DIPARTIMENTO DI MATEMATICA
"Francesco Brioschi"
POLITECNICO DI MILANO

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Collezione dei *Quaderni di Dipartimento*, numero **QDD 98**
Inserito negli *Archivi Digitali di Dipartimento* in data 18-5-2011

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Overdetermined problems for the ∞-Laplacian and web functions

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Abstract

We give necessary and sufficient conditions for functions to be solutions to overdetermined problems for the equation $-\Delta_\infty u = 1$ in a bounded domain of $\mathbb{R}^n$. To this end, we introduce a P-function for the study of the Dirichlet problem and we make use of its properties. In a previous work [9] such conditions were given under the additional assumption that u is a web-function.
Mathematics Subject Classification: 35N25, 35D40.

1 Setting the problem

In this paper we study the overdetermined boundary value problem

$$\begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = -a & \text{on } \partial\Omega\ , \end{cases} \tag{1}$$

where Ω is an open bounded domain of $\mathbb{R}^n$ with a smooth boundary, ν is the unit outer normal to $\partial\Omega$, $a > 0$, and Δ_∞ is the infinity Laplacian operator defined by

$$\Delta_\infty u = \langle D^2 u Du, Du\rangle = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \qquad \text{for all } u \in C^2(\Omega).$$

Since the works by Aronsson [1, 2, 3], the infinity Laplace equation has been extensively studied, see *e.g.* [21] for the homogeneous case and [25] for the inhomogeneous one.
Let Δ_p denote the p-Laplace operator $(1 < p < \infty)$ which we may write as

$$\Delta_p u = \nabla \cdot (|Du|^{p-2} Du) = (p-2)|Du|^{p-4} \left(\Delta_\infty u + \frac{|Du|^2\, \Delta u}{p-2} \right) .$$

Hence, p-harmonic functions u which satisfy $\Delta_p u = 0$ also satisfy $\Delta_\infty u + \frac{|Du|^2\, \Delta u}{p-2} = 0$ so that, if $p \to \infty$, we obtain $\Delta_\infty u = 0$. With a mathematical abuse, this is the reason why problem (1) is usually considered the limit for $p \to \infty$ of the following quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = -a & \text{on } \partial\Omega\, . \end{cases} \tag{2}$$

The first example of overdetermined problem for a second order elliptic equation is due to Serrin [31]; he proved that if $p = 2$ then the linear problem (2) admits a solution if and only if Ω is a ball of radius $R = na$. We also refer to [6, 30, 32] for the same result under weaker regularity assumptions. A similar conclusion was obtained for general $p \in (1, \infty)$, see [19] and the related papers [28, 8, 15, 17, 23]. We also refer to [18] for a geometric approach to (2) and to more general problems.

Since (1) may be seen as the limit of (2) as $p \to \infty$, it is natural to wonder whether the same symmetry result remains valid also in the limit case:

is it true that if (1) admits a solution, then Ω is a ball?

To answer this question it must first be clarified what is meant by solution. As far as we are aware, a complete regularity theory for the solutions to (4) is not yet available. However we cannot expect to have classical C^2 solutions to (1), see (5) and (6) below. And since Δ_∞ is not in divergence form, also a good definition of weak solution seems out of reach. One is so led to define solutions in a different way. Following [11], let us define viscosity solutions to (4). A viscosity subsolution $u \in C^0(\overline{\Omega})$ has the property that $-\Delta_\infty \varphi(x) - 1 \leq 0$ for all $\varphi \in C^2(\Omega)$ such that $\varphi - u$ has a local minimum at x. A viscosity supersolution $u \in C^0(\overline{\Omega})$ has the property that $-\Delta_\infty \psi(x) - 1 \geq 0$ for all $\psi \in C^2(\Omega)$ such that $\psi - u$ has a local maximum at x. A viscosity solution $u \in C^0(\overline{\Omega})$ to problem (4) is both a viscosity sub and supersolution.

Since it is known from [7, 22] that the solutions to (2) converge uniformly to the distance function form the boundary $d = d(x, \partial\Omega)$ as $p \to \infty$, it appears reasonable to expect that the limit problem (1) may have (viscosity) solutions also in some domains other than balls, for instance in domains where d has some special features.

