Nonlinear random perturbations of PDEs.

Gianmario Tessitore (Università di Milano-Bicocca)

Joint work with:

Sandra Cerrai (University of Maryland)
Giuseppina Guatteri (Politecnico di Milano)

Freidlin and Koralov (*PTRF 2010*) have considered the following quasi-linear parabolic problem

$$\begin{cases} \partial_t u_\epsilon(t,x) = \frac{\epsilon}{2}\sum_{i,j=1}^d a_{i,j}(x, u_\epsilon(t,x))\,\partial_{ij} u_\epsilon(t,x) + \sum_{i=1}^d b_i(x)\,\partial_i u_\epsilon(t,x), \\ u_\epsilon(0,x) = g(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{1}$$

Freidlin and Koralov (*PTRF 2010*) have considered the following quasi-linear parabolic problem

$$\begin{cases} \partial_t u_\epsilon(t,x) = \frac{\epsilon}{2}\sum_{i,j=1}^d a_{i,j}(x, u_\epsilon(t,x))\,\partial_{ij} u_\epsilon(t,x) + \sum_{i=1}^d b_i(x)\,\partial_i u_\epsilon(t,x), \\ u_\epsilon(0,x) = g(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{1}$$

together with the randomly perturbed system where (B_t) is a d-dimensional Brownian motion and $a_{ij}(x,r) = (\sigma\sigma^\star)_{ij}(x,r)$.

$$\begin{cases} dX_\epsilon^{t,x}(s) = b(X_\epsilon^{t,x}(s))\,ds + \sqrt{\epsilon}\,\sigma(X_\epsilon^{t,x}(s), u_\epsilon(t-s, X_\epsilon^{t,x}(s)))\,dB_s, \\ X_\epsilon^{t,x}(0) = x, \end{cases} \tag{2}$$

Freidlin and Koralov (*PTRF 2010*) have considered the following quasi-linear parabolic problem

$$\begin{cases} \partial_t u_\epsilon(t,x) = \frac{\epsilon}{2}\sum_{i,j=1}^d a_{i,j}(x, u_\epsilon(t,x))\, \partial_{ij} u_\epsilon(t,x) + \sum_{i=1}^d b_i(x)\, \partial_i u_\epsilon(t,x), \\ u_\epsilon(0,x) = g(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{1}$$

together with the randomly perturbed system where (B_t) is a d-dimensional Brownian motion and $a_{ij}(x,r) = (\sigma\sigma^\star)_{ij}(x,r)$.

$$\begin{cases} dX_\epsilon^{t,x}(s) = b(X_\epsilon^{t,x}(s))\, ds + \sqrt{\epsilon}\, \sigma(X_\epsilon^{t,x}(s), u_\epsilon(t-s, X_\epsilon^{t,x}(s)))\, dB_s, \\ X_\epsilon^{t,x}(0) = x, \end{cases} \tag{2}$$

The PDE (1) and the SDE (2) are related by the relation:

$$u_\epsilon(r,x) = \mathbb{E} g(X_\epsilon^{r,x}(r)), \quad r \geq 0$$

The classical theory of (finite dimensional) parabolic, quasi-linear, PDEs guarantees that equation (1) admits a unique classical solution u_ϵ.

In their framework, Freidlin and Koralov complete a comprehensive program:

- Prove the Large Deviation Principle for the trajectories of $X_\epsilon^{t,x}$ and characterize the action functional.
- Study the exit problem for $(X_\epsilon^{t,x})$ from a fixed domain $D \subset \mathbb{R}^n$.
- Investigate the asymptotic behavior $\lim_{\epsilon \to 0} u_\epsilon(\lambda/\epsilon) := c(\lambda)$.

In their framework, Freidlin and Koralov complete a comprehensive program:

- Prove the Large Deviation Principle for the trajectories of $X_\epsilon^{t,x}$ and characterize the action functional.
- Study the exit problem for $(X_\epsilon^{t,x})$ from a fixed domain $D \subset \mathbb{R}^n$.
- Investigate the asymptotic behavior $\lim_{\epsilon\to 0} u_\epsilon(\lambda/\epsilon) := c(\lambda)$.

Our objective is to start a similar program in the infinite-dimensional case (when equation (2) is a SPDE).

DIFFICULTY: Unlike the finite-dimensional case, the current literature lacks a general Hilbert space theory for quasi-linear equations.

In their framework, Freidlin and Koralov complete a comprehensive program:

- Prove the Large Deviation Principle for the trajectories of $X_\epsilon^{t,x}$ and characterize the action functional.
- Study the exit problem for $(X_\epsilon^{t,x})$ from a fixed domain $D \subset \mathbb{R}^n$.
- Investigate the asymptotic behavior $\lim_{\epsilon\to 0} u_\epsilon(\lambda/\epsilon) := c(\lambda)$.

Our objective is to start a similar program in the infinite-dimensional case (when equation (2) is a SPDE).

DIFFICULTY: Unlike the finite-dimensional case, the current literature lacks a general Hilbert space theory for quasi-linear equations.

Two (divergent) considerations:

- The asymptotic behavior as $\epsilon \to 0$ of the SDE (2) can be studied independently of the PDE (1).
- The PDE (1) is of independent interest. In particular regularity of solutions.

We consider the randomly perturbed partial differential equation defined on a separable **Hilbert space** H, where ϵ is a small parameter.

$$\begin{cases} dX_\epsilon^{t,x}(s) = [AX_\epsilon^{t,x}(s) + b(X_\epsilon^{t,x}(s))]\, dt + \sqrt{\epsilon}\,\sigma(X_\epsilon^{t,x}(s), u_\epsilon(t-s, X_\epsilon^{t,x}(s)))\, dW_s, \\ X_\epsilon^{t,x}(0) = x \in H, \end{cases} \tag{3}$$

where u_ϵ satisfies (at least formally) the quasi-linear equations, $t \geq 0$

$$\begin{cases} D_t u_\epsilon(t,x) = \frac{\epsilon}{2} \mathrm{Tr}\left[\sigma\sigma^\star(x, u_\epsilon(t,x)) D_x^2 u_\epsilon(t,x)\right] + \langle Ax + b(x), Du_\epsilon(t,x)\rangle_H, \\ u_\epsilon(0,x) = g(x), \quad x \in H. \end{cases} \tag{4}$$

and it holds

$$u_\epsilon(t,x) = \mathbb{E} g(X_\epsilon^{t,x}(t))$$

$$\begin{cases} dX_\epsilon^{t,x}(s) = [AX_\epsilon^{t,x}(s) + b(X_\epsilon^{t,x}(s))]\ dt + \sqrt{\epsilon}\,\sigma(X_\epsilon^{t,x}(s), u_\epsilon(t-s, X_\epsilon^{t,x}(s)))\,dW_s, \\ X_\epsilon^{t,x}(0) = x \in\ H, \end{cases}$$

$$\begin{cases} dX_\epsilon^{t,x}(s) = [AX_\epsilon^{t,x}(s) + b(X_\epsilon^{t,x}(s))]\, dt + \sqrt{\epsilon}\,\sigma(X_\epsilon^{t,x}(s), u_\epsilon(t-s, X_\epsilon^{t,x}(s)))\, dW_s, \\ X_\epsilon^{t,x}(0) = x \in H, \end{cases}$$

