á Nagy, *An algorithm for the reconstruction of hv-convex planar bodies by finitely many and noisy measurements of their coordinate X-rays*, Fund. Inf. 141 (2-3) (2015) pp. 169-189.

The set we are looking for.

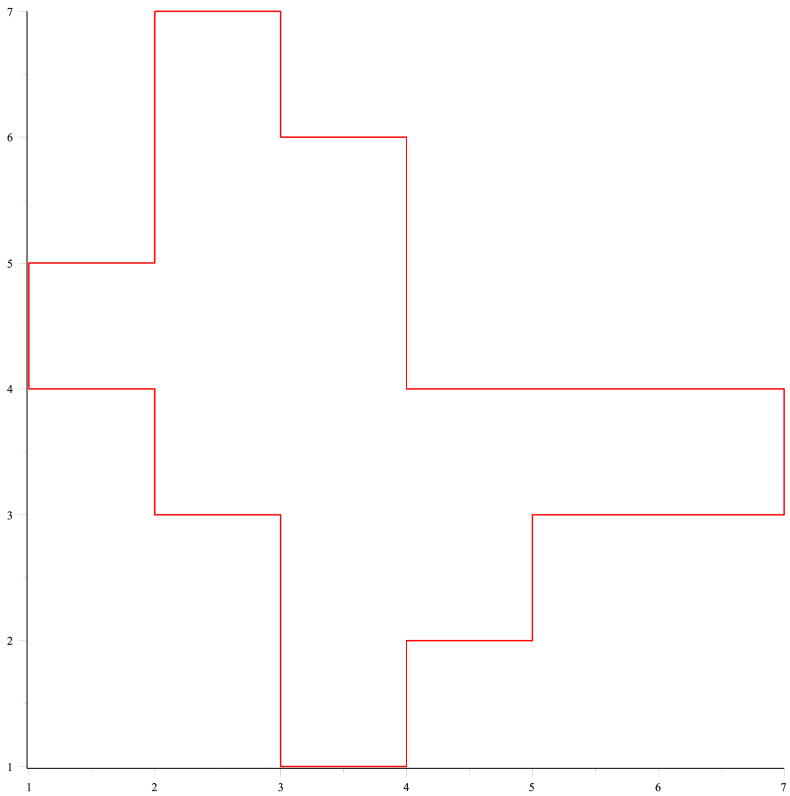


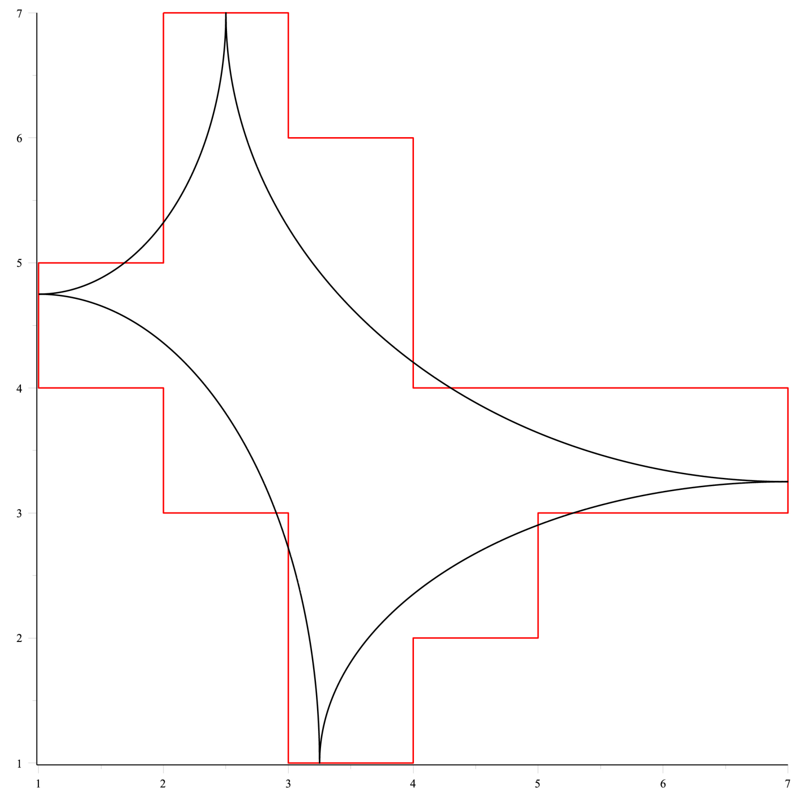
The coordinate X-rays.