Problem (1) was first studied by Buttazzo-Kawohl [9], who proved in particular that there exist indeed some domains Ω different from balls where (1) admits a solution ϕ_Ω. Such a solution turns out to depend only on the distance d to the boundary.

The level lines of d are the so-called inner parallel sets, which were first used in variational problems in the monograph by Pólya and Szegö [29, Section 1.29], and some years later by Makai [27] in order to give a lower bound for the torsional rigidity of planar domains. Functions depending only on d were later named *web functions* in [20] since, in case of planar polygons, their level lines recall the pattern of a spider web. Subsequently, these functions were used in [12, 13] to study the generalized torsion problem $-\Delta_p u = 1$ under homogeneous Dirichlet conditions in planar domains ($n = 2$). In particular, it was shown in [13] that the error made when approximating the minimum in the p-torsion problem by means of web functions tends to zero as $p \to +\infty$. This brings further evidence to the possibility of having viscosity solutions to (1) in domains different from balls.

The purpose of the present paper is to overcome the restriction to web functions considered in [9] where the Authors proved, roughly speaking, that a unique viscosity solution to (1) exists within the class of web functions if and only if Ω satisfies a certain geometrical condition related to its distance function, see Theorem 1.

We prove that the same results holds true *without* the web assumption, thus achieving a precise characterization of the domains where (1) admits a solution, see Theorem 2 below. This result is obtained by exploiting the properties of a suitable P-function, see Theorem 4.

2 Results

For any bounded domain $\Omega \subset \mathbb{R}^n$ let

$$G(\Omega) = \{x \in \Omega;\ \exists ! y \in \partial\Omega \text{ such that } d(x, \partial\Omega) = d(x, y)\} \ .$$

Then the ridge of Ω or cut locus of $\partial\Omega$ is defined by

$$\mathcal{R}(\Omega) = \Omega \setminus G(\Omega) \ .$$

We also set

$$M(\Omega) = \{x \in \Omega;\ d(x, \partial\Omega) = \max_{y \in \Omega} d(y, \partial\Omega)\} \ .$$

In G, the distance $d(x, \partial\Omega)$ to the boundary is at least of class C^1, and also smooth, i.e., of class C^2 or $C^{k,\alpha}$ with $k \geq 2$ and $\alpha \in (0, 1)$ provided $\partial\Omega$ is of the same class, see [14, 24]. It is remarkable that even for a convex plane domain the ridge can have positive measure, see pages 10 and 11 in [26]. Simple examples such as an ellipse or a rectangle show that in general $M(\Omega)$ is a genuine subset of the ridge, but there are many domains with the property $M(\Omega) = \mathcal{R}(\Omega)$. Examples of such domains are for instance a stadium domain (convex hull of two balls of same radius and different center), an annulus, or plane domains which are generated as follows. Let γ be a compact $C^{1,1}$ curve with curvature not exceeding K in modulus and $\Omega = U_b(\gamma) = \{x \in \mathbb{R}2 \mid d(x, \gamma) < b\}$ with $b < 1/K$. Then $M(\Omega) = \mathcal{R}(\Omega)$, see Figure 1.

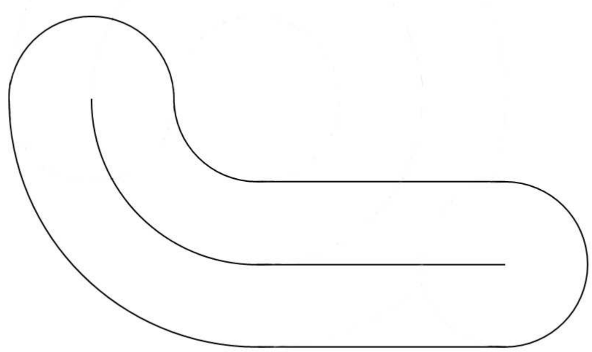

Figure 1: A domain satisfying $M(\Omega) = \mathcal{R}(\Omega)$

With any open bounded domain Ω which satisfies the condition $M(\Omega) = \mathcal{R}(\Omega)$, we associate the web function

$$\phi_\Omega(x) := \frac{1}{4}\Big[a^4 - \Big(a^3 - 3d(x, \partial\Omega)\Big)^{4/3}\Big] , \qquad \text{where} \ \ a := \Big[3 \max_{x \in \Omega} d(x, \partial\Omega)\Big]^{1/3} . \tag{3}$$

