Stochastic control problems with delay: solution through partial smoothing

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Workshop on Stochastic Processes,
Stochastic Optimal Control, and their Applications

Milano, September 26 2024

Outline

Optimal advertising with memory effects

$$\begin{cases} dx(t) = a_0 x(t)dt + \int_{-d}^{0} a_1(\xi)x(t+\xi)d\xi dt + b_0 u(t)dt \\ \qquad + \int_{-d}^{0} b_1(\xi)u(t+\xi)d\xi dt + \sigma dB(t) \\ x(0) = \eta_0,\ x(s) = \eta_1(s),\ u(s) = u_0(s) \quad s \in [-d,0). \end{cases}$$

delay in the state, delay in the control term.

Maximize the functional $\mathbb{E}\left[\int_0^T \ell_0(x(t),u(t))\,dt + \phi(x(T))\right]$,

over the set of admissible strategies (to be fixed!).

Stochastic delay equations in $\mathbb{R}$:

For $\tau \in [0,T]$ and with $d\eta(\theta) = a_0\delta_0(\theta) + a_1(\theta)d\theta$

$$\begin{cases} dy(\tau) = \left[\int_{-d}^{0} d\eta(\theta) y(\tau+\theta)\right] d\tau + \sigma\, dW(\tau), \\ y(0) = h_0 \in \mathbb{R}, \\ y(\theta) = h_1(\theta), \quad \theta \in [-d,0], \quad h_1 \in L^2([-d,0],\mathbb{R}). \end{cases}$$

$y_t(\theta) := y(t+\theta)$. Define $H = \mathbb{R} \oplus L^2([-d,0],\mathbb{R})$ and A by

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} \in H, h_1 \in W^{1,2}([-d,0],\mathbb{R}), h_1(0) = h_0 \right\},$$

$$Ah = A\begin{pmatrix} h_0 \\ h_1 \end{pmatrix} = \begin{pmatrix} \int_{-d}^{0} d\eta(\theta)\, h_1(\theta) \\ dh_1/d\theta \end{pmatrix}.$$

Abstract formulation in $H = \mathbb{R} \oplus L^2([-d,0],\mathbb{R})$: $X_\tau = \begin{pmatrix} y(\tau) \\ y_\tau \end{pmatrix}$,

$$\begin{cases} dX_\tau = AX_\tau d\tau + G\,dW_\tau, & \tau \in [0,T] \\ X_0 = h. \end{cases}$$

$G : \mathbb{R} \longrightarrow H$, $G = \begin{pmatrix} \sigma \\ 0 \end{pmatrix}$, **Controlled delay equations**

$$\begin{cases} dy(\tau) = \left[\int_{-d}^{0} d\eta(\theta)\, y(\tau+\theta)\right] d\tau + u_\tau d\tau + \sigma dW_\tau, & \tau \in [0,T], \\ y(0) = h_0 \in \mathbb{R}, \\ y(\theta) = h_1(\theta), \quad \theta \in [-r,0], & h_1 \in L^2([-d,0],\mathbb{R}). \end{cases}$$

$$\begin{cases} dX_\tau = AX_\tau d\tau + G\sigma^{-1} u_\tau d\tau + G\,dW_\tau, & \tau \in [0,T] \\ X_0 = h. \end{cases}$$

structure condition holds (if the diffusion σ is invertible)

No smoothing for SDEs with delay

$$\begin{cases} dy(t) = dW(t), & \tau \in [0,T], \\ y(0) = h_0,\ y(\theta) = h_1(\theta), & \theta \in [-d,0], \quad a_0 = 0,\ a_1 \equiv 0 \end{cases}$$

$$Ah = A\begin{pmatrix} h_0 \\ h_1 \end{pmatrix} = \begin{pmatrix} 0 \\ dh_1/d\theta \end{pmatrix}.$$

Ornstein-Uhlenbeck process : $dX_t = AX_t dt + GdW_t,\ X_0 = x.$

O-U transition semigroup:

$$R_t[f](x) = \mathbb{E} f(X^x(t)) = \int_H f(z + e^{tA}) \mathcal{N}(0, Q_t)(dz).$$

regularizing property $\leftrightarrow$ null controllability <u>**in** H</u> of

$$\begin{cases} dz(t) = Az(t)dt + Gu(t)dt, & t \in [0,T], \\ z_0 = x. \end{cases}$$

$\rightsquigarrow$ $(e^{tA})_{t\geq 0}$ translation semigroup, it is not possible to steer to 0 $x \in H$ in small times even if $x \in \operatorname{Im} G$

A controlled SDE with delay in the control: the simplest case

- $(\Omega, \mathscr{F}, (\mathscr{F}_s)_{s\geq 0}, \mathbb{P})$, W standard Brownian motion in $\mathbb{R}$, $y \in \mathbb{R}$

$$\begin{cases} dy(s) = \left[a_0 y(s) + b_0 u(s) + \int_{-d}^{0} b_1(\xi) u(s+\xi) d\xi \right] ds + \sigma dW(s), \\ y(0) = y_0 \in \mathbb{R}, u(\xi) = u_0(\xi)\ \xi \in [-d, 0),\ u_0 \in L^2(-d, 0; \mathbb{R}). \end{cases}$$