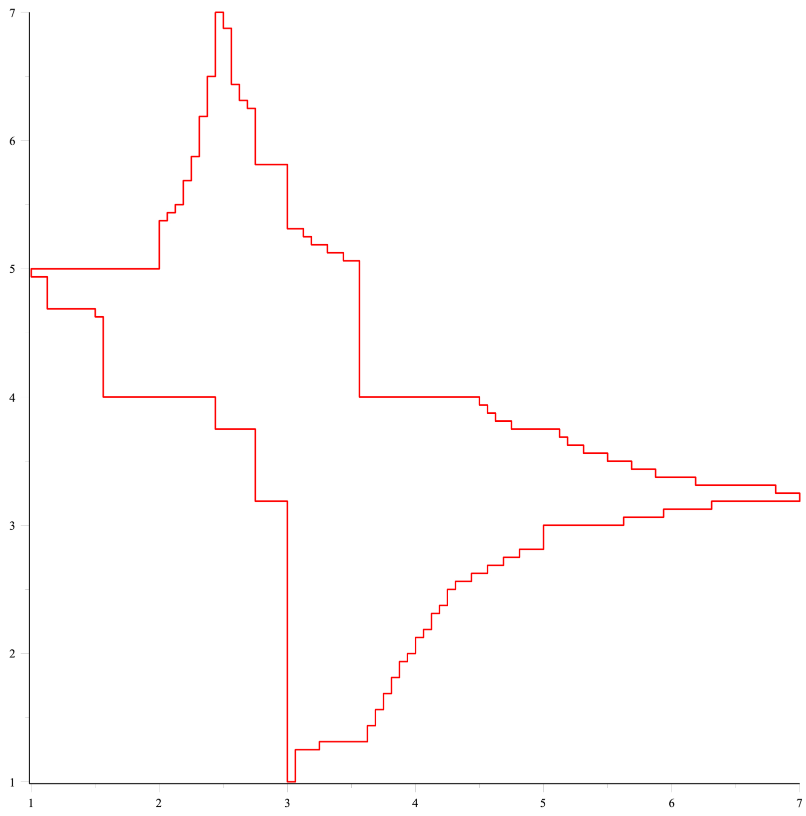


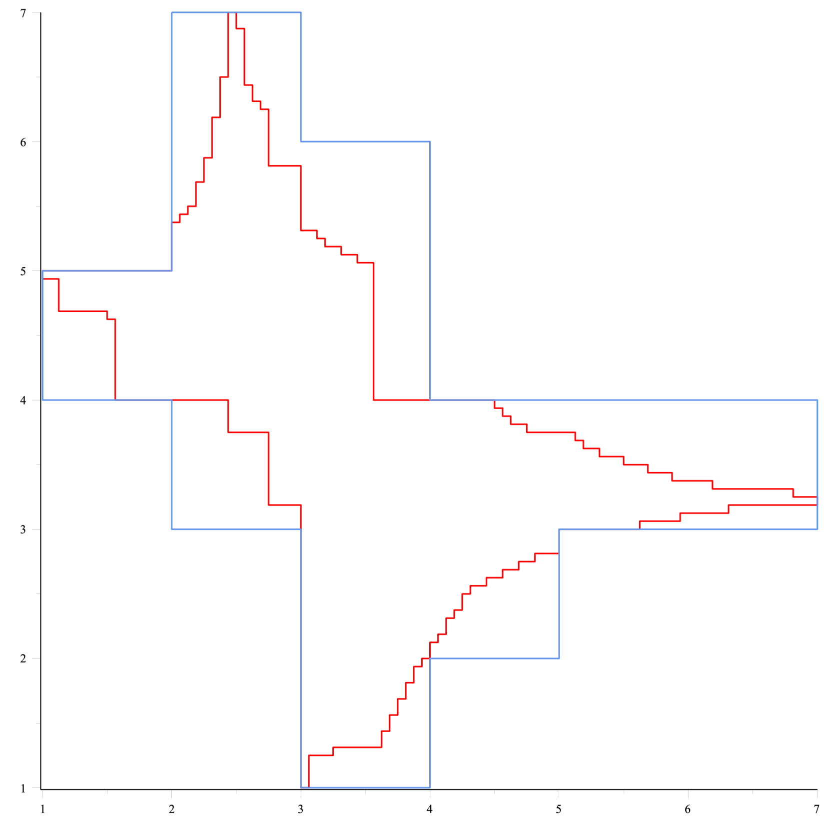
The optimal solution.



