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Recent results on branching random walks

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Abstract

This paper is a collection of recent results on discrete-time and continuous-time branching random walks. Some results are new and others are known. Many aspects of this theory are considered: local, global and strong local survival, the existence of a pure global survival phase and the approximation of branching random walks by means of multitype contact processes or spatially confined branching random walks. Most results are obtained using a generating function approach: the probabilities of extinction are seen as fixed points of an infinite dimensional power series. Throughout this paper we provide many nontrivial examples and counterexamples.

Keywords: branching random walk, survival, phase transition, amenability, critical value, critical behavior, percolation, multitype contact process.
AMS subject classification: 60J80, 60K35.

Contents

1   Introduction

Branching random walks can be considered as processes which simultaneously generalize the concepts of branching process and of random walk. A branching process is a very simple population model (introduced in [17]) where particles breed and die (independently of each other) according to some random law. At any time, this process is completely characterized by the total number of particles alive. Branching random walks (in short, BRWs) add space to this picture: particles live in a spatially structured environment and the reproduction law, which may depend on the location, not only tells how many children the particle has, but also where it places them. The state of the process, at any time, is thus described by the collection of the numbers of particles alive at x, where x varies among the possible sites. Although particles do not actually move, an observer would see a random movement of the population. Moreover if we identify every particle with one of its children (if there are any), then we may view the BRW as a system of random walkers which may disappear (i.e. the corresponding particle has no children) or split into two or more independent walkers (i.e. the corresponding particle has two or more children).

The basic question that one answers studying the branching process is whether it survives (i.e. with positive probability at any time there is someone alive); while the classical question for random walks is whether the walker returns (with positive probability or, equivalently, with probability one) infinitely many times to some fixed site. Transposed into BRW theory, the first question asks whether there is global survival, that is, with positive probability at any time there is someone alive *somewhere*); while the second question deals with local survival, that is, with positive probability the process returns infinitely many times to some fixed site (this event, in contrast with the situation for random walks, may have probability one only in trivial examples).

In the literature one can find BRWs both in continuous and discrete time. The continuous-time setting has been studied by many authors (see [22, 24, 25, 29, 32] just to name a few). As we see in Section 2.2, in this case one studies a family of BRWs which depends on the choice of a parameter λ. There are two (possibly coinciding) parameters of interest: $\lambda_w \leq \lambda_s$. If $\lambda < \lambda_w$ there is almost sure extinction, if $\lambda_w < \lambda \leq \lambda_s$ there is global but not local survival and if $\lambda_s < \lambda$ there is local and global survival (see for instance [36, 38, 6, 7]).

The discrete-time case has been initially considered as a natural generalization of branching processes (see [1, 10, 11, 12, 13, 20]), but, since every continuous-time BRW admits a discrete-time counterpart which has the same behavior, results in this setting naturally extend to continuous time.

In recent years, there has been a growing interest about BRWs in random environment (see for instance [14, 18, 21, 27, 28, 33]). This is an interesting subject that we do not discuss in this paper.

Being at the crossroad between branching process and random walk theories, BRW theory benefits of the techniques of both fields (to be honest, there is a third road at this intersection, since BRWs can also be seen as interacting particle systems – although particles do not interact). Indeed, recalling that the probability of extinction of a branching process is the fixed point of a generating function associated to the offspring distribution, one can associate to the offspring distribution of the BRW a (possibly infinite-dimensional) generating function G (this is what we do in Section 2.5). Moreover it is possible to prove that the extinction probabilities of the BRW are fixed points of this generating function. This is a fundamental tool that we use in Section 4. On the other hand some tools borrowed from random walk theory, such as generating functions of first return probabilities and superharmonic functions, are particularly useful in the no death case (that is, the case where every particle has at least one child).

The paper is a collection of recent results on BRWs: some of them are already known, some are new and their proofs can be found in Section 6. A brief outline of the paper and of main results follows. The paper is divided into five main sections. Section 2 is a short technical introduction to the subject. There is a description of discrete-time and continuous-time BRWs (Sections 2.1 and 2.2 respectively). Classical processes as edge-breeding and site-breeding continuous-time BRWs are discussed and it is shown that, from the point of view of survival vs. extinction, the class of discrete-time BRWs extends the class of continuous-time BRWs. In Section 2.3, other models are presented along with their relation with BRWs. In Sections 2.4 and 2.5 two important tools are discussed: trails and generating functions. While the first one is more important from a theoretical point of view, the second one is repeatedly used to study the behavior of a BRW. In particular a *maximum principle* for solutions of certain inequalities involving these generating functions is proved.

Section 3 presents two particular families of BRWs: $\mathcal{F}$-BRWs (Section 3.1) and BRWs with no death (Section 3.2). The first class (which has been introduced in [42]), is a natural generalization of the classes of continuous-time BRWs on weighted $\mathcal{F}$-graphs (see [7, Proposition 4.5]) and on $\mathcal{F}$-multigraphs (see [6, Definition 3.1]). This class contains *quasi-transitive BRWs* (for instance, edge-breeding continuous-time BRWs on quasi-transitive graphs) and BRWs which are *locally isomorphic to branching processes* (for instance, site-breeding continuous-time BRWs on regular graphs); nevertheless, as Example 3.3 shows, the class of $\mathcal{F}$-BRWs is strictly larger than the class of quasi-transitive BRWs even for edge-breeding continuous-time BRWs (see also Example 4.25). As for BRWs with no death, which are a natural generalization of random walks, we note that even though they represent a limited subclass of BRWs, many results can be extended immediately to the general class of BRWs using a comparison introduced by Harris for branching processes (see [1, Chapter I.12] and Section 3.2).

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Section 4 is devoted to the study of the behavior of BRWs (survival vs. extinction). In Section 4.1 the probabilities of survival are viewed as fixed points of an infinite-dimensional generating function. Local survival is completely characterized through the knowledge of the first-moment matrix M (Theorem 4.3). For global survival we give an equivalent condition in terms of the generating function G (Theorem 4.7(1)). In terms of M we can only provide an equivalent condition for $\mathcal{F}$-BRWs and a necessary one in the general case (Theorem 4.9). Example 4.15 shows that many conjectures about sufficient conditions for global survival are false. In continuous time, Corollary 4.5 identifies λ_s and states almost sure local extinction at $\lambda = \lambda_s$; a characterization of λ_w through the existence of solutions of certain inequalities is given in Theorem 4.12. In the case of $\mathcal{F}$-BRWs, Corollary 4.10 provides a more explicit expression for λ_w (this expression is a lower bound in the general case), and states global extinction at $\lambda = \lambda_w$. Example 4.13 shows that in the general case global survival is possible at λ_w. Clearly local survival implies global survival and the converse is false. When the two events coincide (and have positive probability) we say that there is strong local survival. Proposition 4.19 claims that quasi-transitivity and local survival imply strong local survival; Theorem 4.21 characterizes strong local survival, generalizing [31, Theorem 3.1] which was stated for the no death case. Examples 4.35 and 4.36 show that even when the BRW is locally isomorphic to a branching process (i.e. the reproduction law does not depend on the site) non-strong local survival is possible. Moreover Example 4.31 is an edge-breeding continuous-time BRW on a homogeneous tree with a loop where, for small and large values of λ there is strong local survival while for intermediate values we have non-strong local survival. This shows that, unlike local and global survival, for strong local survival there is no monotonicity in λ. Irreducibility guarantees that if there is local survival at some y (or global survival) starting from some x_0, then there is local survival at any w (or global survival) starting from any x. Clearly the probabilities of global or local survival may depend on the staring point even in the irreducible case (see Example 4.18). Example 4.4 shows that in the reducible case there can be local extinction at x starting from x for all $x \in X$, but local survival at some y starting from some $x \neq y$; in addition, this example shows that in the reducible case it there might be global extinction starting from x but global survival starting from some $y \neq x$. Example 4.32 proves that, even in the irreducible case, if there are vertices where particles have at least one child almost surely, then it might happen that there is strong local survival starting from some vertex and non-strong local survival starting from other vertices. In Example 4.17 we find a BRW which survives globally even if the law at each site gives a branching process which dies out. The main tool that we use in many examples is the discussion in Remark 4.1 which relates the probability of visiting a subset $A \subseteq X$, the probability of local survival at A and the probability of global survival. Section 4.5 is devoted to pure global survival, that is when the process survives globally but not locally. For $\mathcal{F}$-BRWs the existence of a pure global survival phase is equivalent to nonamenability (Corollary 4.24). Theorem 4.23 gives an equivalent condition on the first moment matrix M, in the general case, for nonamenability. Unfortunately, the existence of a pure global survival phase is not equivalent, in general, to nonamenability: Example 4.27 shows that there exists an amenable edge-breeding, continuous-time BRW without pure global survival and, conversely, according to Example 4.28, there exists a nonamenable edge-breeding, continuous-time BRW with pure global survival. Theorem 4.29 gives a sufficient condition for no pure global survival of a continuous-time BRW. In Section 4.6 we treat the

special case where the reproduction law is independent of the site.

In Section 5 the question of the approximation of a BRW is studied. In particular in Section 5.1 we obtain an approximation of a general BRW, which is not necessarily irreducible, by means of a sequence of spatially confined BRWs (Theorem 5.2). This results is a corollary of a generalization of a theorem due to Sarymshakov and Seneta (see [37, Theorem 6.8]) which deals with nonnegative matrices (Theorem 5.1). Here we obtain, as a particular case, that if we have a surviving process, then by confining it to a sufficiently large (possibly finite and not necessarily connected) proper subgraph the resulting BRW survives as well; this result was already known for irreducible BRWs confined to connected subgraphs. In Section 5.2 we study the approximation of the BRW with a sequence of truncated BRWs (which are, in fact, multitype contact processes). The key to obtain such a result is the comparison of our process with a suitable oriented percolation (as explained in [8, 42]). The strategy is then applied to some classes of regular BRWs in discrete-time and continuous-time.

In Section 6 all the proofs of new results can be found, along with some technical lemmas.

2 Basic definitions and preliminaries

2.1 Discrete-time Branching Random Walks

We start with the construction of a generic discrete-time BRW $\{\eta_n\}_{n\in\mathbb{N}}$ (see also [7] where it is called *infinite-type branching process*) on a set X which is at most countable; $\eta_n(x)$ represents the number of particles alive at x at time n. To this aim we consider a family $\mu = \{\mu_x\}_{x\in X}$ of probability measures on the (countable) measurable space $(S_X, 2^{S_X})$ where $S_X := \{f : X \to \mathbb{N} : \sum_y f(y) < \infty\}$. To obtain generation $n+1$ from generation n we proceed as follows: a particle at site $x \in X$ lives one unit of time, then a function $f \in S_X$ is chosen at random according to the law μ_x and the original particle is replaced by $f(y)$ particles at y, for all $y \in X$; this is done independently for all particles of generation n. Note that the choice of f assigns simultaneously the total number of children and the location where they will live. We denote the BRW by the couple (X, μ).

Equivalently we could introduce the BRW by choosing the number of children and then their location. Indeed define $\mathcal{H} : S_X \to \mathbb{N}$ as $\mathcal{H}(f) := \sum_{y\in X} f(y)$ which represents the total number of children associated to f. Denote by ρ_x the measure on $\mathbb{N}$ defined by $\rho_x(\cdot) := \mu_x(\mathcal{H}^{-1}(\cdot))$; this is the law of the random number of children of a particle living at x. For each particle, independently, we pick a number n at random, according to the law ρ_x, and then we choose a function $f \in \mathcal{H}^{-1}(n)$ with probability $\mu_x(f)/\rho_x(n) \equiv \mu_x(f)/\sum_{g\in\mathcal{H}^{-1}(n)} \mu_x(g)$ and we replace the particle at x with $f(y)$ particles at y (for all $y \in X$).

To be precise, to construct the process, pick a family $\{f_{i,n,x}\}_{i,n\in\mathbb{N},x\in X}$ of independent S_X-valued random variables such that, for every $x \in X$, $\{f_{i,n,x}\}_{i,n\in\mathbb{N}}$ have the common law μ_x, and an initial state η_0 such that $\sum_{x\in X} \eta_0(x) < +\infty$. The discrete-time BRW $\{\eta_n\}_{n\in\mathbb{N}}$ is defined iteratively as follows

$$\eta_{n+1}(x) = \sum_{y\in X} \sum_{i=1}^{\eta_n(y)} f_{i,n,y}(x) = \sum_{y\in X} \sum_{j=0}^{\infty} \mathbb{1}_{\{\eta_n(y)=j\}} \sum_{i=1}^{j} f_{i,n,y}(x). \tag{2.1}$$

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Even though in this paper the initial state will always be deterministic, considering a random initial distribution $\sum_{x \in X} \eta_0(x) < +\infty$ a.s., would not be a significant generalization.

While in random walk theory a fundamental role is played by the transition matrix, in BRW theory a similar role is played by the *first-moment matrix* $M = (m_{xy})_{x,y \in X}$, where $m_{xy} := \sum_{f \in S_X} f(y) \mu_x(f)$ is the expected number of particles from x to y (that is, the expected number of children that a particle living at x sends to y). We suppose that $\sup_{x \in X} \sum_{y \in X} m_{xy} < +\infty$; most of the results of this paper still hold without this hypothesis, nevertheless it allows us to avoid dealing with an infinite expected number of offsprings. Note that the expected number of children generated by a particle living at x is $\sum_{y \in X} m_{xy} = \sum_{n \geq 0} n \rho_x(n) =: \bar{\rho}_x$. Given a function f defined on X we denote by Mf the function $Mf(x) := \sum_{y \in X} m_{xy} f(y)$ whenever the RHS makes sense. We denote by $m_{xy}^{(n)}$ the entries of the nth power matrix M^n and we define

$$M_s(x,y) := \limsup_{n \to \infty} \sqrt[n]{m_{xy}^{(n)}}, \quad M_w(x) := \liminf_{n \to \infty} \sqrt[n]{\sum_{y \in X} m_{xy}^{(n)}}, \qquad \forall x,y \in X. \qquad (2.2)$$

From equation (2.1), it is straightforward to prove that the expected number of particles, starting from an initial state η_0, satisfies the recurrence equation $\mathbb{E}^{\eta_0}(\eta_{n+1}(x)) = \sum_{y \in X} m_{yx} \mathbb{E}^{\eta_0}(\eta_n(y))$ hence

$$\mathbb{E}^{\eta_0}(\eta_n(x)) = \sum_{y \in X} m_{yx}^{(n)} \eta_0(y).$$

Remark 2.1. *Note that a BRW can be seen as a random walk on $\mathbb{N}^X$ (to be precise, on $S_X \subseteq \mathbb{N}^X$), where $\mathbf{0}$ is an absorbing state. If $\rho_x(0) > 0$ for all $x \in X$ then every state in $S_X \setminus \{\mathbf{0}\}$ is transient. Basically this is due to the fact that the probability of going into the state $\mathbf{0}$ starting from a state $\eta \in S_X$ in one step is $\prod_{x \in X} \rho_x(0)^{\eta(x)} > 0$; hence the probability of visiting infinitely often the state η without ending in the trap state $\mathbf{0}$ is 0 (for a formal proof in the case of a branching process see [20, Theorem 6.2]).*

We introduce here some terminology borrowed from random walk and graph theory. In general our definitions extend the classical ones which apply to graphs in the following way: a discrete-time counterpart of an edge-breeding continuous-time BRW (see Section 2.2) has the property $\mathcal{P}$ if and only if the underlying graph has the usual property $\mathcal{P}$. The BRW (X, μ) is called *non-oriented* or *symmetric* if $m_{xy} = m_{yx}$ for every $x, y \in X$. (X, μ) is called *nonamenable* if and only if

$$\inf \left\{ \frac{\sum_{x \in S, y \in S^\complement} m_{xy}}{|S|} : S \subseteq X, |S| < \infty \right\} =: \iota_{(X,\mu)} > 0, \qquad (2.3)$$

and it is called *amenable* otherwise. The value $\iota_{(X,\mu)}$ is called *isoperimetric constant* since in the case of a continuous-time, edge-breeding BRW (see the end of Section 2.2) this is the usual isoperimetric constant of the underlying multigraph and, in that case, the nonamenability of the BRW is equivalent to the nonamenability of the multigraph (see [6, Section 3.3] for the definition).

For a generic discrete-time BRW, the set X is not *a priori* a graph; nevertheless, the family of probability measures, $\{\mu_x\}_x$ induces in a natural way a graph structure on X that we denote by (X, E_μ) where $E_\mu := \{(x,y) : m_{xy} > 0\} \equiv \{(x,y) : \exists f \in S_X, \mu_x(f) > 0, f(y) > 0\}$. Roughly speaking, (x,y) is and edge if and only if a particle living at x can send a child at y with positive probability (from now on *wpp*). We say that there is a path from x to y, and we write $x \to y$, if it is possible to find a finite sequence $\{x_i\}_{i=0}^n$ (where $n \in \mathbb{N}$) such that $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in E_\mu$ for all $i = 0, \ldots, n-1$. If $x \to y$ and $y \to x$ we write $x \rightleftharpoons y$. Observe that there is always a path of length 0 from x to itself.

We call the matrix $M = (m_{xy})_{x,y \in X}$ *irreducible* if and only if the graph (X, E_μ) is *connected*, otherwise we call it *reducible*. We denote by $\deg(x)$ the degree of a vertex x, that is, the cardinality of the set $\mathcal{N}_x := \{y \in X : (x,y) \in E_\mu\}$. Note that if (X, μ) is non-oriented then the graph (X, E_μ) is non-oriented (that is, $(x,y) \in E_\mu$ if and only if $(y,x) \in E_\mu$).

Definition 2.2. *The colony can survive in different ways: we say that the colony survives locally wpp at $y \in X$ starting from $x \in X$ if*

$$\mathbb{P}^{\delta_x}(\limsup_{n \to \infty} \eta_n(y) > 0) > 0;$$

we say that it survives globally wpp starting from x if

$$\mathbb{P}^{\delta_x}\left(\sum_{w \in X} \eta_n(w) > 0, \forall n \in N \right) > 0$$

(or, equivalently, $\mathbb{P}^{\delta_x}(\limsup_{n \to \infty} \sum_{w \in X} \eta_n(w) > 0)$).
Let us define the probabilities of extinction $q(x,y) := 1 - \mathbb{P}^{\delta_x}(\limsup_{n \to \infty} \eta_n(y) > 0)$ and $\bar{q}(x) := 1 - \mathbb{P}^{\delta_x}\left(\sum_{w \in X} \eta_n(w) > 0, \forall n \in N \right)$. Following [18], we say that the there is strong local survival wpp at $y \in X$ starting from $x \in X$ if

$$q(x,y) = \bar{q}(x) < 1.$$

Finally we say that the BRW is in a pure global survival phase starting from x if

$$\bar{q}(x) < q(x,x) = 1.$$

From now on when we talk about survival, "wpp" will be tacitly understood. Often we will say simply that local survival occurs "starting from x" or "at x": in this case we mean that $x = y$.

Roughly speaking, there is global survival if there are particles alive somewhere at all times wpp and there is local survival at y if there are particles alive at y at arbitrarily large times wpp. Strong local survival at y starting from x requires that the probability of local survival at y equals the probability of global survival starting from x and that they are both positive. Equivalently, there is strong survival at y starting from x if and only if the probability of local survival at y starting from x conditioned on global survival starting from x is 1. One can show that strong local survival implies that for almost all realizations the process either survives locally and globally or it goes extinct. The typical case (but not the only one) where there is global but no strong local survival is being in a pure global survival phase.

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Clearly local survival at some y starting from x implies global survival starting from x, since $\bar{q}(x) \leq q(x,y)$ for all $x, y \in X$. It is easy to construct examples where $\bar{q}(x) < 1 = q(x,y)$ (see Section 4.5). One may wonder whether it is possible to find examples of BRWs where $\bar{q}(x) < q(x,y) < 1$; according to Examples 4.31 and 4.35 the answer is positive.

We observe that if $x \to y$ then local survival at x implies local survival at y starting from any w such that $w \to x$ ($q(w,y) \leq q(w,x)$ for all $w \in X$). Analogously, if $x \to y$ then global survival starting from y implies global survival starting from x (indeed $\bar{q}(x) \leq \bar{q}(y)$). Moreover if $x \to y$, $w \to w'$ and $q(w',x) < 1$ then $q(w,y) < 1$. In particular if $x \rightleftharpoons y$ then local (resp. global) survival starting from x is equivalent to local (resp. global) survival starting from y. As a consequence, if M is irreducible then the process survives locally (resp. globally) at one vertex if and only if it survives locally (resp. globally) at every vertex. In this case $q(x,y) = q(x,x)$ for all $x, y \in X$ (see Section 4.1 for details). Note that even if in the irreducible case one cannot guarantee that $q(x,x) = q(y,y)$ or $\bar{q}(x) = \bar{q}(y)$ when $x \neq y$ (see for instance Example 4.18).

Assumption 2.3. *We assume henceforth that for all $x \in X$ there is a vertex $y \rightleftharpoons x$ such that $\mu_y(f : \sum_{w:w \rightleftharpoons y} f(w) = 1) < 1$, that is, in every equivalence class (with respect to $\rightleftharpoons$) there is at least one vertex where a particle can have inside the class a number of children different from one wpp.*

Remark 2.4. *The previous assumption guarantees that the restriction of the BRW to an equivalence class is nonsingular (see [20, Definition II.6.2]). There is a technical reason behind the previous assumption. The classical Galton–Watson branching process is a particular BRW where $X := \{x\}$ is a singleton and S_X and μ_x can be identified with $\mathbb{N}$ and a probability measure on $\mathbb{N}$ respectively. It is well-known that*

- *if $\mu_x(1) = 1$ then $m_{xx} = 1$ and there is survival with probability 1;*

- *if $\mu_x(1) < 1$ then there is survival wpp if and only if $m_{xx} > 1$,*

(see also Example 4.2). Hence the condition $m_{xx} > 1$ is equivalent to survival under Assumption 2.3.

For a generic BRW, we call *diffusion matrix* the matrix P with entries $p(x,y) = m_{xy}/\bar{\rho}_x$. Note that P is a stochastic matrix which defines a random walk on X, but it is not true in general that the offsprings are dispersed independently according to P. This last updating rule characterizes a particular, but meaningful, subclass of discrete-time processes that we call *BRWs with independent diffusion*: a particle at site x lives one unit of time and is replaced by a random number of children (with law ρ_x) which are dispersed independently on X, according to a stochastic matrix P. This rule is a particular case of the general one, since here one simply chooses

$$\mu_x(f) = \rho_x \left(\sum_y f(y) \right) \frac{(\sum_y f(y))!}{\prod_y f(y)!} \prod_y p(x,y)^{f(y)}, \quad \forall f \in S_X. \tag{2.4}$$

Clearly in this case the expected number of children at y of a particle living at x is

$$m_{xy} = p(x,y)\bar{\rho}_x. \tag{2.5}$$

2.2 Continuous-time Branching Random Walks

Continuous-time BRWs have been studied extensively by many authors; in this section we make use of a natural correspondence between continuous-time BRWs and discrete-time BRWs which preserves both local and global behaviors.