- $u(\cdot)$ control process $a_0, b_0, \sigma \in \mathbb{R}$, $b_1(\cdot) \in L^2(-d, 0; \mathbb{R})$.
- cost functional to be minimized

$$J_0(y_0, u_0; u(\cdot)) := \mathbb{E}\left[\int_0^T (\ell(s) + \ell_1(u(s)))\, ds + \phi(y(T)) \right].$$

Infinite dimensional representation

([Vinter-Kwong '81, Gozzi-Marinelli '04])

The operators

- $H = \mathbb{R} \times L^2(-d,0;\mathbb{R})$
- $A : D(A) \subset H \to H$

$$D(A) = \{(y_0, y_1) \in H : y_1 \in W^{1,2}([-d,0],\mathbb{R}), y_1(-d) = 0\}.$$
$$A(y_0, y_1) = (a_0 y_0 + y_1(0), -y_1'),$$

- A^* the adjoint operator of A:

$$A^*(y_0, y_1) = (a_0 y_0, y_1'),$$
$$D(A^*) = \{(y_0, y_1) \in H : y_1 \in W^{1,2}([-d,0],\mathbb{R}), y_1(0) = y_0\}.$$

- $B : \mathbb{R} \to H,$ $\quad Bu = (b_0 u, b_1(\cdot) u)$, $u \in \mathbb{R}$.
- $G : \mathbb{R} \to H,$ $\quad Gy = (\sigma y, 0)$, $y \in \mathbb{R}$.

The semigroups

$\{e^{sA}\}_{s\geq 0}$ C_0-semigroup generated by A

$$e^{sA}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} e^{sa_0}y_0 + \int_0^s e^{(s-r)a_0}y_1(-r)1_{[-d,0]}(-r)dr \\ y_1(\cdot - s)1_{[-d+s,s]}(\cdot). \end{pmatrix}$$

first component of the semigroup

$$\left(e^{sA}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}\right)_0 = e^{a_0 s}y_0 + \int_0^{s\wedge d} e^{(s-r)a_0}y_1(-r)dr.$$

$e^{sA^*} = (e^{sA})^*$ C_0-semigroup generated by A^*,

$$e^{sA^*}\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} e^{sa_0^*}z_0 \\ e^{(\cdot+s)a_0^*}z_0 1_{[-s,0]}(\cdot) + z_1(\cdot + s)1_{[-d,-s]}(\cdot) \end{pmatrix}.$$

Stochastic problems with delay in the literature

- Pawels 1977, Feichtinger-Hartl-Sethi 1994, Federico-Tacconi 2013.
- Gozzi-Marinelli 2004, Gozzi-Marinelli-Savin 20009, Gozzi-M-Rosestolato 2024, De Feo 2024
- Gozzi-M 2017, Gozzi-M 2023, Gozzi-M 2023 arxiv
- Fuhrman-M-Tessitore 2010, M-Tessitore, De Feo-Federico-Swiech 2024 ....
- Fuhrman-Pham 2015, Bandini-Cosso, Fuhrman-Pham 2018 ...
- Hu-Peng 1996, Chen-Wu 2010, Guatteri-M 2021-2024-2023 arxiv, Meng-Shi-Wang-Zhang 2023 arxiv......

The infinite dimensional state equation

$(y_0, u_0) \in H$, $u \in \mathcal{U}$, $y(s; 0, y_0, u_0, u)$ solution to the SDE

Define the process $Y = (Y_0, Y_1) \in L^2_{\mathscr{P}}(\Omega \times [0, T], H)$ where

$$Y_0(s) = y(s), \qquad Y_1(s)(\xi) = \int_{-d}^{\xi} b_1(\varsigma) u(\varsigma + s - \xi) d\varsigma.$$

Y unique solution in mild sense of the evolution equation in H

$$\begin{cases} dY(s) = AY(s)dt + Bu(s)dt + G dW(s), & s \in [0, T] \\ Y(0) = x = (x_0, x_1), \end{cases}$$

where $x_0 = y_0$, $x_1(\xi) = \int_{-d}^{\xi} b_1(\varsigma) u_0(\varsigma - \xi) d\varsigma$

The control problem

The objective functional and the value function

- To minimize:

$$J_0(y_0, u_0; u(\cdot)) := \mathbb{E}\left[\int_0^T (\ell(s) + \ell_1(u(s)))\, ds + \phi(y(T))\right]$$

Dynamic Programming approach: initial time $t \in [0, T]$ to vary, Y starting at time t.

$$J(t, x; u(\cdot)) := \mathbb{E}\int_t^T (\ell(s) + \ell_1(u(s)))\; ds + \mathbb{E}\phi(Y_0(T))$$

Note the dependence only on the first component Y_0.