In [9] Buttazzo and Kawohl proved the following result:

Theorem 1. *Let Ω be an open bounded connected domain, with $\partial\Omega$ of class C^2.*

(i) Assume that $M(\Omega) = \mathcal{R}(\Omega)$*. Then* ϕ_Ω *is the unique web viscosity solution of class* $C^1(\overline{\Omega})$ *to problem* (1)*.*

(ii) Conversely, assume that problem (1) *admits a web viscosity solution of class* $C^1(\overline{\Omega})$*. Then* $M(\Omega) = \mathcal{R}(\Omega)$*.*

In this paper we improve Theorem 1 by showing that studying the overdetermined boundary value problem (1) within the class of web functions is in fact not restrictive.
Before introducing our assumptions and in order to justify them, let us make a few remarks on the Dirichlet problem

$$\begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

Inspired by a celebrated example of Aronsson [4], let

$$u(x) = \frac{3^{4/3}}{4\,n^{1/3}} \left(1 - \sum_{i=1}^n x_i^{4/3} \right) \qquad \text{and} \qquad \Omega = \left\{ x \in \mathbb{R}^n;\ \sum_{i=1}^n x_i^{4/3} < 1 \right\}. \tag{5}$$

Then $\partial\Omega \in C^{1,1/3}$ and one can verify that u solves (4) a.e. In fact, u is of class C^2 in a.e. $x \in \Omega$, with only the hyperplanes $x_i = 0$ being excluded. Another explicit example is given by

$$u(x) = \frac{3^{4/3}}{4} \left(1 - |x|^{4/3} \right) \qquad \text{and} \qquad \Omega = \left\{ x \in \mathbb{R}^n;\ |x| < 1 \right\}. \tag{6}$$

These two examples and the results in [16] (which hold for planar domains) suggest that the solution might be of class $C^{1,1/3}(\Omega)$ (as conjectured by Aronsson) and that, denoting by S the sets of points where u fails to be C^2,

$$S \text{ is closed and has measure zero.} \tag{7}$$

Notice also that in these examples the set S coincides with the set where at least one first order partial derivative of the solution vanishes, and that $\Omega \setminus S$ may be disconnected.
We now state our main result.

Theorem 2. *Let* Ω *be an open bounded connected domain, with* $\partial\Omega \in C^2$*.*

(i) Assume that $M(\Omega) = \mathcal{R}(\Omega)$*. Then* ϕ_Ω *is the unique viscosity solution of class* $C^1(\overline{\Omega})$ *to problem* (1)*.*

(ii) Conversely, assume that problem (1) *admits a viscosity solution of class* $C^1(\overline{\Omega}) \cap C^2(\Omega \setminus S)$*, where* S *satisfies* (7)*. Then* $M(\Omega) = \mathcal{R}(\Omega) = S$*.*

Remark 3. Clearly, statement (i) of Theorem 2 improves statement (i) of Theorem 1. Also statement (ii) of Theorem 2 improves statement (ii) of Theorem 1, at least when Ω is convex. Indeed, if problem (1) admits a web viscosity solution $\phi \in C^1(\overline{\Omega})$, then $M(\Omega) = \mathcal{R}(\Omega)$ and $\phi = \phi_\Omega$, which is of class C^2 on the complement of $M(\Omega) = \mathcal{R}(\Omega)$. When Ω is convex and $M(\Omega) = \mathcal{R}(\Omega)$, the set $\mathcal{R}(\Omega)$ satisfies assumption (7).

The proof of Theorem 2 is based on the following result about the Dirichlet problem (4), which has its own interest.