In continuous time each particle has an exponentially distributed random lifetime with parameter 1. The breeding mechanisms can be regulated by means of a nonnegative matrix $K = (k_{xy})_{x,y \in X}$ in such a way that for each particle at x, there is a clock with $Exp(\lambda k_{xy})$-distributed intervals (where $\lambda > 0$), each time the clock rings the particle breeds in y. We say that the BRW has a death rate 1 and a reproduction rate λk_{xy} from x to y. We observe (see Remark 2.5) that the assumption of a nonconstant death rate does not represent a significative generalization. We denote by (X, K) a family of continuous-time BRWs (depending on the parameter $\lambda > 0$), while we use the notation (X, μ) for a discrete-time BRW.

Equivalently, one can associate to each particle at x a clock with $Exp(\lambda k(x))$-distributed intervals ($k(x) = \sum_y k_{xy}$): each time the clock rings the particle breeds and the offspring is placed at random according to a stochastic matrix P (where $p(x, y) = k_{xy}/k(x)$).

To a continuous-time BRW one can associate a discrete-time counterpart; here is the construction. The initial particles represent the generation 0 of the discrete-time BRW; the generation $n + 1$ (for all $n \geq 0$) is obtained by considering the children of all particles of generation n (along with their positions). Clearly the progenies of the original continuous-time BRW and of its discrete-time counterpart are both finite (or both infinite) at the same time. Moreover, almost surely, the two processes have the same local and global behavior. In this sense the theory of continuous-time BRWs, as long as we are interested in the probability of survival (local, strong local and global), is a particular case of the theory of discrete-time BRWs.

Elementary calculations show that each particle living at x, before dying, has a random number of offsprings given by equation (2.4) where

$$\rho_x(i) = \frac{1}{1 + \lambda k(x)} \left(\frac{\lambda k(x)}{1 + \lambda k(x)} \right)^i, \qquad p(x, y) = \frac{k_{xy}}{k(x)}, \qquad (2.6)$$

and this is the law of the discrete-time counterpart; note that the discrete-time counterpart of a continuous-time BRW is a BRW with independent diffusion. Note that ρ_x depends only on $\lambda k(x)$. Using equation (2.5), it is straightforward to show that $m_{xy} = \lambda k_{xy}$ and $\bar{\rho}_x = k(x)$. Note that, for a continuous-time BRW the first-moment matrix M equals λK. From equation (2.6) we have that, for any $\lambda > 0$, the discrete-time counterpart satisfies Assumption 2.3.

Remark 2.5. *The same construction applies to continuous-time BRWs with a death rate $d(x) > 0$ dependent on $x \in X$. In this case the discrete-time counterpart satisfies equation (2.4) where*

$$\rho_x(i) = \frac{d(x)}{d(x) + \lambda k(x)} \left(\frac{\lambda k(x)}{d(x) + \lambda k(x)} \right)^i, \qquad p(x, y) = \frac{k_{xy}}{k(x)}.$$

Hence, from the point of view of local and global survival, this process is equivalent to a continuous-time BRW with death rate 1 and reproduction rate $\lambda k_{xy}/d(x)$ from x to y.

All the definitions given in the previous section extend to the continuous-time case: a continuous-time BRW has a certain property if and only if its discrete-time counterpart has it. In particular, we observe that, for a continuous-time BRW, the isoperimetric constant defined in equation (2.3) equals to $\lambda \iota_{(X,K)}$ where $\iota_{(X,K)}$ is the constant when $\lambda = 1$. Hence either the BRW is nonamenable for all $\lambda > 0$ or it is amenable for all $\lambda > 0$.

Given $x_0 \in X$, two critical parameters are associated to the continuous-time BRW: the *global* (or *weak*) *survival critical parameter* $\lambda_w(x_0)$ and the *local* (or *strong*) *survival critical parameter* $\lambda_s(x_0)$. They are defined as

$$\lambda_w(x_0) := \inf\Big\{\lambda > 0 : \mathbb{P}^{\delta_{x_0}}\Big(\sum_{w \in X} \eta_t(w) > 0, \forall t\Big) > 0\Big\},$$

$$\lambda_s(x_0) := \inf\{\lambda > 0 : \mathbb{P}^{\delta_{x_0}}\big(\limsup_{t\to\infty} \eta_t(x_0) > 0\big) > 0\},$$

where $\mathbf{0}$ is the configuration with no particles at all sites and $\mathbb{P}^{\delta_{x_0}}$ is the law of the process which starts with one individual in x_0. The process is called *globally supercritical*, *critical* or *subcritical* if $\lambda > \lambda_w$, $\lambda = \lambda_w$ or $\lambda < \lambda_w$; an analogous definition is given for the local behavior using λ_s instead of λ_w.

If the graph (X, E_μ) is connected (that is, the BRW is irreducible) then these values do not depend on the initial configuration, provided that this configuration is finite (that is, it has only a finite number of individuals), nor on the choice of x_0. If we have (X, K) and $(Y, \bar{K})$ such that $Y \subseteq X$ and $k_{xy} \geq \bar{k}_{xy}$ for all $x, y \in Y$ then for all $x \in Y$ we have $\lambda_s^X(x) \leq \lambda_s^Y(x)$ and $\lambda_w^X(x) \leq \lambda_w^Y(x)$. In particular we say that there exists a *pure global survival phase* starting from x if the interval $(\lambda_w(x), \lambda_s(x))$ is not empty; clearly, if $\lambda \in (\lambda_w(x), \lambda_s(x))$ then the BRW is in a pure global survival phase according to Definition 2.2.

Given a continuous-time BRW (X, K) we define the following two families of parameters

$$K_s(x, y) := \limsup_{n\to\infty} \sqrt[n]{k_{xy}^{(n)}}, \quad K_w(x) := \liminf_{n\to\infty} \sqrt[n]{\sum_{y \in X} k_{xy}^{(n)}}, \qquad \forall x, y \in X, \qquad (2.7)$$

introduced in [6, 7] where they are called M_s and M_w. Note that supermultiplicative arguments imply that $K_s(x, x) = \lim_n (k_{xx}^{(d(x)n)})^{1/d(x)n}$ where $d(x) := \gcd\{n > 0 : m_{xx}^{(n)} > 0\}$ is the period of $x \in X$ (see [40, Definition 2.19]; hence, for all $x \in X$, we have that $K_s(x, x) \leq K_w(x)$. If the BRW is irreducible then $K_s(x, y)$ and $K_w(x)$ do not depend on $x, y \in X$.

Two special cases are particularly interesting in the continuous-time setting: *site-breeding BRWs* and *edge-breeding BRWs*. We say that a BRW is *site-breeding* if $k(x)$ does not depend on $x \in X$ (cfr. Definition 3.1); on one hand the number of children of a particle is independent of the site, on the other hand, clearly, the diffusion matrix $P = (p(x, y))_{x,y \in X}$ can be inhomogeneous. We say that a BRW is *edge-breeding* if X has a multigraph structure (see [6, Section 2.1] for a formal definition) and k_{xy} is the number of edges from x to y; in this case to each edge there corresponds a constant reproduction rate λ. Thus the total reproduction rate for a particle living at x is $\lambda \cdot \deg(x)$ where $\deg(x)$ is the number of edges *from* x. The diffusion matrix P in this case is the transition matrix of the simple random walk on the underlying multigraph. Note that if the multigraph is *regular* (i.e. $\deg(x)$ does not depend on x) then the edge-breeding BRW is site breeding.

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2.3 Other dynamics

We describe some slightly different dynamics which, in fact, have a natural discrete-time counterpart which has the same local and global behavior.

Multitype BRWs has been studied by some authors (see for instance [26]). In these processes there is a metapopulation which consists of individuals carrying a certain characteristic chosen among a family I of types. A particle of type $i \in I$ living at $x \in X$ generates a random number of children of any type which are randomly placed in X. More precisely, this can be seen as a discrete-time BRW where the space is $X \times I$ and the family of probability measures $\{\mu_{x,i}\}_{x \in X, i \in I}$ are defined on $S_{X \times I}$. Thus, a particle of type i living at x at the end of its lifetime dies and generates $f(y, j)$ children of type j at y (for all $y \in X$ and $j \in I$) with probability $\mu_{x,i}(f)$. For the multitype BRW the global survival of the metapopulation (resp. the local survival of a fixed type) is equivalent to the global (resp. local) survival of the single-type BRW on $X \times I$. On the other hand, the global survival of a fixed type i_0 is equivalent to the survival in the subset $X \times \{i_0\}$, while local survival at x of the metapopulation is equivalent to the survival on $\{x\} \times I$.

Time-dependent BRWs can be described similarly, if we have a time-dependent family of laws $\{\bar{\mu}_{x,n}\}_{x \in X, n \in \mathbb{N}}$ we can construct a BRW on $X \times \mathbb{N}$ where, at the end of its lifetime, a particle at (x, n) dies and generates $f(y)$ children at $(y, n+1)$, for all $y \in X$, with probability $\mu_{x,n}(f)$.

We could define a continuous-time BRW where a particle living at x is endowed with a Poisson clock of parameter $\lambda k(x)$; when this clock rings it picks a function $f \in S_X$ with probability $\mu_x(f)$ and reproduces accordingly. All particles reproduce a random number of times during their exponentially distributed (with mean 1) lifetime. This process has a discrete-time counterpart which can be constructed as in Section 2.2. In this case the discrete-time counterpart in general does not satisfy equation (2.4).

Another definition of BRW, which is used by some authors (see for instance [23] or [41]) is the following. Each particle moves according to a random walk on X. After a random number of steps it dies and is replaced by a random number of children, whose law may depend on the final position. Again this process has a natural counterpart: if in the original BRW a particle starts its life at x and dies at y giving birth to n children then in the discrete-time counterpart this particle does not move and, after one unit of time, it generates n children at y. Note that in terms of local and global survival these two processes are equivalent.

2.4 Reproduction trails

A fundamental tool which allows us to give an alternative construction of the BRW is the reproduction trail (see [36]). We fix an injective map $\phi : X \times X \times \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$. Let the family $\{f_{i,n,x}\}_{i \in \mathbb{Z}, n \geq 0, x \in X}$ be as in Section 2.1 and let η_0 be the initial value. For any fixed realization of the process we call *reproduction trail* to $(x, n) \in X \times \mathbb{N}$ a sequence

$$(x_0, i_0, 1), (x_1, i_1, j_1), \ldots, (x_n, i_n, j_n) \tag{2.8}$$

such that $x = x_n$, $-\eta_0(x_0) \leq i_0 < 0$, $0 < j_l \leq f_{i_{l-1}, l-1, x_{l-1}}(x_l)$ and $\phi(x_{l-1}, x_l, i_{l-1}, j_l) = i_l$, where $0 < l \leq n$. The interpretation is the following: i_n is the identification number of the

particle, which lives at x_n at time n and is the j_n-th offspring of its parent. The sequence $\{x_0, x_1, \ldots, x_n\}$ is the path induced by the trail (sometimes, we say that the trail is based on this path). Given any element (x_l, i_l, j_l) of the trail (2.8), we say that the particle identified by i_n is a descendant of generation $n - l$ of the particle identified by i_l and the trail joining them is $(x_l, i_l, j_l), \ldots, (x_n, i_n, j_n)$. We also say that the trail of the particle i_n is a prolongation of the trail of the particle i_l.

Roughly speaking the trail represents the past history of each single particle back to its original ancestor, that is, the one living at time 0; we note that from the couple (n, i_n), since the map ϕ is injective, we can trace back the entire genealogy of the particle. The random variable $\eta_n(x)$ can be alternatively defined as the number of reproduction trails to (x, n). This construction does not coincide with the one induced by the equation (2.1) but the resulting processes have the same laws.

2.5 Generating functions

Later on we will need some generating functions, both 1-dimensional and infinite dimensional. Define $T_x^n := \sum_{y \in X} m_{xy}^{(n)}$ and $\varphi_{xy}^{(n)} := \sum_{x_1, \ldots, x_{n-1} \in X \setminus \{y\}} m_{xx_1} m_{x_1 x_2} \cdots m_{x_{n-1} y}$ (by definition $\varphi_{xy}^{(0)} := 0$ for all $x, y \in X$). T_x^n is the expected number of particles alive at time n when the initial state is a single particle at x. Roughly speaking, $\varphi_{xy}^{(n)}$ is the expected number of particles alive at y at time n when the initial state is just one particle at x and the process behaves like a BRW except that every particle reaching y at any time $i < n$ is immediately killed (before breeding). In other words $\varphi_{xy}^{(n)}$ is the expected number of particles alive at y at time n whose trail did not hit any (y, i, k) with $k < n$.

Let us consider the following family of 1-dimensional generating functions (depending on $x, y \in X$), where $\lambda \in \mathbb{C}$:

$$\Gamma(x, y | \lambda) := \sum_{n=0}^{\infty} m_{xy}^{(n)} \lambda^n, \qquad \Phi(x, y | \lambda) := \sum_{n=1}^{\infty} \varphi_{xy}^{(n)} \lambda^n.$$

To compare with random walk theory, Γ is the analog of the *Green function* (cfr. [40, Section 1.C]) and Φ is the analog of the generating function of the first-return probabilities (cfr. the function U of [40, Section 1.C]). It is easy to prove that $\Gamma(x, x | \lambda) = \sum_{i \in \mathbb{N}} \Phi(x, x | \lambda)^i$ for all $\lambda > 0$, hence

$$\Gamma(x, x | \lambda) = \frac{1}{1 - \Phi(x, x | \lambda)}, \qquad \forall \lambda \in \mathbb{C} : |\lambda| < M_s(x, x)^{-1},$$

and we have that $M_s(x, x)^{-1} = \max\{\lambda \in \mathbb{R} : \Phi(x, x | \lambda) \leq 1\}$ for all $x \in X$. In particular $\Phi(x, x | 1) \leq 1$ if and only if $M_s(x, x) \leq 1$. The interpretation of $\Gamma(x, y | 1)$ is the expected value of the total number of descendants at y of a common ancestor living at x. On the other hand, $\Phi(x, y | 1)$ is the expected number of descendants at y, of a common ancestor living at x, whose trails start from x and arrive at y for the first time.

The classical approach to branching processes (see for instance [20]) makes use of the one-dimensional generating function of the offspring distribution ρ: $\widetilde{G}(z) = \sum_{n \in \mathbb{N}} \rho(n) z^n$, whose minimal fixed point is the probability of extinction (see also Example 4.2). Inspired by this approach, we associate a generating function $G : [0, 1]^X \to [0, 1]^X$ to the family

$\{\mu_x\}_{x \in X}$ which can be considered as an infinite dimensional power series (see also [7, Section 3]). More precisely, for all $z \in [0,1]^X$ the function $G(z) \in [0,1]^X$ is defined as the following weighted sum of (finite) products

$$G(z|x) := \sum_{f \in S_X} \mu_x(f) \prod_{y \in X} z(y)^{f(y)}. \tag{2.9}$$

Note that G is continuous with respect to the *pointwise convergence topology* of $[0,1]^X$ and nondecreasing with respect to the usual partial order of $[0,1]^X$ (see [7, Sections 2 and 3] for further details). Moreover, G represents the 1-step reproductions; we denote by $G^{(n)}$ the generating function associated to the n-step reproductions, which is inductively defined as $G^{(n+1)}(z) = G^{(n)}(G(z))$.

The generating function G can be explicitly computed in some cases: for instance, for a BRW with independent diffusion (i.e. if equation (2.4) holds). Indeed in this case it is straightforward to show that $G(z|x) = F_x(Pz(x))$ where $F_x(y) = \sum_{n=0}^{\infty} \rho_x(n) y^n$ is the generating function of the number of children and $Pz(x) = \sum_{y \in X} p(x,y) z(y)$ is the transition operator of the corresponding random walk. In particular if $\rho_x(n) = \frac{1}{1+\bar{\rho}_x}(\frac{\bar{\rho}_x}{1+\bar{\rho}_x})^n$ (for instance if we are dealing with the discrete-time counterpart of a continuous-time BRW, see equation (2.6)), we have $G(z|x) = \frac{1}{1+\bar{\rho}_x(1-Pz(x))}$ (see [7, Section 3.1]), that is,

$$G(z) = \frac{\mathbf{1}}{\mathbf{1} + M(\mathbf{1} - z)} \tag{2.10}$$

where $\mathbf{1}(x) := 1$ for all $x \in X$, the ratio is to be intended as coordinatewise and Mv was defined in Section 2.1; in this case m_{xy} is given by equation (2.5), thus $M = \lambda K$.

The following proposition is a sort of *maximum principle* for the function $(z-\bar{q})/(\mathbf{1}-\bar{q})$ where $G(z) \geq z$ (see Section 3.2).

Proposition 2.6. *Let $z \in [0,1]^X$, $z \geq \bar{q}$ be a solution of the inequality $G(z) \geq z$. If $\bar{q} < \mathbf{1}$ and we define $\widehat{z} := (z - \bar{q})/(\mathbf{1} - \bar{q})$ (by definition $\widehat{z}(x) := 1$ for all x such that $\bar{q}(x) = 1$) then for all $x \in X$ such that the set $\mathcal{N}_x = \{y : (x,y) \in E_\mu\}$ is not empty, either $\widehat{z}(y) = \widehat{z}(x)$ for all $y \in \mathcal{N}_x$ or there exists $y \in \mathcal{N}_x$ such that $\widehat{z}(y) > \widehat{z}(x)$. In particular if $\widehat{z}(x) = 1$ then for all $y \in \mathcal{N}_x$ we have $\widehat{z}(y) = 1$. The same results hold if we take the set $\{y \in X : x \to y\}$ instead of $\mathcal{N}_x$.*

The proof, which makes use of some arguments of Section 3.2, can be found in Section 6. We observe that in a finite, final irreducible class (for instance if the BRW is irreducible and the set X is finite) then $\widehat{z}$ is constant if $z \geq \bar{q}$ is a solution of $G(z) \geq z$. Since the probabilities of extinction $\{q(\cdot, A)\}_{A \subseteq X}$ (see Section 4.1 for the definition) are solutions of $G(z) = z$, Proposition 2.6 applies with $z(\cdot) = q(\cdot, A)$ for all $A \subseteq X$. In this case $\widehat{z}(x)$ can be interpreted as the probability of local extinction in A conditioned on global survival (starting from x). Note that if $\mu_x(\mathbf{0}) = 0$ for all $x \in X$ (see Section 3.2) then $\bar{q} = \mathbf{0}$ and $\widehat{z} = z$.

As an application, if we have an irreducible BRW on $\mathbb{N}$ where $m_{xy} > 0$ implies $|x-y| \leq 1$ we get the following behavior of the extinction probabilities: for all $A \subseteq X$, there exists $x_0 \in \mathbb{N} \cup \{+\infty\}$ such that $q(x, A) = q(0, A)$ for all $x \leq x_0$ and $q(x, A) < q(x+1, A)$ for all $x \geq x_0$.

3 Special processes

3.1 $\mathcal{F}$-BRWs

Some results can be achieved if the BRW has some regularity; to this aim we introduce the concept of $\mathcal{F}$-BRW (see also [42, Definition 4.2]).

Definition 3.1. *We say that a BRW (X, μ) is locally isomorphic to a BRW (Y, ν) if there exists a surjective map $g : X \to Y$ such that*

$$\nu_{g(x)}(\cdot) = \mu_x\left(\pi_g^{-1}(\cdot)\right) \tag{3.11}$$

where $\pi_g : S_X \to S_Y$ is defined as $\pi_g(f)(y) = \sum_{z \in g^{-1}(y)} f(z)$ for all $f \in S_X$, $y \in Y$. We say that (X, μ) is a $\mathcal{F}$-BRW if it is locally isomorphic to some BRW (Y, ν) on a finite set Y.

Clearly, if (X, μ) is locally isomorphic to (Y, ν) then

$$G_X(z \circ g|x) = G_Y(z|g(x)) \tag{3.12}$$

for all $z \in [0, 1]^Y$ and $x \in X$. Indeed π_g is surjective and

$$
\begin{aligned}
G_X(z \circ g|x) &= \sum_{f \in S_X} \mu_x(f) \prod_{w \in X} z(g(w))^{f(w)} = \sum_{h \in S_Y} \sum_{f \in \pi_g^{-1}(h)} \mu_x(f) \prod_{w \in X} z(g(w))^{f(w)} \\
&= \sum_{h \in S_Y} \sum_{f \in \pi_g^{-1}(h)} \mu_x(f) \prod_{v \in Y} \prod_{w \in g^{-1}(v)} z(v)^{f(w)} \\
&= \sum_{h \in S_Y} \mu_x(\pi_g^{-1}(h)) \prod_{v \in Y} z(v)^{h(v)} \\
&= \sum_{h \in S_Y} \nu_{g(x)}(h) \prod_{v \in y} z(v)^{h(v)} = G_Y(z|g(x)).
\end{aligned}
$$

By induction on n

$$G_X^{(n+1)}(z \circ g) = G_X(G_X^{(n)}(z \circ g)) = G_X(G_Y^{(n)}(z) \circ g) = G_Y(G_Y^{(n)}(z)) \circ g = G_Y^{(n+1)}(z) \circ g,$$

whence

$$G_X^{(n)}(z \circ g) = G_Y^{(n)}(z) \circ g \tag{3.13}$$

for all $n \in \mathbb{N}$. We note that, since μ is uniquely determined by G, equation (3.12) holds if and only if (X, μ) is locally isomorphic to (Y, ν) and g is the map in Definition 3.1. To see the "only if" part, define $\widehat{\nu}$ by using equation (3.11) (substitute ν with $\widehat{\nu}$), then equation (3.12) holds with $\widehat{G}$ instead of G_Y; thus $\widehat{G} = G_Y$ and this implies that equation (3.11) holds for ν.

Using equation (3.13) and the fact that $\bar{q} = \lim_{n \to \infty} G^{(n)}(\mathbf{0})$ (see equation (4.15) with $A = X$), it is possible to prove that there is global survival for (X, μ) starting from x if and only if there is global survival for (Y, ν) starting from $g(x)$ (see [42, Theorem 4.3]).