Here

- $A : D(A) \subset H \to H$ is the generator of a strongly continuous, **Hilbert-Schmidt**, semigroup $(S(t))$ with:

$$\|S(t)\|_{\mathcal{L}_2(H)} \leq ce^{-\gamma t} \quad \text{for some } 0 < \gamma < 1/2 \text{ and all } t \in [0,1]$$

.

$$\begin{cases} dX_\epsilon^{t,x}(s) = [AX_\epsilon^{t,x}(s) + b(X_\epsilon^{t,x}(s))]\, dt + \sqrt{\epsilon}\,\sigma(X_\epsilon^{t,x}(s), u_\epsilon(t-s, X_\epsilon^{t,x}(s)))\, dW_s, \\ X_\epsilon^{t,x}(0) = x \in H, \end{cases}$$

Here

- $A : D(A) \subset H \to H$ is the generator of a strongly continuous, **Hilbert-Schmidt**, semigroup $(S(t))$ with:

$$\|S(t)\|_{\mathcal{L}_2(H)} \leq ce^{-\gamma t} \quad \text{for some } 0 < \gamma < 1/2 \text{ and all } t \in [0,1]$$

- $b : H \to H$ is a non-linear Lipschitz mapping.

$$\begin{cases} dX_\epsilon^{t,x}(s) = [AX_\epsilon^{t,x}(s) + b(X_\epsilon^{t,x}(s))]\; dt + \sqrt{\epsilon}\,\sigma(X_\epsilon^{t,x}(s), u_\epsilon(t-s, X_\epsilon^{t,x}(s)))\, dW_s, \\ X_\epsilon^{t,x}(0) = x \in\; H, \end{cases}$$

Here

- $A : D(A) \subset H \to H$ is the generator of a strongly continuous, **Hilbert-Schmidt**, semigroup $(S(t))$ with:

$$\|S(t)\|_{\mathcal{L}_2(H)} \leq ce^{-\gamma t} \quad \text{for some } 0 < \gamma < 1/2 \text{ and all } t \in [0,1]$$

- $b : H \to H$ is a non-linear Lipschitz mapping.
- $\sigma : H \times \mathbb{R} \to \mathcal{L}(H,H)$ is some non-linear mapping, Lipchitz in both variables.

$$\begin{cases} dX_\epsilon^{t,x}(s) = [AX_\epsilon^{t,x}(s) + b(X_\epsilon^{t,x}(s))]\, dt + \sqrt{\epsilon}\,\sigma(X_\epsilon^{t,x}(s), u_\epsilon(t-s, X_\epsilon^{t,x}(s)))\, dW_s, \\ X_\epsilon^{t,x}(0) = x \in H, \end{cases}$$

Here

- $A : D(A) \subset H \to H$ is the generator of a strongly continuous, **Hilbert-Schmidt**, semigroup $(S(t))$ with:

$$\|S(t)\|_{\mathcal{L}_2(H)} \leq ce^{-\gamma t} \quad \text{for some } 0 < \gamma < 1/2 \text{ and all } t \in [0,1]$$

- $b : H \to H$ is a non-linear Lipschitz mapping.
- $\sigma : H \times \mathbb{R} \to \mathcal{L}(H, H)$ is some non-linear mapping, Lipchitz in both variables.
- (W_t), $t \geq 0$, is a cylindrical Wiener process in an Hilbert space H.

$$\begin{cases} dX_\epsilon^{t,x}(s) = [AX_\epsilon^{t,x}(s) + b(X_\epsilon^{t,x}(s))]\, dt + \sqrt{\epsilon}\,\sigma(X_\epsilon^{t,x}(s), u_\epsilon(t-s, X_\epsilon^{t,x}(s)))\, dW_s, \\ X_\epsilon^{t,x}(0) = x \in H, \end{cases}$$

Here

- $A : D(A) \subset H \to H$ is the generator of a strongly continuous, **Hilbert-Schmidt**, semigroup $(S(t))$ with:

$$\|S(t)\|_{\mathcal{L}_2(H)} \leq ce^{-\gamma t} \quad \text{for some } 0 < \gamma < 1/2 \text{ and all } t \in [0,1]$$

- $b : H \to H$ is a non-linear Lipschitz mapping.
- $\sigma : H \times \mathbb{R} \to \mathcal{L}(H, H)$ is some non-linear mapping, Lipchitz in both variables.
- (W_t), $t \geq 0$, is a cylindrical Wiener process in an Hilbert space H.
- $g : H \to H$ is a non-linear Lipchitz mapping.

Notice that if

$$v_\epsilon^t(s,x) := u_\epsilon(t-s,x), \quad s \in [0,t]$$

then the above equations rewrite:

$$\begin{cases} D_t v_\epsilon^t(s,x) - \frac{\epsilon}{2}\mathrm{Tr}\left[\sigma\sigma^\star(x, v_\epsilon^t(s,x)) D_x^2 v_\epsilon^t(s,x)\right] + \langle Ax + b(x), Dv_\epsilon^t(s,x)\rangle_H = 0, \\ v_\epsilon^t(t,x) = g(x), \quad x \in H. \end{cases}$$

$$\begin{cases} dX_\epsilon^{t,x}(s) = [AX_\epsilon^{t,x}(s) + b(X_\epsilon^{t,x}(s))]\, dt + \sqrt{\epsilon}\,\sigma(X_\epsilon^{t,x}(s), v_\epsilon^t(s, X_\epsilon^{t,x}(s)))\, dW_s, \\ X_\epsilon^{t,x}(0) = x \in H, \end{cases}$$

and it holds

$$v_\epsilon^t(s,x) = \mathbb{E} g(X_\epsilon^{t-s,x}(t-s))$$

thus by markovianity

$$v_\epsilon^t(s, X_\epsilon^{t,x}(s)) = \mathbb{E}\left(g(X_\epsilon^{t,x}(t) \middle| \mathcal{F}_s^W \right)$$

