Dual Representation of Unbounded Dynamic Concave Utilities

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Overview

dynamic concave utility function

Fix a nonnegative real number $T > 0$ and an integer $d \geq 1$. Assume that $(B_t)_{t\in[0,T]}$ is a standard d-dimensional Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\mathcal{F}_t)_{t\geq 0}$ is its augmented natural filtration with $\mathcal{F}_T = \mathcal{F}$. An equality or inequality between random elements is understood in the sense of $\mathbb{P} - a.s.$. For each $t \in [0, T]$, let $L^\infty(\mathcal{F}_t)$ denote the set of all scalar essentially bounded $\mathcal{F}_t$-measurable random variables, and let $\mathcal{A}(\mathcal{F}_t)$ denote a general linear space of scalar $\mathcal{F}_t$-measurable random variables such that $L^\infty(\mathcal{F}_s) \subset \mathcal{A}(\mathcal{F}_s) \subset \mathcal{A}(\mathcal{F}_t)$ for each $0 \leq s \leq t \leq T$. By a dynamic concave utility function defined on $\mathcal{A}(\mathcal{F}_T)$, we mean a family of time-parameterized operators $\{U_t(\cdot) : \mathcal{A}(\mathcal{F}_T) \to \mathcal{A}(\mathcal{F}_t),\ t \in [0, T]\}$ satisfying the following properties for each $t \in [0, T]$:

dynamic concave utility function

(i) **Monotonicity**: $U_t(\xi) \geq U_t(\eta)$ for each $\xi, \eta \in \mathcal{A}(\mathcal{F}_T)$ such that $\xi \geq \eta$;
(ii) **Translation invariance**: $U_t(\xi + \eta) = U_t(\xi) + \eta$ for each $\xi \in \mathcal{A}(\mathcal{F}_T)$ and $\eta \in \mathcal{A}(\mathcal{F}_t)$;
(iii) **Concavity**: $U_t(\theta\xi + (1-\theta)\eta) \geq \theta U_t(\xi) + (1-\theta)U_t(\eta)$ for all $\xi, \eta \in \mathcal{A}(\mathcal{F}_T)$ and $\theta \in (0, 1)$;
(iv) **Time consistency**: $U_s(\xi) = U_s(U_t(\xi))$ for each $\xi \in \mathcal{A}(\mathcal{F}_T)$ and $s \in [0, t]$.

We denote by $x \cdot y$ the Euclidean inner product of two vectors $x, y \in \mathbb{R}^d$. For a bounded endowment $\xi \in L^\infty(\mathcal{F}_T)$ and for a lower semi-continuous convex function $f : \mathbb{R}^d \to \mathbb{R}_+ \cup \{+\infty\}$ such that

$$f(0) = 0 \quad \text{and} \quad \liminf_{|q| \to +\infty} \frac{f(q)}{|q|^2} > 0,$$

we define

$$U_t(\xi) := \operatorname{essinf} \left\{ \mathbb{E}_{\mathbb{Q}^q} \left[\xi + \int_t^T f(q_s) \mathrm{d}s \middle| \mathcal{F}_t \right] \middle| \mathbb{Q}^q \sim \mathbb{P} \right\}, \quad t \in [0, T], \tag{1.1}$$

where $\mathbb{Q}^q$ is equivalent to $\mathbb{P}$ such that

$$\mathbb{E} \left[\frac{\mathrm{d}\mathbb{Q}^q}{\mathrm{d}\mathbb{P}} \middle| \mathcal{F}_t \right] = \exp \left\{ \int_0^t q_s \cdot \mathrm{d}B_s - \frac{1}{2} \int_0^t |q_s|^2 \mathrm{d}s \right\}, \quad t \in [0, T],$$

and $\mathbb{E}_{\mathbb{Q}^q}[\cdot|\mathcal{F}_t]$ is the expectation operator conditioned on the σ-field $\mathcal{F}_t$ under the probability measure $\mathbb{Q}^q$.

It is easy to verify that the time-parameterized operator $\{U_t(\cdot), t \in [0, T]\}$ defined via (1.1) constitutes a dynamic concave utility function defined on $L^\infty(\mathcal{F}_T)$. On the other hand, Delbaen et al. (DelbaenPengRosazza2010FS) show that all dynamic concave utility functions are of a similar form under some mild additional assumptions. Furthermore, Theorems 2.1-2.2 in DelbaenHuBao2011PTRF show that there exists an $(\mathcal{F}_t)$-progressively measurable square-integrable $\mathbb{R}^d$-valued process $(Z_t)_{t\in[0,T]}$ such that $(U_t(\xi), Z_t)_{t\in[0,T]}$ is the unique bounded solution of the following scalar backward stochastic differential equation (BSDE in short):

$$Y_t = \xi - \int_t^T g(Z_s)\,\mathrm{d}s + \int_t^T Z_s \cdot \mathrm{d}B_s, \quad t \in [0, T], \tag{1.2}$$

where

$$g(z) := \sup_{q \in \mathbb{R}^d} [z \cdot q - f(q)] \geq 0, \quad \forall z \in \mathbb{R}^d$$

is the conjugate concave function of f with

$$g(0) = 0 \quad \text{and} \quad \limsup_{|z| \to +\infty} \frac{g(z)}{|z|^2} < +\infty.$$