In particular note that the total offspring generating functions $G(t\mathbf{1}|x)$ satisfy $G_X(t\mathbf{1}_X|x) = G_X(t\mathbf{1}_Y \circ g|x) = G_Y(t\mathbf{1}_Y|g(x))$, hence the offspring distribution of a particle at x behaving

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according to (X, μ) is the same of the offspring distribution of a particle at $g(x)$ behaving according to (Y, ν).

If (X, μ) is locally isomorphic to a BRW (Y, ν) where Y is a singleton (equivalently, if the law of the total number of children ρ_x does not depend on $x \in X$) then we say that the BRW is *locally isomorphic to a branching process* (this case is discussed in details in Section 4.6).

It is easy to prove that $\sum_{w \in X} m^X_{xw} v(x) = \frac{d}{dt} G_X(\mathbf{1} - (1-t)v|x)|_{t=1}$ for all $x \in X$ and $v \in [0,1]^X$, hence, using equation (3.13), we have $\sum_{w \in X} m^X_{xw} z(g(x)) = \sum_{y \in Y} m^Y_{g(x)y} z(y)$, for all $x \in X$ and $z \in [0,1]^Y$. This, in turn, implies that $\sum_{w \in X} m^{X,(n)}_{xw} z(g(w)) = \sum_{y \in Y} m^{Y,(n)}_{g(x)y} z(y)$ for all $n \in \mathbb{N}$. In particular, when $n = 1$ and $z = \mathbf{1}$ we have $\bar{\rho}^X_x = \bar{\rho}^Y_{g(x)}$.

In continuous time (see [7]) we say that (X, K) is *locally isomorphic* to $(Y, \widetilde{K})$ if and only if there exists a surjective map $g : X \to Y$ such that $\sum_{z \in g^{-1}(y)} k_{xz} = \widetilde{k}_{g(x)y}$ for all $x \in X$ and $y \in Y$. We observe that (X, K) is locally isomorphic to $(Y, \widetilde{K})$ if and only if the discrete-time counterparts satisfy Definition 3.1; this can easily be checked by proving that the equation (3.12) holds when G_X and G_Y satisfy equation (2.10). Note that a continuous-time BRW is site-breeding if and only if it is locally isomorphic to a branching process. On the other hand a continuous-time, edge-breeding BRW is locally isomorphic to another edge-breeding BRW if and only if the underlying multigraphs satisfy [6, Definition 3.1].

Definition 3.2. *Let $\gamma : X \to X$ be an injective map. We say that $\mu = \{\mu_x\}_{x \in X}$ is γ-invariant if for all $x, y \in X$ and $f \in S_X$ we have $\mu_x(f) = \mu_{\gamma(x)}(f \circ \gamma^{-1})$.*

Moreover (X, μ) is quasi transitive if and only if there exists a finite subset $X_0 \subseteq X$ such that for all $x \in X$ there exists a bijective map $\gamma : X \to X$ and $x_0 \in X_0$ satisfying $\gamma(x_0) = x$ and μ is γ-invariant.

The previous definition generalizes the usual one (which applies to graphs) in the following way: a discrete-time counterpart of an edge-breeding continuous-time BRW is quasi transitive if and only if the underlying graph is quasi transitive (that is, the action of the group of automorphisms has only finitely many orbits).

We note that every quasi-transitive BRW is an $\mathcal{F}$-BRW. Indeed, consider the equivalence relation $x \sim y$ if and only if there exists a bijective map $\gamma : X \to X$ such that $\gamma(x) = y$. Clearly if $Y := X/_\sim$ then $\#Y \leq \#X_0$. Let g be the usual projection from X onto $X/_\sim$ and $\nu_{g(x)}(\cdot) := \mu_x(\pi_g^{-1}(\cdot))$. We have to show that the last definition is well-posed. Note that if μ is γ invariant then $g = g \circ \gamma$ which implies $\pi_g(f) = \pi_{g \circ \gamma}(f) = \pi_g(f \circ \gamma^{-1})$; indeed

$$\pi_g(f)(y) = \sum_{z \in g^{-1}(y)} f(z) = \sum_{z \in (g \circ \gamma)^{-1}(y)} f(z) = \sum_{z \in g^{-1}(y)} f(\gamma^{-1}(z))$$

whence for all $h \in S_Y$

$$\mu_x(\pi_g^{-1}(h)) = \mu_{\gamma(x)}(\{f \circ \gamma^{-1} : \pi_g(f) = h\})$$
$$= \mu_y(\{f \circ \gamma^{-1} : \pi_g(f \circ \gamma^{-1}) = h\}) = \mu_y(\pi_g^{-1}(h)).$$

The class of $\mathcal{F}$-BRWs is strictly larger than the class of quasi-transitive BRWs. An example is given by the BRW described in Example 4.35. Another one is the following (a further example can be found in [6, Example 3.2]).
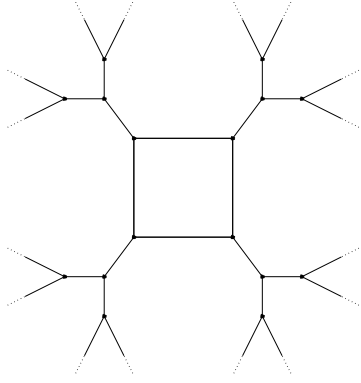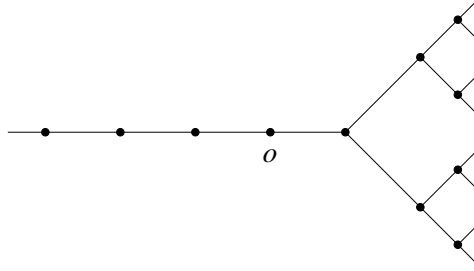
Figure 1: The regular graph of Example 3.3.

Figure 2: The graph X of Example 4.27.

Figure 1:

Example 3.3. *[6, Example 3.1] Take a square and attach to every vertex a branch of a homogeneous tree of degree 3, obtaining a regular graph (of degree 3) which is not quasi transitive (see Figure 1). If we attach now to each vertex a new edge with a new endpoint we obtain a non-oriented, nonamenable $\mathcal{F}$-graph $(X, E(X))$ (see [6, Definition 3.1]) which is neither regular nor quasi transitive. It is easily seen (see [6, Lemma 3.2]) to be locally isomorphic to a multigraph with adjacency matrix*

$$N = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding continuous-time, edge-breeding BRW is an $\mathcal{F}$-BRW which is defined on a (nonamenable) multigraph which is neither regular nor quasi transitive (thus, the process is not quasi transitive).

3.2   BRWs with and without death: a comparison

Some authors (see for instance [14, 18, 26, 33, 34] and some results of [31]) have extensively studied the special case where $\mu_x(\mathbf{0}) = 0$ for all $x \in X$ or, that is the same, $\rho_x(0) = 0$ for all $x \in X$. We call this kind of process a *BRW with no death*; to be honest, in the usual interpretation each particle still dies but it has at least one descendant almost surely. On the other hand one can think that particles never die (in this case, $\rho_x(n)$ must be interpreted as the probability of having $n - 1$ children). We stick with the first interpretation.

Being the global survival trivial in this case, it is interesting to explore the local behavior of the BRW. This situation is the closest one to the random walk theory. In particular, the process can become extinct locally at x with probability one (some authors call the process *transient* in this case) or survive locally with positive probability. While in the general situation the colony cannot always survive with probability one (if $\rho_x(0) > 0$ the process starting from x might die at the first step), a BRW with no death can survive locally at

x either with probability one (strong local survival or *recurrent BRW*) or with a strictly positive probability different from one (*weakly recurrent BRW*).

In this section we want to discuss how we may interpret results on BRWs with no death in the general case. The first idea, which was introduced by Harris in the case of a branching process (see for instance [19] or [1, Chapter I.12]), is to condition on global survival. It is clear that a generic BRW, such that $\bar{q}(x) < 1$ for all $x \in X$, conditioned on global survival is not a BRW with no death (it is not even a BRW). Indeed, for the conditioned process, the behaviors of different particles of the same generation are not independent: the probability of dying without breeding is strictly positive but the probability that all particles in the same generation die out without breeding is 0. Nevertheless it is possible to associate to a generic irreducible BRW, with a fixed starting configuration, a BRW with no death. Given a generic irreducible BRW $\{\eta_n\}_{n\in\mathbb{N}}$ consider the event $\Omega_\infty = \{\sum_{x\in X} \eta_n(x) > 0, \forall n \in \mathbb{N}\}$ and define the process $\{\widehat{\eta}_n\}_{n\in\mathbb{N}}$ as follows: $\widehat{\eta}_n(x,\omega)$ equals the number of particles in $\eta_n(x,\omega)$ with an infinite line of descent when $\omega \in \Omega_\infty$ and it equals 0 when $\omega \notin \Omega_\infty$. It can be shown that this process, restricted to Ω_∞ is a BRW (that we call again $\{\widehat{\eta}_n\}_{n\in\mathbb{N}}$) and its generating function is

$$\widehat{G}(z|x) = \frac{G(v(z)|x) - \bar{q}(x)}{1 - \bar{q}(x)} \tag{3.14}$$

where $\bar{q}(x)$ is the probability of global extinction starting from $x \in X$ (see Sections 2.1 and 4.1), G is the generating function of the original BRW and $v : [0,1]^X \to [0,1]^X$ is defined as $v(z|x) := \bar{q}(x) + z(x)(1 - \bar{q}(x))$. In a more compact way equation (3.14) can be written as $\widehat{G} = T_{\bar{q}}^{-1} \circ G \circ T_{\bar{q}}$ where $T_w : [0,1]^X \to \{z \in [0,1]^X : w \le z\}$ is defined as $T_w z(x) := z(x)(1 - w(x)) + w(x)$; note that T_w is nondecreasing and, if $w(x) < 1$ for all $x \in X$, bijective. More explicitly

$$\widehat{G}(z|x) = \sum_{f\in S_X} \widehat{\mu}_x(f) \prod_{y\in X} z(y)^{f(y)},$$

$$\text{where } \widehat{\mu}_x(f) = \begin{cases} \dfrac{\sum_{g\in S_X : g\ge f} \mu_x(g) \prod_{y\in X} \binom{g(y)}{f(y)} \bar{q}(y)^{g(y)-f(y)}(1-\bar{q}(y))^{f(y)}}{1 - \bar{q}(x)} & \text{if } f \ne \mathbf{0} \\ 0 & \text{if } f = \mathbf{0}. \end{cases}$$

Indeed $\{\widehat{\eta}_n\}_{n\in\mathbb{N}}$ is a Markov process and if we consider the offsprings of a particle living at x at some time n, then its reproduction law conditioned on the global survival of the process and on having an infinite line of descent is equal to the reproduction law of an initial particle at y conditioned on $\mathcal{A}_x$, that is

$$\mathbb{P}(\widehat{\eta}_1 = f | \mathcal{A}_x) = \begin{cases} \dfrac{\mathbb{P}(\widehat{\eta}_1 = f)}{1 - \bar{q}(x)} & f > \mathbf{0} \\ 0 & f = \mathbf{0} \end{cases}$$

where $\mathcal{A}_x$ is the event that there is global survival (i.e. the first particle has an infinite line of descent) starting from one particle at $x \in X$ (clearly $\mathbb{P}(\mathcal{A}_x) = 1 - \bar{q}(x)$). Hence, if $f > \mathbf{0}$,

$$\mathbb{P}(\widehat{\eta}_1 = f | \mathcal{A}_x) = \frac{1}{1 - \bar{q}(x)} \sum_{g\in S_X : g\ge f} \mu_x(g) \prod_{y\in Y} \binom{g(y)}{f(y)} (1 - \bar{q}(y))^{f(y)} \bar{q}(y)^{g(y)-f(y)}.$$

This implies that

$$\widehat{G}(z|x) = \sum_{f \in S_X} \mathbb{P}(\widehat{\eta} = f | \mathcal{A}_x) \prod_{y \in Y} z(y)^{f(y)}$$

$$= \sum_{g \in S_X : g > \mathbf{0}} \frac{\mu_x(g)}{1 - \bar{q}(x)} \sum_{f \in S_X : \mathbf{0} < f \leq g} \prod_{y \in Y} \binom{g(y)}{f(y)} (z(y)(1 - \bar{q}(y)))^{f(y)} \bar{q}(y)^{g(y) - f(y)}$$

$$= \sum_{g \in S_X : g > \mathbf{0}} \frac{\mu_x(g)}{1 - \bar{q}(x)} \left[\prod_{y \in Y} (T_{\bar{q}} z(y))^{g(y)} - \prod_{y \in Y} \bar{q}(y)^{g(y)} \right]$$

$$= \frac{1}{1 - \bar{q}(x)} \left[G(T_{\bar{q}} z|x) - G(\bar{q}|x) \right] = \frac{1}{1 - \bar{q}(x)} \left[G(T_{\bar{q}} z|x) - \bar{q}(x) \right].$$

It can be shown, following [1], that many results about survival are true for $\{\eta_n\}_{n \in \mathbb{N}}$ if and only if they are true for $\{\widehat{\eta}_n\}_{n \in \mathbb{N}}$. Proving this equivalence in details goes beyond the purpose of this paper. Nevertheless we observe that, if $\bar{q} < \mathbf{1}$ then $T_{\bar{q}}$ is a bijective map from the set of fixed points of $\widehat{G}$ to the set of fixed points of G. Moreover, since $\{\widehat{\eta}_n\}_{n \in \mathbb{N}}$ is obtained by $\{\eta_n\}_{n \in \mathbb{N}}$ by removing all the particles with finite progeny, which are clearly irrelevant in view of the survival due to the fact that $\bar{q}(x) < 1$ for all $x \in X$, we have immediately that the probability of local survival of $\{\widehat{\eta}_n\}_{n \in \mathbb{N}}$ in A (for all $A \subseteq X$), starting from x is equal to the same probability for $\{\eta_n\}_{n \in \mathbb{N}}$, that is, $q(x, A)$. In particular the probability of local survival at A starting from x conditioned on $\mathcal{A}_x$ is $1 - (T_{\bar{q}}^{-1} q(\cdot, A))(x) = (1 - q(x, A))/(1 - \bar{q}(x))$.

We call the process $\{\widehat{\eta}_n\}_{n \in \mathbb{N}}$ conditioned on $\mathcal{A}_x$ the *no-death BRW associated to* $\{\eta_n\}_{n \in \mathbb{N}}$ starting from $x \in X$.

4 Survival

4.1 Probabilities of extinction

Define $q_n(x, A)$ as the probability of extinction before generation $n + 1$ in A starting with one particle at x, namely $q_n(x, A) = \mathbb{P}(\eta_k(x) = 0, \forall k \geq n + 1, \forall x \in A)$. It is clear that $\{q_n(x, A)\}_{n \in \mathbb{N}}$ is a nondecreasing sequence satisfying

$$\begin{cases} q_n(\cdot, A) = G(q_{n-1}(\cdot, A)), & \forall n \geq 1 \\ q_0(x, A) = 0, & \forall x \in A, \end{cases} \tag{4.15}$$

hence there is a limit $q(x, A) = \lim_{n \to \infty} q_n(x, A) \in [0, 1]^X$ which is the probability of local extinction in A starting with one particle at x. Note that equation (4.15) defines completely the sequence $\{q_n(\cdot, A)\}_{n \in \mathbb{N}}$ only when $A = X$ (otherwise one needs the values $q_0(x, A)$ for $x \notin A$). Since G is continuous we have that $q(\cdot, A) = G(q(\cdot, A))$, hence these probabilities are fixed points of G (and Proposition 2.6 applies). Note that $q(\cdot, \emptyset) = \mathbf{1}$, $q(\cdot, X) = \bar{q}(\cdot)$ and $q(\cdot, \{y\}) = q(\cdot, y)$ (see Definition 2.2). It can be shown (see [7, Corollary 2.2]) that $\bar{q}$ is the smallest fixed point of $G(z)$ in $[0, 1]^X$, since it is $\bar{q} = \lim_{n \to \infty} G^{(n)}(\mathbf{0})$. Using the same arguments, one can prove that $\bar{q}$ is the smallest fixed point of $G^{(m)}$ for all $m \in \mathbb{N}$.

Note that $A \subseteq B$ implies $q(\cdot, A) \geq q(\cdot, B)$. In particular, $q(\cdot, y) \geq \bar{q}$ for all $y \in X$. Since for all finite $A \subseteq X$ we have $q(x, A) \geq 1 - \sum_{y \in A}(1 - q(x, y))$ then, for any given finite $A \subseteq X$, $q(x, A) = 1$ if and only if $q(x, y) = 1$ for all $y \in A$.

Moreover, given a BRW (X, μ) and $Y \subseteq X$, consider (Y, ν) obtained by killing all particles outside Y; in this case $q^X(x, A) \leq q^Y(x, A)$ for all $x \in Y$, $A \subseteq Y$.

If $x \to x'$ and $A \subseteq X$ then $q(x', A) < 1$ implies $q(x, A) < 1$; as a consequence, if $x \rightleftharpoons x'$ and $y \rightleftharpoons y'$ then $q(x, A) < 1$ if and only if $q(x', A) < 1$ and $q(x, y) = q(x, y')$. In the irreducible case $q(x, A) < 1$ for some $x \in X$ if and only if $q(w, A) < 1$ for all $w \in X$; in particular $\bar{q}(x) < 1$ for some $x \in X$ if and only if $\bar{q}(w) < 1$ for all $w \in X$. Moreover $q(x, A) < 1$ for some $x \in X$ and a finite $A \subseteq X$ if and only if $q(w, B) < 1$ for all $w \in X$ and all finite $B \subseteq X$. Indeed, in the irreducible case, one can prove that $q(x, A) = q(x, x)$ for all $x \in X$ and every finite $A \subseteq X$: since surviving in a finite subset A is equivalent to surviving in at least one of its points, then it is enough to prove it in the case $A := \{y\}$ for $y \in X$; in this case the conclusion follows from a Borel-Cantelli argument.

In the irreducible case, if $\rho_x(0) > 0$ for all $x \in X$, we have that $\bar{q}(x) = q(x, A)$ for some $x \in X$ and a finite subset $A \subseteq X$ if and only if $\bar{q}(y) = q(y, B)$ for all $y \in X$ and all finite subsets $B \subseteq X$. Indeed, if $\bar{q}(x) = 1$ then $q(y, B) = 1$ for all $y \in X$ and $B \subseteq X$ and there is nothing to prove. Suppose that $\bar{q}(x) = q(x, A) < 1$ and $\bar{q}(y) < q(y, B)$ for some $x, y \in X$ and $A, B \subseteq X$ finite. By irreducibility $q(x, A) = q(x, x) = q(x, B)$ hence we can assume that $A = B$. We know that there is a positive probability that the process, starting from x has at least one descendant at y. There is also a positive probability that all the particles (except one at y) die out and the progeny of the surviving particle survives globally but not locally at A. Thus, there is a positive probability, starting from x, of surviving globally but not locally at A and this is a contradiction. Observe that if we drop the assumption $\rho_x(0) > 0$ for all $x \in X$, we might actually have $\bar{q}(x) = q(x, A) < 1$ and $\bar{q}(y) < q(y, A)$ for some $x, y \in X$ and a finite $A \subseteq X$ (see Example 4.32).

Remark 4.1. *We observe that the following assertions are equivalent for every nonempty subset $A \subseteq X$.*

(1) $q(x, A) = \bar{q}(x)$, for all $x \in X$;

(2) $q_0(x, A) \leq \bar{q}(x)$, for all $x \in X$;

(3) the probability of visiting A at least once starting from x is larger than the probability of global survival starting from x, for all $x \in X$:

(4) for all $x \in X$, either $\bar{q}(x) = 1$ or the probability of visiting A at least once starting from x conditioned on global survival starting from x is 1 (strong local survival in A starting from x);

(5) for all $x \in X$, either $\bar{q}(x) = 1$ or the probability of local survival in A starting from x conditioned on global survival starting from x is 1.

Indeed, since $\{q_n(\cdot, A)\}_{n \in \mathbb{N}}$ is non decreasing, $q_n(\cdot, A) = G(q_{n-1}(\cdot, A))$ and $\bar{q}$ is the smallest fixed point of G, we have immediately that

$$q(\cdot, A) = \bar{q}(\cdot) \Longleftrightarrow q_0(\cdot, A) \leq \bar{q}(\cdot), \tag{4.16}$$

that is, (1)$\Longleftrightarrow$(2). Moreover the event "local survival in A starting from x" implies both "global survival starting from x" and "visiting A at least once starting from x", hence $q(x, A) = \bar{q}(x)$ if and only if the probability of visiting A infinitely many times starting from x conditioned on global survival is 1 and (1)$\Longleftrightarrow$(5)$\Longrightarrow$(4). Trivially (2)$\Longleftrightarrow$(3) and (4)$\Longrightarrow$(3). This proves the equivalence.

Hence if there exists $x \in X$ such that $q(x, A) > \bar{q}(x)$ (that is, there is a positive probability of global survival and nonlocal survival in A starting from x) then there exists $y \in X$ such that $q_0(y, A) > \bar{q}(y)$ (that is, there is a positive probability that the colony survives globally starting from y without ever visiting A). Of course, $q_0(x, A) > \bar{q}(x)$ implies $q(x, A) > \bar{q}(x)$ but the converse is not true. In particular for a BRW with no death there is strong local survival in A starting from x for all $x \in X$ if and only if the probability of visiting A is 1 starting from every vertex. This is the BRW counterpart of an analogous result in random walk theory; a vertex x is transient if and only if there exists a vertex y such that with positive probability the walker never visits x starting from y.

We note that, a priori, there is not an order relation between the events "visiting A at least once starting from x" and "global survival starting from x". Nevertheless if, for all $x \in X$, the probability of "visiting A at least once starting from x" is larger or equal to the probability of "global survival starting from x" then, using equation (4.16), we have that the probability of "global survival starting from x never visiting A" is 0.

An application of the previous remark to the construction of a BRW which survives globally and locally, but not strong locally, is given in Example 4.31.

If G has only one fixed point $z < \mathbf{1}$ then $q(\cdot, y) = \mathbf{1}$ or $q(\cdot, y) = \bar{q}(\cdot)$. More precisely, if one can prove that $q(\cdot, y) < \mathbf{1}$ then $q(\cdot, y) = q(\cdot, A) = \bar{q}(\cdot)$ for all $A \ni y$. In this case, global survival starting from x (i.e. $\bar{q}(x) < 1$) is equivalent to local survival at y starting from x and it implies strong local survival at y starting from x. If for some $y \in X$ we have $q(\cdot, y) = \bar{q}$ then the global survival starting from x implies the strong local survival at y starting from x.