The equations for $X_\epsilon^{t,x}$ can be rewritten in a closed form as:

$$\begin{cases} dX_\epsilon^{t,x}(s) = [AX_\epsilon^{t,x}(s) + b((X_\epsilon^{t,x}(s))]\, ds + \\ \qquad \sqrt{\epsilon}\,\sigma\left(X_\epsilon^{t,x}(s), \mathbb{E}\left(g(X_\epsilon^{t,x})(t)\middle|\mathcal{F}_s^W\right)\right) dW_s, \\ X_\epsilon^{t,x}(0) = x \in H, \end{cases} \tag{5}$$

The equations for $X_\epsilon^{t,x}$ can be rewritten in a closed form as:

$$\begin{cases} dX_\epsilon^{t,x}(s) = [AX_\epsilon^{t,x}(s) + b((X_\epsilon^{t,x}(s))]\ ds + \\ \qquad\qquad \sqrt{\epsilon}\,\sigma\left(X_\epsilon^{t,x}(s), \mathbb{E}\left(g(X_\epsilon^{t,x})(t)\middle|\mathcal{F}_s^W\right)\right)\ dW_s, \\ X_\epsilon^{t,x}(0) = x \in\ H, \end{cases} \tag{5}$$

or, if you prefer, as:

$$\begin{cases} dX_\epsilon^{t,x}(s) = [AX_\epsilon^{t,x}(s) + b((X_\epsilon^{t,x}(s))]\ ds + \sqrt{\epsilon}\,\sigma\left(X_\epsilon^{t,x}(s), Y_\epsilon^{t,x}(s)\right)\ dW_s, \\ dY_\epsilon^{t,x}(s) = Z_\epsilon^{t,x}(s) dW_s \\ X_\epsilon^{t,x}(0) = x \in\ H, \\ Y_\epsilon^{t,x}(t) = g(X_\epsilon^{t,x}(t)) \end{cases}$$

The well posedness of the above equation **for small** ϵ can be easily established (essentially by Itô isometry and contraction principle):

Theorem

Fix $T > 0$ *there exists* $\bar{\epsilon}(T) > 0$ *such that for all* $t \leq T$ *and* $\epsilon \leq \bar{\epsilon}(T)$ *there exists a unique solution* $(X_\epsilon^{t,x})(s)_{s\in[0,t]})$ *with continuous trajectories of equation*

$$\begin{cases} dX_\epsilon^{t,x}(s) = [AX_\epsilon^{t,x}(s) + b((X_\epsilon^{t,x}(s))]\, ds + \\ \qquad\qquad\qquad \sqrt{\epsilon}\,\sigma\left(X_\epsilon^{t,x}(s), \mathbb{E}\left(g(X_\epsilon^{t,x})(t)\middle|\mathcal{F}_s^W\right)\right)\, dW_s, \\ X_\epsilon^{t,x}(0) = x \in H. \end{cases}$$

Moreover

$$\mathbb{E}(\sup_{s\in[0,t]} |X_\epsilon^{t,x}(s)|^2) \leq C(T)(1 + |x|^2)$$

$$\mathbb{E}(\sup_{s\in[0,t]} |X_\epsilon^{t,x}(s) - X_\epsilon^{t,x'}(s)|^2) \leq C(T)|x - x'|^2$$

The well posedness of the above equation **for small** ϵ can be easily established (essentially by Itô isometry and contraction principle):

Theorem

Fix $T > 0$ *there exists* $\bar{\epsilon}(T) > 0$ *such that for all* $t \le T$ *and* $\epsilon \le \bar{\epsilon}(T)$ *there exists a unique solution* $(X_\epsilon^{t,x})(s)_{s\in[0,t]})$ *with continuous trajectories of equation*

$$\begin{cases} dX_\epsilon^{t,x}(s) = [AX_\epsilon^{t,x}(s) + b((X_\epsilon^{t,x}(s))]\, ds + \\ \qquad\qquad \sqrt{\epsilon}\,\sigma\left(X_\epsilon^{t,x}(s), \mathbb{E}\left(g(X_\epsilon^{t,x})(t)\middle|\mathcal{F}_s^W\right)\right)\, dW_s, \\ X_\epsilon^{t,x}(0) = x \in H. \end{cases}$$

Moreover

$$\mathbb{E}(\sup_{s\in[0,t]} |X_\epsilon^{t,x}(s)|^2) \le C(T)(1+|x|^2)$$

$$\mathbb{E}(\sup_{s\in[0,t]} |X_\epsilon^{t,x}(s) - X_\epsilon^{t,x'}(s)|^2) \le C(T)|x-x'|^2$$

Remark: if A is dissipative then $\bar{\epsilon} > 0$ can be chosen independently on T

When $\epsilon \to 0$ the laws of $(X_\epsilon^{t,x})$ converge to the Dirac measure centerd in in Z^x where

$$\frac{d}{ds} Z^x(s) = AZ^x(s) + b(Z^x(s)); \qquad Z_\epsilon^x(0) = x$$

The events $\Gamma \subset C([0,t]; H)$ that do not contain Z^x describe a **deviant** behavior.

We want to know how deviant a particular event Γ such that $Z^x \notin \bar{\Gamma}$ is, more precisely we want to compute the exponential rate at which $\mathcal{L}(X_\epsilon^{t,x})(\Gamma)$ goes to zero.

When $\epsilon \to 0$ the laws of $(X_\epsilon^{t,x})$ converge to the Dirac measure centerd in in Z^x where

$$\frac{d}{ds} Z^x(s) = AZ^x(s) + b(Z^x(s)); \qquad Z_\epsilon^x(0) = x$$

The events $\Gamma \subset C([0,t]; H)$ that do not contain Z^x describe a **deviant** behavior.

We want to know how deviant a particular event Γ such that $Z^x \notin \bar{\Gamma}$ is, more precisely we want to compute the exponential rate at which $\mathcal{L}(X_\epsilon^{t,x})(\Gamma)$ goes to zero.

A family of probability measures $\{\mu_\epsilon\}_{\epsilon>0}$ on a Polish space E satisfies a large deviation principle, with speed $1/\epsilon$ and **action functional** $I : E \to [0, +\infty]$ if

- for every $A \subset E$ open $\liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(A) \geq -\inf_{a \in A} I(a)$,
- for every $C \subset E$ closed $\limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(C) \leq -\inf_{a \in C} I(a)$.
- for every $s \geq 0$, the set $\{I(a) \leq s\}$ is compact in E,

Theorem (S. Cerrai, G. Guatteri, G.T.)