The infimum is attained: there exists $\tilde{q}$ such that

$$Y_t = \mathbb{E}_{\mathbb{Q}^{\tilde{q}}} \left[\xi + \int_t^T f(s, \tilde{q}_s) \mathrm{d}s \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \tag{1.3}$$

The dynamic concave utility function defined on $L^\infty(\mathcal{F}_T)$ via (1.1) is a dual representation as the solution to BSDE (1.2). While for an unbounded endowment ξ, if it has finite exponential moments of arbitrary order, BSDE (1.2) are known to still admit a unique adapted solution $(Y_t, Z_t)_{t\in[0,T]}$ such that $\sup_{t\in[0,T]}|Y_t|$ has finite exponential moments of arbitrary order, and the associated comparison theorem and stability theorem on the solutions of preceding BSDEs hold; see Briand and Hu (BriandHu2006PTRF, BriandHu2008PTRF) for details. Then, the following questions are naturally asked: for an endowment ξ admitting finite exponential moments of arbitrary order, is the formula (1.1) still well-defined? And if yes, is it still represented as the solution of BSDE (1.2)? Are there larger linear spaces of possibly unbounded endowments connected to different features of functions f where the preceding duality remains to be true? In the paper, we shall give some affirmative answers to these questions. In particular, we consider the general case of core function f, which might be time-varying and random.

Note that up to a sign, a dynamic concave utility is actually equivalent to a dynamic convex risk measure. They are both popular notions in finance mathematics, and have received an extensive attention for example in Bion-Nadal2008FS, Bion-Nadal2009SPA, Cheridito-Delbaen-Kupper2004SPA, Cheridito-Delbaen-Kupper2005FS, Delbaen2000LN, Delbaen2002AFS, DelbaenPengRosazza2010FS, DetlefsenScandolo2005FS, DuffieEpstein1992Econometrica, ElKarouiPengQuenez1997MF, FollmerPenner2006SD, FollmerSchied2002FS, FollmerSchied2002AFS, FrittelliRosazza2004book, Jiang2008AAP, KloppelSchweizer2007MF, Peng2004LectureNotes, Rosazza2006IME, SmithBickel2022OR. Consequently, all assertions of the present paper are available for the dynamic convex risk measures.

First of all, we introduce some notations and spaces used in this paper. Let $\mathbb{R}_+ := [0, +\infty)$, and for $a, b \in \mathbb{R}$, let $a \wedge b := \min\{a, b\}$, $a^+ := \max\{a, 0\}$ and $a^- := -\min\{a, 0\}$. Let $\mathbf{1}_A(x)$ represent the indicator function of set A. For a convex function $f : \mathbb{R}^d \to \mathbb{R}$, denote by $\partial f(z_0)$ its subdifferential at z_0, which is the non-empty convex compact set of elements $u \in \mathbb{R}^d$ such that

$$f(z) - f(z_0) \geq u \cdot (z - z_0), \quad \forall z \in \mathbb{R}^d.$$

For $p, \mu > 0$ and $t \in [0, T]$, we denote by $L^p(\mathcal{F}_t)$ the set of $\mathcal{F}_t$-measurable scalar random variables η such that $\mathbb{E}[|\eta|^p] < +\infty$, and define the following three spaces of $\mathcal{F}_t$-measurable scalar random variables:

$$L(\ln L)^p(\mathcal{F}_t) := \{\eta \in \mathcal{F}_t \,|\mathbb{E}\left[|\eta|(\ln(1+|\eta|))^p\right] < +\infty\},$$

$$L\exp[\mu(\ln L)^p](\mathcal{F}_t) := \{\eta \in \mathcal{F}_t \,|\mathbb{E}\left[|\eta| \exp\left(\mu(\ln(1+|\eta|))^p\right)\right] < +\infty\}$$

and

$$\exp(\mu L^p)(\mathcal{F}_t) := \{\eta \in \mathcal{F}_t \,|\mathbb{E}\left[\exp\left(\mu|\eta|^p\right)\right] < +\infty\}.$$

It is not hard to verify that these spaces become smaller when the parameter μ or p increases, and for each $\mu, \bar{\mu}, r > 0$ and $0 < p < 1 < q$,

$$L^\infty(\mathcal{F}_t) \subset \exp(\mu L^r)(\mathcal{F}_t) \subset L\exp[\bar{\mu}(\ln L)^q](\mathcal{F}_t) \subset L\exp[\mu \ln L](\mathcal{F}_t) = L^{1+\mu}$$

and

$$L^q(\mathcal{F}_t) \subset L\exp[\mu(\ln L)^p](\mathcal{F}_t) \subset L(\ln L)^r(\mathcal{F}_t) \subset L^1(\mathcal{F}_t).$$

For each $p, \mu > 0$ and $0 < \bar{p} \leq 1 < \tilde{p}$, it is clear that the following spaces

$$L\exp[\mu(\ln L)^{\bar{p}}](\mathcal{F}_t), \quad \bigcap_{\bar{\mu}>0} L\exp[\bar{\mu}(\ln L)^{\bar{p}}](\mathcal{F}_t), \quad \bigcup_{\bar{\mu}>0} L\exp[\bar{\mu}(\ln L)^{\tilde{p}}](\mathcal{F}_t)$$

and

$$\bigcap_{\bar{\mu}>0} \exp(\bar{\mu}L^p)(\mathcal{F}_t)$$

are all linear spaces containing $L^\infty(\mathcal{F}_t)$. Note that under the conditions without causing confusion, the σ-algebra $(\mathcal{F}_T)$ is usually omitted in these notations on the spaces of random variables.