If equation (2.4) holds and $\rho(n) = \frac{1}{1+\bar{\rho}_x}(\frac{\bar{\rho}_x}{1+\bar{\rho}_x})^n$, we have that the survival probability in A, $v_A := \mathbf{1} - q(\cdot, A)$, satisfies the equality $M v_A = v_A/(\mathbf{1} - v_A)$ (see equation (2.10)). In particular in the continuous-time case we have $\lambda K v_A = v_A/(\mathbf{1} - v_A)$.

Example 4.2. *In the case of a Galton–Watson branching process, the generating function is $G(z) = \sum_{n \in \mathbb{N}} \mu(n) z^n$ and its smallest fixed point $\bar{q} \in [0, 1]$ satisfies $\bar{q} < 1$ if and only if $1 < \frac{d}{dz} G(z)|_{z=1} = \sum_{n \in \mathbb{N}} n\mu(n) = m_{xx}$ (cfr. Assumption 2.3 and Remark 2.4) .*

If we are dealing with a continuous-time branching process with reproduction rate λk_{xx} then $G(z) = 1/(1 + \lambda k_{xx}(1 - z))$ and $\bar{q} = \min(1, 1/\lambda k_{xx})$. We see that, according to the general case, $\bar{q} < 1$ if and only if $\lambda k_{xx} > 1$.

4.2 Local survival

The fact that there is local survival or not, depends only on the first-moment matrix M. Indeed we have the following characterization which contains [42, Theorem 4.1] (some hints about the proof can be found in Section 6). We note that the following result still holds without the hypothesis $\sup_{x \in X} \sum_{y \in X} m_{xy} < +\infty$.

Theorem 4.3. *Let (X, μ) be a BRW.*

(1) *There is local survival starting from x if and only if $M_s(x, x) > 1$.*

(2) *If $\sup_{w \in X : x \to w \to y} M_s(w, w) > 1$ then there is local survival at y starting from x. Moreover if the cardinality of $w \in X : x \to w \to y$ is finite (for instance if X is finite) the converse is true.*

We recall that a useful characterization $M_s(x, x) = (\max\{\lambda \in \mathbb{R} : \Phi(x, x|\lambda) \le 1\})^{-1}$ was given in Section 2.5. Clearly, since $t \mapsto \Phi(x, x|t)$ is left continuous and strictly increasing, $M_s(x, x) > 1$ if and only if $\Phi(x, x|1) > 1$, which is another condition equivalent to local survival at x. Note that $M_s(x, x)$ depends only on the values $\{m_{wy}\}_{w, y \rightleftharpoons x}$. Thus the BRW survives locally at x if and only if it does so when restricted to the irreducibility class of x. If the BRW is irreducible then $M_s(x, x) = M_s(w, y)$ for all $x, w, y \in X$; hence there is local survival at y starting from x if and only if $M_s(w, w) > 1$ for some $w \in X$ (equivalently, for all $w \in X$).

To compare with random walk theory, the reader might recall the definition of spectral radius of a random walk on X with transition matrix P as $1/\limsup_{n \to \infty} \sqrt[n]{p^{(n)}(x, x)}$ (see the discussion after Proposition 4.33 and [40, Section 2.C]). If $\limsup_{n \to \infty} \sqrt[n]{p^{(n)}(x, x)} < 1$ then the random walk is transient, i.e. it returns to x a finite number of times almost surely (where $p^{(n)}(x, x)$ are the n-step return probabilities); on the other hand if $\limsup_{n \to \infty} \sqrt[n]{p^{(n)}(x, x)} = 1$ then the process may be either recurrent or transient. We observe that, while for a random walk the probability of returning to a site infinitely many times obeys a 0-1 law, a BRW can survive locally with any probability in $[0, 1]$. Of course, survival with probability one is possible only in the no-death case.

We show that the sufficient condition $\sup_{w \in X : x \to w \to y} M_s(w, w) > 1$ stated in Theorem 4.3 is not necessary in the reducible case when the cardinality of X is infinite.

Example 4.4. *Let $X := \mathbb{N} \times \{0, 1\}$ (see Figure 1), fix $p > 1/2$ and consider the BRW with the following reproduction rules:*

(a) *every particle at $(i, 0)$ has 2 children at $(i + 1, 0)$ and 1 child at $(i, 1)$ with probability p and no children with probability $1 - p$ (for all $i \in \mathbb{N}$);*

(b) *every particle at $(i, 1)$ has 2 children at $(i - 1, 1)$ with probability p and no children with probability $1 - p$ (for all $i \ge 1$);*

(c) *every particle at $(0, 1)$ has 1 child at $(0, 1)$ with probability p and no children with probability $1 - p$.*

Clearly $\limsup_{n \to \infty} \sqrt[n]{m_{ww}^{(n)}} = 0$ for all $x \in X$. Nevertheless, there is local survival at $(i, 1)$ for all $i \in \mathbb{N}$ starting from $(j, 0)$ for any fixed $j \in \mathbb{N}$ (note that there is no local survival at any $(i, 0)$ for every starting point).

In the case of continuous-time BRWs with rates $\{\lambda k_{xy}\}_{x, y \in X}$ one is also interested in the characterization of the local critical parameter $\lambda_s(x)$. Moreover one may also wonder whether at $\lambda = \lambda_s(x)$ there is survival or extinction. We already observed that the behavior of the continuous-time BRW is equivalent to the behavior of its discrete-time counterpart (that is, the BRW with independent diffusion where $\{\mu_x\}_{x \in X}$ is given by equations (2.4) and (2.6)). If we apply Theorem 4.3 then we obtain the following corollary, which gives the strong critical value and states that at the local critical value there is local extinction a.s.
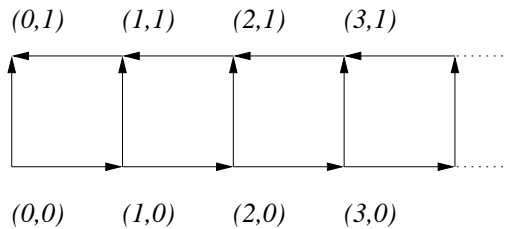
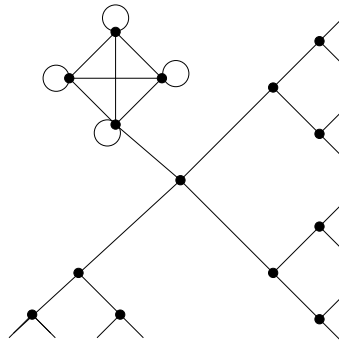Figure 1: The graph $\mathbb{N} \times \{0, 1\}$ of Example 4.4.

Figure 2: The graph of Example 4.28 (the circles are loops).

Figure 2:

Corollary 4.5. *[7, Theorems 4.1 and 4.7] Given a continuous-time BRW (X, K), $\lambda_s(x) = 1/K_s(x, x) = 1/\limsup_{n \to \infty} \sqrt[n]{k_{xx}^{(n)}}$. If $\lambda = \lambda_s(x)$ then there is local extinction at x.*

It is worth mentioning that, for a edge-breeding, irreducible BRW the critical value λ_s (not depending on x) was already identified in [36, Lemma 3.1], even though the critical behavior was not known.

Another result about local survival, in the context of irreducible BRWs with no death, is the following which relies on the existence of positive superharmonic functions; this one has a well-known counterpart in the random walk theory (see [40, Theorem 6.21]).

Theorem 4.6. *[33, Theorem 2.1] Let (X, μ) be an irreducible BRW such that $\rho_x(0) = 0$ for all $x \in X$. There is local extinction if and only if there exists a strictly positive function f on X such that $Mf \leq f$.*

The proof of the previous theorem is inspired by the proof of Theorem 4.21. Note that in the irreducible case with no death, one can obtain Theorem 4.3 directly from Theorem 4.6 Indeed, see [39, Section 7.A], $M_s = \min\{t > 0 : \exists f : X \to (0, +\infty), Mf \leq tf\}$ (remember that in the irreducible case $M_s(x, y)$ does not depend on $x, y \in X$).

4.3 Global survival

The classical approach to estimate the probability of extinction of a branching process uses the fact that this probability is the minimal fixed point in $[0, 1]$ of the generating function of the law of the number of children. Theorem 4.7 extends this approach to BRWs; indeed in this case the vector of global extinction probability $\bar{q} \in [0, 1]^X$ is the smallest fixed point of the infinite dimensional generating function G (see Section 4.1). Global survival is equivalent to the existence of a fixed point of G strictly smaller than $\mathbf{1}$. We remark here that given $v, w \in [0, 1]^X$ by $v < w$ we mean, as usual, that $v \leq w$ and $v \neq w$, that is $v(x) \leq w(x)$ for all $x \in X$ and $v(x_0) < w(x_0)$ for some $x_0 \in X$. Theorem 4.7 gives an equivalent condition for global survival and a necessary one.

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Theorem 4.7. *[42, Theorem 4.1] Let (X, μ) be a discrete-time BRW.*

(1) *There is global survival starting from x if and only if there exists $z \in [0,1]^X$, $z(x) < 1$, such that $G(z|y) \le z(y)$, for all $y \in X$ (equivalently, such that $G(z|y) = z(y)$, for all $y \in X$).*

(2) *If there is global survival starting from x, then there exists $v \in [0,1]^X$, $v(x) > 0$, such that $Mv \ge v$. Moreover, for all y, $Mv(y) = v(y)$ if and only if $G(\mathbf{1} - (1-t)v|y) = 1 - (1-t)v(y), \forall t \in [0,1]$.*

It is easy to show (see for instance [7, Section 3]) that the first condition implies that $\bar{q} \le z < \mathbf{1}$, that is, z is an upper bound for the probabilities of global extinction. As for the second condition, one has that if $\bar{q} < \mathbf{1}$ then taking $v = \mathbf{1} - \bar{q}$ (the probabilities of global survival) one obtains the inequality $Mv \ge v$. This is the analog for a BRW of the well-known result which states that a branching process survives if and only if the expected number of children is strictly larger than one. Nevertheless, for a BRW, even if we prove that $Mv(y) > v(y)$ for all $y \in X$ and some $v \ne \mathbf{0}$ this does not suffice for global survival: a counterexample is given by Example 4.16.

In particular cases we can characterize global survival in terms of M: the following corollary follows easily from Theorem 4.7(1) and equation (2.10) by taking $z = \mathbf{1} - v$.

Corollary 4.8. *Suppose that the generating function of (X, μ) satisfies equation (2.10) (for instance, if (X, μ) is a BRW with independent diffusion where $\rho_x(n) = \frac{1}{1+\bar{\rho}_x}(\frac{\bar{\rho}_x}{1+\bar{\rho}_x})^n$) then there is global survival starting from $x \in X$ if and only if there exists $v \in [0,1]^X$, $v(x) > 0$ such that*

$$Mv \ge v/(\mathbf{1} - v), \qquad (equivalently, \ Mv = v/(\mathbf{1} - v))$$

(where the ratio is taken coordinatewise).

Observe that the solution v in the previous corollary provides a lower bound for the probabilities of global survival $\mathbf{1} - \bar{q}$; moreover, among all the solutions of either equations, the largest one is $\mathbf{1} - \bar{q}$.

Now we notice that in order to characterize local survival we studied the behavior of the expected number of n-th generation offsprings coming back to x, that is $m_{xx}^{(n)}$, see Theorem 4.3. For the global survival problem, we are naturally lead to investigate the behavior of the expected number of n-th generation offsprings whose ancestor is a single particle at x, that is $\sum_y m_{xy}^{(n)}$. First of all, observe that it is easy to show, by using supermultiplicative arguments, that $M_w(x) \ge M_s(x, x)$. moreover, for an irreducible BRW, $M_w(x)$ does not depend on $x \in X$.

A complete description of global survival in terms of $M_w(x)$ is not possible in general (we have a necessary condition); indeed Example 4.16 shows that global survival does not depend on M alone. Nevertheless, a characterization of global survival by means of $M_w(x)$ holds for the class of $\mathcal{F}$-BRWs.

Theorem 4.9. *[42, Theorems 4.1 and 4.3] Let (X, μ) be a discrete-time BRW.*

(1) *If there is global survival starting from x, then $M_w(x) \ge 1$.*

(2) If (X, μ) is an $\mathcal{F}$-BRW then there is global survival for (X, μ) starting from x if and only if $M_w(x) > 1$.

We already observed that if the BRW is irreducible then $\bar{q} < \mathbf{1}$ implies $\bar{q}(x) < 1$ for all $x \in X$; Example 4.4 shows that in the reducible case it might happen that $\bar{q} < \mathbf{1}$ and $\bar{q}(x) = 1$ for some $x \in X$.

In the case of continuous-time BRWs, one is interested in identifying $\lambda_w(x)$ and the behavior of the process when $\lambda = \lambda_w(x)$. The following result is a corollary of Theorems 4.7 and 4.9. It characterizes $\lambda_w(x)$ in the case of $\mathcal{F}$-BRWs and gives a lower bound in the general case.

Corollary 4.10. *[7, Theorems 4.3 and 4.8, Proposition 4.5] Consider a continuous-time BRW (X, K).*

(1) $\lambda_w(x) \geq 1/K_w(x) = 1/\liminf_{n \in \mathbb{N}} \sqrt[n]{\sum_{y \in X} k_{xy}^{(n)}}$.

(2) If (X, μ) is an $\mathcal{F}$-BRW then $\lambda_w(x) = 1/K_w(x)$. Moreover, if $\lambda = \lambda_w(x)$ then there is global extinction starting from x.

(3) Suppose that for all $y \in X$ there exists $x \in X$ such that $x \to y$. If there is global survival starting from x, then there exists $v \in [0,1]^X$, $v(x) > 0$, such that $\lambda K v > v$.

Note the difference between Theorem 4.7(2) and Corollary 4.10(3): in the second case we have the strict inequality $Mv = \lambda K v > v$. As a consequence of the previous corollary we can compute the global critical value for two frequently used classes of continuous-time BRWs which are locally isomorphic to a branching process: for a site-breeding BRW $\lambda_w(x) = 1/k$ for all $x \in X$ (where $k(x) = k$ for all $x \in X$), while for an edge-breeding BRW on a regular graph of degree d we have that $\lambda_w(x) = 1/d$ for all $x \in X$.

Remark 4.11. *Consider a BRW (X, μ) where X is finite. Following [7, Remark 4.4] one can prove that $M_w(x) = \max_{w:x \to w} M_s(w, w)$; remember that global survival starting from x is equivalent to global (hence local) survival in some class (hence at some point w) since X is finite. For a continuous-time BRW, this means that $K_w(x) = \max_{w:x \to w} K_s(w, w)$ and hence $\lambda_w(x) = \min_{w:x \to w} \lambda_s(w)$. If the BRW is finite and irreducible then there is only one class of irreducibility and the previous results hold without $\max$ and $\min$.*

Still in the case of continuous-time BRWs, we have a characterization of $\lambda_w(x)$, which makes use of the existence of a solution of certain systems of inequalities.

Theorem 4.12. *[7, Theorem 4.2] Let (X, K) be a continuous-time BRW and let $x \in X$.*

(1) For any fixed $\lambda > 0$ there is global survival starting from $x \in X$ if and only if there exists a solution $v \in [0,1]^X$ of the inequality $\lambda K v \geq v/(\mathbf{1} - v)$ such that $v(x) > 0$.

(2) If $\lambda \leq \lambda_w(x)$ and $v \in [0,1]^X$ is such that $\lambda K v \geq v/(\mathbf{1} - v)$ then $\inf_{y:x \to y, v(y) > 0} v(y) = 0$.

(3)
$$\lambda_w(x) = \inf\{\lambda \in \mathbb{R} : \exists v \in l_+^\infty(X), v(x) > 0 \text{ such that } \lambda K v \geq v/(\mathbf{1} - v)\}.$$

(4) For all $n \in \mathbb{N}$, $n \geq 1$ we have

$$\lambda_w(x) = \inf\{\lambda \in \mathbb{R} : \exists v \in l_+^\infty(X), v(x) > 0 \text{ such that } \lambda^n K^n v \geq v\}.$$

We note that, by taking $n = 1$ in Theorem 4.12(4), we have that $\lambda_w(x) = \inf\{\underline{r}_K(v) : v \in l^\infty(X), v(x) = 1\}$ where $\underline{r}_K(v)$ is the *lower Collatz-Wielandt number of v* (see [15], [16] and [30]). In particular, according to Theorem 4.12(4), we have that for a continuous-time BRW

$$\lambda_w(x) = \inf\{\lambda > 0 : \exists v \in l_+^\infty(X), v(x) > 0 \text{ such that } \lambda K v \geq v\}$$
$$= \inf\{\lambda > 0 : \exists v \in l_+^\infty(X), v(x) > 0 \text{ such that } \lambda K v = v\}$$

while, according to Theorem 4.6, if the BRW is irreducible,

$$\lambda_s := \max\{\lambda > 0 : \exists v > 0 \text{ such that } \lambda K v \leq v\}.$$

Finally, we know that a continuous-time BRW dies out locally at x when $\lambda = \lambda_s(x)$ (see Theorem 4.3). Theorem 4.12(2) states that the vector of probabilities of survival v of a generic BRW at the critical point $\lambda_w(x)$ if it is not equal to $\mathbf{0}$ it satisfies $\inf_{y \in X} v(y) = 0$. This proves immediately that an irreducible $\mathcal{F}$-BRW dies out globally when $\lambda = \lambda_w$ (which is independent of $x \in X$). Theorem 4.12(2) is the most reasonable result we can expect in full generality; indeed, here is an example of an irreducible BRW which survives globally when $\lambda = \lambda_w$.

Example 4.13. *[7, Example 3] Let $X := \mathbb{N}$ and K be defined by $k_{0\,1} := 2$, $k_{n\,n+1} := (1 + 1/n)^2$, $k_{n\,n-1} := 1/3^n$ (for all $n \geq 1$) and 0 otherwise. Note that the corresponding continuous-time BRW is irreducible. In order to show that $\lambda_w = 1$ we look for solutions of the inequality $\lambda K v \geq v/(\mathbf{1} - v)$. The system becomes*

$$\begin{cases} 2\lambda v(1) \geq v(0)/(1 - v(0)) \\ \lambda(v(n+1)(1+1/n)^2 + v(n-1)/3^n) \geq v(n)/(1 - v(n)) & \text{for all } n \geq 1. \end{cases}$$

Clearly, for all $\lambda \geq 1$, $v(0) = 1/2$ and $v(n) := 1/(n+1)$ (for all $n \geq 1$) is a solution; this implies, according to Theorem 4.12(1), that there is global survival for $\lambda \geq 1$, thus $\lambda_w \leq 1$. If $\lambda < 1$ then there are no solutions in $l_+^\infty(X)$. Indeed one can prove by induction that any solution must satisfy $v(n+1)/v(n) \geq \frac{1}{\lambda}\left(\frac{n}{n+1}\right)^2\left(1 - \frac{1}{2^n}\right)$ for all $n \geq 2$. Thus $v(n+1)/v(n)$ is eventually larger than $1 + \varepsilon$ for some $\varepsilon > 0$, hence either $v = \mathbf{0}$ or $\lim_n v(n) = +\infty$. This implies that $\lambda_w = 1$ and there is global survival if $\lambda = \lambda_w$.

Another result, which applies to edge-breeding, irreducible, continuous-time BRWs and which deals with the relation between K_w and λ_w is the following.

Theorem 4.14. *[6, Theorem 3.3] Let (X, K) be an edge-breeding, irreducible, continuous-time BRW on a multigraph X; let us suppose that there exists $x_0 \in X$, $Y \subseteq X$ and $n_0 \in \mathbb{N}$ such that*

(1) for all $x \in X$ we have that $B^+(x, n_0) \cap Y \neq \emptyset$;

(2) for all $y \in Y$ there exists an injective map $\varphi_y : X \to X$, such that $\varphi_y(x_0) = y$ and $k_{\varphi_y(x)\varphi_y(z)} \geq k_{xz}$ for all $x, z \in X$,

where $B^+(x, n_0)$ is the set of all vertices which can be reached from x in at most n_0 steps. Then $\lambda_w = 1/K_w$.

The previous theorem is based on the following result (see [6, Theorem 3.1]) which gives an interesting sufficient condition for the equality $\lambda_w = 1/K_w$. If the multigraph satisfies this geometrical condition:

$$\forall \varepsilon > 0 \, \exists \bar{n} = \bar{n}(\varepsilon) : \sup_{n \leq \bar{n}} \sqrt[n]{T_x^n} \geq K_w - \varepsilon, \, \forall x \in X$$

then $\lambda_w = 1/K_w$. Note that, by definition of K_w, for all fixed $\varepsilon > 0$ and $x \in X$, there exists n_x such that $\sqrt[n_x]{T_x^{n_x}} \geq K_w - \varepsilon$. The above condition is a request of uniformity in x. An application of Theorem 4.14 is given in the following example.

Example 4.15. *[6, Example 3.3] Given a sequence of positive natural numbers $\{m_k\}_{k \geq 1}$ we construct a non-oriented, rooted tree $\mathbb{T}$ (with root o) such that if $x \in \mathbb{T}$ satisfies $d(o, x) = k$ then it has m_{k+1} neighbors y such that $d(o, y) = k + 1$ (where d is the natural distance of a non-oriented graph). We call this radial graph $T_{\{m_k\}}$-tree. If the sequence is periodic of period b, then Theorem 4.14 applies with $x_0 = o$, $n_0 = b$, $Y := \cup_{n \in \mathbb{N}} S(o, nb)$ (where $S(o, nb)$ is the sphere with center o and radius nb with respect to the distance d) and φ_y (where $y \in Y$) maps isomorphically the tree $\mathbb{T}$ onto the subtree branching from y; in this case the global critical parameter λ_w of the (irreducible) edge-breeding, continuous-time BRW on the tree equals $1/K_w$. Note that for every periodic sequence the BRW is not an $\mathcal{F}$-BRW, hence Corollary 4.10(2) does not apply.*

We already observed that local survival depends only on the first moment matrix M. It is clear that if we investigate the global survival in a class of BRWs where there is a one-to-one correspondence between first moment matrices and processes (as in the case of continuous-time BRWs), then also the global survival depends only on M. This is also true, by Corollary 4.8, for a BRW with independent diffusion satisfying equation (2.6). On the other hand, for a generic BRW, according to the following example, the global survival does not depend exclusively on M; in particular, even $M_w(x) > 1$ does not imply global survival starting from x.