Assume that $(S(t))$ is an analytic semigroup then the family $\{\mathcal{L}(X_\epsilon^{t,x})\}_{\epsilon \in (0,\bar{\epsilon})}$ satisfies a ***LDP*** *in $C([0,t];H)$ governed by the action functional*

$$I_{t,x}(f) = \frac{1}{2} \inf \left\{ \int_0^t \|\varphi(s)\|_H^2 \, ds \,:\, f(s) = X_\varphi^{t,x}(s),\ s \in [0,t] \right\},$$

where f ranges over continuous function $[0,t] \to H$ with $f(0) = x$ and $X_\varphi^{t,x}$ is the unique mild solution of problem

$$\begin{cases} (X_\varphi^{t,x}(s))' = AX_\varphi^{t,x}(s) + b(X_\varphi^{t,x}(s)) + \sigma\left(X_\varphi^{t,x}(s), g(Z^{X_\varphi^{t,x}(s)}(t-s))\right)\varphi(s) \\ X_\varphi^{t,x}(0) = x \in H, \end{cases}$$

We recall that for every $y \in H$, (Z^y) verifies:

$$Z^y(s) = e^{sA}y + \int_0^s e^{(s-r)A} b(Z^y(r))\, dr.$$

By general results, [A. Budhiraja, P. Dupuis, V. Maroulas, *Ann. Probab.* 2008] **a LDP with rate function** $I_{t,x}(f) = \frac{1}{2} \inf \left\{ \int_0^t \|\varphi(s)\|_H^2 \, ds \,:\, f = X_\varphi^{t,x} \right\}$, **holds** when the conditions below are verified

By general results, [A. Budhiraja, P. Dupuis, V. Maroulas, *Ann. Probab.* 2008] **a LDP with rate function** $I_{t,x}(f) = \frac{1}{2}\inf\left\{\int_0^t \|\varphi(s)\|_H^2\,ds \,:\, f = X_\varphi^{t,x}\right\}$, **holds** when the conditions below are verified

- for every $t, R > 0$, the level sets $\{I_{t,x} \leq R\}$ are compact in $C([0,t];H)$.
- For every $M > 0$ denote by $\Lambda_{t,M}$ the set of progressively measurable processes φ such that $\|\varphi\|_{L^2(0,t;H)} \leq M$, $\mathbb{P}-$a.s..

  For all $\{\varphi_\epsilon\}_{\epsilon>0} \subseteq \Lambda_{t,M}$ and $\varphi \in \Lambda_{t,M}$:

  if $\varphi_\epsilon \rightharpoonup \varphi$ **weakly in** $L^2(0,t;H)$ **in distribution**

  then $X^{t,x}_{\varphi_\epsilon,\epsilon} \to X^{t,x}_\varphi$ **strongly in** $C([0,t];H)$ **in distribution**

Where $X^{t,x}_{\varphi_\epsilon,\epsilon}$ solves:

$$\begin{cases} dX^{t,x}_{\varphi_\epsilon,\epsilon}(s) = \left[AX^{t,x}_{\varphi_\epsilon,\epsilon}(s) + b((X^{t,x}_{\varphi_\epsilon,\epsilon}(s))\right]ds + \\ \qquad \sqrt{\epsilon}\,\sigma\left(X^{t,x}_{\varphi_\epsilon,\epsilon}(s), v^t_\epsilon(s, X^{t,x}_{\varphi_\epsilon,\epsilon}(s))\right)dW_s + \sigma\left(X^{t,x}_{\varphi_\epsilon,\epsilon}(s), v^t_\epsilon(s, X^{t,x}_{\varphi_\epsilon,\epsilon}(s))\right)\varphi_\epsilon(s)ds, \\ X^{t,x}_{\varphi_\epsilon,\epsilon}(0) = x \in H, \end{cases}$$

and $v^t_\epsilon(s,x) = \mathbb{E}g(X^{t-s,x}_\epsilon(t-s))$

The controlled SPDE

We will concentrate on the second condition.
It is worth noticing that the controlled SPDE

$$\begin{cases} dX^{t,x}_{\varphi_\epsilon,\epsilon}(s) = \left[AX^{t,x}_{\varphi_\epsilon,\epsilon}(s) + b((X^{t,x}_{\varphi_\epsilon,\epsilon}(s))\right] ds + \\ \qquad \sqrt{\epsilon}\,\sigma\left(X^{t,x}_{\varphi_\epsilon,\epsilon}(s), v^t_\epsilon(s, X^{t,x}_{\varphi_\epsilon,\epsilon}(s))\right) dW_s + \sigma\left(X^{t,x}_{\varphi_\epsilon,\epsilon}(s), v^t_\epsilon(s, X^{t,x}_{\varphi_\epsilon,\epsilon}(s))\right) \varphi_\epsilon(s) ds, \\ X^{t,x}_{\varphi_\epsilon,\epsilon}(0) = x \in H, \end{cases}$$

can be rewritten as

$$\begin{cases} dX^{t,x}_{\varphi_\epsilon,\epsilon}(s) = \left[AX^{t,x}_{\varphi_\epsilon,\epsilon}(s) + b((X^{t,x}_{\varphi_\epsilon,\epsilon}(s))\right] ds + \\ \qquad \sqrt{\epsilon}\,\sigma\left(X^{t,x}_{\varphi_\epsilon,\epsilon}(s), \mathbb{E}^{\mathbb{P}^\epsilon}\left(g(X^{t,x}_{\varphi_\epsilon,\epsilon})(t) \middle| \mathcal{F}^W_s\right)\right) dW^\epsilon_s, \\ X^{t,x}_\epsilon(0) = x \in H, \end{cases}$$

where $W^\epsilon_s := W_s + \frac{1}{\sqrt{\epsilon}} \int_0^s \varphi(r) dr$
and $\mathbb{P}^\epsilon$ is the probability under which (W^ϵ) is a Wiener process.

Sketch of the proof

We have to prove that for all $\{\varphi_\epsilon\}_{\epsilon>0}$ with $|\varphi_\epsilon|_{L^2(0,t;H)} \leq M$, $\mathbb{P}$-a.s.
if $\varphi_\epsilon \rightharpoonup \varphi$ weakly in $L^2(0,t;H)$ in distribution then $X^{t,x}_{\varphi_\epsilon,\epsilon} \to X^{t,x}_{\varphi}$ strongly in $C([0,t];H)$ in distribution.

We have to prove that for all $\{\varphi_\epsilon\}_{\epsilon>0}$ with $|\varphi_\epsilon|_{L^2(0,t;H)} \leq M$, $\mathbb{P}$-a.s.
if $\varphi_\epsilon \rightharpoonup \varphi$ weakly in $L^2(0,t;H)$ in distribution then $X^{t,x}_{\varphi_\epsilon,\epsilon} \to X^{t,x}_{\varphi}$ strongly in $C([0,t];H)$ in distribution.