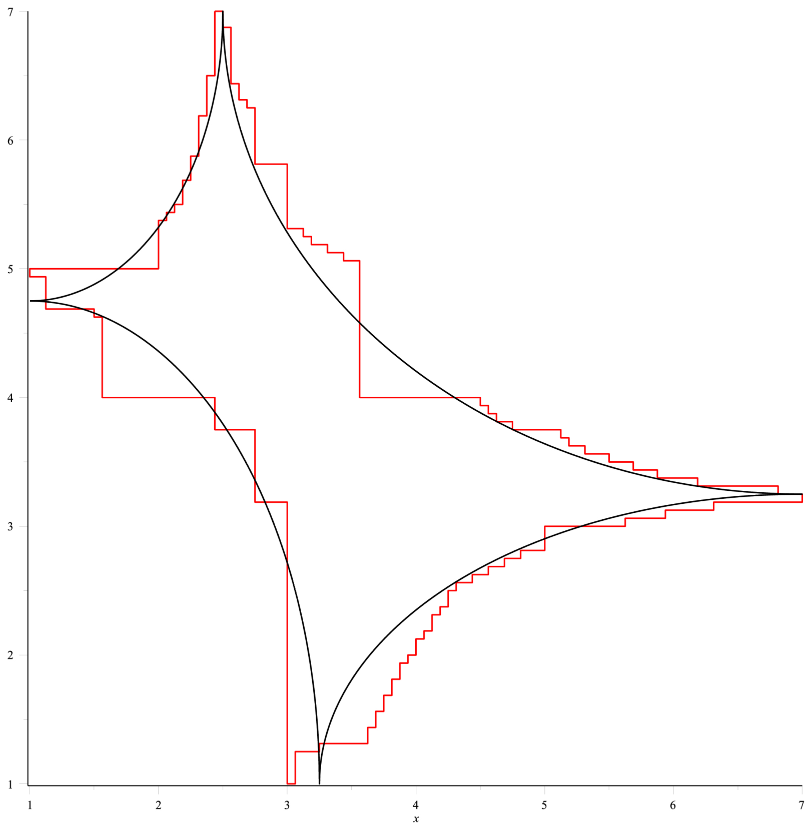


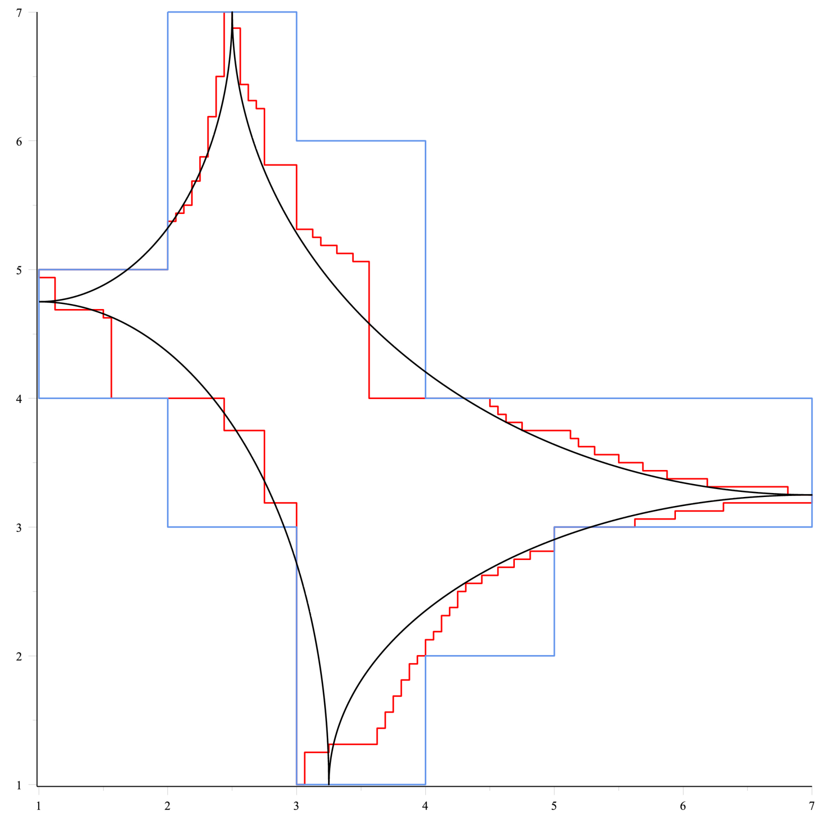
The greedy version.





Comparing figures.





5. **Reconstruction by the least average values**. The discrete version of the presented tomographic tools can be found in

Cs. Vincze, On the taxicab distance sum function and its applications in discrete tomography, Period. Math. Hung. 79 (2019), pp. 177–190.