- value function

$$v(t, x) := \inf_{u(\cdot) \in \mathcal{U}} J(t, x; u(\cdot)), \qquad t \in [0, T], x \in \mathcal{H}.$$

Outline

The HJB equation for the value function

$$\begin{cases} -\frac{\partial v(t,x)}{\partial t} = \frac{1}{2} Tr\, G^* G D^2 v(t,x) + \langle Ax, Dv(t,x)\rangle \\ \qquad + H_{min}(Dv(t,x)) + \ell(t), \qquad t \in [0,T], x = (x_0, x_1) \in H, \\ v(T,x) = \phi(x_0), \end{cases}$$

Hamiltonian function

$$H_{min}(p) := \inf_{u \in \mathbb{R}} H_{CV}(p;u) := \inf_{u \in \mathbb{R}} \{< p, Bu >_{\mathcal{H}} + \ell_1(u)\} := g_{min}(B^* p).$$

Goal: to find a solution with **“enough” regularity** for verification theorem and optimal feedback map of the type

$$u^*(s) = G(s, Y^*(s)), \quad s \in [t,T], \qquad (u^*, Y^*) \text{ optimal pair}$$

“Enough” $\leftrightarrow$ *argmin* of H_{CV} make sense, i.e. $B^* Dv(t,x)$ exists.

Overview on known results

- value function v "smooth" (e.g. $C^{1,2}$) $\Rightarrow$ v solves the HJB equation.
- difficult to prove directly regularity results for the value function going beyond the continuity.
- good concept of solution for HJB equations seems to be the concept of ***viscosity solution*** (Crandall and Lions, '80). Problem: regularity not required..
- *Problem:* Regularity results very rare for problems with delay in the control: partial results for the first order deterministic case ([Federico-Goldys-Gozzi '11, Federico-Tacconi '13])
- However we want **"enough" regularity** ....

Overview on known results

To prove existence of "smooth" solutions three standard tools:

- by fixed point arguments by means of smoothing properties of transition semigroup (see e.g. [Cannarsa-Da Prato 1991] and many others); no smoothing for the transition semigroup (as usual for problems with delay);
- representing the solution with a suitable Backward SDE (see e.g. [Fuhrman-Tessitore '02]) $ImB \subset ImG$;
- by suitable ad hoc change of variables or explicit representation formulae only in very specific cases (see e.g. [Da Prato-Debussche '99]).

Here

Partial smoothing

structure of the problem

- J only depend on the "present" variable x_0;
- only a directional derivative $B^* Dv(t,x)$ is needed

Method and results: $\mathscr{L}\psi(x) := \frac{1}{2} Tr\, G^* G D^2\psi + \langle Ax, D\psi\rangle;$

Ornstein-Uhlenbeck process: $dX(t) = AX(t)dt + GdW(t)$

$\mathscr{L}$ generator the O-U transition semigroup:

$$R_t[\psi](x) = \mathbb{E}\left[\psi\left(e^{tA}x + \int_0^t e^{(t-s)A} dW(s)\right)\right]$$
$$= \int_H \psi(e^{tA}x + y)\mathscr{N}_{Q_t}(dy),\ Q_t := \int_0^t e^{sA} GG^* e^{sA^*} ds.$$

Method and results

- R_t to rewrite the HJB in mild form:

$$v(t,x) = R_{T-t}[\phi](x) + \int_t^T R_{s-t}[H_{min}(Dv(s,\cdot)) + \ell(s)](x)\, ds$$
$$= R_{T-t}[\phi](x) + \int_t^T R_{s-t}[g_{min}(B^* Dv(s,\cdot)) + \ell(s)](x)\, ds.$$

- fixed point theorem using regularizing properties of the O-U semigroup R_t.

Main idea: to exploit the fact that the final datum data ϕ in J depends only on the "present" component of the state.

$\rightsquigarrow$ it is enough to prove a partial smoothing property of the O-U semigroup.

- "Partial" smoothing property for the O-U semigroup: it maps B_b continuous functions depending only on the first components into differentiable functions or, "at least", into functions for which $B^* Dv$ is well defined (see e.g. [Lunardi, '97] for a similar result in finite dimension, and [M., '05] for similar results in infinite dimensions).

- To prove the the "partial" smoothing property for the O-U semigroup: study the "present" subsystem:

$$dy(s) = \left[a_0 y(s) + b_0 u(s) + \int_{-d}^{0} b_1(\xi) u(s+\xi) d\xi \right] ds + \sigma dW(s), \quad y \in \mathbb{R}$$

"present" subsystem

$$dy(s) = a_0 y(s) ds + \sigma dW(s), \quad y \in \mathbb{R}$$

- The HJB equation admits a unique solution in a suitable space to be chosen carefully when the datum ϕ depends only on the "present".

Statements of results

Infinite dimensional refromulation: controlled equation

$$dY(t) = AY(t)dt + Bu(t)dt + GdW(t)$$

uncontrolled O-U process:

$$dY(t) = AY(t)dt + GdW(t), \qquad G = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

Lemma Let $Q_t := \int_0^t e^{sA} GG^* e^{sA*}\, ds, \qquad Q_t^0 := \int_0^t e^{sa_0} \sigma\sigma^* e^{sa_0^*}\, ds.$

$$Q_t \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} Q_t^0 x_0 \\ 0 \end{pmatrix} \rightsquigarrow \operatorname{Im} Q_t = \operatorname{Im} Q_t^0 \times \{0\} \subseteq \mathbb{R} \times \{0\}$$

and $\forall \overline{\phi} \in B_b(\mathbb{R}, \mathbb{R})$, setting $\phi(x) = \overline{\phi}(x_0)$

$$R_t[\phi](x) = \int_{\mathbb{R}} \overline{\phi}(z_0 + (e^{tA}x)_0) \mathcal{N}(0, Q_t^0)(dz_0).$$