We proceed as follows:

- Use Skorohod's Theorem to pass from convergence in law to $\mathbb{P}$-a.s. (weak) convergence $\varphi_\epsilon \rightharpoonup \phi$ in $L^2(0,t;H)$

Sketch of the proof

We have to prove that for all $\{\varphi_\epsilon\}_{\epsilon>0}$ with $|\varphi_\epsilon|_{L^2(0,t;H)} \leq M$, $\mathbb{P}$-a.s.
if $\varphi_\epsilon \rightharpoonup \varphi$ weakly in $L^2(0,t;H)$ in distribution then $X^{t,x}_{\varphi_\epsilon,\epsilon} \to X^{t,x}_{\varphi}$ strongly in $C([0,t];H)$ in distribution.

We proceed as follows:

- Use Skorohod's Theorem to pass from convergence in law to $\mathbb{P}$-a.s. (weak) convergence $\varphi_\epsilon \rightharpoonup \phi$ in $L^2(0,t;H)$
- Prove that $v^t_\epsilon(s,x) = \mathbb{E}g(X^{t-s,x}_\epsilon(t-s))$ is lipschitz in x uniformly with respect to $s \in [0,t]$ and $\epsilon \leq \bar{\epsilon}$

We have to prove that for all $\{\varphi_\epsilon\}_{\epsilon>0}$ with $|\varphi_\epsilon|_{L^2(0,t;H)} \leq M$, $\mathbb{P}$-a.s. if $\varphi_\epsilon \rightharpoonup \varphi$ weakly in $L^2(0,t;H)$ in distribution then $X^{t,x}_{\varphi_\epsilon,\epsilon} \to X^{t,x}_{\varphi}$ strongly in $C([0,t];H)$ in distribution.

We proceed as follows:

- Use Skorohod's Theorem to pass from convergence in law to $\mathbb{P}$-a.s. (weak) convergence $\varphi_\epsilon \rightharpoonup \phi$ in $L^2(0,t;H)$
- Prove that $v^t_\epsilon(s,x) = \mathbb{E}g(X^{t-s,x}_\epsilon(t-s))$ is lipschitz in x uniformly with respect to $s \in [0,t]$ and $\epsilon \leq \bar{\epsilon}$
- prove that for some $\delta > 0$, $\alpha > 0$

$$\|X^{t,x}_{\varphi_\epsilon} - X^{t,x}_{\varphi}\|_{C^\delta([0,t];\mathcal{D}((-A)^\alpha))} \leq \|\varphi_\epsilon - \varphi\|_{L^2(0,t;H)}$$

  where $C^\delta([0,t];\mathcal{D}((-A)^\alpha))$ is the space of δ-Holder continuous functions with values in $\mathcal{D}((-A)^\alpha)$

We have to prove that for all $\{\varphi_\epsilon\}_{\epsilon>0}$ with $|\varphi_\epsilon|_{L^2(0,t;H)} \leq M$, $\mathbb{P}$-a.s.
if $\varphi_\epsilon \rightharpoonup \varphi$ weakly in $L^2(0,t;H)$ in distribution then $X^{t,x}_{\varphi_\epsilon,\epsilon} \to X^{t,x}_{\varphi}$ strongly in $C([0,t];H)$ in distribution.

We proceed as follows:

- Use Skorohod's Theorem to pass from convergence in law to $\mathbb{P}$-a.s. (weak) convergence $\varphi_\epsilon \rightharpoonup \phi$ in $L^2(0,t;H)$
- Prove that $v^t_\epsilon(s,x) = \mathbb{E}g(X^{t-s,x}_\epsilon(t-s))$ is lipschitz in x uniformly with respect to $s \in [0,t]$ and $\epsilon \leq \bar{\epsilon}$
- prove that for some $\delta > 0$, $\alpha > 0$

$$\|X^{t,x}_{\varphi_\epsilon} - X^{t,x}_{\varphi}\|_{C^\delta([0,t];\mathcal{D}((-A)^\alpha))} \leq \|\varphi_\epsilon - \varphi\|_{L^2(0,t;H)}$$

  where $C^\delta([0,t];\mathcal{D}((-A)^\alpha))$ is the space of δ-Holder continuous functions with values in $\mathcal{D}((-A)^\alpha)$
- exploit the compact embedding $C^\delta([0,t];\mathcal{D}((-A)^\alpha)) \hookrightarrow C([0,t];H)$ to obtain strong convergence of $X^{t,x}_{\varphi_\epsilon,\epsilon} \to X^{t,x}_{\varphi}$ in $C([0,t];H)$.

□

We want to extend our results to an explicit nonlinear reaction diffusion equation

$$\begin{cases} dX_\epsilon^{t,x}(s,\xi) = \Delta_\xi X_\epsilon^{t,x}(s,\xi)ds + b_0(\xi, X_\epsilon^{t,x}(s,\xi))ds \\ \qquad\qquad +\sqrt{\epsilon}\,\sigma_0\left(\xi, X_\epsilon^{t,x}(s,\xi), \mathbb{E}\left(g(X_\epsilon^{t,x}(s))\middle|\mathcal{F}_s^W\right)\right)\dot{W}(s,\xi)ds, \\ \\ X_\epsilon^{t,x}(0,\xi) = x(\xi) \quad \xi \in [0,1]; \qquad X_\epsilon^{t,x}(s,0) = X_\epsilon^{t,x}(s,1) \;\; s \in [0,T] \end{cases}$$

where $\sigma_0 : \mathbb{R}^3 \to \mathbb{R}$ and $g : C([0,1]) \to \mathbb{R}$ are Lipschitz and

$$b_0(\xi, x) = -x^{2m+1} + a_m(\xi)x^{2m} + ... + a_0(\xi)$$

is an odd dissipative polynomial function.

Existence and uniqueness (even for ϵ small) is non-trivial: we can not localize in time.

Step 1: Solve

$$\begin{cases} dX_\epsilon^{t,x}(s,\xi) = \Delta_\xi X_\epsilon^{t,x}(s,\xi)ds \\ \qquad\qquad + \sqrt{\epsilon}\,\sigma_0\left(\xi, X_\epsilon^{t,x}(s,\xi), \mathbb{E}\left(g(X_\epsilon^{t,x}(s))\middle|\mathcal{F}_s^W\right)\right)\dot{W}(s,\xi)ds, \\ X_\epsilon^{t,x}(0,\xi) = x(\xi) \quad \xi\in[0,1]; \qquad X_\epsilon^{t,x}(s,0) = X_\epsilon^{t,x}(s,1) \;\; s\in[0,T] \end{cases}$$

in the space $L^p(\Omega; C([0,T]; C_0([0,1])))$ see [S. Cerrai PTRF 2003]