Denote by Σ_T the set of all $(\mathcal{F}_t)$-stopping times τ taking values in $[0, T]$. For an $(\mathcal{F}_t)$-adapted scalar process $(X_t)_{t\in[0,T]}$, we say that it belongs to class (D) if the family of random variables $\{X_\tau : \tau \in \Sigma_T\}$ is uniformly integrable. Finally, let us recall some updated results on scalar BSDEs. Consider the following scalar BSDE:

$$Y_t = \xi - \int_t^T g(s, Z_s)\mathrm{d}s + \int_t^T Z_s \cdot \mathrm{d}B_s, \quad t \in [0, T], \tag{2.1}$$

where ξ is called the terminal value, which is an $\mathcal{F}_T$-measurable scalar random variable, the random function

$$g(\omega, t, z) : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R},$$

which is $(\mathcal{F}_t)$-progressively measurable for each $z \in \mathbb{R}^d$, is called the generator of (2.1), and the pair of $(\mathcal{F}_t)$-progressively measurable processes $(Y_t, Z_t)_{t\in[0,T]}$ taking values in $\mathbb{R} \times \mathbb{R}^d$ is called an adapted solution of (2.1) if $\mathbb{P} - a.s.$, $t \mapsto Y_t$ is continuous, $t \mapsto |g(t, Z_t)| + |Z_t|^2$ is integrable, and (2.1) holds.

Throughout the whole paper, we will be given two $(\mathcal{F}_t)$-progressively measurable nonnegative scalar processes $(h_t)_{t\in[0,T]}$ and $(\bar{h}_t)_{t\in[0,T]}$, and four positive constants $\gamma, \lambda, c > 0$ and $\alpha \in (1,2)$. Let $\alpha^* > 2$ represent the conjugate of α, i.e.,

$$\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1.$$

Assumptions

(H0) $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $g(\omega, t, \cdot)$ is convex.

(H1) g has a quadratic growth in z, i.e.,

$$|g(\omega, t, z)| \leq \bar{h}_t(\omega) + \frac{\gamma}{2}|z|^2.$$

(H2) g has a sub-quadratic growth in z, i.e., $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $z \in \mathbb{R}^d$, we have

$$|g(\omega, t, z)| \leq \bar{h}_t(\omega) + \gamma|z|^\alpha.$$

(H3) g has a super-linear growth in z, i.e., $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $z \in \mathbb{R}^d$, we have

$$|g(\omega, t, z)| \leq \bar{h}_t(\omega) + \gamma|z| \left(\ln(e + |z|)\right)^\lambda.$$

(H4) g has a linear growth in z, i.e., $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $z \in \mathbb{R}^d$, we have

$$|g(\omega, t, z)| \leq \bar{h}_t(\omega) + \gamma|z|.$$

Remark

Remark 2.1

Both assumptions (H0) and (H4) yield that $\mathrm{d}\mathbb{P} \times \mathrm{d}t -$ *a.e., for each* $\theta \in (0,1)$ *and* $z_1, z_2 \in \mathbb{R}^d$*, we have*

$$\begin{aligned} g(\omega, t, z_1) &= g\left(\omega, t, \theta z_2 + (1-\theta)\frac{z_1 - \theta z_2}{1-\theta}\right) \\ &\leq \theta g(\omega, t, z_2) + (1-\theta)\left(\bar{h}_t(\omega) + \gamma\frac{|z_1 - \theta z_2|}{1-\theta}\right) \\ &= \theta g(\omega, t, z_2) + (1-\theta)\bar{h}_t(\omega) + \gamma|z_1 - \theta z_2|, \end{aligned}$$

and then, by letting $\theta \to 1$ *in the last inequality,*

$$|g(\omega, t, z_1) - g(\omega, t, z_2)| \leq \gamma|z_1 - z_2|,$$

i.e., g *is Lipschitz with respect to* z*.*

Proposition 1

Assume that the generator g satisfies assumptions (H0) and (H1), and that

$$|\xi| + \int_0^T \bar{h}_t \mathrm{d}t \in \bigcap_{\mu>0} \exp(\mu L).$$

Then, BSDE (2.1) admits a unique adapted solution $(Y_t, Z_t)_{t\in[0,T]}$ such that

$$\sup_{t\in[0,T]} |Y_t| \in \bigcap_{\mu>0} \exp(\mu L).$$

Proposition 2

Assume that the generator g satisfies assumptions (H0) and (H2), and that

$$|\xi| + \int_0^T \bar{h}_t \mathrm{d}t \in \bigcap_{\mu>0} \exp(\mu L^{\frac{2}{\alpha^*}}).$$

Then, BSDE (2.1) admits a unique adapted solution $(Y_t, Z_t)_{t\in[0,T]}$ such that

$$\sup_{t\in[0,T]} |Y_t| \in \bigcap_{\mu>0} \exp(\mu L^{\frac{2}{\alpha^*}}).$$

Proposition 3

Assume that the generator g satisfies assumptions (H0) and (H3), and that

$$|\xi| + \int_0^T \bar{h}_t \mathrm{d}t \in \bigcap_{\mu>0} L\exp[\mu(\ln L)^{(\frac{1}{2}+\lambda)\vee(2\lambda)}].$$