Theorem (Gozzi-M 2017)

$R_t[\phi]$ is Fréchet differentiable and $\forall\, h \in H$,

$$D(R_t[\phi])(x)h = \int_{\mathbb{R}} \phi\left(z_0 + (e^{tA}x)_0\right)$$

$$\left\langle (Q_t^0)^{-1/2}\left(e^{tA}h\right)_0, (Q_t^0)^{-1/2} z_0 \right\rangle \mathcal{N}(0, Q_t^0)(dz_0)$$

$$\rightsquigarrow |D(R_t[\phi])(x)h| \le \|\overline{\phi}\|_\infty \left\| (Q_t^0)^{-1/2}\left(e^{tA}\right)_0 \right\| |h|,$$

$$\rightsquigarrow |DR_t[\phi](x)h| \le C t^{-r-\frac{1}{2}} \|\overline{\phi}\|_\infty |h|.$$

r the Kalman exponent, it is 0 if and only if σ is onto.

Note: control theoretic interpretation of this fact using null controllability of the "present" subsystem.

The spaces

- $\alpha \in (0,1)$, $T>0$, $C^{0,1}_{\alpha,B}([0,T]\times H$ space of

$f \in C_b([0,T]\times \mathcal{H}) \cap C^{0,1}([0,T)\times H)$ s.t. $\forall\, t\in(0,T], x\in H$

$(t,x)\mapsto (T-t)^{\alpha} B^* Df(t,x), [0,T)\times H^*$ bounded and continuous

- Banach space with the norm

$$\|f\|_{C^{0,1}_{\alpha,B}} = \sup_{(t,x)\in[0,T]\times H} |f(t,x)| + \sup_{(t,x)\in[0,T)\times H} (T-t)^{\alpha} \|B^* Df(t,x)\|_{\mathbb{R}}.$$

- $v:[0,T]\times H \to \mathbb{R}$ is a mild solution of the HJB equation if
- $v \in C^{0,1}_{\frac{1}{2},B}([0,T]\times H,\mathbb{R})$;
- v satisfies

$$v(t,x) = R_{T-t}[\phi](x) + \int_t^T R_{s-t}[g_{min}(B^* Dv(s,\cdot)) + \ell_0(s)](x)\, ds$$

Solution of the HJB

Theorem

Let $g_{min}(\cdot) : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous: $\exists K > 0$ s.t. $\forall p_1, p_2, \in \mathbb{R}$

$$|g_{min}(p_1) - g_{min}(p_2)| \leq K\left(|p_1 - p_2|\right).$$

Then the HJB equation admits a mild solution which turns out to be unique in a suitable space $\Sigma_T(A, B)$.

Proof. Apply the contraction mapping principle in the suitable space $\Sigma_T(A, B)$ using the smoothing property of R_t and careful estimates on the integral convolution term.
Then prove uniqueness in $\Sigma_T(A, B)$ using a Gronwall-like estimate.

Verification Theorem and Optimal Feedback Controls

- Prove that the mild solution can be seen as the limit, in the sense of π or $\mathcal{K}$-convergence, of classical solutions.
- Apply Ito formula to the approximating solutions and to pass to the limit to prove the Verification Theorem (sufficient condition for optimality) through the so-called *Fundamental Identity*.
- We need to solve the Closed Loop Equation (CLE), and for this we need to prove some further regularity of the solutions under some further assumptions.

Verification Theorem

Theorem (Verification Theorem - Gozzi-M 2017)

Let v be the mild solution of the HJB equation, $(t,x) \in [0,T] \times H$. Then, for every admissible control $u(\cdot)$, we have the fundamental identity

$$v(t,x) = J(t,x;u(\cdot))$$
$$- \mathbb{E} \int_t^T [g_{CV}(B^* Dv(s,Y(s));u(s)) - g_{min}(B^* Dv(s,Y(s)))]\, ds$$

$$\overline{u}(s) \in \arg\min_{u \in U} g_{CV}(B^* Dv(s,Y(s));u), \quad \overline{u} \text{ admissible}$$

$\overline{u}(\cdot)$ is optimal and the value function $V(t,x) = v(t,x)$.

Verification Theorem and Optimal Feedback Controls

Theorem (Optimal Feedbacks - Gozzi-M 2017)

Let v be the mild solution of the HJB equation. Assume g_{min} have Lipschitz continuous derivative and that there exists a Lipschitz continuous selection γ of the map

$$p \mapsto \arg\min_{u\in U} g_{CV}(p;u)$$

*Fix $(t,x) \in [0,T] \times H$. Then, the **closed loop equation***

$$\begin{cases} dY(s) = AY(s)dt + B\gamma(B^* Dv(s,Y(s)))dt + GdW(s), \\ Y(0t) = x = (x_0, x_1), \end{cases}$$

has a unique solution Y^*. Setting $u^*(s) = \gamma(B^* Dv(s,Y^*(s)))$ the couple $(u^*(\cdot), Y^*(\cdot))$ is optimal.