Theorem 4. *Let Ω be an open bounded connected domain, with $\partial\Omega \in C^2$. Assume that problem (4) admits a viscosity solution u of class $C^1(\overline{\Omega}) \cap C^2(\Omega \setminus S)$, where S satisfies (7). Then the P-function*

$$P(x) := \frac{|\nabla u(x)|^4}{4} + u(x) \tag{8}$$

attains both its maximum and minimum over $\overline{\Omega}$ on $\partial\Omega$. In particular,

$$\min_{\partial\Omega} \frac{|\nabla u|^4}{4} \le \|u\|_\infty \le \max_{\partial\Omega} \frac{|\nabla u|^4}{4} \ .$$

Remark 5. In the case of example (5) the P-function (8) becomes

$$P(x) = \frac{3^{4/3}}{4\,n^{4/3}} \left(\sum_{i=1}^n x_i^{2/3} \right)^2 + \frac{3^{4/3}}{4\,n^{1/3}} \left(1 - \sum_{i=1}^n x_i^{4/3} \right) .$$

Moreover,

$$\min_{\partial\Omega} \frac{|\nabla u|^4}{4} = \frac{3^{4/3}}{4\,n^{4/3}} \ , \qquad u(0) = \|u\|_\infty = \max_{\partial\Omega} \frac{|\nabla u|^4}{4} = \frac{3^{4/3}}{4\,n^{1/3}} \ .$$

The equality of the last two terms is simply explained by Step 1 in the proof of Theorem 4: one can connect the origin 0 (where u attains its maximum in Ω) with the point $(\frac{1}{n^{3/4}}, \frac{1}{n^{3/4}}, ..., \frac{1}{n^{3/4}})$ (where $|\nabla u|$ attains its maximum on $\partial\Omega$) with the steepest descent line $x_1 = x_2 = ... = x_n$ of u.

As possible variants of Theorem 2, we point out the following problems.

– Consider the overdetermined boundary value problem

$$\begin{cases} -\Delta_\infty u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = -a & \text{on } \partial\Omega \ . \end{cases}$$

Then statement of Theorem 2 (ii) remains true. Moreover, also statement (i) remains true provided f satisfies assumptions which guarantee the uniqueness of viscosity solutions to the corresponding Dirichlet problem.

– Consider the overdetermined boundary value problem

$$\begin{cases} -\Delta_\infty^N u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = -a & \text{on } \partial\Omega \ , \end{cases} \tag{9}$$

where Δ_∞^N denotes the "normalized" ∞-Laplacian defined by

$$\Delta_\infty^N u := \langle D^2 u Du, Du \rangle |Du|^{-2} \ .$$

It is the formal limit as $p \to \infty$ of the normalized p-Laplacian

$$\Delta_p^N u := \frac{\Delta_p u}{(p-2)|Du|^{p-2}} = \Delta_\infty^N u + \frac{\Delta u}{p-2}.$$

Then the statement of Theorem 2 remains true, provided also Theorem 4 remains true when the operator Δ_∞ is changed into Δ_∞^N. We point out that the normalized ∞-Laplacian is related to a "tug of war"- game in game theory. The right hand side 1 in (9) represents the running cost of the game. For details we refer the interested reader to [33] and references therein.

3 Proofs

Proof of Theorem 4.
Let u be the viscosity solution to (4) of class $C^1(\overline{\Omega}) \cap C^2(\Omega \setminus S)$ and let P be as in (8).

Step 1: P is constant along the steepest descent lines of u as long as they do not intersect S.
Using the fact that u is both a viscosity sub and supersolution and (7), one sees that u satisfies the pde in $\Omega \setminus S$, that is, almost everywhere in Ω. In particular, we point out that

$$\nabla u \neq 0 \qquad \text{in } \Omega \setminus S \ , \tag{10}$$

because $\nabla u(x) = 0$ implies $\Delta_\infty u(x) = 0$ and the equation would not be satisfied at x.
Incidentally, notice that the converse inclusion $S \subseteq \{\nabla u = 0\}$ is in general false, see example (6).
Then, denoting by σ the (unit) direction $-\nabla u/|\nabla u|$ of steepest descent of u, the equation $-\Delta_\infty u = 1$ can be rewritten as