Then, BSDE (2.1) admits a unique adapted solution $(Y_t, Z_t)_{t\in[0,T]}$ such that for each $\mu > 0$, the process $\left(|Y_t| \exp\left(\mu\left(\ln(1+|Y_t|)\right)^{(\frac{1}{2}+\lambda)\vee(2\lambda)}\right)\right)_{t\in[0,T]}$ belongs to *class*(D).
Moreover, if there exists a constant $\bar{\mu} > 0$ such that

$$|\xi| + \int_0^T \bar{h}_t \mathrm{d}t \in L\exp[\bar{\mu}(\ln L)^{1+2\lambda}],$$

then for any positive constant $\tilde{\mu} < \bar{\mu}$, we have

$$\sup_{t\in[0,T]} |Y_t| \in L\exp[\tilde{\mu}(\ln L)^{1+2\lambda}].$$

Proposition 4

Assume that the generator g satisfies assumptions (H0) and (H4), and that for some constant $\mu > \mu_0 := \gamma\sqrt{2T}$, we have

$$|\xi| + \int_0^T \bar{h}_t \mathrm{d}t \in L\exp[\mu(\ln L)^{\frac{1}{2}}].$$

Then, BSDE (2.1) admits a unique adapted solution $(Y_t, Z_t)_{t\in[0,T]}$ such that for some $\tilde{\mu} > 0$,

$$\text{the process } \left(|Y_t|\exp\left(\tilde{\mu}\sqrt{\ln(1+|Y_t|)}\right)\right)_{t\in[0,T]} \text{ belongs to class (D).}$$

Moreover, if

$$|\xi| + \int_0^T \bar{h}_t \mathrm{d}t \in \bigcap_{\bar{\mu}>0} L\exp[\bar{\mu}(\ln L)^{\frac{1}{2}}],$$

then for any $\bar{\mu} > 0$, we have

$$\text{the process } \left(|Y_t|\exp\left(\bar{\mu}\sqrt{\ln(1+|Y_t|)}\right)\right) \text{ belongs to class (D).}$$

Proof

Proof.

Proposition 1 is a direct consequence of Corollary 6 in BriandHu2008PTRF, and Proposition 2 follows immediately from Theorem 3.1 in FanHu2021SPA. Furthermore, according to Theorem 2.4 of FanHuTang2023SPA together with Doob's maximal inequality for martingales, we can easily derive Proposition 3. Finally, in view of 2.1, Proposition 4 can be obtained by applying Theorem 3.1 of HuTang2018ECP and Theorem 2.5 of BuckdahnHuTang2018ECP. The interested readers are also refereed to FanHu2019ECP,OKimPak2021CRM,FanHuTang2023arXiv for more details. The proof is complete. □

We always assume that the random core function

$$f(\omega, t, q) : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$$

is $(\mathcal{F}_t)$-progressively measurable for each $q \in \mathbb{R}^d$, and that f satisfies the following assumption:

(A0) $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $f(\omega, t, \cdot)$ is lower semi-continuous and convex, and there exists an $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^d$-valued process $(\bar{q}_t)_{t \in [0,T]}$ and a constant $k \geq 0$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$,

$$|\bar{q}_t(\omega)| \leq k \ \text{ and } \ f(\omega, t, \bar{q}_t(\omega)) \leq h_t(\omega).$$

Recalling that $(h_t)_{t \in [0,T]}$ is a given $(\mathcal{F}_t)$-progressively measurable nonnegative scalar process.

In particular, if $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $f(\omega, t, \cdot)$ is a convex function taking values in $\mathbb{R}$ and $f(\omega, t, 0) \equiv 0$, then assumption (A0) is naturally satisfied.

For each endowment $\xi \in \mathcal{A}(\mathcal{F}_T)$ and random core function f, we define the following process space $\mathcal{H}(\xi, f)$

$$\Big\{ \quad (q_t)_{t\in[0,T]} \text{ is an } (\mathcal{F}_t)\text{-progressively measurable } \mathbb{R}^d\text{-valued process :}$$
$$\int_0^T |q_s|^2 \mathrm{d}s < +\infty,\ \mathbb{E}_{\mathbb{Q}^q}\Big[|\xi| + \int_0^T (h_s + |f(s, q_s)|)\mathrm{d}s\Big] < +\infty,$$
$$\text{with } L_t^q := \exp\Big(\int_0^t q_s \cdot \mathrm{d}B_s - \frac{1}{2}\int_0^t |q_s|^2 \mathrm{d}s\Big),\ t \in [0, T],$$
$$\text{being a uniformly integrable martingale, and } \frac{\mathrm{d}\mathbb{Q}^q}{\mathrm{d}\mathbb{P}} := L_T^q \quad \Big\}. \tag{3.2}$$

