Regularity and Geometric aspects of nonlinear PDEs, Milan 2025
Abstracts

L. Abatangelo, *On solutions to a class of degenerate equations with the Grushin operator*

The Grushin Laplacian $-\Delta_\alpha$ is a degenerate elliptic operator in $\mathbb{R}^{h+k}$ with singular set $\{0\} \times \mathbb{R}^k$. We consider weak solutions of $-\Delta_\alpha u = Vu$ for $\alpha \in \mathbb{N}$ in an open bounded connected domain Ω which intersects the singular set.
I will present some recent results obtained in collaboration with Alberto Ferrero and Paolo Luzzini (UniPO) about asymptotic behavior of weak solutions at singular points. From these, we are able to deduce some unique continuation principles for solutions.
I will try to focus mainly on the general framework of the operator involved, in particular on the role of the space geometry attached to it. Special attention will be devoted to the so-called *homogeneous dimension* $Q = h + (1 + \alpha)k$, that has got relevant consequences in our context.

D. Bucur, *A sharp quantitative nonlinear Poincaré inequality on convex domains*

We give a new inequality for the first nontrivial Neumann eigenvalue of the p-Laplacian on a convex domain with a power-concave weight. Our result improves the classical estimate in terms of the diameter, first stated in a seminal paper by Payne and Weinberger: we add in the lower bound an extra term depending on the second largest John semi-axis of the set (equivalent to a power of the width in the two dimensional case). The power exponent in the extra term is sharp, and the constant in front of it is explicitly tracked, thus enlightening the interplay between space dimension, nonlinearity and power-concavity. As an intermediate result, we establish a sharp, explicit upper bound estimate for the p-Laplacian eigenfunctions.
This is a joint work with V. Amato and I. Fragalà.

M. M. Fall, *Uniqueness and non-degeneracy properties of the Generalized Benjamin-Ono equation*

Solitary waves of the Benjamin-Ono equation are completely described by nontrivial solutions to the equation $(-\Delta)^{\frac{1}{2}}u + u = u^2$ on $\mathbb{R}^N$. The uniqueness, up to translations, to this equation is a classical result by Amick and Toland (1991) and an alternative proof was obtained by Albert (1992). A generalization to this equation is $(-\Delta)^s u + u = |u|^p$ on $\mathbb{R}^N$. A uniqueness, up to translation to this equation for s not equal to $1/2$ and p not equal to 2 has been a long open problem, besides uniqueness of ground-state solutions for a more general equation has been proved by Frank and Lenzmann. It still remains an open problem about the uniqueness of all nontrivial solutions, not only the ground-state for s not equal to 2. The talk is about the uniqueness and non degeneracy properties in the case $(-\Delta)^s u + u = u^2$ on $\mathbb{R}^N$, for the optimal values of the parameter s. Joint work with Tobias Weth.

V. Felli, *Neumann eigenvalues in domains with small holes*

The behavior of simple eigenvalues of the Neumann Laplacian in domains with little holes is discussed. I present an asymptotic expansion for all the eigenvalues of the perturbed problem which are converging to simple eigenvalues of the limit one. The eigenvalue variation turns out to depend on a geometric quantity resembling the notion of (boundary) torsional rigidity. In the particular case of a hole shrinking to a point, a fine blow-up analysis identifies the exact vanishing order of such a quantity and establishes some connections between the location of the hole and the sign of the eigenvalue variation.
The results presented in the talk have been obtained in collaboration with L. Liverani and R. Ognibene.

X. Fernández-Real, *Stable Free Boundaries in Dimension 3: Bernoulli and Allen-Cahn*

In this talk, we present a forthcoming work on the classification of global stable solutions to the Bernoulli problem in $\mathbb{R}^3$. In particular, this yields local universal curvature bounds for the free boundary for the local problem.
By means of this result, we prove the free boundary Allen–Cahn stability conjecture in dimension 3: global stable solutions to the free boundary analogue of the Allen–Cahn equation are one dimensional in dimension 3. This solves a long-standing conjecture in the free boundary case.
This is a joint work with H. Chan, A. Figalli, and J. Serra.

A. Iacopetti, *Shape Optimization and Overdetermined Problems in Cones and Cylinders*

In this talk, we present some recent results concerning partially overdetermined problems in unbounded regions. In particular, we focus on the cases of cones and cylinders, investigating the stability and instability of certain classes of solutions that are naturally connected to the geometry of the container. Moreover, we discuss the existence of minimizers of the torsional energy under a volume constraint and their geometric and topological properties.
These results are collected in a series of joint works with Prof. F. Pacella (Univ. of Rome “La Sapienza”), Prof. T. Weth (Univ. of Frankfurt), Dott. D. Gregorin (Univ. of Urbino), and Prof. P. Caldiroli (Univ. of Turin).