$$-u_{\sigma\sigma}|u_\sigma|^2 = 1 \qquad \text{in } \Omega \setminus S \ , \tag{11}$$

where $u_{\sigma\sigma}$ is the second directional derivative in direction σ, that is $\langle D^2 u \ \sigma, \sigma \rangle$.
We multiply both sides of (11) by u_σ. We obtain

$$-u_\sigma^3 \, u_{\sigma\sigma} = u_\sigma \qquad \text{in } \Omega \setminus S \ ,$$

or equivalently

$$\frac{\partial}{\partial \sigma}\Big(\frac{u_\sigma^4}{4} + u\Big) = 0 \qquad \text{in } \Omega \setminus S \ .$$

Therefore the P-function is constant along the steepest descent lines of u, as long as they do not intersect S.

Step 2: finding a point y where P varies on the the steepest descent line.
We are going to prove Theorem 4 arguing by contradiction. Assume that,

$$\mu = \min_{\overline{\Omega}} P < \min_{\partial\Omega} P = m$$

and consider the (open nonempty) set

$$A := \{x \in \Omega; \ \mu < P(x) < m\}.$$

By assumption (7) and by (10) it cannot be $\nabla u \equiv 0$ in A. So, take $\overline{x} \in A$ such that $\nabla u(\overline{x}) = 0$ and consider the corresponding level set of P:

$$\Gamma := \{x \in \Omega; \ P(x) = P(\overline{x})\}.$$

Let $y \in \Gamma$ be an absolute minimum of u constrained on Γ. Then we have

$$\frac{|\nabla u(y)|^4}{4} + u(y) = P(y) = P(\overline{x}) = \frac{|\nabla u(\overline{x})|^4}{4} + u(\overline{x}) > u(\overline{x}).$$

In turn, this yields

$$\frac{|\nabla u(y)|^4}{4} > u(\overline{x}) - u(y) \geq 0.$$

Therefore, $\nabla u(y)$ is a nontrivial vector orthogonal to Γ at the point y (because y is an absolute minimum).

Step 3: conclusion.
Let $y \in A$ be as in Step 2 so that $y \notin \partial\Omega$. Therefore, by continuity of ∇u and P, there exists $r > 0$ such that if $|x - y| < r$, then: $x \in \Omega$, $\nabla u(x) \neq 0$, and the steepest descent line crosses the level set of P containing x transversally at x. Hence, if $|x - y| < r$, P is not constant along the steepest descent line. In view of Step 1, this means that any such x belongs to S, contradicting (7).

Proof of Theorem 2.
(i) By [25, Theorems 1 and 3], the Dirichlet boundary value problem (4) admits a unique viscosity solution in $C(\overline{\Omega})$. Hence, if problem (1) admits a viscosity solution in $C^1(\overline{\Omega})$, such solution is necessarily unique. Thanks to the assumption $M(\Omega) = \mathcal{R}(\Omega)$, by Theorem 1 (i), we infer that $\phi_\Omega \in C^1(\overline{\Omega})$ and it is a viscosity solution to problem (1). Therefore ϕ_Ω is the unique viscosity solution in $C^1(\overline{\Omega})$ to problem (1).

(ii) Let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega \setminus S)$ be a viscosity solution to problem (1), and let P be as in (8). By the two boundary conditions satisfied by u, we know that $P(x) = a^4/4$ for every $x \in \partial\Omega$. Hence, By Theorem 4, we have

$$|\nabla u| = (a^4 - 4u)^{1/4} =: g(u) \qquad \text{in } \Omega\ . \tag{12}$$

Since the function ϕ_Ω in (3) is a classical solution to (12), by [10, Theorem 1.2] it is also a viscosity solution. Then it is the unique viscosity solution (see *e.g.* [5, Theorem III.1] applied to $H(u,p) = |p|^4 - (a^4 - 4u)$). This means that $u = \phi_\Omega$ so that u is a web function. By Theorem 1 (ii), we deduce that $M(\Omega) = \mathcal{R}(\Omega)$. $\square$

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