Step 1: Solve

$$\begin{cases} dX_\epsilon^{t,x}(s,\xi) = \Delta_\xi X_\epsilon^{t,x}(s,\xi)ds \\ \qquad\qquad +\sqrt{\epsilon}\,\sigma_0\left(\xi, X_\epsilon^{t,x}(s,\xi), \mathbb{E}\left(g(X_\epsilon^{t,x}(s))\middle|\mathcal{F}_s^W\right)\right)\dot{W}(s,\xi)ds, \\ X_\epsilon^{t,x}(0,\xi) = x(\xi) \quad \xi\in[0,1]; \qquad X_\epsilon^{t,x}(s,0) = X_\epsilon^{t,x}(s,1) \;\; s\in[0,T] \end{cases}$$

in the space $L^p(\Omega; C([0,T]; C_0([0,1])))$ see [S. Cerrai PTRF 2003]

Step 2: Show that the map Φ that asociates to $Z \in C([0,T]; C_0([0,1]))$ the solution of equation

$$\Phi(Z)_t = \int_0^t e^{(t-s)\Delta} b_0(\Phi(Z)_s)ds + Z_t$$

is Lipschitz in the sup norm (see [M. Salins, SPDEs A&C 2021])

Step 1: Solve

$$\begin{cases} dX_\epsilon^{t,x}(s,\xi) = \Delta_\xi X_\epsilon^{t,x}(s,\xi)ds \\ \qquad\qquad + \sqrt{\epsilon}\,\sigma_0\left(\xi, X_\epsilon^{t,x}(s,\xi), \mathbb{E}\left(g(X_\epsilon^{t,x}(s))\middle|\mathcal{F}_s^W\right)\right)\dot{W}(s,\xi)ds, \\ X_\epsilon^{t,x}(0,\xi) = x(\xi) \quad \xi \in [0,1]; \qquad X_\epsilon^{t,x}(s,0) = X_\epsilon^{t,x}(s,1) \ \ s \in [0,T] \end{cases}$$

in the space $L^p(\Omega; C([0,T]; C_0([0,1])))$ see [S. Cerrai PTRF 2003]

Step 2: Show that the map Φ that asociates to $Z \in C([0,T]; C_0([0,1]))$ the solution of equation

$$\Phi(Z)_t = \int_0^t e^{(t-s)\Delta} b_0(\Phi(Z)_s)ds + Z_t$$

is Lipschitz in the sup norm (see [M. Salins, SPDEs A&C 2021])

Large Deviation Principle: To Do

We come back to the PDE in infinite variables

$$\begin{cases} D_t u_\epsilon(t,x) = \frac{\epsilon}{2}\mathrm{Tr}\left[\sigma\sigma^\star(x,u_\epsilon(t,x))D_x^2 u_\epsilon(t,x)\right] + \langle Ax + b(x), Du_\epsilon(t,x)\rangle_H, \\ u_\epsilon(0,x) = g(x), \quad x \in H. \end{cases}$$

We come back to the PDE in infinite variables

$$\begin{cases} D_t u_\epsilon(t,x) = \frac{\epsilon}{2}\mathrm{Tr}\left[\sigma\sigma^\star(x,u_\epsilon(t,x))D_x^2 u_\epsilon(t,x)\right] + \langle Ax + b(x), Du_\epsilon(t,x)\rangle_H, \\ u_\epsilon(0,x) = g(x), \quad x \in H. \end{cases}$$

We are now interested into "classical and regular" solutions. We will only able to consider the case when there exist a bounded non-negative symmetric operator Q and a continuous mapping $F : H \times \mathbb{R} \to \mathcal{L}_1^+(H)$ such that

$$\sigma^\star\sigma(x,r) = Q + \delta\, F(x,r), \qquad x \in H, \quad r \in \mathbb{R}.$$

We come back to the PDE in infinite variables

$$\begin{cases} D_t u_\epsilon(t,x) = \frac{\epsilon}{2}\mathrm{Tr}\left[\sigma\sigma^\star(x,u_\epsilon(t,x))D_x^2 u_\epsilon(t,x)\right] + \langle Ax + b(x), Du_\epsilon(t,x)\rangle_H, \\ u_\epsilon(0,x) = g(x), \quad x \in H. \end{cases}$$

We are now interested into "classical and regular" solutions. We will only able to consider the case when there exist a bounded non-negative symmetric operator Q and a continuous mapping $F : H \times \mathbb{R} \to \mathcal{L}_1^+(H)$ such that

$$\sigma^\star\sigma(x,r) = Q + \delta\, F(x,r), \qquad x \in H, \quad r \in \mathbb{R}.$$

$\delta > 0$ will be chosen **small enough**. On A and Q we assume:

- The semigroup (S) generated by A is of negative type
- For every $t \geq 0$, $Q_t := \displaystyle\int_0^t S(r)QS^\star(r)\,dr$ is trace class (that is $\in \mathcal{L}_1^+(H)$).

We come back to the PDE in infinite variables

$$\begin{cases} D_t u_\epsilon(t,x) = \frac{\epsilon}{2}\mathrm{Tr}\left[\sigma\sigma^\star(x,u_\epsilon(t,x))D_x^2 u_\epsilon(t,x)\right] + \langle Ax+b(x), Du_\epsilon(t,x)\rangle_H, \\ u_\epsilon(0,x) = g(x), \quad x \in H. \end{cases}$$

We are now interested into "classical and regular" solutions. We will only able to consider the case when there exist a bounded non-negative symmetric operator Q and a continuous mapping $F : H \times \mathbb{R} \to \mathcal{L}_1^+(H)$ such that

$$\sigma^\star\sigma(x,r) = Q + \delta\, F(x,r), \qquad x \in H, \quad r \in \mathbb{R}.$$

$\delta > 0$ will be chosen **small enough**. On A and Q we assume:

- The semigroup (S) generated by A is of negative type
- For every $t \geq 0$, $Q_t := \int_0^t S(r)QS^\star(r)\,dr$ is trace class (that is $\in \mathcal{L}_1^+(H)$).
- For every $t > 0$ $S(t)H \subset Q_t^{1/2}H$.

We come back to the PDE in infinite variables

$$\begin{cases} D_t u_\epsilon(t,x) = \frac{\epsilon}{2}\text{Tr}\left[\sigma\sigma^\star(x,u_\epsilon(t,x))D_x^2 u_\epsilon(t,x)\right] + \langle Ax + b(x), Du_\epsilon(t,x)\rangle_H, \\ u_\epsilon(0,x) = g(x), \quad x \in H. \end{cases}$$

We are now interested into "classical and regular" solutions. We will only able to consider the case when there exist a bounded non-negative symmetric operator Q and a continuous mapping $F : H \times \mathbb{R} \to \mathcal{L}_1^+(H)$ such that

$$\sigma^\star\sigma(x,r) = Q + \delta\, F(x,r), \quad x \in H, \quad r \in \mathbb{R}.$$

$\delta > 0$ will be chosen **small enough**. On A and Q we assume:

- The semigroup (S) generated by A is of negative type
- For every $t \geq 0$, $Q_t := \int_0^t S(r)QS^\star(r)\,dr$ is trace class (that is $\in \mathcal{L}_1^+(H)$).
- For every $t > 0$ $S(t)H \subset Q_t^{1/2}H$.
- If we define $\Lambda_t := Q_t^{-1/2}S(t)$ there exists some $\lambda > 0$ such that
  $\|\Lambda_t\|_{\mathcal{L}(H)} \leq c\,(t \wedge 1)^{-1/2}e^{-\lambda t}, \quad t > 0.$

An example

Let $H = L^2(\mathcal{O})$, for some bounded interval $\mathcal{O} \subset \mathbb{R}$, and let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal basis of H contained in $L^\infty(\mathcal{O})$.