We aim to study that under what conditions on the endowment ξ and the random core function f, the following time-parameterized operator

$$U_t(\xi) := \operatorname*{ess\,inf}_{q\in\mathcal{H}(\xi,f)} \mathbb{E}_{\mathbb{Q}^q}\left[\xi + \int_t^T f(s,q_s)\mathrm{d}s \,\middle|\, \mathcal{F}_t\right], \quad t\in[0,T], \tag{3.3}$$

is well-defined, and admits a dual representation via the adapted solution of the following BSDE

$$Y_t = \xi - \int_t^T g(s,Z_s)\mathrm{d}s + \int_t^T Z_s\cdot \mathrm{d}B_s, \quad t\in[0,T], \tag{3.4}$$

where the generator g of BSDE (3.4) is the Legendre-Fenchel transform of f, i.e.,

$$g(\omega,t,z) := \sup_{q\in\mathbb{R}^d}\left(z\cdot q - f(\omega,t,q)\right), \quad \forall(\omega,t,z)\in\Omega\times[0,T]\times\mathbb{R}^d. \tag{3.5}$$

Furthermore, set $\tilde{q}_s \in \partial g(s, Z_s)$ for each $s \in [0, T]$, we prove that $(\tilde{q}_t)_{t\in[0,T]} \in \mathcal{H}(\xi, f)$ and

$$Y_t = \mathbb{E}_{\mathbb{Q}^{\tilde{q}}}\left[\xi + \int_t^T f(s, \tilde{q}_s)\mathrm{d}s \,\middle|\, \mathcal{F}_t\right], \quad t \in [0, T]. \tag{3.6}$$

Hence, the operator $\{U_t(\cdot),\ t \in [0, T]\}$ defined via (3.3) constitutes a dynamic concave utility function defined on some linear spaces bigger than $L^\infty(\mathcal{F}_T)$. It is clear from (3.5) and (A0) that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $g(\omega, t, \cdot)$ is a convex function defined on $\mathbb{R}^d$, and

$$g(\omega, t, z) \geq z \cdot \bar{q}_t(\omega) - f(\omega, t, \bar{q}_t(\omega)) \geq -k|z| - h_t(\omega), \quad \forall z \in \mathbb{R}^d. \tag{3.7}$$

Let us further present the following assumptions on the core function f, all of which can ensure that the dual function g of f satisfies that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $g(\omega, t, z) < +\infty$ for each $z \in \mathbb{R}^d$.

(A1) $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $q \in \mathbb{R}^d$, we have

$$f(\omega, t, q) \geq \frac{1}{2\gamma}|q|^2 - h_t(\omega).$$

(A2) $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $q \in \mathbb{R}^d$, we have

$$f(\omega, t, q) \geq \gamma^{-\frac{1}{\alpha-1}}|q|^{\alpha^*} - h_t(\omega).$$

(A3) $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $q \in \mathbb{R}^d$, we have

$$f(\omega, t, q) \geq c \exp\left(2\gamma^{-\frac{1}{\lambda}}|q|^{\frac{1}{\lambda}}\right) - h_t(\omega).$$

(A4) $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $q \in \mathbb{R}^d$, we have

$$f(\omega, t, q) \geq -h_t(\omega) \ \text{ and } \ f(\omega, t, q) \equiv +\infty \text{ in the case of } |q| > \gamma.$$

Theorem 1

Assume that the core function f satisfies assumptions (A0) and (A1), and that $|\xi| + \int_0^T h_t \mathrm{d}t \in \bigcap_{\mu>0} \exp(\mu L)$. Then, the process $\{U_t(\xi),\ t \in [0, T]\}$ defined via (3.3) is well-defined, the generator g defined in (3.5) satisfies assumptions (H0) and (H1) with $\bar{h}_t = h_t + k^2/(2\gamma)$, and there exists an $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^d$-valued process $(Z_t)_{t\in[0,T]}$ such that $(Y_t := U_t(\xi), Z_t)_{t\in[0,T]}$ is the unique adapted solution of BSDE (3.4) satisfying $\sup_{t\in[0,T]} |Y_t| \in \bigcap_{\mu>0} \exp(\mu L)$. Moreover, the operator $\{U_t(\cdot),\ t \in [0, T]\}$ defined via (3.3) constitutes a dynamic concave utility function defined on $\cap_{\mu>0} \exp(\mu L)$.

$$Y_t = \mathbb{E}_{\mathbb{Q}^{\tilde{q}}}\left[\xi + \int_t^T f(s, \tilde{q}_s)\mathrm{d}s \,\middle|\, \mathcal{F}_t\right], \quad t \in [0, T]. \qquad (3.8)$$

Theorem 2

Assume that the core function f satisfies assumptions (A0) and (A2), and that $|\xi| + \int_0^T h_t \mathrm{d}t \in \bigcap_{\mu>0} \exp(\mu L^{\frac{2}{\alpha^*}})$. Then, the process $\{U_t(\xi),\ t \in [0, T]\}$ defined via (3.3) is well-defined, the generator g defined in (3.5) satisfies assumptions (H0) and (H2) with $\bar{h}_t = h_t + \gamma^{-\frac{1}{\alpha-1}} k^{\alpha^*}$, and there exists an $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^d$-valued process $(Z_t)_{t\in[0,T]}$ such that $(Y_t := U_t(\xi), Z_t)_{t\in[0,T]}$ is the unique adapted solution of BSDE (3.4) satisfying $\sup\limits_{t\in[0,T]} |Y_t| \in \bigcap\limits_{\mu>0} \exp(\mu L^{\frac{2}{\alpha^*}})$. Moreover, the operator $\{U_t(\cdot),\ t \in [0, T]\}$ defined via (3.3) constitutes a dynamic concave utility function defined on $\cap_{\mu>0} \exp(\mu L^{\frac{2}{\alpha^*}})$.