Example 4.16. *[42, Example 4.4] Let $X = \mathbb{N}$ and consider the family of BRWs $(\mathbb{N}, \mu)$ with $\mu_i = p_i \delta_{n_i \mathbb{1}_{\{i+1\}} + \mathbb{1}_{\{i-1\}}} + (1 - p_i) \delta_{\mathbf{0}}$ (for all $i \geq 1$) and $\mu_0 = p_0 \delta_{n_0 \mathbb{1}_{\{1\}}} + (1 - p_0) \delta_{\mathbf{0}}$. Roughly speaking, each particle at $i \geq 1$ has n_i children at $i + 1$ and 1 at $i - 1$ with probability p_i and no children at all with probability $1 - p_i$; each particle at 0 has n_0 children at 1 with probability p_0 and no children at all otherwise.*

According to Theorem 4.7(1) global survival starting from 0 is equivalent to the existence of $z \in [0, 1]^{\mathbb{N}}$, $z(0) < 1$, such that $G(z|i) \leq z(i)$, for all i where

$$G(z|i) = \begin{cases} p_i z(i+1)^{n_i} z(i-1) + 1 - p_i & i \geq 1 \\ p_0 z(1)^{n_0} + 1 - p_0 & i = 0. \end{cases}$$

The trick is to choose the sequences $\{p_i\}_{i\in\mathbb{N}}$ and $\{n_i\}_{i\in\mathbb{N}}$ such that $p_i \to 0$ fast enough and $p_i n_i = 2$ for all $i \in \mathbb{N}$; this way, the unique solutions of $G(z) \le z$ is $z = \mathbf{1}$. All the details can be found in [42, Example 4.4].

On the other hand, if the BRW is given by $\mu_i = 1/2\,\delta_{4\mathbb{1}_{\{i+1\}}} + p_i\delta_{\mathbb{1}_{\{i-1\}}} + (1/2-p_i)\delta_{\mathbf{0}}$ (for all $i \ge 1$) and $\mu_0 = 1/2\delta_{4\mathbb{1}_{\{1\}}} + 1/2\delta_{\mathbf{0}}$ (where p_i is the same as before) then the first-moment matrix M is the same as before, but in this case the process survives globally (the total number of particles dominates a branching process with $\bar{\rho} = 2$). Moreover, $M_w(x_0) \ge 2$ since at each step the expected number of children $\bar{\rho}_x$ is at least 2 for all $x \in X$.

Another legitimate question arises from Theorem 4.9: is it true that $\sum_{y\in X} m_{xy} < 1$ for all $x \in X$ implies global extinction? According to the following example (see also [7, Example 1]), the answer is negative.

Example 4.17. *We start by considering a reducible BRW. Let $X = \mathbb{N}$, $\{p_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1]$ and suppose that a particle at n has one child at $n+1$ with probability p_n and no children with probability $1-p_n$. The generating function of this process is $\widetilde{G}(z|n) = 1 - p_n + p_n z(n+1)$. The probability of extinction of this process, starting with one particle at n, equals $z(n) = 1 - \prod_{i=n}^{\infty} p_i$ (z is the smallest solution of $\widetilde{G}(z) = z$); hence it survives wpp, if and only if $\sum_{i=1}^{\infty}(1-p_i) < +\infty$.*

This process is stochastically dominated by the irreducible BRW where each particle at $n \ge 1$ has one child at $n+1$ with probability p_n, one child at $n-1$ with probability $(1-p_n)/2$ (if $n = 0$ then it has one child at 0 with probability $(1-p_0)/2$) and no children at all with probability $(1-p_n)/2$. The generating function G can be explicitly computed

$$G(z|n) = \begin{cases} \frac{1-p_n}{2} + \frac{1-p_n}{2}z(n-1) + p_n z(n+1) & n \ge 1 \\ \frac{1-p_0}{2} + \frac{1-p_0}{2}z(0) + p_0 z(1) & n = 0. \end{cases}$$

By coupling this process with the previous one (see [42, Section 3.3]) or, simply, by applying Theorem 4.7(1) ($z(n) = 1 - \prod_{i=n}^{\infty} p_i$ is a solution of $G(z) \le z$) one can prove that $\sum_{i=1}^{\infty}(1-p_i) < +\infty$ implies global survival. Note that here $\sum_{j\in\mathbb{N}} m_{ij} = (1+p_i)/2 < 1$; clearly, $M_w(i) = 1$.

The following example shows how to apply the results of this section to the study of a couple of interesting branching processes. The first one appears for instance in the proof of [34, Proposition 3.6] (see also Corollary 4.38).

Example 4.18. *Let ρ be a measure on $\mathbb{N}$ with generating function $\phi(z) := \sum_{n\in\mathbb{N}} \rho(n)z^n$ and denote by $\bar{\rho} = \frac{\mathrm{d}}{\mathrm{d}z}\phi(z)|_{z=1}$ (suppose that $\rho(0) < 1$). Consider the following Galton-Watson branching processes. BP_1 is the process where each particle gives birth to n children with probability $\rho(n)$ and each newborn particle is killed (independently) with probability $1-p$. BP_2 is the process where each particle is killed (independently) before breeding with probability $1-p$, otherwise it gives birth to n children with probability $\rho(n)$. We suppose that $p \in (0,1)$ to avoid trivial situations.*

In order to study these two branching processes simultaneously, consider the BRW $\{\eta_n\}_{n\in\mathbb{N}}$ defined by $X := \{1,2\}$ and

$$\mu_1(a,b) := \begin{cases} 0 & \text{if } a \ne 0 \\ \rho(b) & \text{if } a = 0 \end{cases} \qquad \mu_2(a,b) := \begin{cases} 0 & \text{if } b \ne 0 \text{ or } a \ge 2 \\ 1-p & \text{if } a = b = 0 \\ p & \text{if } a = 1, b = 0, \end{cases}$$

where $\mu_j(a, b) \equiv \mu_j(f : f(1) = a, f(2) = b)$. Clearly

$$G(z_1, z_2) = (\phi(z_2), pz_1 + 1 - p), \qquad \forall z_1, z_2 \in [0, 1].$$

Note that $\{\eta_{2n}(1)\}_{n \in \mathbb{N}}$ and $\{\eta_{2n}(2)\}_{n \in \mathbb{N}}$ are realizations of BP_1 and BP_2 respectively. Indeed the 2-step generating function of $\{\eta_n\}_{n \in \mathbb{N}}$ is given by $G^{(2)}(z_1, z_2|1) = \phi(pz_1 + 1 - p)$ and $G^{(2)}(z_1, z_2|2) = p\phi(z_2) + 1 - p$; $z \mapsto \phi(pz + 1 - p)$ is the generating function of BP_1 and $z \mapsto p\phi(z) + 1 - p$ is the generating function of BP_2. Note that there is no distinction between local and global survival for the BRW since it is irreducible and X is finite. Moreover the survivals of the BRW, the BP_1 and the BP_2 are all equivalent and, in turn, they are equivalent, for instance, to $1 < \frac{\mathrm{d}}{\mathrm{d}z}\phi(pz + 1 - p)|_{z=1} = p\bar{\rho}$ (see Example 4.2). The vector of extinction probabilities satisfies $\bar{q} = G(\bar{q})$ that is

$$\begin{cases} \phi(\bar{q}(2)) = \bar{q}(1) \\ p\bar{q}(1) + 1 - p = \bar{q}(2). \end{cases}$$

Note that $\bar{q}(1)$ (resp. $\bar{q}(2)$) is also the extinction probability of BP_1 (resp. BP_2) since $\bar{q} = (\bar{q}(1), \bar{q}(2))$ is also the smallest fixed point of $G^{(2)}$ (see Section 4.1). Clearly $\bar{q}(1) < 1$ if and only if $\bar{q}(2) < 1$ and, in this case, $p(\bar{q}(2) - \bar{q}(1)) = (1 - p)(1 - \bar{q}(2)) > 0$. This implies that if there is survival then the probability of survival of BP_1 is strictly larger than the probability of survival of BP_2. The same result can be obtained by convexity and by the fact that $\phi(1) = 1$ which implies the following order relation between the generating functions of BP_1 and BP_2: $\phi(pz + 1 - p) < p\phi(z) + 1 - p$ whenever $z \neq 1$ and $p \in (0, 1)$, whence $\bar{q}(1) < \bar{q}(2)$. Finally if we denote by $\bar{\alpha}$ the probability of extinction of BP_1 when $p = 1$, that is, the smallest solution in $[0, 1]$ of $\phi(z) = z$, from the inequality $\phi(\bar{q}(2)) = \bar{q}(1) < \bar{q}(2)$ we have also $\phi(\bar{q}(1)) < \phi(\bar{q}(2)) = (q(1))$. Hence $\bar{\alpha} < \min(\bar{q}(1), \bar{q}(2))$.

4.4 Strong local survival

The main result of this section is the following proposition.

Proposition 4.19. *Let (X, μ) be an irreducible and quasi-transitive BRW. Then the existence of $x \in X$ such that there is local survival at x (i.e. $q(x, x) < 1$) implies that there is strong local survival at y starting from w for every $w, y \in X$ (i.e $q(w, y) = \bar{q}(w)$).*

In the particular case of a quasi-transitive, irreducible BRW with no death and with independent diffusion, Proposition 4.19 was proved in [34, Theorem 3.7]. The proof we give in Section 6 is of a different nature and it is a corollary of the following result which describes some properties of fixed points of G in the case of an $\mathcal{F}$-BRW.

Theorem 4.20. *Let (X, μ) be an $\mathcal{F}$-BRW. Then, there exists at most one fixed point z for G such that $\sup_{x \in X} z(x) < 1$, namely $z = \bar{q}$. Hence for all $x \in X$, $q(\cdot, x) = \bar{q}(\cdot)$ or $\sup_{w \in X} q(w, x) = 1$. In particular when (X, μ) is irreducible then it is either $q(x, x) = \bar{q}(x)$ for all $x \in X$ or $\sup_{x \in X} q(x, x) = 1$.*

The proof of this theorem, which can be found in Section 6, relies on Lemmas 6.1 and 6.2 which guarantee the strict convexity of the function G evaluated on a line in $[0, 1]^X$. The

existence of an example of an irreducible $\mathcal{F}$-BRW where $\bar{q}(x) < q(x,x) < 1$ for all $x \in X$ is given in Example 4.35.

In particular we can describe the case when X is finite (not necessarily irreducible). Clearly in this case $\bar{q}(w) = \min_{x \in X: w \to x} q(w,x)$, hence for all w such that $\bar{q}(w) < 1$ there exists x such that $q(w,x) = \bar{q}(w)$. Moreover, using Theorem 4.20, for all $x \in X$ we have that it is either $q(\cdot, x) = \bar{q}(\cdot)$ or there exists $w \in X$ such that $q(w,x) = 1$. If the BRW is irreducible (and X is finite) then it is $\bar{q}(w) = q(w,w)$ for all $w \in X$ or $q(w,x) = 1$ for all $w, x \in X$. Recall that, in the irreducible case, if $\rho_x(0) > 0$ for all $x \in X$, then strong local survival is a common property of all vertices as local and global survival are (see discussion in . Section 4.1). This is clearly false in the reducible case but it might be false as well in the irreducible case if we drop the assumption $\rho_x(0) > 0$ for all $x \in X$ as Example 4.32 shows.

If we are dealing with a continuous-time BRW, it might happen that if λ is small enough or large enough there is strong local survival but in a intermediate interval for λ there might be global and local survival with different probabilities. You can find this behavior in the BRW of Example 4.31 which is inspired by Remark 4.1. In particular this shows that, unlike local and global survival, strong local survival is not *monotonic*.

The following result is a natural generalization of [31, Theorem 3.1]. We give a sketch of the proof in Section 6.

Theorem 4.21. *Let (X, μ) be an irreducible, globally surviving BRW. Then there is no strong local survival if and only if there exists a finite, nonempty set $A \subseteq X$ and a function $v \in [0,1]^X$ such that $\bar{q} \leq v$ and*

$$\begin{cases} G(v|x) \geq v(x), & \forall x \in A^{\complement}, \\ (\tau_{\bar{q}} v)(x_0) > \max_{x \in A}(\tau_{\bar{q}} v)(x) & \text{for some } x_0 \in A^{\complement}, \end{cases} \tag{4.17}$$

where $\tau_{\bar{q}} v = (v - \bar{q})/(1 - \bar{q})$ is the inverse of the map $T_{\bar{q}}$ defined in Section 3.2 (and the ratio is taken coordinatewise).

It is worth mentioning at least one result for irreducible BRWs with no death. The following proposition gives a general criterion for the strong local survival of a BRW with no death and with independent diffusion.

Proposition 4.22. *[34, Lemma 3.4] Let (X, μ) be an irreducible BRW with independent diffusion where $\rho_x(0) = 0$ for all $x \in X$. If for some $c > 0$ the set $C := \{x : q(x,x) \leq 1 - c\}$ is visited infinitely often by the BRW, then there is strong local survival.*

As a corollary one can prove Proposition 4.38 concerning BRWs which are locally isomorphic to branching processes. Note that, since in the previous proposition we deal with BRWs with independent diffusion, it is possible to substitute the hypothesis that "the BRW visits infinitely often the set C" with "the set C is recurrent for the random walk (X, P)".

In view of the discussion of Section 3.2, a reasonable hypothesis for a generalization of the previous result to a BRW where $\rho_x(0) \neq 0$ for some $x \in X$, could be the fact that the BRW visits infinitely often the set $C := \{x : (1 - q(x,x))/(1 - \bar{q}(x)) \geq c\}$ for some $c > 0$.

4.5 Pure global survival

The idea of *pure global survival* has been introduced in continuous-time BRW theory (and, more generally, in interacting particle theory) to define the situation where $\lambda_s(x) > \lambda_w(x)$. In this case for every $\lambda \in (\lambda_w(x), \lambda_s(x)]$ there is a positive probability of global survival starting from x but the colony dies out locally at x almost surely. We know that when $\lambda = \lambda_s(x)$ there is local extinction at x due to Corollary 4.5, while if $\lambda = \lambda_w(x)$ both global extinction or global survival starting from x are possible (due to Example 4.13 and Corollary 4.10(2) respectively). Hence it is conceivable that when $\lambda = \lambda_w(x) = \lambda_s(x)$ it might happen that the process dies out locally but survives globally (we do not know of any example though). From now on we agree with many authors by defining the phase of *pure global survival* at x when $\lambda_w(x) < \lambda_s(x)$. A necessary condition for the existence of a pure global survival phase starting from x is clearly that $K_s(x, x) < K_w(x)$; indeed, according to Corollary 4.10(1), if $K_s(x, x) = K_w(x)$ then there is no pure global survival starting from $x \in X$ since $1/K_w(x) \le \lambda_w(x) \le \lambda_s(x) = 1/K_s(x, x)$. In some cases this condition is also sufficient (see for instance Corollary 4.10(2) and Theorem 4.14).

We note that for an irreducible BRW, given $A \subseteq X$, $q(x, A) < 1$ for some $x \in X$ if and only if $q(y, A) < 1$ for all $y \in X$; thus, the existence of pure global survival does not depend on the starting vertex. It has been observed that the existence of a pure global survival phase is in many cases associated with nonamenability. We start with a characterization of nonamenability for irreducible, non-oriented discrete-time BRWs.

Recall that for an irreducible BRW $M_s(x, y) = M_s$ and $M_w(x) = M_w$ for all $x, y \in X$. Analogously $\lambda_w(x) = \lambda_w$ and $\lambda_s(x) = \lambda_s$ for all $x \in X$ in the case of an irreducible continuous-time BRW.

Theorem 4.23. *Let (X, μ) be an irreducible, non-oriented $\mathcal{F}$-BRW. Then the following claims are equivalent:*

(1) the BRW is nonamenable;

(2) $M_s < M_w$.

A sketch of the proof of this result can be found in Section 6. Note that an irreducible discrete-time $\mathcal{F}$-BRW is in a pure global survival regime (i.e. there is global survival and local extinction) if and only if $M_s \le 1 < M_w$. Hence, the equivalent conditions in Theorem 4.23 are necessary but not sufficient for global survival.

On the other hand, in the irreducible continuous-time case, the existence of a phase of pure global survival is equivalent to $\lambda_w < \lambda_s$. According to Corollaries 4.5 and 4.10, for an $\mathcal{F}$-BRW, $\lambda_s = 1/K_s = \lambda/M_s$ and $\lambda_w = 1/K_w = \lambda/M_w$, thus we have the following corollary.

Corollary 4.24. *[6, Theorem 3.6] Let $(X, E(X))$ be an irreducible and non-oriented, continuous-time $\mathcal{F}$-BRW. Then $\lambda_w < \lambda_s$ if and only if $(X, E(X))$ is nonamenable.*

We observe that in the irreducible case in the pure global survival phase, the colony survives globally by clearing all finite subsets $A \subseteq X$. We might think of it as a sort of drifting towards some sort of boundary of the graph (we do not want to give more details on this).

Example 4.25. *The classical examples of amenable and nonamenable graphs are $\mathbb{Z}^d$ (for all $d \in \mathbb{N}$) and $\mathbb{T}_d$ (with $d \geq 3$) respectively. These properties can be verified by computing explicitly λ_s and λ_w for the edge-breeding continuous-time BRWs on these graphs.*

Let us consider the Euclidean lattice $X = \mathbb{Z}^d$ and the edge-breeding BRW on X. In this case Corollaries 4.5 and 4.10 apply and $\lambda_s = \lambda_w = 1/2d$ since $\bar{\rho} = 2\lambda d$. Indeed $\mathbb{Z}^d$ is an amenable graph.

If we consider an edge-breeding BRW on the homogeneous tree $\mathbb{T}_d$, where each edge has $d \geq 3$ neighbors, the situation is different. Easy computations show that $\lambda_w = 1/d$ (since the graph is regular) and $\lambda_s = 1/2\sqrt{d-1}$. Indeed, observe that, in this case, $\lambda_s = r/d$ where $r = \max\{t \in \mathbb{R} : F(x,x|t) \leq 1\}$ (as explained in Section 4.6 after Proposition 4.33). In this case it is easy to find the explicit expression of the function $F(x,x|t) = (d - \sqrt{d^2 - 4(d-1)t^2})/(2(d-1)t)$ (see, for instance, the proof of [39, Lemma 1.24]), whence $r = d/2\sqrt{d-1}$. We have verified by direct computation that $\lambda_w < \lambda_s$.

Pure global survival is a fragile property of a BRW. Finite modifications, such as for an edge-breeding BRW attaching a complete finite graph to a vertex or removing a set of vertices and/or edges, can create it or destroy it as we show in the following remark.

Remark 4.26. *[6, Remark 3.2] For simplicity, in this remark we consider only edge-breeding BRWs on multigraphs, that is, continuous-time BRWs on X where k_{xy} is the number of edges from x to Y for all $x, y \in X$. Note that if $(Y, E(Y))$ is a submultigraph of $(X, E(X))$ then $\lambda_w^X \leq \lambda_w^Y$, $\lambda_s^X \leq \lambda_s^Y$, $K_w^X \geq K_w^Y$, $K_s^X \geq K_s^Y$.*

Suppose that there exists a finite subset $S \subseteq X$ such that $X \setminus S$ is the disjoint union of a finite number of connected multigraphs $X_1, \ldots, X_n$ then, the existence of a pure global survival phase on X, implies the existence of a pure global survival phase on some X_i. Indeed for every $\lambda \in (\lambda_w(X), \lambda_s(X))$ the λ-BRW leaves eventually a.s. the subset S. Hence it survives (globally but not locally) at least on one connected component; this means that, although $\lambda_s(X_i) \geq \lambda_s(X)$, $\lambda_w(X_i) \geq \lambda_w(X)$ for all $i = 1, \ldots, n$, there exists i_0 such that $\lambda_w(X_{i_0}) = \lambda_w(X)$. The existence of a pure global survival phase on X_{i_0} follows from $\lambda_s(X_{i_0}) \geq \lambda_s(X) > \lambda_w(X) = \lambda_w(X_{i_0})$.

Moreover if there exists a subset S as above such that $\lambda_w(X_i) > \lambda_w(X)$ for all i, then there is no pure global survival for the BRW on X. Take for instance a graph $(X', E(X'))$ and $k \in \mathbb{N}$ such that $1/k < \lambda_w(X')$. Attach a complete graph of degree k to a vertex of X' by an edge (a complete graph is a finite set where every couple (x, y) is an edge), we obtain a new graph X such that $\lambda_s(X) = \lambda_w(X) \leq 1/k < \lambda_w(X')$; hence even if the BRW on X' has a pure global survival phase, the BRW on X has none. Nevertheless, using the same arguments as in Example 4.31, if the original BRW has a pure global survival phase, the new one has non-strong local survival.

There are examples of irreducible amenable BRWs with pure global survival (see Example 4.27) and of irreducible nonamenable BRWs with no pure global survival (see Example 4.28 which makes use of Remark 4.26). Recall that, for an edge-breeding BRW on a graph (or a multigraph), nonamenability is equivalent to the usual nonamenability of the graph.

Example 4.27. *Consider an irreducible, edge-breeding continuous-time BRW on the (non-oriented) graph X obtained by attaching to a copy of $\mathbb{N}$ one branch T of the homogeneous*

tree $\mathbb{T}_3$ *(see Figure 2). The BRW is amenable by the presence of* $\mathbb{N}$. *We claim that* $\lambda_s^X = \lambda_s^{\mathbb{T}_3}$ *and* $\lambda_w^X = \lambda_w^{\mathbb{T}_3}$. *Indeed* $T \subset X \subset \mathbb{T}_3$, *hence* $\lambda_s^T \geq \lambda_s^X \geq \lambda_s^{\mathbb{T}_3}$ *and* $\lambda_w^T \geq \lambda_w^X \geq \lambda_w^{\mathbb{T}_3}$. *But by approximation,* $\lambda_s^T = \lambda_s^{\mathbb{T}_3}$. *Indeed* $\lambda_s^T \geq \lambda_s^{\mathbb{T}_3}$ *and does not depend on the starting vertex; moreover* T *contains arbitrarily large balls isomorphic to balls of* $\mathbb{T}_3$, *hence by Theorem 5.2 their critical local parameters coincide. Note that by Remark 4.26 since* $\mathbb{T}$ *is a disjoint union of three copies of* T, *then* $\lambda_w^T = \lambda_w^{\mathbb{T}_3}$. *Using Example 4.25 we have* $\lambda_w^X = \lambda_w^T \leq \lambda_w^{\mathbb{T}_3}$

It is worth mentioning an alternative proof of $\lambda_w^T = \lambda_w^{\mathbb{T}_3}$ *which makes use of Theorem 4.12(4). Indeed* $\lambda_w^{\mathbb{T}_3} = 1/3$ *and it is clear that the function* $v \in l_+^\infty(\mathbb{T}_3)$ *defined by*

$$v(x) := \begin{cases} 1 & x = o \\ 5 - 2^{2-n} & d(x,o) = n \end{cases}$$

(where o *is the root of* T *and* $d(x,o)$ *denotes the natural distance on the graph between* x *and* o*) is a solution of* $\frac{1}{3}Kv = v$; *this implies* $\lambda_w^T \leq 1/3$.