Control problems SDEs with delay in the state

$$dy(t) = \left[\int_{-d}^{0} y(t+\theta)\, d\eta(\theta) \right] dt + \sigma u(s)\, ds + \sigma\, dW(t)$$

Controlled SDE: Mild form: structure condition

$$X_t = e^{tA} x_{t_0} + \int_0^t e^{(t-s)A} G\, u(s)\, ds + \int_0^t e^{(t-s)A} G\, dW(s).$$

Cost functional with $\phi = \overline{\phi} \circ P$:

$$J(t,x,u) = \mathbb{E} \int_t^T g(u(s))\, ds + \mathbb{E}\phi(X_T^u).$$

where

$$P : H \to \mathbb{R}, \quad P \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \alpha_0 n x_0 + \int_{-d}^{0} f(\theta) x_1(\theta) d\theta$$

where $\alpha_0 \in \mathbb{R}$ and $f \in L^2([-d,0],\mathbb{R})$.

The HJB equation

Stochastic optimal control problem: minimize over all admissible controls the cost functional:

$$J(t,x,u) = \mathbb{E}\int_t^T g(u(s))\,ds + \mathbb{E}\phi(X_T^u),\ \phi = \overline{\phi}\circ P.$$

Value function: $V(t,\xi) := \inf_{u\in\mathscr{A}^d} J(t,\xi,u)$

The HJB equation for the value function

$$\begin{cases} -\frac{\partial v(t,x)}{\partial t} = \frac{1}{2} Tr\ G^* G D^2 v(t,x) + \langle Ax, Dv(t,x)\rangle \\ \qquad\qquad + H_{min}(\nabla v(t,x)), \qquad t\in[0,T], x=(x_0,x_1)\in H, \\ v(T,x) = \phi(x), \end{cases}$$

look at the regularizing properties of the O-U transition semigroup

$$dX_t = AX_t\,dt + G\,dW_t, \qquad t \in [0,T], X(0) = x.$$

$$\begin{aligned} R_t[f](x) &= \mathbb{E} f(X_t^x) = \int_H f(z)\mathcal{N}(e^{tA}x, Q_t)(dz) \\ &= \int_H f(z + e^{tA})\mathcal{N}(0, Q_t)(dz). \end{aligned}$$

regularization for special functions

$$P : H \to \mathbb{R}, \quad P\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \alpha_0 x_0 + \int_{-d}^{0} x_1(\theta) f(\theta)\,d\theta,\ f \in L^2([-d,0],\mathbb{R})$$

Given $\overline{\phi} : \mathbb{R} \to \mathbb{R}$ define, $\phi : H \to \mathbb{R}$

$$\phi(x) = \overline{\phi}(P(x)) \quad \forall x = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \in H,\ \phi = \overline{\phi} \circ P$$

Theorem (M-Tessitore, JDE 2022)

Let X O-U process, R_t O-U transition semigroup, $\phi = \overline{\phi} \circ P$,, $\overline{\phi} \in C_b(\mathbb{R})$ with α_0 and σ invertible. Then $R_t[\phi] : H \to \mathbb{R}$ is differentiable and moreover the following estimate holds true:

$$|\nabla R_t[\phi](x)h| \leq C \frac{|h|}{\sqrt{t}} \|\overline{\phi}\|_\infty, \quad h \in H$$

$$R_t[\phi](x) = \int_H \phi(P(y + e^{tA}x))\mathcal{N}(0, Q_t)(dy)$$
$$= \int_{\mathbb{R}^n} \overline{\phi}(z + Pe^{tA}x)\mathcal{N}(0, PQ_tP^*)(dz)$$

and

$$R_t[\phi](x + h) = \int_{\mathbb{R}^n} \overline{\phi}(z + Pe^{tA}(x + h))\mathcal{N}(0, PQ_tP^*)(dz)$$
$$= \int_{\mathbb{R}^n} \overline{\phi}(z + Pe^{tA}x)\mathcal{N}(Pe^{tA}h, PQ_tP^*)(dz)$$

change of variable $z = Py$, $P^*\xi = \begin{pmatrix} \alpha_0^*\xi \\ f^*(\cdot)\xi \end{pmatrix} \in H$

covariance operator $\overline{Q}_t = PQ_tP^*$ of the Gaussian measures

$\mathcal{N}(0, PQ_tP^*)(dz)$ and $\mathcal{N}(Pe^{tA}h, PQ_tP^*)(dz)$ in $\mathbb{R}^n$:

$$\overline{Q}_t = \int_0^t \alpha_0 e^{sa_0}\sigma\sigma^* e^{sa_0^*}\alpha_0^* \, ds + o(t) \text{ as } t \to 0.$$

and

$$|\overline{Q}_t^{\frac{1}{2}}|^2 = \int_0^t |\sigma^* e^{sa_0^*}\alpha_0^* y|^2 \, ds + o(t),$$

$\mathcal{N}(0, PQ_tP^*)(dz)$ and $\mathcal{N}(Pe^{tA}h, PQ_tP^*)(dz)$ equivalent if α_0, σ invertible

$\rightsquigarrow$ regularization of the semigroup on special functions

Method and results

- Fixed point theorem that exploits the regularization properties of the O-U semigroup R_t. **Main idea: ϕ depends only on the "P" component of the state: $\phi = \overline{\phi} \circ P$ So it is enough to prove a partial smoothing property of the O-U semigroup.**
- O-U semigroup R_t has a "partial" smoothing property: it maps bounded continuous functions depending only on the "P"components into differentiable functions or, "at least", into functions for which ∇v is well defined, it would be enough ∇vG
- The HJB equation admits a unique solution in a suitable space (to be chosen carefully) when the datum ϕ depends only on the "P" component.