It is a special case of the general theory with counting measure in the integral formulas.

5.1. **Summary**. Let $F = \{x_i \in \mathbb{R}^n \mid i = 1, \ldots, m\}$ be a finite set of different points in the coordinate space and consider the taxicab distance sum function

$$f(x) := \sum_{i=1}^{m} d_1(x, x_i) = \sum_{i=1}^{m} \sum_{j=1}^{n} |x^j - x_i^j|. \tag{20}$$

Introducing the one-sided partial derivatives

$$D_j^+ f(x) := \lim_{\varepsilon \to 0^+} \frac{f(x^1, \ldots, x^j + \varepsilon, \ldots, x^n) - f(x)}{\varepsilon},$$

$$D_j^- f(x) := \lim_{\varepsilon \to 0^-} \frac{f(x^1, \ldots, x^j + \varepsilon, \ldots, x^n) - f(x)}{\varepsilon}$$

we have the following collection of formulas:

$$D_j^+ f(x) = \left|F \leq_j x^j\right| - \left|F >_j x^j\right|,$$

$$D_j^- f(x) = \left|F <_j x^j\right| - \left|F \geq_j x^j\right|,$$

where

$$F >_j t := \{x_i \in F \mid x_i^j > t\},\ F =_j t := \{x_i \in F \mid x_i^j = t\},$$

$$F <_j t := \{x_i \in F \mid x_i^j < t\},$$

$$F \geq_j t := \{x_i \in F \mid x_i^j \geq t\},\ F \leq_j t := \{x_i \in F \mid x_i^j \leq t\},$$

$$\frac{D_j^+ f(x) - D_j^- f(x)}{2} = \left|F =_j x^j\right| \quad (j = 1, \ldots, n).$$

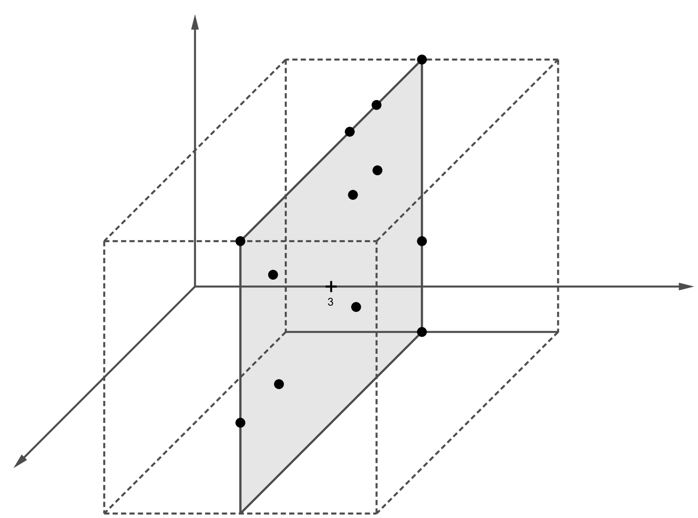
The cardinality $\left|F =_j x^j\right|$ is the number of the points in the intersection of F with the hyperplane $x + H_j$, where $H_j := \{x \in \mathbb{R}^n \mid x^j = 0\}$. The $(n-1)$-dimensional X-ray function parallel to the coordinate hyperplane H_j is defined as

$$X_j\colon \mathbb{R} \to \mathbb{R},\ \ X_j(t) := \Big| F =_j t \Big| \qquad (j = 1, \ldots, n). \tag{21}$$

X-rays take the zero value except at finitely many $t \in \mathbb{R}$. In terms of X-rays

$$f(x) = \sum_{i=1}^{m} d_1(x, x_i) = \sum_{i=1}^{m} \sum_{j=1}^{n} |x^j - x_i^j| = \sum_{j=1}^{n} \sum_{t \in \mathbb{R}} X_j(t) |x^j - t| \qquad (x \in \mathbb{R}^n).$$

Therefore the taxicab distance sum function accumulates the coordinate X-ray information.



A 2 - dimensional X-ray in 3D: $X_2(3) = 12$.

5.2. **The least avarege value principle: reconstruction of finite planar sets by the coordinate X-rays**. Taking a finite set of points $F = \{x_1, \ldots, x_m\} \subset \mathbb{R}^2$ in the plane let

$$G = \{t^1_1, \ldots, t^1_{m_1}\} \times \{t^2_1, \ldots, t^2_{m_2}\} \subset \mathbb{R}^2$$

be the grid determined by the coordinates of the focal points in F. The problem is to reconstruct F by the number of points

$$X_1(t^1_1), \ldots, X_1(t^1_{m_1}) \quad \text{and} \quad X_2(t^2_1), \ldots, X_2(t^2_{m_2})$$

along the vertical and the horizontal directions, respectively.