$$Y_t = \mathbb{E}_{\mathbb{Q}^{\tilde{q}}}\left[\xi + \int_t^T f(s, \tilde{q}_s)\mathrm{d}s \middle| \mathcal{F}_t\right], \quad t \in [0, T]. \tag{3.9}$$

Theorem 3

Assume that the core function f satisfies assumptions (A0) and (A3), and that $|\xi| + \int_0^T h_t \mathrm{d}t \in \bigcup_{\mu>0} L\exp[\mu(\ln L)^{1+2\lambda}]$. Then, the process $\{U_t(\xi),\ t\in[0,T]\}$ defined via (3.3) is well-defined, the generator g defined in (3.5) satisfies assumptions (H0) and (H3) with $\bar{h}_t = h_t + \exp\left(2\gamma^{-\frac{1}{\lambda}}|k|^{\frac{1}{\lambda}}\right) + C_{c,\gamma,\lambda}$, where $C_{c,\gamma,\lambda}$ is a positive constant depending only on (c,γ,λ), and there exists an $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^d$-valued process $(Z_t)_{t\in[0,T]}$ such that $(Y_t := U_t(\xi), Z_t)_{t\in[0,T]}$ is the unique adapted solution of BSDE (3.4) satisfying

$$\sup_{t\in[0,T]} |Y_t| \in \bigcup_{\mu>0} L\exp[\mu(\ln L)^{1+2\lambda}].$$

Moreover, the operator $\{U_t(\cdot),\ t\in[0,T]\}$ defined via (3.3) constitutes a dynamic concave utility function defined on $\cup_{\mu>0} L\exp[\mu(\ln L)^{1+2\lambda}]$.

Theorem 4

Assume that the core function f satisfies assumptions (A0) with $k \leq \gamma$ and (A4), and that $|\xi| + \int_0^T h_t \mathrm{d}t \in \bigcap_{\bar{\mu}>0} L \exp[\bar{\mu}(\ln L)^{\frac{1}{2}}]$. Then, the process $\{U_t(\xi),\ t \in [0, T]\}$ defined via (3.3) is well-defined, the generator g defined in (3.5) satisfies assumptions (H0) and (H4) with $\bar{h}_t = h_t$, and there exists an $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^d$-valued process $(Z_t)_{t\in[0,T]}$ such that $(Y_t := U_t(\xi), Z_t)_{t\in[0,T]}$ is the unique adapted solution of BSDE (3.4) such that for each $\bar{\mu} > 0$,

$$\text{the process } \left(|Y_t| \exp\left(\bar{\mu}\sqrt{\ln(1+|Y_t|)}\right)\right)_{t\in[0,T]} \text{ belongs to class (D).}$$

Moreover, the operator $\{U_t(\cdot),\ t \in [0, T]\}$ defined via (3.3) constitutes a dynamic concave utility function defined on $\cap_{\bar{\mu}>0} L \exp[\bar{\mu}(\ln L)^{\frac{1}{2}}]$.

One idea of Proof

Proposition 3.1

Let $(q_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^d$-valued process such that $\mathbb{P}-$ a.s., $\int_0^T |q_s|^2 \mathrm{d}s < +\infty$. Then, we have

$$\mathbb{E}\left[L_T^q \ln(1+L_T^q)\right] \le \frac{1}{2}\mathbb{E}\left[\int_0^T L_t^q |q_t|^2 \mathrm{d}t\right] + \ln 2 \qquad (3.10)$$

and

$$\mathbb{E}\left[L_T^q \left[\ln(1+L_T^q)\right]^{\frac{\alpha^*}{2}}\right] \le C_{\alpha,T}\mathbb{E}\left[\int_0^T L_t^q |q_t|^{\alpha^*} \mathrm{d}t\right] + C_{\alpha,T}, \qquad (3.11)$$

$$\mathbb{E}\left[L_T^q \exp\left(\mu \left[\ln(1+L_T^q)\right]^{\frac{1}{1+2\lambda}}\right)\right] \le \varepsilon\mathbb{E}\left[\int_0^T L_s^q \exp\left(2\gamma^{-\frac{1}{\lambda}}|q_s|^{\frac{1}{\lambda}}\right) \mathrm{d}s\right] + \tilde{C}_{\mu} \qquad (3.12)$$

Example 1

We would like to introduce several practical examples of unbounded endowments in a financial market. Let $(X_t)_{t\in[0,T]}$ be the unique adapted solution of the following SDE:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\cdot \mathrm{d}B_t, \quad t \in [0, T]; \quad X_0 = x_0,$$

where $x_0 \in \mathbb{R}$ is a given constant and both $b(t,x) : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma(t,x) : [0, T] \times \mathbb{R} \to \mathbb{R}^d$ are measurable functions satisfying that for each $x_1, x_2 \in \mathbb{R}^2$ and $t \in [0, T]$, we have

$$|b(t,x_1)-b(t,x_2)|+|\sigma(t,x_1)-\sigma(t,x_2)| \le c|x_1-x_2|, |b(t,0)|+|\sigma(t,0)| \le c.$$