Example 4.28. *Consider a nonamenable graph* X' *such that the corresponding edge-breeding continuous-time BRW has a pure global survival phase (take for instance* $X' := \mathbb{T}_3$ *the homogeneous tree with degree 3). Following Remark 4.26, attach to a vertex of* X' *a complete graph with degree* $k > 1/\lambda_w^{X'}$ *by an edge (see Figure 2). It is easy to show that the resulting graph* X *is still nonamenable, nevertheless, according to Remark 4.26, there is no pure global survival for the corresponding edge-breeding BRW. Roughly speaking, since* $\lambda_w^X \leq 1/k < 1/3$, *then for every* $\lambda \in (\lambda_w^X, 1/3)$ *the process cannot survive globally in* $X' := \mathbb{T}_3$ *hence it hits infinitely often with positive probability the complete graph, hence* $\lambda_s^X = \lambda_w^X$.

The following result gives a useful sufficient condition for the absence of pure global survival for a continuous-time BRW (X, K) which is based only on the geometry of the graph (X, E_K) generated by the BRW (where $E_K := \{(x,y) \in X^2 : k_{xy} > 0\}$). It is a slight generalization of [6, Proposition 2.1].

Theorem 4.29. *Let* (X, K) *be a continuous-time non-oriented BRW and let* $x_0 \in X$. *Suppose that there exists* $\kappa \in (0, 1]^X$ *and* $\{c_n\}_{n \in \mathbb{N}}$ *such that, for all* $n \in \mathbb{N}$

$$\begin{cases} \kappa(y)/\kappa(x_0) \leq c_n & \forall y \in B(x_0, n) \\ \kappa(x)k_{xy} = \kappa(y)k_{yx} & \forall x, y \in X, \end{cases}$$

where $B(x, n)$ *is the ball of center* x *and radius* n *w.r. to the natural distance of the graph* (X, E_K). *If* $\sqrt[n]{c_n} \to 1$ *and* $\sqrt[n]{|B(x_0, n)|} \to 1$ *as* $n \to 1$ *then* $K_s(x_0, x_0) = K_w(x_0)$ *and there is no pure global survival starting from* x_0.

The condition $\sqrt[n]{|B(x, n)|} \to 1$ as $n \to 1$ is usually called *subexponential growth*. The previous result applies, for instance to BRWs on $\mathbb{Z}^d$ or d-dimensional combs (see [5] for the definition). This result extends easily to discrete-time non-oriented BRWs using $M_w(x)$ and $M_s(x, x)$ instead of $K_w(x)$ and $K_s(x, x)$ respectively.

Remark 4.30. *We can apply the previous arguments to the family of rooted trees in Example 4.15.* X *is nonamenable if and only if* T_1 *is nonamenable, that is, if and only if there exists* i *such that* $m_i \geq 2$. *In this case, according to Theorem 4.23,* $\lambda_w^X < \lambda_s^X$, *hence by*

Remark 4.26 (considering $X \setminus \widetilde{Y}$) there exists i such that $\lambda_w^{T_i} < \lambda_s^{T_i}$. This means that for all $i = 1, \ldots, b$ we have $\lambda_w^{T_i} < \lambda_s^{T_i}$ and there is pure global survival on T_i. On the other hand, if $m_i \equiv 1$ for all $i = 1, \ldots, b$ the graph has subexponential growth, then there is no pure global survival.

Finally we construct an example of a continuous-time BRW, where if λ is small enough or large enough there is strong local survival but in a intermediate interval for λ there is global and local survival with different probabilities. This is obtained by modifying the edge-breeding BRW on a particular graph, namely the homogeneous tree $\mathbb{T}_d$. The crucial property that we need here is the existence of a pure global survival phase, thus the procedure applies to every BRW with such a phase.

Example 4.31. *Consider the edge-breeding continuous-time BRW on the homogeneous tree $\mathbb{T}_d$ with degree $d \geq 3$. We know from Example 4.25 that if $\lambda \leq 1/d$ the probabilities of survival are 0, if $\lambda > 1/2\sqrt{d-1}$ there is strong local survival (according to Proposition 4.19) and if $\lambda \in (1/d, 1/2\sqrt{d-1}]$ the probability of global survival is positive and independent from the starting point and the probability of local survival at any finite $A \subseteq X$ is 0.*

Fix $\lambda \in (1/d, 1/2\sqrt{d-1}]$. According to Remark 4.1, there exists $x \in X$ such that there is a positive probability of global survival starting from x without visiting A. In this case, any modification of the rates in the subset A provides a new BRW such that there is still a positive probability of global survival starting from x without ever visiting A (since, the original BRW and the new one coincide until the first hitting time on A). On the other hand, if there is $y \in A$ such that $x \to y$ and we add a loop in y and a rate $k_{yy} > 1/\lambda$ then $\bar{q}(x) < q(x,y) < 1$; the first inequality holds by the discussion above on local modifications and the second one holds since $\lambda k_{yy} > 1$ implies local survival at y (then irreducibility implies local survival at y starting from x). This means that, for this fixed value of λ, we obtained a locally and globally (but not strong-locally) surviving BRW at y starting from x.

Suppose now that $k_{yy} > d$; then (see Remark 4.26) we have a new BRW such that $\lambda'_w = \lambda'_s \leq 1/k_{yy}$. In this case, when $\lambda \leq \lambda'_w$ there is global extinction. When $\lambda > 1/2\sqrt{d-1}$ there is strong local survival for the original BRW (by Proposition 4.19) which implies strong local survival for the new one (the probability of hitting x conditioned on global survival is 1 for both processes and Remark 4.1 applies). If $\lambda \in (\lambda'_w, 1/d]$ there is local and global survival with the same probability since in order to survive globally, the process must visit x infinitely many times (it cannot survive globally in the branches of $\mathbb{T}_d$). If $\lambda \in (1/d, 1/2\sqrt{d-1}]$ then, according to the previous discussion, there is non-strong local survival for the new BRW.

We show that even in the irreducible case, if $\rho_x(0) = 0$ for some $x \in X$, we might have strong local survival starting from some vertices and not from others.

Example 4.32. *Let us consider a modification of the discrete-time counterpart of the edge-breeding BRW on $\mathbb{T}_d$ with degree $d \geq 3$ and $\lambda \in (1/d, 1/2\sqrt{d-1}]$. Let us fix a vertex y; in this modified version we add, with probability one, one child at y for every particle at y. In this case $\bar{q}(y) = q(y,A) = 0$ for all $A \subseteq X$. On the other hand according to the discussion in Example 4.31, there is a vertex y such that $\bar{q}(x) < q(x,y)$.*

4.6 An application: BRW locally isomorphic to a branching process

This class of BRWs is very easy to study and it gives an immediate connection with the theory of random walk. Recall that a BRW (X, μ) is *locally isomorphic to a branching process* if and only if the laws of the offspring number $\rho_x = \rho$ is independent of $x \in X$ (see Definition 3.1). In this case the branching process can be constructed as the BRW $(\{0\}, \nu)$, where $\nu_0 := \rho$. All the results of this section also apply to continuous-time BRWs where ρ_x is independent of $x \in X$, since their discrete-time counterpart is locally isomorphic to a branching process. In particular, for a continuous-time BRW (X, K), ρ_x is uniquely determined by $k(x) \equiv \sum_{y \in X} k_{xy}$ (and by λ), hence $(X; K)$ is locally isomorphic to a branching process if and only if $k(x)$ does not depend on $x \in X$.

The following result characterizes local and global survival for this class. Remember the definition of the diffusion matrix given in Section 2.1 as $p(x, y) := m_{xy}/\bar{\rho}_x$; note that, for a BRW with independent diffusion, the diffusion matrix coincides with the transition probability matrix P. In this proposition we denote by F the generating function of the hitting probabilities, (cfr. the function U of [40, Section 1.C]).

Proposition 4.33. *Let the BRW be locally isomorphic to a branching process and denote by ρ the common offspring law. Then*

(1) there is global survival if and only if $\bar{\rho} > 1$;

(2) there is local survival at x if and only if $\bar{\rho} > 1/\limsup_{n \to \infty} \sqrt[n]{p^{(n)}(x, x)}$;

(3) there is local survival at x if and only if either $F(x, x|\bar{\rho}) > 1$ or $F(x, y|\bar{\rho})$ diverges.

A sketch of the proof can be found in Section 6. We note that there is local survival at x if and only if
$$\bar{\rho} > \max\{t \in \mathbb{R} : F(x, x|t) \le 1\} \equiv r(x, x)$$
where $r(x, x) = 1/\limsup_{n \to \infty} \sqrt[n]{p^{(n)}(x, x)}$ is the spectral radius of the random walk P (see [40, Section 2.C]). As a corollary one derives the critical parameters for continuous-time BRWs which are locally isomorphic to a branching process: $\lambda_w = 1/k$ and $\lambda_s(x) = r(x, x)/k$ (where $k = k(x)$ for all $x \in X$).

It is clear that, in the irreducible case, there is pure global survival (see Section 4.5) if and only if $1 < \bar{\rho} \le r$ (where $r = r(x, x)$ in this case does not depend on $x \in X$ due to the irreducibility). This is possible if and only if $r > 1$ which is equivalent to nonamenability (see Theorem 4.23) since in this case $M_s(x, y) = \bar{\rho}/r$ and $M_w(x) = \bar{\rho}$. Notice that if there is pure global survival then P defines a transient random walk but the converse is not true: if P is the simple random walk on $\mathbb{Z}^d$ there is no pure global survival for any $\bar{\rho}$ and d.

In general there is no strong local survival, even if the BRW has independent diffusion as Examples 4.35 and 4.36 show. Before discussing the examples we need an easy lemma, whose proof can be found in Section 6.

Lemma 4.34. *Let $\{\alpha_i\}_{i \in \mathbb{N}}$ and $\{k_i\}_{i \in \mathbb{N}}$ be such that $\alpha_i \in (-\infty, 1)$ and $k_i \ge 0$ for all $i \in \mathbb{N}$. Then*
$$\sum_{i \in \mathbb{N}} k_i \alpha_i < +\infty \Longleftarrow \prod_{i \in \mathbb{N}} (1 - \alpha_i)^{k_i} > 0.$$

Moreover if $\alpha_i \in [0, 1)$ and $k_i \geq 1$ eventually as $i \to \infty$ then

$$\sum_{i \in \mathbb{N}} k_i \alpha_i < +\infty \iff \prod_{i \in \mathbb{N}} (1 - \alpha_i)^{k_i} > 0.$$

Example 4.35. *Fix $X := \mathbb{N}$ and consider a BRW with the following reproduction probabilities. Every particle has two children with probability $3/4$ and no children with probability $1/4$. Each newborn particle is dispersed independently according to a nearest neighbor matrix P on $\mathbb{N}$. More precisely*

$$p(i, j) := \begin{cases} p_i & \text{if } j = i + 1 \\ 1 - p_i & \text{if } j = i - 1, \end{cases}$$

and $p_0 = 1$. The process described above is an irreducible $\mathcal{F}$-BRW for every choice of the set $\{p_i\}_{i \in \mathbb{N} \setminus \{0\}}$.

The generating function of the total number of children is $z \mapsto 3z^2/4 + 1/4$ and its minimal fixed point is $1/3 = \bar{q}(x)$ (for all $x \in \mathbb{N}$).

Choose $p_1 < 5/9$; it is easy to show that the process confined to $\{0, 1\}$ survives (since the expected number of children at 0 every two generations (starting from 0) is $(3/2)^2(1 - p_1) > 1$). By irreducibility this implies that $q(x, y) < 1$ and $\bar{q}(x) < 1$ for all $x, y \in \mathbb{N}$.

Choose the p_is such that $\prod_{i=1}^{\infty} p_i^{2^i} > 0$, which, according to Lemma 4.34, is equivalent to $\sum_{i=1}^{\infty} 2^i(1 - p_i) < +\infty$. Consider the branching process N_n representing the total number of particles alive at time n: for all n, $N_n \leq 2^n$ almost surely. The probability, conditioned on global survival, that every particle places its children (if any) to its right, is the conditioned expected value of $\prod_{i=1}^{\infty} p_i^{N_i}$. But $\prod_{i=1}^{\infty} p_i^{N_i} \geq \prod_{i=1}^{\infty} p_i^{2^i} > 0$ almost surely. Hence, conditioning on global survival there is a positive probability of non-local survival. This implies $q(\cdot, y) \neq \bar{q}$ for every $y \in \mathbb{N}$. Note that, according to Theorem 4.20, $\sup_{x \in \mathbb{N}} q(x, x) = 1$.

The key in the previous example is that the total number of particles alive at time n is bounded. This is not an essential assumption. The following example shows that, given any law ρ of a surviving branching process (that is, $\bar{\rho} = \sum_{n \in \mathbb{N}} \rho(n) > 1$), it is possible to construct an irreducible BRW which is locally isomorphic to a branching process with no strong local survival.

Example 4.36. *Let $X = \mathbb{N}$ and $\rho_x := \rho$ for all $x \in \mathbb{N}$; ρ being the law of a surviving branching process. We know that $\bar{q}(x) \equiv \bar{q}$ for all $x \in \mathbb{N}$ where $\bar{q} < 1$ is the smallest fixed point of $z \mapsto \sum_{n \ in \mathbb{N}} \rho(n) z^n$. Pick a sequence of natural numbers $\{N_i\}_{i \in \mathbb{N}}$ satisfying*

$$\prod_{i \in \mathbb{N}} \rho([0, N_{i+1}])^{\prod_{j=0}^{i} N_j} > \bar{q}, \tag{4.18}$$

where $N_0 := 1$. Note that the probability of the event $\mathcal{A}$ where every particle alive at time i has at most N_{i+1} children for all $i \in \mathbb{N}$ is bounded from below by the LHS of equation (4.18). Thus, from equation (4.18), with a probability larger than $\prod_{i \in \mathbb{N}} \rho([0, N_{i+1}])^{\prod_{j=0}^{i} N_j} - \bar{q} > 0$ the colony survives globally and the total size of the population at time n is not larger than $\prod_{j=0}^{n} N_j$ (i.e. the intersection between $\mathcal{A}$ and global survival has positive probability).

We define a BRW with independent diffusion where P is as follows

$$p(i,j) := \begin{cases} p_i & j = i+1,\ i \geq 0 \\ 1 - p_i & j = i-1,\ i \geq 1 \\ 1 - p_0 & i = j = 0. \end{cases}$$

Let p_0 such that $(1-p_0)\bar\rho > 1$; this implies local survival. We choose the sequence $\{p_i\}_{i \in \mathbb{N}}$, where $p_i \in (0,1)$ in such a way that

$$\prod_{i \in \mathbb{N}} p_i^{\prod_{j=0}^{i} N_j} > 0. \tag{4.19}$$

Using equation (4.19), if we condition on $\mathcal{A}$, the probability that, every particle places its children (if any) to its right is bounded from below by $\prod_{i \in \mathbb{N}} p_i^{\prod_{j=0}^{i} N_j}$. This implies that there is a positive probability of global, non-local survival.

The choice of the sequences $\{N_i\}_{i \in \mathbb{N}}$ and $\{p_i\}_{i \in \mathbb{N}}$ satisfying equations (4.18) and (4.19) respectively can be done as follows. Choose a sequence $\{\alpha_i\}_{i \in \mathbb{N}}$ such that $\alpha_i \in (0,1)$ for all $i \in \mathbb{N}$ and $\prod_{i \in \mathbb{N}} \alpha_i > 1 - \bar{q}$. Then, iteratively, if we fixed $N_0, \ldots, N_k$, since $\lim_{x \to \infty} \rho([0,x]) = 1$ there exists $N_{k+1} \in \mathbb{N}$ such that $\rho([0, N_{k+1}]) > \alpha_{k+1}^{1/\prod_{j=0}^{k} N_j}$. Moreover, according to Lemma 4.34, equation (4.19), is equivalent to $\sum_{i \in \mathbb{N}} (1 - p_i)\prod_{j=0}^{i} N_j < \infty$, hence let us take, for instance, $p_i > 1/(i \cdot \prod_{j=0}^{i} N_j)$.

We note that the class constructed in this example includes discrete-time counterparts of continuous-time BRWs where ρ can be chosen as in equation (2.6) where $k(x)$ does not depend on x, $k_{xy} := k(x)p(x,y)$ (where P is defined as before) and $\lambda > \lambda_s$ is fixed. Finally we observe that this example extends naturally to an example of a site-breeding BRW on a radial tree where the number of branches of a vertex at distance k from the root is at least $1/p(k, k+1)$.

The following easy theorem gives another sufficient condition for the strong local survival of a BRW which is locally isomorphic to a branching process. The proof can be found in Section 6.

Theorem 4.37. *Suppose that (X, μ) is an irreducible BRW with independent diffusion such that P is the transition matrix of a recurrent random walk and $\rho_x = \rho$ for all $x \in X$. Then, global survival starting from some $x \in X$ implies strong local survival at y starting from w for all $w, y \in X$.*

In case of a BRW with no death and with independent diffusion one can prove the following proposition which makes use of Proposition 4.22. By $U(x, y|z)$ we mean the usual generating function of the first-return probabilities of the random walk P as defined in [40, equation (1.26)]; in particular $U(x, y)$ is the probability of visiting y *after* starting from x.

Proposition 4.38. *[34, Proposition 3.6]* *Let (X, μ) be an irreducible BRW with independent diffusion where $\rho_x = \rho$ for all $x \in X$ and $\rho(0) = 0$. If $\bar\rho > \sup_{x \in X} 1/U(x,x)$ then there is strong local survival.*

A result like this one could be proved without the no-death hypothesis using the comparison described in Section 3.2; in this case a reasonable hypothesis should be $\bar{\rho} > \sup_{x\in X} 1/U(x, x|1 - \rho(0))$.

Remark 4.39. *In order to extend some results, as Proposition 4.38 for instance, from the homogeneous case ($\rho_x = \rho$ for all $x \in X$) to the inhomogeneous case, we have to find sufficient conditions such that the infimum of the probabilities of survival of the branching processes with laws $\{\rho_x\}_{x\in X}$ is strictly larger than 0. Observe that if $\rho \succeq \widehat{\rho}$ then for all nondecreasing, positive, measurable function f we have $\sum_{n\in\mathbb{N}} f(n)\rho(n) \geq \sum_{n\in\mathbb{N}} f(n)\widehat{\rho}(n)$ which, in turn, implies $\sum_{n\in\mathbb{N}} z^n\rho(n) \leq \sum_{n\in\mathbb{N}} z^n\widehat{\rho}(n)$ for all $z \in [0, 1]$. Hence if $\sum_{n\in\mathbb{N}} z^n\widehat{\rho}(n) \leq z$ then $\sum_{n\in\mathbb{N}} z^n\rho(n) \leq z$. Thus if $\rho_x \succeq \widehat{\rho}$ for all $x \in X$ then the probabilities of extinction of the branching processes associated to the ρ_xs are all dominated by the probability of extinction of the branching process associated to $\widehat{\rho}$. Hence one possibility would be to assume that for all $x \in X$, ρ_x dominates some law of a surviving branching process $\widehat{\rho}$. This way the results for inhomogeneous BRWs are simply corollaries of the homogeneous case. Clearly this is not a significant improvement.*

One might guess that if the expected number of offsprings $\bar{\rho}_x$ is sufficiently large then the supremum of the probabilities of extinction of the branching processes with laws $\{\rho_x\}_{x\in X}$ is strictly smaller than 1. But clearly, if $\sup_{x\in X} \rho_x(0) = 1$ then the supremum of the probabilities of extinction is 1. Even bounding $\rho_x(0), \ldots, \rho_x(k)$ (for all $x \in X$) is not sufficient. Indeed consider the set $\mathcal{A}$ of all probability generating functions and a subset $\mathcal{A}_{a_0,a_1;\ldots,a_k,m}$ defined as

$$\mathcal{A} := \{\phi(z) = \sum_i \rho(i)z^i : \rho(i) \geq 0 \; \forall i, \; \sum_i \rho(i) = 1\}$$

$$\mathcal{A}_{a_0,a_1,\ldots,a_k,m} := \{\phi \in \mathcal{A} : \phi(z) = \sum_i \rho(i)z^i, \; \rho(i) \leq a_i \; \forall i \leq k, \; \bar{\rho} \geq m\}$$

(where $a_i \in [0, 1]$ and $m \in \mathbb{R}$, $m > 1$). In this case, either there is a surviving branching process (which does not necessarily belong to $\mathcal{A}_{a_0,a_1,\ldots,a_k,m}$) whose law is stochastically dominated by all $\rho \in \mathcal{A}_{a_0,a_1,\ldots,a_k,m}$ or the supremum of the probabilities of extinction is 1.

In order to prove this claim, let $k_0 := \max\{i = 0, \ldots, k : \sum_{j=0}^k a_i < 1\}$ and define $\widehat{\phi}(z) := \sum_{i=0}^{k_0} a_i z^i + (1 - \sum_{i=0}^{k_0} a_i)z^{k_0+1}$. Clearly the branching process corresponding to $\widehat{\phi}$ (which might not belong to $\mathcal{A}_{a_0,a_1,\ldots,a_k,m}$) is dominated by every branching process in $\mathcal{A}_{a_0,a_1,\ldots,a_k,m}$ and $\widehat{\phi} \leq \phi$ in $[0, 1]$ for every $\phi \in \mathcal{A}_{a_0,a_1,\ldots,a_k,m}$. This implies that the minimal fixed point, $c \leq 1$, of $\widehat{\phi}$ is an upper bound of the minimal fixed point of $\phi \in \mathcal{A}_{a_0,a_1,\ldots,a_k,m}$. If $c < 1$, that is, $\sum_{i=0}^{k_0} ia_i + (1 - \sum_{i=0}^{k_0} a_i)(k_0 + 1) > 1$ (i.e. the branching process survives) there is nothing to prove. On the other hand, if $c = 1$ then consider

$$\phi_N = \sum_{i=0}^{k_0} a_i z^i + w_N z^{k_0+1} + (1 - w_N - \sum_{i=0}^{k_0} a_i)z^N$$

where $w_N = (\sum_{i=0}^{k_0} ia_i - m + (1 - \sum_{i=0}^{k_0} a_i)N)/(N - (k_0 + 1)) \in [0, 1 - \sum_{i=0}^{k_0} a_i]$ eventually as $N \to \infty$ (remember that $c = 1$ if and only if $1 \geq \frac{\mathrm{d}}{\mathrm{d}z}\widehat{\phi}(z)|_{z=1} \equiv \sum_{i=0}^{k_0} ia_i + (1 - \sum_{i=0}^{k_0} a_i)(k_0 + 1)$). Then $\frac{\mathrm{d}}{\mathrm{d}z}\phi_N(z)|_{z=1} = m$, $\phi_N \in \mathcal{A}_{a_0,a_1,\ldots,a_k,m}$ and $\lim_{N\to\infty} \phi_N \equiv \widehat{\phi}$. It is straightforward to show that the minimal fixed point c_N of ϕ_N converges to $c = 1$ as $N \to \infty$.