At first we reconstruct the taxicab distance sum function

$$f(x) = \sum_{j_1=1}^{m_1} X_1(t^1_{j_1})|x^1 - t^1_{j_1}| + \sum_{j_2=1}^{m_2} X_2(t^2_{j_2})|x^2 - t^2_{j_2}|. \tag{22}$$

associated with F. The least avarege value principle means that points of the grid with low taxicab sum values are preferred to be focal points provided that the X-rays show free positions in the rows (or columns). For the reconstruction we use the following subsequent steps:

(i) Focal points due to the least average distance: choose the point (points) of the grid, where the taxicab distance sum function attains its least value at,

(ii) Focal points due to the X-rays: after choosing a point due to the least average value, we use the X-rays to choose or disqualify the points in the row and the column of the selected point - it depends on the number of the free positions,

(iii) Repeat (ii) as far as possible. Otherwise return to (i).

The algorithm (i)-(iii) has polynomial time complexity because it is based on the quicksort of the values of the taxicab distance sum function.

Example. Let the taxicab distance sum function be given as

$$f(x^1, x^2) = 2\left|x^1 - 0\right| + \left|x^1 - 1\right| + 2\left|x^1 - 3\right| + 3\left|x^1 - 4\right| + 4\left|x^1 - 5\right| +$$

$$\left|x^2 - 0\right| + 4\left|x^2 - 1.5\right| + 2\left|x^2 - 2\right| + 2\left|x^2 - 3\right| + 3\left|x^2 - 4.2\right|,$$

i.e.

$$t_1^1 = 0, t_2^1 = 1, t_3^1 = 3, t_4^1 = 4, t_5^1 = 5,$$

$$t_1^2 = 0, t_2^2 = 1.5, t_3^2 = 2, t_4^2 = 3, t_5^2 = 4.2$$

and

$$X_1(0) = 2,\ X_1(1) = 1,\ X_1(3) = 2,\ X_1(4) = 3,\ X_1(5) = 4,$$

$$X_2(0) = 1,\ X_2(1.5) = 4,\ X_2(2) = 2,\ X_2(3) = 2,\ X_2(4.2) = 3.$$

The matrix

$$M := \begin{bmatrix} 60.8 & 52.8 & 40.8 & 38.8 & 42.8 \\ 53.6 & 45.6 & 33.6 & 31.6 & 35.6 \\ 51.6 & 43.6 & 31.6 & 29.6 & 33.6 \\ 52.6 & 44.6 & 32.6 & 30.6 & 34.6 \\ 67.6 & 59.6 & 47.6 & 45.6 & 49.6 \end{bmatrix}.$$

shows the values of the taxicab distance sum function f at the points of the grid

$$G = \{0, 1, 3, 4, 5\} \times \{0, 1.5, 2, 3, 4.2\}.$$

In the sense of (i), the steps 1-3 are

$$\begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & 29.6 & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix} \longrightarrow \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & 29.6 & x \\ x & x & x & 30.6 & x \\ x & x & x & x & x \end{bmatrix} \longrightarrow \begin{bmatrix} x & x & x & x & x \\ x & x & x & 31.6 & x \\ x & x & 31.6 & 29.6 & x \\ x & x & x & 30.6 & x \\ x & x & x & x & x \end{bmatrix},$$

where the least values of M are choosen.

Since $X_1(4) = 3$ and $X_2(2) = 2$ we can not choose further values from the fourth column and the third row:

$$\begin{bmatrix} x & x & x & 0 & x \\ x & x & x & 31.6 & x \\ 0 & 0 & 31.6 & 29.6 & 0 \\ x & x & x & 30.6 & x \\ x & x & x & 0 & x \end{bmatrix}$$

Since $X_1(5) = 4,\ X_2(0) = 1,\ X_2(3) = 2,\ X_1(0) = 2$ we have to choose all positions in the corresponding rows and columns:

$$\begin{bmatrix} 60.8 & x & x & 0 & 42.8 \\ 0 & 0 & 0 & 31.6 & 35.6 \\ 0 & 0 & 31.6 & 29.6 & 0 \\ 52.6 & x & x & 30.6 & 34.6 \\ 0 & 0 & 0 & 0 & 49.6 \end{bmatrix}.$$