We consider an endowment ξ which equals to X_T. By classical theory of SDEs we know that

$$\xi := X_T \in L^2 \subset \cap_{\mu>0} L\exp[\mu(\ln L)^{\frac{1}{2}}].$$

Example 1-continued

Moreover, if it is also supposed that $|\sigma(t,x)| \leq c$ for each $(t,x) \in [0,T] \times \mathbb{R}$, then there exists two positive constants c_1 and c_2 depending only on (c,T) such that $\mathbb{E}\left[\exp\left(c_1 \sup_{t\in[0,T]} |X_t|^2\right)\right] \leq c_2 \exp(c_2|x_0|^2)$, which indicates that for each $\lambda > 0$, $\xi := X_T \in \bigcap\limits_{\mu>0} \exp(\mu L) \subset \bigcap_{\mu>0} \exp(\mu L^{\frac{2}{\alpha^*}}) \subset \cup_{\mu>0} L \exp[\mu(\ln L)^{1+2\lambda}] \subset L^2$. Finally, if we let $d = 1$, $b(t,x) := bx$ and $\sigma(t,x) := \sigma x$ for two positive constants b and σ, then we have

$$X_t = x_0 \exp\left(bt - \frac{1}{2}\sigma^2 + \sigma B_t\right), \quad t \in [0,T].$$

Thus, for a typical European call option ξ defined by $(X_T - K)^+$ with the previously agreed strike price $K > 0$, we can conclude that for each $\lambda \in (0, 1/2]$,

$$\xi := (X_T - K)^+ \in \cup_{\mu>0} L \exp[\mu(\ln L)^{1+2\lambda}].$$

Example 2

We present the following several examples to which Theorems can apply.

Let $(\bar{q}_t)_{t\in[0,T]}$ be an $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^d$-valued process such that $\mathrm{d}\mathbb{P}\times\mathrm{d}t - a.e.$, $|\bar{q}_t|\leq\gamma$ and let the core function f be defined as follows: $\forall(\omega,t,q)\in\Omega\times[0,T]\times\mathbb{R}^d$, $f(\omega,t,q) := \begin{cases} h_t(\omega), & q=\bar{q}_t(\omega); \\ +\infty, & otherwise. \end{cases}$ It is clear that f satisfies assumptions (A0) and (A4), and if

$$\mathbb{E}_{\mathbb{Q}^{\bar{q}}}\left[\xi+\int_0^T h_s\mathrm{d}s\right] < +\infty, \textit{then} \tag{3.13}$$

$$\begin{aligned} U_t(\xi) &:= \operatorname*{ess\,inf}_{q\in\mathcal{H}(\xi,f)} \mathbb{E}_{\mathbb{Q}^q}\left[\xi+\int_t^T f(s,q_s)\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}^{\bar{q}}}\left[\xi+\int_t^T h_s\mathrm{d}s \,\middle|\, \mathcal{F}_t\right], \quad t\in[0,T]. \end{aligned} \tag{3.14}$$

Example 2 - continued

On the other hand, it is easy to verify that the convex conjugate function of f is the following:

$$\forall(\omega, t, z) \in \Omega\times[0, T]\times\mathbb{R}^d,\ \ g(\omega, t, z) := \sup_{q\in\mathbb{R}^d} (z\cdot q - f(\omega, t, q)) = \bar{q}_t(\omega)\cdot z$$

According to Girsanov's theorem, we know that when (3.13) is satisfied, the process $\{U_t(\xi)\}_{t\in[0,T]}$ in (3.14) is just the first process of the unique solution of BSDE (3.4) with this generator g. We remark that Theorem 4 indicates that the above assertion holds when (3.13) is replaced with

$$|\xi| + \int_0^T h_t \mathrm{d}t \in \bigcap_{\mu>0} L\exp[\mu(\ln L)^{\frac{1}{2}}].$$

Example 3

Let the core function f be defined as follows:
$\forall(\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}^d$,

$$f(\omega, t, q) := \begin{cases} 0, & |q| \leq \gamma; \\ +\infty, & |q| > \gamma. \end{cases}$$

It is clear that f satisfies assumptions (A0) and (A4) with $h_t \equiv 0$, and for each $\xi \in L^2$, we have

$$\begin{aligned} U_t(\xi) &:= \operatorname*{ess\,inf}_{q \in \mathcal{H}(\xi, f)} \mathbb{E}_{\mathbb{Q}^q} \left[\xi + \int_t^T f(s, q_s) \mathrm{d}s \middle| \mathcal{F}_t \right] \\ &= \operatorname*{ess\,inf}_{q. \in \mathbb{R}^d : |q.| \leq \gamma} \mathbb{E}_{\mathbb{Q}^{\bar{q}}} \left[\xi | \mathcal{F}_t \right], \quad t \in [0, T]. \end{aligned} \tag{3.15}$$

Example 3 - continued

On the other hand, it is easy to verify that the convex conjugate function of f is the following:

$$\forall(\omega, t, z) \in \Omega\times[0, T]\times\mathbb{R}^d,\ \ g(\omega, t, z) := \sup_{q\in\mathbb{R}^d} (z\cdot q - f(\omega, t, q)) = \gamma|z|.$$

It follows from Lemma 3 of ChenPeng2000SPL that for each $\xi \in L^2$, the process $\{U_t(\xi)\}_{t\in[0,T]}$ in (3.15) is just the first process of the unique solution of BSDE (3.4) with this generator g. We remark that Theorem 4 indicates that the above assertion holds when $\xi \in \cap_{\mu>0} L\exp[\mu(\ln L)^{\frac{1}{2}}]$.