5 Approximation

5.1 Spatial approximation

The first kind of approximation is based upon a result on approximation of nonnegative matrices which is interesting in itself. Recall the usual classification of indices of a matrix $M = (m_{xy})_{x,y \in X}$ (which is supposed to be nonnegative throughout this section) as described in [37, Chapter 1]. For any index x we denote by $[x]$ its *class*, that is, the set of indices which communicate with x. We define the convergence parameters $R(x,y) := M_s(x,y)^{-1}$ and $R := \inf_{x,y \in X} R(x,y)$. It is straightforward to show that $M_s(x,y) = M_s(x_1, y_1)$ if $[x] = [x_1]$ and $[y] = [y_1]$; this implies that for irreducible matrices, $R(x,y)$ is independent of $x, y \in X$.

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of subsets of X and denote by $_nR$ the convergence parameter of $M_n = (m_{xy})_{x,y \in X_n}$; clearly, if the sequence $\{X_n\}_{n \in \mathbb{N}}$ is nondecreasing, we have that $_nR \geq _{n+1}R$. The following theorem generalizes [37, Theorem 6.8] (note that the submatrices $\{M_n\}_{n \in \mathbb{N}}$ are not necessarily irreducible); it is the key to prove our main result about spatial approximation (Theorem 5.2).

Theorem 5.1. [42, Theorem 5.1][1] *Let $\{X_n\}_{n \in \mathbb{N}}$ be a general sequence of subsets of X such that $\liminf_{n \to \infty} X_n = X$ and suppose that $M = (m_{xy})_{x,y \in X}$ is a nonnegative matrix. Then for all $x_0 \in X$ we have $_nR(x_0, x_0) \to R(x_0, x_0)$. Moreover if M is irreducible and $M_n = (m_{xy})_{x,y \in X_n}$ then $_nR \to R$ as $n \to \infty$ and, in particular, for all $x_0 \in X$ we have $_nR(x_0, x_0) \to R$.*

Note that in the previous theorem the subsets $\{X_n\}_{n \in \mathbb{N}}$ can be chosen arbitrarily; in particular they may be finite proper subsets. Moreover the result extends easily to the case of a sequence of nonnegative matrices $M_n = (m(n)_{xy})_{x,y \in X_n}$ where $\liminf_{n \to \infty} X_n = X$, $0 \leq m(n)_{xy} \leq m_{xy}$ for all $x, y \in X_n$ and $\lim_{n \to \infty} m(n)_{xy} = m_{xy}$ for all $x, y \in X$ (note that $m(n)_{xy}$ is eventually well defined for all $x, y \in X$ as $n \to \infty$). The idea of the proof is essentially contained in [42, Theorem 5.2].

Given a sequence of BRWs $\{(X_n, \mu_n)\}_{n \in \mathbb{N}}$ such that $\liminf_{n \to \infty} X_n = X$, we define $m(n)_{xy} := \sum_{f \in S_{X_n}} f(y)\mu_{n,x}(f)$ and the corresponding sequence of matrices $\{M_n\}_{n \in \mathbb{N}}$. Note that in the following result we are not assuming that the BRW is irreducible.

Theorem 5.2. [42, Theorem 5.2] *Let us fix a vertex $x_0 \in X$. If $\liminf_{n \to \infty} X_n = X$ and $m(n)_{xy} \leq m_{xy}$ for all $x, y \in X_n$, $n \in \mathbb{N}$ and $m(n)_{xy} \to m_{xy}$ as $n \to \infty$ then*

(1) (X, μ) dies out locally (resp. globally) a.s. starting from $x_0 \implies (X_n, \mu_n)$ dies out locally (resp. globally) a.s starting from x_0 for all $n \in \mathbb{N}$;

(2) (X, μ) survives locally starting from $x_0 \implies (X_n, \mu_n)$ survives locally starting from x_0 eventually as $n \to \infty$.

Clearly the discrete-time counterpart of a spatially confined BRW in continuous-time is obtained by spatially confining the discrete-time counterpart of the continuous-time BRW. Hence, Theorem 5.2 yields an analogous result for BRWs in continuous time.

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[1]We observe that in [42, Section 5.1] the hypotheses that M is a nonnegative matrix is missing, even though it is implicitly used.

Corollary 5.3. *[7, Theorem 3.1] Let (X, K) be a continuous-time BRW and let us consider a sequence of continuous-time BRWs $\{(X_n, K_n)\}_{n \in \mathbb{N}}$ such that $\limsup_{n \to \infty} X_n = X$. Let us suppose that $k_{xy}(n) \le k_{xy}$ for all $n \in \mathbb{N}$, $x, y \in X_n$ and $k_{xy}(n) \to k_{xy}$ as $n \to \infty$ for all $x, y \in X$. Then $\lambda_s(X_n, K_n) \ge \lambda_s(X, K)$ and $\lambda_s(X_n, K_n) \to \lambda_s(X, K)$ as $n \to \infty$.*

Among all possible choices of the sequence $\{(X_n, \mu_n)\}_{n \in \mathbb{N}}$ there is one which is *induced* by (X, μ) on the subsets $\{X_n\}_{n \in \mathbb{N}}$; more precisely, one can take $\mu_n(g) := \sum_{f \in S_X : f|_{X_n} = g} \mu_x(f)$ for all $x \in X_n$ and $g \in S_{X_n}$. Roughly speaking, this choice means that all reproductions outside X_n are suppressed. In this case it is simply $m(n)_{xy} = m_{xy}$ for all $x, y \in X_n$.

Since Theorem 5.2 deals with local survival, one can wonder what can be said about global survival. First of all, if the process (X, μ) survives globally and locally then eventually (X_n, μ_n) survives locally and thus globally. The question is nontrivial when (X, μ) survives globally but not locally, which we assume henceforth in this brief discussion. In this last case, if, for instance, X_n is finite for every $n \in \mathbb{N}$ and the graph (X_n, E_{μ_n}) is connected then, by Theorem 5.2(1), (X_n, μ_n) dies out (locally and globally) a.s. for all values of $n \in \mathbb{N}$. On the other hand, the case where X_n is finite for every $n \in \mathbb{N}$ and the graph (X_n, E_{μ_n}) is not connected is more complicated and can be treated as in [7, Remark 4.4]. When X_n is infinite for infinitely many values of n, it is possible that there is no global survival for infinitely many values of n. An example in the discrete-time case can be found in [42, Remark 5.3] while an example in the continuous-time case can be constructed using [6, Remark 3.2]. Taking the couple (X_n, μ_n) random, some results can be achieved as an application of Theorem 5.2 (see for instance [8, Theorem 7.1] or [18, Theorem 2.4]).

Example 5.4. *Consider the edge-breeding (continuous-time) BRW on $\mathbb{Z}^d$. We saw in Example 4.25 that if $\lambda > \lambda_s = 1/2d$ then there is local survival. Suppose that $d > 1$ and that $\lambda \le 1/2$ and consider the infinite cylinder $X_n := \{x \in \mathbb{Z}^d : |x(i)| \le n, \forall i = 2, \ldots, d\}$. It is clear that there is no local survival for the BRW restricted to $X_0 = \mathbb{Z} \subseteq \mathbb{Z}^d$, nevertheless according to Corollary 5.3 there exists n_0 such that there is local survival on X_n for all $n \ge n_0$. This shows a difference between random walks and BRWs: the simple random walk on X_n is recurrent for all $n \in \mathbb{N}$ (as the simple random walk on $\mathbb{Z}$); on the other hand, while the BRW restricted to $X_0 = \mathbb{Z}$ dies out locally, it survives when restricted to X_n if n is sufficiently large (in some sense the BRW on X_n approaches the BRW on $\mathbb{Z}^d$ as n tends to infinity).*

Another consequence of Theorem 5.2 is the following (see also [42, Section 5]): consider the edge-breeding continuous-time BRW on $\mathbb{Z}^d$ (but the argument extends easily to any translation invariant BRW in discrete and continuous time). Let us choose a connected subset $Y \subset \mathbb{Z}^d$ such that every finite subset $A \subset \mathbb{Z}^d$ is a subset of a suitable translation of Y in $\mathbb{Z}^d$. Then the strong critical parameter λ'_s of the BRW restricted to Y is equal to $\lambda_s = 1/2d$. A possible choice is $Y := \{y \in \mathbb{Z}^d : \langle y, y_0 \rangle \ge \alpha \|y\| \cdot \|y_0\|\}$ for some fixed nontrivial $y_0 \in \mathbb{Z}^d$ and $\alpha < 1$ (where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ represent the usual scalar product and norm of $\mathbb{Z}^d$ respectively).

5.2 Approximation by truncated BRWs

The family of discrete-time BRWs can be extended to the more general class of *truncated BRWs* where a maximum of $m \in \mathbb{N} \cup \{\infty\}$ particles per site is allowed. We denote this

process as a BRW$_m$. The dynamics is described by the following recursive relation

$$\eta_{n+1}^m(x) = m \wedge \sum_{y \in X} \sum_{i=1}^{\eta_n^m(y)} f_{i,n,y}(x) = m \wedge \sum_{y \in X} \sum_{j=0}^{\infty} \mathbb{1}_{\{\eta_n^m(y)=j\}} \sum_{i=1}^{j} f_{i,n,y}(x).$$

Clearly the BRW$_\infty$ is the usual BRW and the BRW$_1$ is the well-known *contact process*. Note that while for the BRW the reproductions of two particles are independent, this is not true for the BRW$_m$ which is a truly interacting particle system.

There is an analogous class of continuous-time processes that we still call truncated BRWs which are subject to the same constraint (see [8]).

Remark 5.5. *Even though the BRW$_m$ is the only generalization that we need here, there is a more general class of continuous-time processes, called restrained BRWs, which is worth mentioning and which has been introduced and studied in [4] (an analogous construction in discrete-time is straightforward and we omit it). Consider an infinite connected graph X and let $\eta(x)$ be the number of individuals living at the site $x \in X$. The lifespan of each individual is an exponential random variable of mean 1. During its lifetime each individual tries to reproduce following a Poisson process of intensity λ. Every time the clock associated to the Poisson process rings, the individual tries to send an offspring to a randomly chosen target neighboring site. The target neighboring site is chosen using the transition matrix $P = (p(x,y))_{x,y \in X}$ of a nearest neighbor random walk on X. Call the target site y. The reproduction on y is effective only with probability $c(\eta(y))/\lambda$, where $c : \mathbb{N} \to \mathbb{R}^+$ is a non-increasing and nonnegative function with $c(0) = \lambda$. In this case the population living at y increases by one individual, otherwise nothing happens.*

Observe that the restrained BRW is a Markov process and the continuous-time truncated BRW$_m$ are special cases ($c \equiv \lambda$ if $m = \infty$ and $c = \lambda \mathbb{1}_{\{0,1,\ldots,m-1\}}$) otherwise). Results about survival and stationary measures of this process can be found in [4]; it is worth noting that this process may have an ecological equilibrium, that is, a phase of local survival where the expected number of individuals per site is bounded from above. This is not possible for the BRW where local survival implies almost surely an unbounded population (see the proof of [42, theorem 4.1(1)] and [20, Theorem 6.2]).

We observe that the discrete-time counterpart of a continuous-time truncated BRW is not a discrete-time truncated BRW. Indeed, in order to construct the discrete-time counterparts we lose the original time scale: on one hand, particles which are in the same generation in the discrete-time process might have disjoint lifespan intervals in the continuous-time process and, on the other hand, particles living at the same time in the continuous-time process might belong to disjoint generations in the discrete-time counterpart. Hence the results about approximation of continuous-time BRWs by means of continuous-time truncated BRWs (see [8]) cannot be considered as particular cases of the analogous results for discrete-time processes (see [42]). Nevertheless, the techniques used are very similar and the results essentially coincides.

The goal of this section is to study the approximation of a BRW $\{\eta_n\}_{n \in \mathbb{N}}$ by means of the sequence of truncated BRWs $\{\{\eta_n^m\}_{n \in \mathbb{N}}\}_{m \in \mathbb{N}}$. It is clear, by stochastic domination (see [42, Section 3.3]), that if the BRW dies out locally (resp. globally) a.s. then any truncated BRW dies out locally (resp. globally). We are going to prove here a result similar to Theorem 5.2

as m tends to infinity. For discrete-time BRWs this has been done in [42] while the results for continuous-time BRWs can be found in [8].

From now on we make some assumptions on (X, μ). First, we assume that X is countable. Indeed, the finite case is uninteresting since the the truncated BRW $\{\eta_n^m\}_{n \in \mathbb{N}}$ do not survive for any $m \in \mathbb{N}$ in discrete and continuous time (by standard Markov chain arguments, being $\mathbf{0}$ an absorbing state). Second, we require that the graph (X, E_μ) has finite geometry, that is, $\sup_{x \in X} \deg(x) < +\infty$ and that the matrix M is irreducible- We denote its convergence parameter by R_μ. We observe that, using this notation, according to Theorem 4.3, local survival is equivalent to $R_\mu < 1$. For a continuous-time BRW (X, K) we denote the convergence parameter of the matrix K by R_K and we observe that according to Corollary 4.5, local survival is equivalent to $1/R_K < \lambda$.

Finally, we suppose that

$$\sup_{x \in X} \rho_x([n, +\infty)) \to 0, \qquad \text{as } n \to +\infty. \tag{5.20}$$

This assumption allows to use the measure ρ defined as

$$\rho(n) = \sup_{x \in X} \rho_x([n, +\infty)) - \sup_{x \in X} \rho_x([n+1, +\infty)),$$

to stochastically dominate all the laws $\{\rho_x\}_{x \in X}$. Indeed equation (5.20) is equivalent to the existence of a measure which dominates the ρ_xs. The measure ρ has finite second (hence first) moment if and only if $\int_0^\infty \sup_{x \in X} \rho_x([\sqrt{t}, +\infty)) \mathrm{d}t < +\infty$ (that we assume henceforth). For a continuous-time BRW (X, K) we simply assume that $\sup_{x \in X} k(x) < +\infty$ and this implies immediately the stochastic domination by means of a continuous-time branching process with parameter $\sup_{x \in X} k(x)$ (see [8] for details).

We start with the result on the approximation of the local behavior of a BRW by means of the sequence of truncated BRWs $\{\{\eta_n^m\}_{n \in \mathbb{N}}\}_{m \in \mathbb{N}}$.

Theorem 5.6. *[42, Theorem 6.3]*
Suppose that at least one of the following conditions holds

(1) (X, μ) is quasi transitive and irreducible;

(2) (X, μ) is irreducible and there exists γ bijection on X such that

    (a) μ is γ-invariant;

    (b) for some $x_0 \in X$ we have $x_0 = \gamma^n x_0$ if and only if $n = 0$.

If if $\{\eta_n\}_{n \in \mathbb{N}}$ survives locally (starting from x_0) then $\{\eta_n^m\}_{n \in \mathbb{N}}$ survives locally (starting from x_0) eventually as $m \to +\infty$.

In the continuous-time case there is an analogous result which gives an approximation of λ_s by λ_s^m, where λ_s^m is the local critical parameter of the truncated BRW with (at most) m particles per site.

Theorem 5.7. *[8, Theorem 5.1]*
Let (X, K) be a continuous-time BRW and suppose that at least one of the following conditions holds

(1) (X, K) *is quasi transitive;*

(2) (X, K) *is irreducible and there exists γ bijection on X such that*

     (a) μ is γ-invariant;

     (b) for some $x_0 \in X$ we have $x_0 = \gamma^n x_0$ if and only if $n = 0$.

Then

$$\lim_{m \to +\infty} \lambda_s^m = \lambda_s \geq \lim_{m \to +\infty} \lambda_w^m \geq \lambda_w.$$

Moreover if $\lambda_s = \lambda_w$ then $\lambda_w^m \downarrow_{m \to +\infty} \lambda_w$.

Let us consider now the global behavior; as before, we start with a discrete-time process. We take (X, μ) with $X = \mathbb{Z} \times Y$ (for some set Y) and we denote by $g : X \to \mathbb{Z}$ the usual projection from X onto $\mathbb{Z}$, namely $g(n, y) := n$. In the following we use the same notation as in Section 3.1. We suppose that $\nu = \mu \circ g^{-1}$ is translation invariant (that is, γ-invariant according to Definition 3.2 for every translation operator γ on $\mathbb{Z}$) and we denote the common distribution and the expected number of offsprings of the BRW by ρ and $\bar{\rho} = \sum_{y \in X} m_{xy}$; observe that they. do not depend on $x \in X$ or $i \in \mathbb{Z}$ since ν is translation invariant.

Theorem 5.8. *[42, Theorem 6.5] Let $X = \mathbb{Z} \times Y$ and suppose that the BRW (X, μ) is locally isomorphic to $(\mathbb{Z}, \nu)$ where ν is translation invariant. If $m_{xy} = 0$ whenever $|g(x) - g(y)| > 1$ then*

(1) the BRW survives globally starting from x if and only if $\bar{\rho} = \sum_{y \in \mathbb{Z}} m_{xy} > 1$;

(2) if the BRW survives globally (starting from x) then then the BRW_m survives globally (starting from x) for every sufficiently large m.

Note that the hypotheses that we made in the previous theorem implies that the BRW $(\mathbb{Z}, \nu)$ is "nearest neighbor" in the sense that reproductions are possible only in the same site or towards neighboring sites (in the usual graph $\mathbb{Z}$). Theorem 5.8 applies to translation invariant BRWs on two particular graphs: $\mathbb{Z}^d$ and the homogeneous tree $\mathbb{T}_r$ with degree r. In particular the application to $\mathbb{T}_r$ is possible since the product $\mathbb{Z} \times Y$ is meant as a set product and not a graph product; indeed the set of vertices of $\mathbb{T}_r$ can be seen as $\mathbb{Z}^2$ and the projection g as the horocyclic map (see [39, Section 12.13]).

Corollary 5.9. *[42, Corollary 6.6] If the BRW $(\mathbb{Z}^d, \mu)$ is translation invariant and there exists a projection g on one of the coordinates such that $m_{xy} = 0$ whenever $|g(x) - g(y)| > 1$, then*

(1) the BRW survives globally (starting from x) if and only if $\bar{\rho} = \sum_{y \in \mathbb{Z}} m_{xy} > 1$;

(2) if the BRW survives globally (starting from x) then the BRW_m survives globally (starting from x) for every sufficiently large m.

Corollary 5.10. *[42, Corollary 6.7] Let $\mathbb{T}_r$ be a homogeneous tree and suppose that the BRW $(\mathbb{T}_r, \mu)$ is γ-invariant for every automorphism γ of $\mathbb{T}_r$. If $\mu_x(f) \neq 0$ implies $\mathrm{supp}(f) \subseteq B(x, 1)$ (where $B(x, 1)$ is the usual ball of radius 1 and center x of the graph $\mathbb{T}_r$) then*

(1) the BRW survives globally (starting from x) if and only if $\bar{\rho} = \sum_{y \in \mathbb{Z}} m_{xy} > 1$;

(2) if the BRW survives globally (starting from x) then the BRW_m survives globally (starting from x) for every sufficiently large m.

Let us consider now the continuous-time case; there are analogous results which give an approximation of λ_w by λ_w^m, where λ_w^m is the global critical parameter of the truncated BRW with (at most) m particles per site. From now on we deal with a site-breeding BRW; thus, $k(x) = k$ for all $x \in X$, that is, we set $k_{xy} = kp(x,y)$ where P is a transition matrix of a random walk. We stress that in this case $\lambda_w = 1/k$. We are concerned with the question whether $\lambda_w^m \downarrow \lambda_w = 1/k$ or not. Under the hypotheses of Theorem 5.7, this is the case when the BRW has no pure global survival phase (i.e. $r = 1$ where r is the spectral radius of the random walk P). The interesting case is $r > 1$. Most natural examples are drifting random walks on $\mathbb{Z}^d$ and the simple random walk on homogeneous trees: as for discrete-time processes, in both cases we show that $\lambda_w^m \overset{m \to \infty}{\longrightarrow} \lambda_w$.

Theorem 5.11. *[8, Theorem 6.1] Let P be a transition matrix of a nearest-neighbor, translation invariant random walk on $\mathbb{Z}$. Then $\lim_{m \to +\infty} \lambda_w^m = 1/k = \lambda_w$.*

This result immediately extends to the case of a class of more general spaces (including multidimensional lattices $\mathbb{Z}^d$) in the following way.

Corollary 5.12. *[8, Corollary 6.1] Let us consider the BRW $(Y \times \mathbb{Z}, K_\alpha)$ where $K_\alpha = \alpha(\mathbb{I}^Y \times P) + (1 - \alpha)(Q \times \mathbb{I}^{\mathbb{Z}})$ and Q and P are transition matrices of a random walk on Y and of a translation invariant random walk on $\mathbb{Z}^d$ (as in Theorem 5.11) respectively. Then $\lim_{k \to +\infty} \lambda_w^k = 1/k = \lambda_w$.*

For a homogeneous tree the following result holds.

Theorem 5.13. *[8, Theorem 6.2] If $X = \mathbb{T}_d$ is the homogeneous tree of degree d and P is the simple random walk on X then $\lim_{m \to +\infty} \lambda_w^m = 1/k = \lambda_w$.*

Observe that Theorem 5.13 can be immediately extended to the edge-breeding BRW on $\mathbb{T}_d$ (on regular graphs, the edge-breeding BRW is a particular site-breeding BRW). In the edge-breeding case we have that $\lim_{m \to +\infty} \lambda_w^m = 1/d = \lambda_w$; on the other hand, according to Theorem 5.7, $\lim_{m \to +\infty} \lambda_s^m = 1/2\sqrt{d-1} = \lambda_s$. Thus $\lambda_w^m < \lambda_s^m$ eventually as $m \to \infty$ (pure global survival phase for truncated BRWs). In particular in [35] it was shown that $\lambda_w^1 < \lambda_s^1$, hence we conjecture that $\lambda_w^m < \lambda_s^m$ for all $m \in \mathbb{N}$.