Since the X-rays enforce no more steps we return to the least average value 32.6 to continue the process:

$$\begin{bmatrix} 60.8 & x & x & 0 & 42.8 \\ 0 & 0 & 0 & 31.6 & 35.6 \\ 0 & 0 & 31.6 & 29.6 & 0 \\ 52.6 & x & 32.6 & 30.6 & 34.6 \\ 0 & 0 & 0 & 0 & 49.6 \end{bmatrix}$$

Since $X_1(3) = 2,\ X_2(1.5) = 4,\ X_1(1) = 1$ we have the solution matrix

$$\begin{bmatrix} 60.8 & 52.8 & 0 & 0 & 42.8 \\ 0 & 0 & 0 & 31.6 & 35.6 \\ 0 & 0 & 31.6 & 29.6 & 0 \\ 52.6 & 0 & 32.6 & 30.6 & 34.6 \\ 0 & 0 & 0 & 0 & 49.6 \end{bmatrix}.$$

To develop the method to be a plain enumeration of the possible solutions we have to admit shifting and skipping. Shifting refers to the starting position, i.e. we use the same process by starting with 30.6 (for example):

$$\begin{bmatrix} 60.8 & 0 & 0 & 38.8 & 42.8 \\ 0 & 0 & 0 & 31.6 & 35.6 \\ 0 & 0 & 31.6 & 0 & 33.6 \\ 0 & 44.6 & 32.6 & 30.6 & 34.6 \\ 67.6 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is a different solution of the problem. Skipping means that we use the process (i)-(iii) by starting with 29.6 (for example) but skipping 30.6:

$$\begin{bmatrix} 60.8 & 0 & 0 & 38.8 & 42.8 \\ 0 & 0 & 0 & 31.6 & 35.6 \\ 0 & 0 & 31.6 & 29.6 & 0 \\ 52.6 & 44.6 & 32.6 & 0 & 34.6 \\ 0 & 0 & 0 & 0 & 49.6 \end{bmatrix}$$

We have a new solution of the problem again*

Concluding remarks. The least avarege value principle transforms the coordinate X-rays into some geometric information by the values of the taxicab distance sum function. They are working as probability-like quantities whenever the subsequent step of the algorithm is not determined by the X-rays. Completing the method with shifting and skipping it is a plain enumeration of the possible solutions.

*By a theorem due to H. J. Ryser, two matrices of zeros and ones with equal row and column sum vectors can be transformed into each other by changing the alternating zeros and ones in 2 by 2 submatrices.

H. J. Ryser, *Combinatorial properties of matrices of zeros and ones*, Canad. J. Math. 9, 371-377 (1957).

H. J. Ryser, *Matrices of zeros and ones* Bull. Amer. Math. Soc. 66 (6), 442-464 (1960).

II. **Euclidean distance-mean functions and their geometric applications: remetrization results for closed non-transitive subgroups in the orthogonal group**.

In the preamble to his fourth problem presented at the International Mathematical Congress in Paris (1900) Hilbert suggested the examination of geometries standing next to Euclidean one in the sense that they satisfy much of Euclidean's axioms except some (tipically one) of them. In the classical non-Euclidean geometry the axiom taking to fail is the fameous parallel postulate. Another type of geometry standing next to Euclidean one is the geometry of normed spaces or, in a more general context, the geometry of Minkowski spaces*. The congruence due to the various size of the group of linear isometries is of special interest.

Let G be a closed subgroup in the Euclidean orthogonal group. It is transitive if any two points on the Euclidean unit sphere can be joined by an orbit under G. In other words, the orbit of a single unit element covers the entire Euclidean unit sphere.

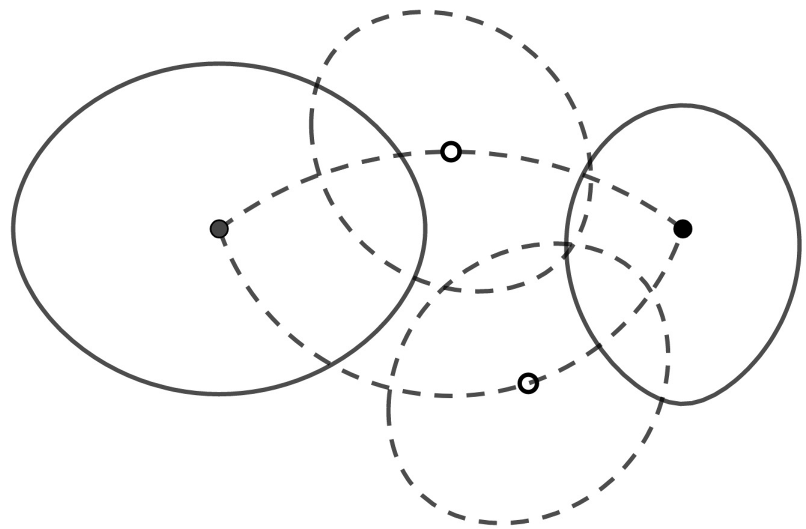
*J. C. Á. Paiva and A. Thompson, *On the Perimeter and Area of the Unit Disc*, Amer. Math. Monthly Vol. 112, No. 2 (2005), pp. 141-154.