Example 4

$\forall(\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}^d$,

$$f(\omega, t, q) := \begin{cases} \dfrac{|q|^2}{2}, & |q| \le \gamma; \\ +\infty, & |q| > \gamma. \end{cases}$$

It is clear that f satisfies assumptions (A0) and (A4) with $h_t \equiv 0$. It is not hard to verify that the convex conjugate function of f is the following: $\forall(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$,

$$g(\omega, t, z) := \sup_{q \in \mathbb{R}^d} (z \cdot q - f(\omega, t, q)) = \begin{cases} \dfrac{|z|^2}{2}, & |z| \le \gamma; \\ \gamma|z| - \dfrac{\gamma^2}{2}, & |z| > \gamma, \end{cases}$$

and that this generator g satisfies assumptions (H0) and (H4). Thus, if $\xi \in \cap_{\mu>0} L \exp[\mu(\ln L)^{\frac{1}{2}}]$, the conclusions in Theorem 4 can be applied.

Example 5

Let the core function f be defined as follows:
$\forall(\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}^d$,

$$f(\omega, t, q) := e^{|q|} + h_t(\omega).$$

It is clear that f satisfies assumptions (A0) and (A3) with $(c, \gamma, \lambda) = (1, 2, 1)$. It is not hard to verify that the convex conjugate function of f is the following:
$\forall(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$,

$$g(\omega, t, z) := \sup_{q \in \mathbb{R}^d} (z \cdot q - f(\omega, t, q)) = |z|(\ln |z| - 1) - h_t(\omega),$$

and that g satisfies assumptions (H0) and (H3). Thus, if $\xi + \int_0^T h_t \mathrm{d}t \in \cup_{\mu > 0} L \exp[\mu(\ln L)^3]$, then the conclusions in Theorem 3 can be applied.

Example 6

Let the core function f be defined as follows:
$\forall(\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}^d$,

$$f(\omega, t, q) := \frac{1}{4}|q|^4 + h_t(\omega).$$

It is clear that f satisfies assumptions (A0) and (A2) with $(\alpha, \alpha^*, \gamma) = (\frac{4}{3}, 4, \sqrt[3]{4})$. It is not hard to verify that the convex conjugate function of f is the following:
$\forall(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$,

$$g(\omega, t, z) := \sup_{q \in \mathbb{R}^d} (z \cdot q - f(\omega, t, q)) = \frac{3}{4}|z|^{\frac{4}{3}} - h_t(\omega),$$

and that this generator g satisfies assumptions (H0) and (H2). Thus, if $\xi + \int_0^T h_t \mathrm{d}t \in \cap_{\mu>0} \exp(\mu L^{\frac{1}{2}})$, then the conclusions in Theorem 2 can be applied.

Example 7

$\forall(\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}^d$, $f(\omega, t, q) := \frac{1}{2\gamma}|q|^2$. It is clear that f satisfies assumptions (A0) and (A1) with $h_t \equiv 0$. On the other hand, it is easy to verify that the convex conjugate function of f is the following: $\forall(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}^d$, $g(\omega, t, z) := \sup_{q\in\mathbb{R}^d}(z \cdot q - f(\omega, t, q)) = \frac{\gamma}{2}|z|^2$. It follows from BriandHu2006PTRF that for each $\xi \in \exp(\gamma L)$, BSDE (3.4) with the above generator g admits a unique adapted solution $(Y_t, Z_t)_{t\in[0,T]}$ such that $\{\exp(\gamma|Y_t|)\}_{t\in[0,T]}$ belongs to class (D), and the process Y can be explicitly expressed as follows: $Y_t = \frac{1}{\gamma} \ln\left(\mathbb{E}\left[\exp(\gamma\xi)|\mathcal{F}_t\right]\right), \quad t \in [0, T]$. Thus, according to Theorem 1, we can conclude that for each $\xi \in \cap_{\mu>0} \exp(\mu L)$,

$$U_t(\xi) := \operatorname*{ess\,inf}_{q\in\mathcal{H}(\xi,f)} \mathbb{E}_{\mathbb{Q}^q}\left[\xi + \frac{1}{2\gamma}\int_t^T |q_s|^2 \mathrm{d}s \,\Big|\, \mathcal{F}_t\right] = \frac{1}{\gamma} \ln \mathbb{E}\left[\exp(\gamma\xi)|\mathcal{F}_t\right].$$

Example 8

Let $d = 1$ and the core function f be defined as follows: $\forall(\omega, t, q) \in \Omega \times [0, T] \times \mathbb{R}$,

$$f(\omega, t, q) := \begin{cases} +\infty, & q < 1; \\ q - 1, & 1 \leq q \leq 2; \\ \frac{q^2}{4}, & q > 2. \end{cases}$$

It is clear that f satisfies assumptions (A0) and (A1) with $\gamma = 2$ and $h_t \equiv 0$. It is not hard to verify that the convex conjugate function of f is the following: $\forall(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}$,

$$g(\omega, t, z) := \sup_{q \in \mathbb{R}^d} (z \cdot q - f(\omega, t, q)) = \begin{cases} z, & z < 1; \\ z^2, & z \geq 1, \end{cases}$$

and that this generator g satisfies assumptions (H0) and (H1). Thus, if $\xi \in \cap_{\mu > 0} \exp(\mu L)$, then the conclusions in Theorem 1 can be applied.

Happy Birthday, Marco.