S. Jarohs, *The exterior Bernoulli problem for the half Laplacian*

Given a smooth bounded domain K in $\mathbb{R}^N$, and a parameter $\lambda > 0$, the exterior Bernoulli problem (EBP) for the half Laplacian is to find a function $u : \mathbb{R}^N \to \mathbb{R}$ and a smooth open subset Ω of $\mathbb{R}^N$ such that $\overline{K} \subset \Omega$ and u is a solution to the problem

$$(-\Delta)^{1/2} u = 0 \quad \text{in } \Omega \setminus \overline{K}, \quad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \quad u = 1 \quad \text{in } \overline{K},$$

with

$$D_\Omega^{1/2} u(\theta) = \lim_{t \to 0^+} \frac{u(\theta + t\nu(\theta))}{t^{1/2}} = \lambda \quad \text{for all } \theta \in \partial\Omega,$$

where $\nu(\theta)$ denotes the interior unit normal at $\theta \in \partial\Omega$.

In this talk, the existence of a solution to the EBP with its geometric properties and resulting regularity is discussed. Furthermore qualitative properties related to the asymptotic behavior of the free boundary of solutions when the *Bernoulli's gradient parameter* λ tends to 0^+ or to $+\infty$ are presented.
The talk is based on two joint works with Tadeusz Kulczycki and Paolo Salani.

M. Medina de la Torre, *Blowing-up solutions to critical competitive systems in dimensions 3 and 4*

We will analyze the existence and the structure of different sign-changing solutions to the Yamabe equation in the whole space and we will use them to find positive solutions to critical competitive systems in dimensions 3 and 4.

B. Pellacci, *Optimization problems arising in populations dynamic*

We will discuss some recent results concerning optmization problems arising in populations dynamic models. The optimization of the distribution of resources in logistc models leads to minimize a principal eigenvalue with respect to a sign-changing weight. Important qualitative properties of the positivity set of the optimal weight, such as being connected, as well as its location, are still not known in general. We will present some new achievements in the asymptotical study regarding these properties. When the model predicts survival for every diffusion coefficient, it becomes relevant the optimization of the total population, which is more related to the nonlinear problem.
Joint works with Francesca Gladiali (Università di Sassari), Iula Martina Bulai (Università di Sassari), Dario Mazzoleni (Università di Pavia), Lorenzo Ferreri (Scuola Normale Superiore di Pisa), Gianmaria Verzini (Politecnico di Milano).

X. Ros Oton, *Boundary regularity for semilinear elliptic PDE*

We study the boundary regularity for one of the simplest nonlinear elliptic PDEs,

$$\Delta u = u^\alpha, \text{ with } \alpha \in (-1, 1).$$

This question arises in the Alt-Phillips free boundary problem, for which we prove for the first time that free boundaries are C^∞ near regular points.

D. Ruiz, *Compactly supported solutions to the stationary 2D Euler equations with non-circular streamlines*

In this talk we are interested in compactly supported solutions of the steady Euler equations. In 3D the existence of this type of solutions has been an open problem for a long time, until the recent result of Gavrilov (2019). In 2D, instead, it is easy to construct solutions via radially symmetric stream functions. Nonradial weak solutions with low regularity have also been found in the literature, but even the C^1 case was open. In this talk we construct nonradial compactly supported solutions with C^k regularity, for any natural number k. For the proof, we look for stream functions which are solutions to non-autonomous semilinear elliptic equations with non-Lipschitz nonlinearities. In this framework we prove local bifurcation of nonradial solutions from a suitable 1-parameter family of radial solutions.

This is joint work with A. Enciso (ICMAT, Madrid) and Antonio J. Fernández (UAM, Madrid).

E. Vecchi, *Mixed local-nonlocal singular problems*

In this talk I will focus on boundary value problems of mixed type, meaning that the leading operator is given by the superposition of $-\Delta$ and $(-\Delta)^s$ with $s \in (0,1)$, in presence of singular nonlinearities of the form $u^{-\gamma}$ with $\gamma > 0$. I will discuss a few regularity results in the spirit of Boccardo and Orsina and multiplicity results of positive solutions when a critical power term is added to the singular nonlinearity.
The talk is mainly based on a joint work with S. Biagi (Politecnico di Milano).