It is clear that there are no alternatives of Euclidean geometry for such a group because any invariant convex body must be a Euclidean ball.

In contrast, for any closed non-transitive subgroup in the Euclidean orthogonal group there exists an invariant generalized conic body containing the origin in its interior such that it is not a unit ball with respect to any inner product (ellipsoid-problem) and its boundary is a smooth hypersurface (regularity condition). Working as a unit ball such a generalized conic induces a non-Euclidean Minkowski functional such that the elements of G are still linear isometries: Minkowski geometry is an alternative of Euclidean geometry for any closed non-transitive subgroup G in the Euclidean orthogonal group.

In the context of Riemannian geometry G is the closure of the holonomy group of a metric linear connection ∇. If the closure of the holonomy group is a not transitive subgroup in the Euclidean orthogonal group, then we can construct a holonomy-invariant generalized conic body in the tangent space at a single point such that it is not a unit ball with respect to any inner product and its boundary is a smooth hypersurface.

Using parallel transports with respect to ∇, a holonomy-invariant body can be translated to any point of a connected manifold in a consistent way (independently of the connecting path).



The translates of the generalized conic induce Minkowski functionals in the tangent spaces instead of the Riemannian inner products to measure the length of tangent vectors. Such a smoothly varying family of Minkowski functionals is called a Finsler metric on the manifold: Finsler geometry is an alternative of Riemannian geometry for the linear connection ∇ because the parallel transports obviously preserve the Finslerian length of tangent vectors.

Example (finite groups and polyellipses). Let M be a flat compact Riemannian manifold. Bieberbach's theorem* states that the holonomy group of the Lévi-Civita connection ∇ is finite. Therefore we can find a finite invariant system of elements working as the focal set of an invariant polyellipsoid in the tangent space at a single point of the manifold. Extension by parallel transports provides a Finslerian environment for ∇.

Finsler geometry is a non-Riemannian geometry in a finite number of dimensions. The differentiable structure is the same as the Riemannian one but distance is not uniform in all directions. Instead of the euclidean spheres in the tangent spaces, the unit vectors form the boundary of general convex sets containing the origin in their interiors. (M. Berger)

In general the holonomy group of a metric linear connection is not finite. To adopt the polyellipsoids to the general situation we should develop the theory of conics with infinitely many focal points.

Example (circular conics). Consider the parametrization $c(t) := (\cos t, \sin t, 0)$ of the circle S_1 in the Euclidean space of dimension three. The function

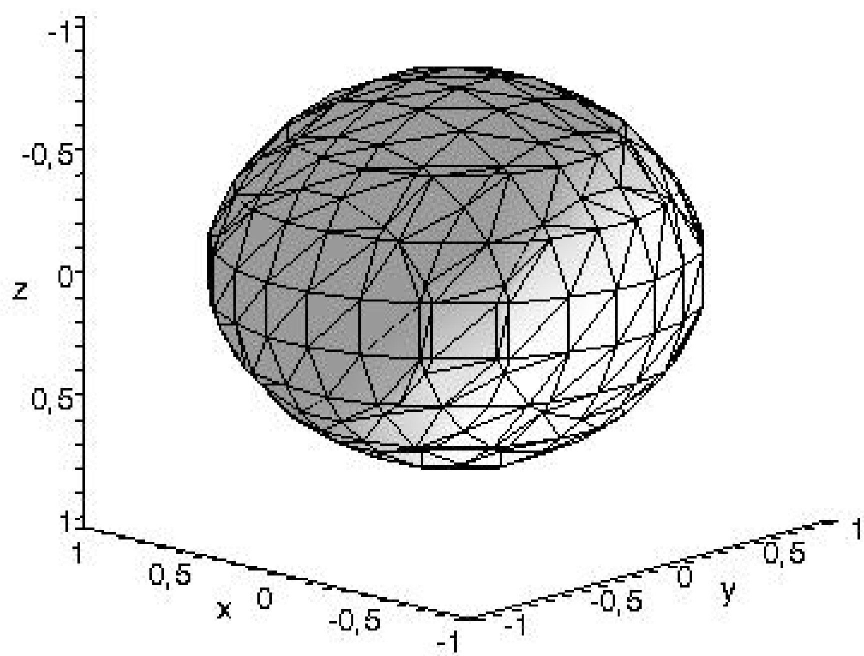
$$f(x, y, z) := \frac{1}{2\pi} \int_0^{2\pi} \sqrt{(x - \cos t)^2 + (y - \sin t)^2 + z^2}\, dt$$