starting papers on partial smoothing Gozzi-M SICON 2017 & 2023

The spaces:

- $C^0_{1/2}([0,T]\times H)$ functions $f \in C_b([0,T]\times H)\cap C^{0,1}([0,T)\times H)$ such that for all $t\in(0,T]$, $x\in H$ the map

$$(t,x)\mapsto (T-t)^{1/2}\nabla f(t,x) \qquad \text{is bounded and continuous.}$$

- $\Sigma^1_{T,1/2}\subset C^{0,1}_{1/2}([0,T]\times H)$ the space of functions $g\in C_b([0,T]\times H)$ such that

$$g(t,x)=\bar f\left(t,Pe^{tA}x\right), \qquad \forall (t,x)\in[0,T]\times H,\ f:[0,T]\times\mathbb{R}^n\to\mathbb{R}$$

if, for any $t\in(0,T]$ the function $g(t,\cdot)$ is G-Fréchet differentiable such that

$$t^{\frac12}\nabla g(t,x)=\bar f\left(t,Pe^{tA}x)\right), \qquad \forall (t,x)\in[0,T]\times H,\ \bar f\in C_b((0,T]\times\mathbb{R}^n)$$

- $\Sigma^1_{T,1/2}$ is a closed subspace of $C^{0,1}_{1/2}([0,T]\times H)$

Theorem (M-Tessitore JDE 2022)

Let ψ Lipschitz continuous. Then HJB equation admits a mild solution, which turns out to be unique in $\Sigma^1_{T,1/2} \subset C^{0,1}_{1/2}([0,T] \times H)$.

Idea of the proof For $f \in \Sigma^1_{T,1/2}$, $t \in [0,T]$, $x \in H$ let $\mathscr{C}$

$$\mathscr{C}(f)(t,x) := R_t[\phi](x) + \int_0^t R_{t-s}[H_{min}(\nabla f(s,\cdot))](x)\, ds.$$

$\mathscr{C}$ well defined in $\Sigma^1_{T,1/2}$ with values in $\Sigma^1_{T,1/2}$.

$\Sigma^1_{T,1/2}$ is a closed subspace of $C^{0,1}_{1/2}([0,T] \times H)$ + $\mathscr{C}$ contraction $\rightsquigarrow$ Contraction Mapping Principle there exists a unique (in $\Sigma^1_{T,1/2}$) fixed point of the map $\mathscr{C}$, $\rightsquigarrow$ this gives a mild solution

Verification Theorem and Optimal Feedback Controls

- The problem satisfies the so called structure condition.
- Smooth the final datum $\overline{\phi}$ and prove the so called *Fundamental Identity* by means of BSDEs
- Pass to the limit to prove the Verification Theorem (sufficient condition for optimality) through the so-called *Fundamental Identity*.
- We need to solve the Closed Loop Equation (CLE), and for this we need to prove some further regularity of the solutions under some further assumptions.

Outline

Motivation from control

Motivation from stochastic optimal control: cost functional with $\phi = \overline{\phi} \circ P$, $\ell_0(s,\cdot) = \overline{\ell_0}(s,\cdot) \circ P$:

$$J(t,x,u) = \mathbb{E}\int_t^T \left[g\left(u(s)\right) + \ell_0(s, X_s^u)\right] ds + \mathbb{E}\phi\left(X_T^u\right).$$

HJB equation associated

$$\begin{cases} -\dfrac{\partial v(t,x)}{\partial t} = \mathscr{L}[v(t,\cdot)](x) + \ell_0(t,x) + H_{min}(\nabla^G v(t,x)), \\ v(T,x) = \phi(x). \end{cases}$$

in mild form

$$v(t,x) = R_{T-t}[\phi](x) + \int_t^T R_{s-t}\left[H_{min}(\nabla^G v(s,\cdot))\right](x)\, ds$$
$$+ \int_t^T R_{s-t}\left[\ell_0(s,\cdot)\right](x)\, ds,$$

Problem: even if $\ell_0(t,x) = \overline{\ell}_0(t,Px)$ it follows that

$$\int_0^t R_{t-s}\ell_0(s,\cdot)(x)\,ds$$
$$= \int_0^t \int_H \overline{\ell}_0(s, P(z + e^{(t-s)A}x))\mathcal{N}(0,Q_t)(dz)ds$$
$$= \overline{f}(t, P((e^{rA}x)_0)_{r\in[0,T]}$$

This convolution term cannot be written as a $g(t, Pe^{tA}x)$

$$\mathscr{C}(f)(t,x) := R_t[\phi](x) + \int_0^t R_{t-s}\left[H_{min}(\nabla^G f(s,\cdot)) + \ell_0(s,\cdot)\right](x)\,ds.$$

$\mathscr{C}$ does not take values in $\Sigma^1_{T,1/2}$
$\rightsquigarrow$ we need to introduce a *lifted setting*. (M-Gozzi arxiv 2023)