- A is the realization of the Laplace operator with Dirichlet boundary conditions in $\mathcal{O}$
- $Q = I$.

Let $H = L^2(\mathcal{O})$, for some bounded interval $\mathcal{O} \subset \mathbb{R}$, and let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of H contained in $L^\infty(\mathcal{O})$.

- A is the realization of the Laplace operator with Dirichlet boundary conditions in $\mathcal{O}$
- $Q = I$.

We fix non-negative numbers $\{\lambda_i\}_{i\in\mathbb{N}}$, and we assume that $\sum_{i=1}^{\infty} \lambda_i \|e_i\|_{L^\infty(\mathcal{O})} < \infty$.

- For every $x \in H$, $r \in \mathbb{R}$, and $i \in \mathbb{N}$, we define

  $$[F(x,r)e_i](\xi) = \mathfrak{f}_i(x(\xi), r)\lambda_i e_i(\xi), \qquad \xi \in \mathcal{O},$$

  for some nice smooth functions $\mathfrak{f}_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

The well-posedness result for the quasi-linear equation

We rewrite the quasi-linear equation as a **perturbation** of Ornstein-Uhlenbeck (ϵ is now irrelevant since we have δ small; we set $\epsilon = 1$ and simplify notation)

$$\begin{cases} D_s u(s,x) = \mathcal{L}u(s,x) + \frac{\delta}{2}\mathrm{Tr}\left[F(x,u(s,x))D_x^2 u(s,x)\right] + \langle b(x), Du(s,x)\rangle_H, \\ u(0,x) = g(x), \quad x \in H, \end{cases}$$

$\mathcal{L}\varphi(x) = \mathrm{Tr}\left[QD_x^2\varphi(x)\right] + \langle Ax, D_x\varphi(x)\rangle_H$ is the Ornstein-Uhlenbeck operator.

We rewrite the quasi-linear equation as a **perturbation** of Ornstein-Uhlenbeck (ϵ is now irrelevant since we have δ small; we set $\epsilon = 1$ and simplify notation)

$$\begin{cases} D_s u(s,x) = \mathcal{L}u(s,x) + \frac{\delta}{2}\mathrm{Tr}\left[F(x,u(s,x))D_x^2 u(s,x)\right] + \langle b(x), Du(s,x)\rangle_H, \\ u(0,x) = g(x), \quad x \in H, \end{cases}$$

$\mathcal{L}\varphi(x) = \mathrm{Tr}\left[QD_x^2\varphi(x)\right] + \langle Ax, D_x\varphi(x)\rangle_H$ is the Ornstein-Uhlenbeck operator.

Strategy: treat the red terms on the right hand side as perturbations of $\mathcal{L}$.

We rewrite the quasi-linear equation as a **perturbation** of Ornstein-Uhlenbeck (ϵ is now irrelevant since we have δ small; we set $\epsilon = 1$ and simplify notation)

$$\begin{cases} D_s u(s,x) = \mathcal{L}u(s,x) + \frac{\delta}{2}\mathrm{Tr}\left[F(x,u(s,x))D_x^2 u(s,x)\right] + \langle b(x), Du(s,x)\rangle_H, \\ u(0,x) = g(x), \quad x \in H, \end{cases}$$

$\mathcal{L}\varphi(x) = \mathrm{Tr}\left[QD_x^2\varphi(x)\right] + \langle Ax, D_x\varphi(x)\rangle_H$ is the Ornstein-Uhlenbeck operator.

Strategy: treat the red terms on the right hand side as perturbations of $\mathcal{L}$.

Theorem (S. Cerrai, G. Guatteri, G.T., *JFA* 2024)

Fix $\frac{1}{2} < \eta < 1$, $0 < \vartheta < \frac{\eta - 1}{2}$, let $\varrho = \frac{1-\eta+\vartheta}{2}$ and assume that $g \in C_b^\eta(H)$. There exists $\bar{\delta} > 0$ such that for every $\delta \leq \bar{\delta}$, $t > 0$ there exists a unique classical solution $u \in C([0,t], H)$. Moreover denoting Hölder norms by $\|\cdot\|_\alpha$.

$$\sup_{s\in(0,t]}\left(\|u(s,\cdot)\|_\eta + (s\wedge 1)^\varrho\|D_x u(s,\cdot)\|_\vartheta + (s\wedge 1)^{\varrho+\frac{1}{2}}\|D_x^2 u(s,\cdot)\|_\vartheta\right) \leq c_\delta\,\|g\|_\eta,$$

for some constant $c_\delta > 0$ independent of t.

Step 1.

We write the quasi-linear PDE in mild form as

$$u(s,x) = R_s g(x) + \frac{\delta}{2}\int_0^s R_{s-r}\mathrm{Tr}\left[\mathfrak{F}(u(r,\cdot))D_x^2 u(r,\cdot)\right](x)\,dr + \int_0^s R_{s-r}\langle b(\cdot), Du(r,\cdot)\rangle_H(x)\,dr.$$

where $\mathfrak{F}(\psi)(x) = F(x,\psi(x))$ and

$$R_s\psi := \int_H \psi(S(s)x+y)\mathcal{N}_{Q_s}(dy), \qquad \psi \in B_b(H)$$

is the Ornstein-Uhlenbeck semigroup.