*L. S. Charlap, *Bieberbach groups and flat manifolds*, Springer 1986.

measures the distance-mean from the elements of the focal set S_1. The surface of the form

$$f(x,y,z) = \frac{8}{2\pi} \tag{23}$$

is a generalized conic. According to the invariance of the focal set under the rotations about the z-axis, equation (23) gives a revolution surface.

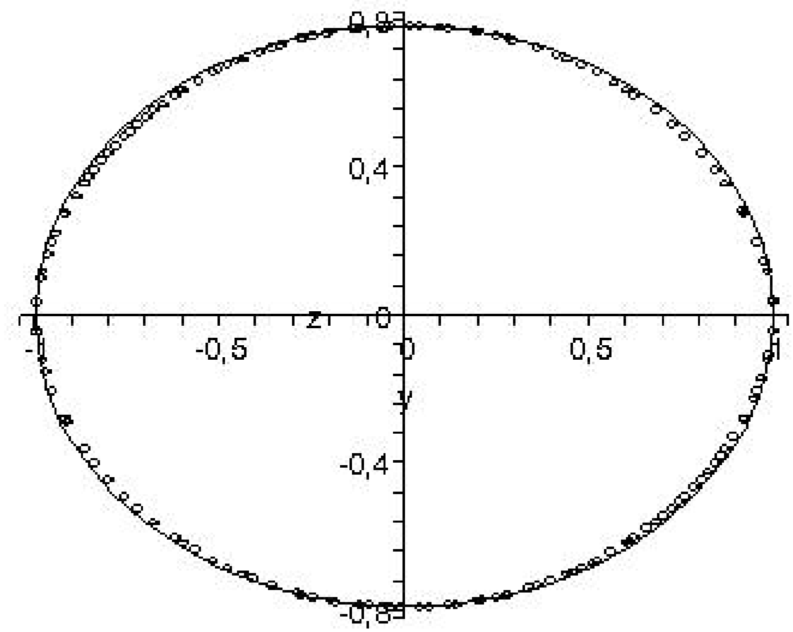


The generalized conic surface (23).

The generatix and its approximating ellipse can be distinguished by sharp power mean estimations for elliptic integrals in 3D and the Gaussian hyperbolic function in higher dimensional spaces.

H. Alzer and S.-L- Qui, *Monotonicity theorems and inequalities for the complete elliptic integrals*, Journal of Computational and Applied Mathematics **172** (2004), pp. 289-312.

K. C. Richards, *Sharp power mean bounds for the Gaussian hypergeometric function*, J. Math. Anal. Appl. **308** (2005), pp. 303-313.



The generatrix (pointstyle) and its approximating ellipse.

Concluding remarks. A generalized conic body of type (23) induces a non-Euclidean Minkowski functional* such that the one-parameter subgroup of Euclidean rotations about the z-axis are still linear isometries. The presented examples (finite and reducible subgroups) are the prototypes of non-transitive subgroups in the Euclidean orthogonal group. Suppose that M is a connected Riemannian manifold and ∇ is a metric linear connection on M. If the closure of the holonomy group of ∇ is a not transitive subgroup in the Euclidean orthogonal group then there is a non-Riemannian Finsler metric such that the parallel transports with respect to ∇ preserve the Finslerian length of tangent vectors and the unit balls in the tangent spaces are generalized conic bodies. Especially, Riemannian unit balls are conics (ellipsoids) in the classical sense.

Cs. Vincze and Á. Nagy, *Examples and notes on generalized conics and their applications*, Acta Math. Acad. Paedagog. Nyházi **26** (2010), pp. 359-575.

Cs. Vincze and Á. Nagy, *An introduction to the theory of generalized conics and their applications*, Journal of Geom. and Phys. **61** (2011), pp. 815-828.

Cs. Vincze, *Lazy orbits: an optimization problem on the sphere*, J. of Geom. and Phys. Vol. 124, pp. 180-198 (2018). arXiv:1709.06410.

*To provide smoothness we need a slight modification of the level rate for the focal circle to be contained entirely in the interior of the level set.