Lifting partial smoothing

Recall: $\ell_0(s,x) = \overline{\ell}_0(s,Px)$

$$\int_0^t R_{t-s}\ell_0(s,\cdot)(x)\,ds = \overline{f}(t,(Pe^{rA}x)_{r\in[0,T]})$$

$$\mathscr{C}_A^P((0,T];H) := \Big\{f \in C((0,T];H) : f(t) = Pe^{tA}x,\ \forall t \in (0,T]\Big\}.$$

$$y_x^P(t) = Pe^{tA}x, \qquad \Upsilon_T^P : H \to \mathscr{C}_A^P((0,T];H),\ \Upsilon_T^P(x) = y_x^P|_{[0,T]}$$

$$\mathscr{S}_{\eta,prog}^P((T_0,T]\times H) := \Big\{f : \exists F_f \in B_b((T_0,T]\times\mathscr{C}_A^P((0,+\infty);H);\mathbb{R}) :$$

$$(t,x) \mapsto (t-T_0)^\eta F_f(t,x) \text{ bounded},\ f(t,x) = F_f(t,y_x^P(\cdot\wedge t))\Big\}$$

$$\mathscr{S}_{\eta,prog}^{1,P}((T_0,T]\times H)) := \Big\{f \in \mathscr{S}^P((T_0,T]\times H) \cap C_\eta^{0,1,G}([T_0,T]\times H) :$$

$$(t,x) \mapsto (t-T_0)^\eta \nabla^G f(t,x) \in \mathscr{S}_{prog}^P((T_0,T]\times H;K)\Big\}.$$

Partial smoothing in the "lifted" setting

Assume that

$$\Upsilon_s^P e^{(t-s)A} G k \in \mathrm{Im}\left(\Upsilon_s^P Q_{t-s} (\Upsilon_s^P)^*\right)^{1/2}.$$

$$\|\left(\Upsilon_s^P Q_{t-s} (\Upsilon_s^P)^*\right)^{-1/2} \Upsilon_s^P e^{tA} G\| \le \kappa_0((t-s)^{-\gamma} \vee 1), \qquad \forall t > s \ge 0.$$

then $\forall f \in \mathscr{S}^P_{\eta,prog}((0,T] \times H, \mathbb{R})$

$$g(t,x) := \int_0^t R_{t-s}[f(s,\cdot)](x) ds \text{ well defined in } \mathscr{S}^{1,P}_{\gamma,prog}((0,T] \times H; K)$$

$$|\left\langle \nabla^G [g(t,\cdot)](x), k \right\rangle| \le C \|f\|_\infty$$

Lifted partial smoothing for SDDEs (M- Orrieri-Zanco)

$a_0 = 0$, $a_1 \equiv 0$; prove that $\Upsilon_s^P e^{(t-s)A} Gk \in \mathrm{Im}\left(\Upsilon_s^P Q_{t-s} (\Upsilon_s^P)^*\right)^{1/2}$ i. e.

$$\| G^* e^{(t-s)A^*} (\Upsilon_s^P)^* z \|_H \le \kappa_{T_1} (t-s)^{-\gamma} . \| \left(\Upsilon_s^P Q_{t-s} (\Upsilon_s^P)^*\right)^{1/2} z \|$$

$$\| G^* e^{(t-s)A^*} (\Upsilon_s^P)^* z \|_H^2$$

$$= \| \int_0^s G^* \begin{pmatrix} \alpha_0 z_0(r) + \int_{-d \vee -(t-s+r)}^0 z_0(r) f(\theta) d\theta \\ z_0(r) f(\cdot - (t-s+r)) 1_{[-d+t-s+r, 0]}(\cdot) \end{pmatrix} dr \|_H^2$$

$$= \| \int_0^s \begin{pmatrix} \alpha_0 z_0(r) + \int_{-d \vee -(t-s+r)}^0 z_0(r) f(\theta) d\theta \\ 0 \end{pmatrix} dr \|_H^2$$

$$\le C (t-s)^{-1/2} . \| \left(\Upsilon_s^P Q_{t-s} (\Upsilon_s^P)^*\right)^{1/2} z \|$$

Solution of the semilinear Kolmogorov equation

$$v(t,x) = R_{T-t}[\phi](x) + \int_t^T R_{s-t}\left[H_{min}(\nabla^G v(s,\cdot))\right](x)\,ds + \int_t^T R_{s-t}[\ell_0(s,\cdot)](x)\,ds$$

Theorem (M-Orrieri-Zanco)

The semilinear Kolmogorov equation admits a mild solution v, which is unique among the functions w such that $w(T-\cdot,\cdot) \in \mathscr{S}^{1,P,G}_{\gamma,prog}([0,T]\times H)$ and it satisfies, for $C_T > 0$

$$\|v(T-\cdot,\cdot)\|_{\mathscr{S}^{1,P,G}_{\gamma}} \le C_T\left(\|\phi\|_\infty + \|\ell_0\|_\infty\right).$$

Further work

- Lipschitz continuity of $\nabla^G v$ (M - Orrieri - Zanco)
- Partial smoothing for MFG (Calvia - Gozzi - M)
- Smoothing in case of common noise for MFG (Calvia - Gozzi - M -Tessitore)
- partial smoothing for delay equations with multiplicative noise

THANKS A LOT FOR YOUR ATTENTION!