Notice that if $\xi(s,x) = [R_s\psi](x)$ then ξ is a classical solution of the linear PDE in H:

$$\begin{cases} \frac{d}{ds}\xi(s,x) = \mathcal{L}\,\xi(s,x) \\ \xi(0,x) = \psi(x) \end{cases}$$

By [Da Prato, Zabczyk 2002] we know that the operator R_s, $s > 0$ is smoothing. For instance for every $0 \leq \beta \leq \alpha$ there exist some $c_{\alpha,\beta} > 0$ such that

$$\|R_s\varphi\|_\alpha \leq c_{\alpha,\beta}\,(s \wedge 1)^{-\frac{\alpha-\beta}{2}}\,\|\varphi\|_\beta, \qquad t > 0.$$

By [Da Prato, Zabczyk 2002] we know that the operator R_s, $s > 0$ is smoothing. For instance for every $0 \leq \beta \leq \alpha$ there exist some $c_{\alpha,\beta} > 0$ such that

$$\|R_s\varphi\|_\alpha \leq c_{\alpha,\beta}\,(s \wedge 1)^{-\frac{\alpha-\beta}{2}}\,\|\varphi\|_\beta, \qquad t > 0.$$

Moreover R_s, $s > 0$ maps $C^0(H)$ into $C^\infty(H)$. In particular if we set

$$\|\varphi\|_{s,\theta} := \left(\|\varphi\|_0 + e^{-\omega\theta s}\,[\varphi]_\theta\right)$$

where $[\varphi]_\theta$ is the θ-Hölder seminorm then il holds for every $n \in \mathbb{N}$ and $0 \leq \theta \leq \rho \leq 1$

$$\|D^n R_s\varphi\|_\theta \leq c_{n,\theta,\rho}\,(s \wedge 1)^{-\frac{n-(\rho-\theta)}{2}}\,e^{-\lambda ns}\|\varphi\|_{s,\rho}, \qquad s > 0.$$

Step 2: We try to establish a contraction argument, we have to take into account that:

- we deal with a second order nonlinear term $\mathrm{Tr}\left[F(u(r,\cdot))D_x^2 u(r,\cdot)\right]$ in our function space we have to go up to **second order differentiability**.
- g is not smooth we have to cope with **explosions** of norms when $s \searrow 0$.

Thus we chose to work with the space of smooth Hölder continuous functions in H endowed with a slight modification of the norm:

$$\sup_{s\in(0,t]}\left(\|u(s,\cdot)\|_\eta + (s\wedge 1)^\varrho \|D_x u(s,\cdot)\|_\vartheta + (s\wedge 1)^{\varrho+\frac{1}{2}}\|D_x^2 u(s,\cdot)\|_\vartheta\right) := |||u|||_{\eta,\rho,\theta,t}$$

Step 2: We try to establish a contraction argument, we have to take into account that:

- we deal with a second order nonlinear term $\mathrm{Tr}\left[F(u(r,\cdot))D_x^2 u(r,\cdot)\right]$ in our function space we have to go up to **second order differentiability**.
- g is not smooth we have to cope with **explosions** of norms when $s \searrow 0$.

Thus we chose to work with the space of smooth Hölder continuous functions in H endowed with a slight modification of the norm:

$$\sup_{s\in(0,t]}\left(\|u(s,\cdot)\|_\eta + (s\wedge 1)^\varrho \|D_x u(s,\cdot)\|_\vartheta + (s\wedge 1)^{\varrho+\frac{1}{2}}\|D_x^2 u(s,\cdot)\|_\vartheta\right) := \|\!|\!| u |\!|\!\|_{\eta,\rho,\theta,t}$$

We prove, exploiting smoothing estimates for (R), that if

$$\Gamma[u](s,x) := R_s g(x) + \frac{\delta}{2}\int_0^s R_{s-r}\mathrm{Tr}\left[\mathfrak{F}(u(r,\cdot))D_x^2 u(r,\cdot)\right](x)\,dr + \int_0^s R_{s-r}\langle b(\cdot), Du(r,\cdot)\rangle_H(x)\,dr.$$

and **if δ and t are sufficiently small** then Γ is a contraction with respect to norm $\|\!|\!|\cdot|\!|\!\|_{\eta,\rho,\theta,t}$. Thus **a unique local mild solution exists.**

Step 3: We show, again exploiting the smoothing of R_s, that any local mild solution u is in fact a **classical** solution. In particular,

- $u(s,\cdot) \in C_b^2(H)$, for every $s \in (0,t]$,
- $QD_x^2 u(t,x) \in \mathcal{L}_1(H)$ (is a trace-class operator),
- $u(\cdot,x) \in C^1(0,+\infty)$, for every $x \in D(A)$,

Step 3: We show, again exploiting the smoothing of R_s, that any local mild solution u is in fact a **classical** solution. In particular,

- $u(s,\cdot) \in C_b^2(H)$, for every $s \in (0,t]$,
- $QD_x^2 u(t,x) \in \mathcal{L}_1(H)$ (is a trace-class operator),
- $u(\cdot,x) \in C^1(0,+\infty)$, for every $x \in D(A)$,

Step 4: We show that local solution satisfies an *a-priori* bound, that is lies in a ball with respect to the norm $|||u|||_{\eta,\rho,\theta,t}$.

Notice that the estimate if the C^0 part of the norm, that is the **maximum principle**

$$\|u(s,\cdot)\|_0 \leq \|g\|_0$$

comes form stochastic representation of the solution u.

We conclude **global existence** by a standard iterative process. □

We remark that, even in the case in which F only depends on x, that is for equation:

$$\begin{cases} D_t u(t,x) = \mathcal{L}u(t,x) + \operatorname{Tr}\left[\mathrm{f}(x) D_x^2 u(t,x)\right] + \langle b(x), Du(t,x)\rangle_H, \\ u(0,x) = g(x), \quad x \in H, \end{cases} \tag{6}$$

existence of classical solutions is not a trivial result, see [Cannarsa, Da Prato 1996] or [Zambotti 1999].

We remark that, even in the case in which F only depends on x, that is for equation:

$$\begin{cases} D_t u(t,x) = \mathcal{L}u(t,x) + \operatorname{Tr}\left[\mathrm{f}(x) D_x^2 u(t,x)\right] + \langle b(x), Du(t,x)\rangle_H, \\ u(0,x) = g(x), \quad x \in H, \end{cases} \tag{6}$$

existence of classical solutions is not a trivial result, see [Cannarsa, Da Prato 1996] or [Zambotti 1999].

In particular, the regularity estimates for u, obtained in the above papers, depending on the regularity of f, are not good enough (for us).
Namely we can not attack our PDE by a contraction argument following the schema:

$$u \mapsto F \circ u := f \text{ and then } f \mapsto u$$

where the last map is indeed given by equation (6).

Some References

- A. Budhiraja, P. Dupuis, V. Maroulas, Annals of Probability (2008).
- M. Freidlin, L. Koralov, Probability Theory Related Fields, (2010).
- S. Cerrai, G. Guatteri & GT, Journal of Functional Analysis, (2024).
- P. Cannarsa, G. Da Prato, Advances in Differential Equations, (1996).
- M. Freidlin, A. Wentzell, RANDOM PERTURBATIONS OF DYNAMICAL SYSTEMS, 2012.
- E. Pardoux, S. Tang, Probability Theory Related Fields, (1999).
- L. Zambotti, Stochastic Analysis and Applications, (1999).

Grazie

e

Tanti auguri!