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Solution of the semilinear Kolmogorov equation

coming back to applications to regularization by noise

$$v(t,x) = \int_t^T R_{s-t}\left[e^{-(s-t)A_n} GB(s,\cdot)\right](x)\,ds + \int_t^T R_{s-t}\left[e^{-(s-t)A_n}\nabla^G v(s,\cdot)B(s,\cdot)\right](x)\,ds$$

Theorem

The semilinear Kolmogorov equation admits a mild solution v, which is unique among the functions w such that $w(T-\cdot,\cdot) \in \mathcal{S}^{2,P,G}_{\gamma,prog}([0,T]\times H)$ and it satisfies, $C_T > 0$,

$$\|v(T-\cdot,\cdot)\|_{C^{0,2,G}} \le C_T\left(\|B\|_\infty\right).$$

Motivation from regularization by noise

stochastic evolution equation in H separable Hilbert space

$$dX_t = AX_t dt + GB(t,X_t)dt + GdW_t, \quad X_0 = x \in H, t \in [0,T]$$

$B : [0,T] \times H \to H$ bounded continuous and $\exists K > 0\, \beta \in (0,1)$ s.t.

$$|B((t,x) - B(t,y)|_U \le K|x-y|_H^\beta, \quad x,y \in H, \ t \in [0,T].$$

weak (mild) solution $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}, W, X)$,

$$X_t = e^{(t-s)A}x + \int_0^t e^{(t-s)A} GB(s,X_s)ds + \int_0^t e^{(t-s)A} GdW_s$$

$\rightsquigarrow$ existence of a (weak) mild solution by the Girsanov theorem.
pathwise uniqueness: X^1, X^2 solutions on $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}, W,)$

$$\mathbb{P}(X_t^1 = X_t^2 \text{ a.e. } t \in [0,T]) = 1$$

- S. Watanabe, T. Yamada, J. Math. Kyoto Univ. (1971) weak existence + pathwise uniqueness = strong existence
- M. Ondreját Dissertationes Math. (Rozprawy Mat.) (2004) infinite dimensional version of Yamada-Watanabe result

Idea of the method: ODEs

• A variant of the Zvonkin-Veretennikov approach: the Ito-Tanaka trick for SDEs (cf. Flandoli-Gubinelli-Priola 2010) :

SDDEs with irregular drift

$$\begin{cases} dy(t) = \bar{b}\left(\alpha_0 y(t) + \int_{-d}^{0} f(\theta) y(t+\theta) d\theta\right) dt + \sigma dW_t, & t \in [0,T] \\ y(0) = x_0 \\ y(\theta) = x_1(\theta), \quad \theta \in [-d,0) \text{ a.e.}, \end{cases}$$

rewritten as

$$\begin{cases} dX_t = AX_t dt + GB(t, X_t) + GdW_t, & t \in [0,T], \\ X_0 = x. \end{cases}$$

perturbed Ornstein-Uhlenbeck process with $B(t,z) = \bar{b}(t, Pz)$

$\bar{b}$ only α Holder continuous.

The BSDEs approach (Addona-M-Priola 2023)

PDE of Kolmogorv type in mild form

$$v(t,x) = \int_t^T R_{s-t}\left[e^{-(s-t)A_n} GB(s,\cdot)\right](x)\, ds + \int_t^T R_{s-t}\left[e^{-(s-t)A_n}\nabla^G v(s,\cdot)B(s,\cdot)\right](x)\, ds,$$

$$\mathscr{L}_t^n[f](x) = \frac{1}{2} Tr(\, GG^* \,\nabla^2 f(x)) + \langle A_n x, \nabla f(x)\rangle + \langle GB(t,x), \nabla f(x)\rangle;$$

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} + \mathscr{L}_t^n[v(t,\cdot)](x) = Av(t,x) - GB(t,x), x \in H, \;\; t \in [0,T], \\ v(T,x) = 0. \end{cases}$$

$$X_t = e^{tA}x + \int_0^t \left(e^{(t-s)A} - e^{(t-s)A_n}\right) GB(s,X_s)ds + \int_0^t e^{(t-s)A} G dW_s + u_n^t(0,x) + \int_0^t \nabla^G u_n^t(s, X_s^{0,x}) dW, \quad \mathbb{P}\text{-a.s.},$$

The associated HJB equation in $\mathbb{R}^n$

Calling $\mathscr{L}$ the generator of the above SDE when $u(\cdot) \equiv 0$:

$$[\mathscr{L}\psi](z) := \frac{1}{2} TrGG^* \partial_{zz}\psi(z) + \langle Az, \partial_z \psi(z) \rangle_{\mathbb{R}^n},$$

the HJB equation for the value function V is

$$-\partial_t V(t,z) - \mathscr{L} V(t,z) = H_{MAX}(\partial_z V(t,z)) + \ell_0(t,z),$$

where

$$H_{MAX}(p) = \sup_{u \in U} \{ \langle Bu, p \rangle_{\mathbb{R}^n} + \ell_1(u) \}$$

with terminal condition $V(T,z) = \phi(z)$. The candidate feedback map is

$$F(t,z) := [\ell_1']^{-1} (B^* \partial_z V(t,z))$$