Discussing the details of the proofs goes beyond the purpose of this paper. We just observe that they rely on a comparison between the processes and a suitable oriented percolation. Such a strategy has been introduced in [9] and widely used since then. The difference in our case is that the percolation is not even one-dependent and this brings some additional difficulties from a technical point of view (see [8, Section 4] and [42, Section 6.1]). Some applications and a slight generalization of these results of approximation can be found in [2, Theorem 3.4] and [3, Theorem 1].

6 Proofs

Here we sketch the proofs of the new results.

Proof of Proposition 2.6. Without loss of generality we can suppose that $\bar{q}(x) < 1$ for all $x \in X$. Indeed, given x_0 such that $\bar{q}(x_0) = 1$ then for all $x \in \mathcal{N}_{x_0}$ we have $\bar{q}(x) = 1$. Since we defined $\hat{z}(x) := 1$ whenever $\bar{q}(x) = 1$ we can remove these vertices obtaining a new set $X' \subseteq X$. Consider the restricted BRW on X' (obtained by killing all the particle going outside X'). The generating function G' of the new BRW satisfies $G'((z|_{X'})|x) \geq G(z|x)$ for all $x \in X'$, hence $G(z) \geq z$ implies $G'(z|_{X'}) \geq z|_{X'}$. Moreover $\hat{z}$ satisfies the conclusions of the proposition if and only if $\widehat{z|_{X'}} \equiv \hat{z}|_{X'}$ does. Thus, it is enough to prove the result for the BRW restricted to X'.

We use the notation of Section 3.2. Note that $\hat{z} := T_{\bar{q}}^{-1}(z)$, thus $G(z) \geq z$ is equivalent to $\hat{G}(\hat{z}) \geq \hat{z}$. Hence it is enough to prove the proposition when $\mu_x(\mathbf{0}) = 0$ for all $x \in X$ which implies $\bar{q} = \mathbf{0}$ and $\hat{z} = z$. Suppose that $\mathcal{N}_x$ is nonempty, $z(y) \leq z(x)$ for all $y \in \mathcal{N}_x$ and $z(y_0) < z(x)$ for some $y_0 \in \mathcal{N}_x$. Then, using the fact that $z \leq \mathbf{1}$ and that $\prod_{y \in X} z(y)^{f(y)} \leq z(x)$ if $\mathcal{H}(f) \geq 1$, we have that $z(x) \leq G(z|x) \leq \sum_{f \in S_X : f(y_0)=0} \mu_x(f)z(x) + \sum_{f \in S_X : f(y_0)>0} \mu_x(f)z(y_0) < z(x)$ which is a contradiction. As for the second part, since $z(y) \leq 1 = z(x)$ for all $y \in X$ then we have $z(y) = 1$ for all $y \in X$. Finally, by induction we obtain the result for the set $\{y \in X : x \to y\}$. $\square$

Proof of Theorem 4.3. The first part of the theorem is [42, Theorem 4.1]. The sufficient condition in the second part follows easily from the first part. Clearly, it is equivalent to study the BRW restricted to $Y := \{w : x \to w \to y\}$ which is finite. In this case $q(w,w) = 1$ for all w in a finite irreducible class implies a.s. extinction in the class; if the number of classes is finite then $q(x,y) = 1$. $\square$

Before proving Proposition 4.19 and Theorem 4.20 we need two technical lemmas.

Lemma 6.1. *Let (X,μ) be a BRW and fix $z, v \in [0,1]^X$ such that $z + \varepsilon v \in [0,1]^X$ for some $\varepsilon > 0$. Then the function $t \mapsto G(z + vt|x)$ is strictly convex if and only if*

$$\exists f : \mu_x(f) > 0, \sum_{y \in \text{supp}(v)} f(y) \geq 2, \text{supp}(z) \cup \text{supp}(v) \supseteq \text{supp}(f). \tag{6.21}$$

Proof of Lemma 6.1. Let us evaluate the function G on the line $t \mapsto z + tv$ where $t \in [0,T)$ and $T := \sup\{s > 0 : z + sv \in [0,1]^X\}$.

$$G(z+tv|x) = \sum_{f\in S_X}\mu_x(f)\prod_{y\in X}\sum_{i=0}^{f(y)}\binom{f(y)}{i}z(y)^{f(y)-i}v(y)^it^i$$

$$= \sum_{f\in S_X}\mu_x(f)\sum_{g\in S_X:g\le f}\prod_{y\in X}\binom{f(y)}{g(y)}z(y)^{f(y)-g(y)}v(y)^{g(y)}t^{g(y)}$$

$$= \sum_{f\in S_X}\mu_x(f)\sum_{g\in S_X:g\le f}t^{\mathcal{H}(g)}\prod_{y\in X}\binom{f(y)}{g(y)}z(y)^{f(y)-g(y)}v(y)^{g(y)}$$

$$= \sum_{f\in S_X}\mu_x(f)\sum_{i=0}^{\infty}\sum_{g\in S_X:\mathcal{H}(g)=i,g\le f}t^i\prod_{y\in X}\binom{f(y)}{g(y)}z(y)^{f(y)-g(y)}v(y)^{g(y)}$$

$$= \sum_{i=0}^{\infty}t^i\left(\sum_{f,g\in S_X:\mathcal{H}(g)=i,g\le f}\mu_x(f)\prod_{y\in X}\binom{f(y)}{g(y)}z(y)^{f(y)-g(y)}v(y)^{g(y)}\right)$$

The strict convexity of a power series in t with nonnegative coefficients is equivalent to the strict positivity of at least one coefficient corresponding to t^i with $i\ge 2$. Hence it is easy to show that each of the following assertions is equivalent to the next one and that they are all equivalent to the strict convexity of $t\mapsto G(z+vt|x)$

1. $\exists f,g:\mathcal{H}(g)\ge 2,\ f\ge g,\ \mu_x(f)>0:\operatorname{supp}(v)\supseteq\operatorname{supp}(g),\ \operatorname{supp}(z)\supseteq\operatorname{supp}(f-g)$;

2. $\exists f,g:\mathcal{H}(g)\ge 2,\ f\ge g,\ \mu_x(f)>0:g=f\mathbb{1}_{\operatorname{supp}(v)},\ \operatorname{supp}(z)\supseteq\operatorname{supp}(f)\setminus\operatorname{supp}(v)$;

3. $\exists f:\mu_x(f)>0:\sum_{y\in\operatorname{supp}(v)}f(y)\ge 2,\ \operatorname{supp}(z)\supseteq\operatorname{supp}(f)\setminus\operatorname{supp}(v)$;

4. $\exists f:\mu_x(f)>0:\sum_{y\in\operatorname{supp}(v)}f(y)\ge 2,\ \operatorname{supp}(z)\cup\operatorname{supp}(v)\supseteq\operatorname{supp}(f)$;

5. $\exists f:\mu_x(f)>0:\sum_{y\in\operatorname{supp}(v)}f(y)\ge 2,\ \operatorname{supp}(z+v)\supseteq\operatorname{supp}(f)$.

$\square$

Lemma 6.2. *Let (X,μ) be a BRW and fix $x_0\in X$. Suppose that for some $\bar{x}$ in the same irreducible class of x_0 and $f\in S_X$ we have that $\mu_{\bar{x}}(f)>0$, $\sum_{w:w\rightleftharpoons x_0}f(w)\ge 2$. We can fix $\bar{n}\in\mathbb{N}$ such that if the process starts with one particle at $x_0\in X$ then we have at least 2 particles at x_0 in the generation $\bar{n}$ wpp.*

Proof of Lemma 6.2. Consider a path $x_0,x_1,\ldots,x_m=\bar{x}$ and let $f\in S_X$ be such that $\mu_{\bar{x}}(f)>0$ and $\sum_{w:w\rightleftharpoons x_0}f(w)\ge 2$. We can have two cases.
(a). There exists $x_{m+1}\in X$ such that $x_{m+1}\rightleftharpoons x_0$ and $f(x_{m+1})\ge 2$; in this case consider the closed path $x_0,x_1,x_2,\ldots,x_m,x_{m+1},\ldots,x_n=x_0$ and take $\bar{n}:=n$. Since any particle at x_i has at least one child at x_{i+1} wpp and a particle at $\bar{x}$ has at least 2 children at x_{m+1} wpp, then any particle at x_0 has at least 2 descendants at x_0 in the $\bar{n}$th generation. Indeed, denote by $f_i\in S_X$ such that $\mu_{x_i}(f_i)>0$, $f_i(x_{i+1})\ge 1$ for all $i=0,\ldots\bar{n}-1$ (f_m being f), then the probability that a particle at x_0 has at least 2 particle at x_0 in the $\bar{n}$th generation is bounded from below by $\prod_{i=0}^m\mu_i(f_i)\prod_{j=m+1}^{\bar{n}-1}\mu_j(f_i)^2$.

(b). There exists a couple of different vertices x_{m+1}, y_{m+1} such that $x_{m+1}, y_{m+1} \rightleftharpoons x_0$ and $f(x_{m+1}), f(y_{m+1}) \geq 1$; in this case consider the paths $x_0, x_1, \ldots x_m, x_{m+1}, \ldots, x_{n_1} = x_0$ and $x_0, x_1, \ldots x_m, y_{m+1}, \ldots, y_{n_2} = x_0$ and take $\bar{n} := GCD(n_1, n_2)$ (the conclusion is similar as before). $\qquad\square$

Proof of Theorem 4.20. For every z fixed point of G, we know that $z \geq \bar{q}$ and $z \leq \mathbf{1}_X$; this implies that if $\sup_{x \in X} z(x) < 1$ for some fixed point then necessarily $\sup_{x \in X} \bar{q}(x) < 1$. Hence, if $\bar{q} = \mathbf{1}$ there is nothing to prove. Otherwise, we show that if $G(z) = z$ and $z \neq \bar{q}$ then $\sup_{w \in X} z(w) = 1$. Suppose that the BRW is locally isomorphic to (Y, ν) through the map g and define $h(y) := \sup_{w \in g^{-1}(y)} z(w)$. Clearly $h \in [0, 1]^Y$ and $h \circ g \geq z$ which implies that $G_Y(h) \geq h$. Indeed

$$G_Y(h|y) = \sup_{x \in g^{-1}(y)} G_Y(h|g(x)) = \sup_{x \in g^{-1}(y)} G(h \circ g|x)$$
$$\geq \sup_{x \in g^{-1}(y)} G(z|x) = \sup_{x \in g^{-1}(y)} z(x) = h(y).$$

If Y finite then we can choose $\widetilde{y} \in Y$ which minimizes

$$t(y) := \frac{1 - \bar{q}^Y(y)}{h(y) - \bar{q}^Y(y)}$$

(where $t(y) := +\infty$ if $h(y) = \bar{q}^Y(y)$); note that $t(y) \geq 1$ for all $y \in Y$ and $t(\widetilde{y}) < +\infty$. By applying the maximum principle (Proposition 2.6) to the function $1/t(y)$ (where y is ranging in the set $\{w : \bar{q}^Y(w) < 1\}$) we have that it is constant on $\{y : \widetilde{y} \to y\}$. Since $\bar{q}^Y(\widetilde{y}) < 1$ and Y is finite, then there exists y_0 such that $\widetilde{y} \to y_0$ and there is local survival at y_0 starting from y_0. Since (Y, ν) satisfies Assumption 2.3 then there exists $\bar{y} \rightleftharpoons y_0$ such that a particle living at $\bar{y}$ wpp has at least 2 children in the irreducible class of y_0. Then by taking y_0 instead of x_0 in Lemma 6.2 we have that we can find $\bar{n} \in \mathbb{N}$ such that the function

$$\phi(t) := G_Y^{(\bar{n})}(\bar{q}^Y + t(h - \bar{q}^Y)|y_0) - \bar{q}^Y(y_0) - t(h(y_0) - \bar{q}^Y(y_0))$$

is strictly convex by Lemma 6.1. Indeed $G_Y^{(\bar{n})}$ is the generating function of the BRW constructed by considering the n-th generations of the original BRW where $\bar{n}|n$ and, under our hypotheses, it satisfies equation (6.21).

Note that ϕ is well defined in $[0, t(y_0)]$ since

$$r_t(y) := \bar{q}^Y(y) + t(h(y) - \bar{q}^Y(y)) \leq \bar{q}^Y(y) + t(y_0)(h(y) - \bar{q}^Y(y)) \leq 1$$

hence $r_t \in [0, 1]^Y$ for all $t \in [0, t(y_0)]$.

Clearly every fixed point of G_Y is a fixed point of $G_Y^{(\bar{n})}$; in particular, $G^{(\bar{n})}(z) = z$ and $G_Y^{(\bar{n})}(\bar{q}^Y) = \bar{q}^Y$, whence $\phi(0) = 0$ and $\phi(1) = G_Y^{(\bar{n})}(h|y_0) - h(y_0)$. Now, using equation (3.13), $G_Y^{(\bar{n})}(h) \geq h$ and this, in turn, implies $\phi(1) \geq 0$. Since ϕ is strictly convex we have that $\phi(t) > 0$ for all $t \in (1, t(y_0)]$. If $t(y_0) > 1$ then $0 < \phi(t(y_0)) = G_Y^{(\bar{n})}(r_{t(y_0)}|y_0) - 1$ but this is a contradiction since $r_{t(y_0)} \in [0, 1]^Y$ and $G_Y^{(\bar{n})}(r_{t(y_0)}) \in [0, 1]^Y$. In the end $t(y_0) = 1$, thus $1 = h(y_0) = \sup_{w \in X} z(w)$. $\qquad\square$

Note that, from the previous proof, if the BRW on Y is irreducible then by the maximum principle we have that $(h - \bar{q}^Y)/(\mathbf{1} - \bar{q}^Y)$ is a constant function, thus $h(y) = \sup_{w \in g^{-1}(y)} z(w) = 1$ for all $y \in Y$.

Proof of Proposition 4.19. Since (X, E_μ) is irreducible we have that $q(x, y) = q(x, x)$ for all $x, y \in X$ and if $\bar{q} < \mathbf{1}$ (resp. $q(\cdot, y) < \mathbf{1}$) then $\bar{q}(x) < 1$ (resp. $q(x, y) < 1$) for all $x \in X$. Moreover, quasi transitivity implies that if $q(\cdot, y) < \mathbf{1}$ then $\sup_{x \in X} q(x, y) < 1$. Thus, according to Theorem 4.20, $q(\cdot, y) \neq \mathbf{1}$ implies $q(\cdot, y) = \bar{q}$. $\qquad\square$

Proof of Theorem 4.21. According to Section 3.2, there is strong local survival at y starting from x for the BRW $\{\eta_n\}_{n \in \mathbb{N}}$ if and only if there is a.s. local survival at y for the associated BRW with no death, that is, $\{\widehat{\eta}_n\}_{n \in \mathbb{N}}$ conditioned on $\mathcal{A}_x$ (global survival starting from x). Moreover v satisfies equation (4.17) if and only if $v_1 := T_{\bar{q}}^{-1} v$ satisfies

$$\begin{cases} \widehat{G}(v_1 | x) \geq v_1(x), & \forall x \in A^\complement, \\ v_1(x_0) > \max_{x \in A} v_1(x) & \text{for some } x_0 \in A^\complement, \end{cases}$$

which is equation (4.17) in the case of the associated BRW with no death. Hence it is enough to prove the result for the case $\rho_x(0) = 0$ for all $x \in X$.

For completeness, we sketch the proof of [31, Theorem 3.1]. Suppose that there exist a function v and a set A as in the statement of the theorem. Recall the definition of G given by equation (2.9), define $\widetilde{Q}_n := \prod_{x \in X} v(x)^{\eta_n(x)}$ and $\sigma := \min\{n \in \mathbb{N} : \sum_{x \in A} \eta_n(x) > 0\}$, where $\{\eta_n\}_{n \in \mathbb{N}}$ is a realization of the BRW. As usual $\min \emptyset := +\infty$. Let $Q_n := \widetilde{Q}_{n \wedge \sigma}$. If $\bar{v}(x) := \mathbb{E}[\widetilde{Q}_{n+1} | \eta_n = \delta_x]$ then it is easy to show that $\bar{v} = G(v)$. Using the same arguments as in [31, Theorem 3.1] we can show that $\{Q_n\}_{n \in \mathbb{N}}$ is a nonnegative supermartingale, hence there exists $Q_\infty := \lim_{n \to \infty} Q_n$ in L^1 and almost surely. Clearly $\mathbb{E}[Q_\infty] \geq \mathbb{E}[Q_0]$. If $\eta_0 := \delta_{x_0}$ where $x_0 \notin A$ satisfies the hypotheses of Theorem 4.21 and if there were strong local survival then at least one particle would hit A a.s., thus $Q_\infty \leq \max_{x \in A} v(x) < v(x_0)$ which is a contradiction. This yields the first part of the proof.

Assume now that there is no strong local survival. Fix $\bar{x} \in X$ and $A := \{\bar{x}\}$. Define $v(x) := q_0(x, A)$, the probability of never hitting A starting from x. Since the BRW is irreducible, then there is no strong local survival if and only if $v(x) > 0$ for some x. Clearly $v(\bar{x}) = 0 < v(x_0)$ for some $x_0 \notin A$ and

$$v(x) = \sum_{g \in S_X : g(\bar{x}) = 0} \mu_x(g) \prod_{y \in X} v(y)^{g(y)} \leq G(v | x), \quad \forall x \neq \bar{x}$$

and the theorem is proved. $\qquad\square$

Proof of Theorem 4.23. The proof is essentially the same as in [6, Section 3.3]. We just sketch the main steps. Let Y be the finite set onto which X can be mapped by definition of $\mathcal{F}$-BRW. Instead of the operators N and $\widetilde{N}$ we use $Mf(x) := \sum_{w \in X} m_{xw} f(w)$ (for all $x \in X$) and $\widetilde{M}f(y) := \sum_{w \in Y} \widetilde{m}_{yw} f(w)$ (for all $y \in Y$) where $m_{xw} = m_{xw}^X$ and $\widetilde{m}_{yw} = m_{yw}^Y$. These are well-defined, bounded, linear operators from $l^2(X)$ and $l^2(Y)$ into itself respectively. One can prove that $\|M\| = \rho(M) = M_s$ where $\rho(M)$ is the spectral radius of the operator (this can be done as in [6, Lemma 3.3] and [38, Lemma 2.2]). Similarly $\|\widetilde{M}\| = \rho(\widetilde{M}) = \widetilde{M}_s = \widetilde{M}_w = M_w$ (here we use the finiteness of Y).

Moreover, it can be shown that for nonamenable BRWs there exists $c > 0$ such that, for all $f \in l^2(X)$,
$$\|f\|_{D(2)} \geq c\|f\|_2,$$
where the Dirichlet norm is defined as
$$\|f\|_{D(2)} = \left(\sum_{x,y \in X} m_{xy}|f(x) - f(y)|^2 \right)^{1/2}.$$

The proof of this inequality is analogous to the one of [38, Theorem 2.6], the only difference being the presence of m_{xy} which can be easily dealt with.

The rest of the proof is the same as [6, Theorem 3.6] using the term $\sum_{x \in S, y \in S^{\complement}} m_{xy}$ instead of $|\partial_E S|$ and using the graph G^2 induced by M^2. $\qquad\square$

Proof of Theorem 4.29. Note that $\kappa(x)k_{xy}^{(n)} = \kappa(y)k_{yx}^{(n)}$ for all $x, y \in X$, $n \in \mathbb{N}$. Moreover, by the Cauchy-Schwartz inequality, for all $n \in \mathbb{N}$,

$$K_s(x_0, x_0)^{2n} \geq k_{x_0 x_0}^{(2n)} = \sum_{y \in X} k_{x_0 y}^{(n)} k_{yx_0}^{(n)} = \sum_{y \in B(x_0, n)} (k_{x_0 y}^{(n)})^2 \frac{\kappa(x_0)}{\kappa(y)} \geq \frac{\left(\sum_y k_{x_0 y}^{(n)} \right)^2}{c_n |B(x_0, n)|}.$$

Hence

$$K_w(x_0) = \liminf_n \sqrt[n]{\sum_y k_{x_0 y}^{(n)}} = \liminf_n \sqrt[2n]{\frac{\left(\sum_y k_{x_0 y}^{(n)} \right)^2}{c_n |B(x_0, n)|}} \leq K_s(x_0, x_0).$$

$\qquad\square$

Proof of Proposition 4.33. (1) and (2) follow easily from Theorems 4.3(1) and 4.9(2). As for (3), we note that $m^{(n)}(x, y) = \bar{\rho}^n p^{(n)}(x, y)$ and that the generating function Φ defined in Section 2.5 satisfies $\Phi(x, y|t) = F(x, y|t\bar{\rho})$. Thus, $\Phi(x, x|1) > 1$, which is equivalent to local survival at x, is equivalent to $F(x, x|\bar{\rho}) > 1$. $\qquad\square$

Proof of Lemma 4.34. Clearly $\prod_{i \in \mathbb{N}} (1 - \alpha_i)^{k_i} > 0$ if and only if $\sum_{i \in \mathbb{N}} k_i \log(1 - \alpha_i) > -\infty$. Observe that $\log(1 - x) \leq -x$ for all $x < 1$ hence

$$\sum_{i \in \mathbb{N}} k_i \alpha_i \leq -\sum_{i \in \mathbb{N}} k_i \log(1 - \alpha_i) < \infty.$$

If $\alpha_i \in [0, 1)$ and $k_i \geq 1$ eventually as $i \to \infty$ then there is no loss of generality by assuming that $\alpha_i \in [0, 1)$ and $k_i \geq 1$ for all i. In this case, since $k_i \geq 1$ both sides imply $\alpha_i \to 0$. Thus $\log(1 - \alpha_i) \sim -\alpha_i$ and

$$\sum_{i \in \mathbb{N}} k_i \log(1 - \alpha_i) > -\infty \iff \sum_{i \in \mathbb{N}} k_i \alpha_i < \infty.$$

$\qquad\square$

Proof of Theorem 4.37. Since μ satisfies equation (2.4) then $G(z|x) = \sum_{n=0}^{\infty} \rho_x(n)(Pz(x))^n$ where $Pz(x) = \sum_{y \in X} p(x,y)z(y)$. On the other hand, $\rho_x = \rho$ for all $x \in X$, thus $\bar{q}(x) = \bar{q}$ for all $x \in X$ where $\bar{q}$ is the smallest fixed point in $[0,1]$ of $t \mapsto G(t\mathbf{1}|x) \equiv \sum_{n=0}^{\infty} \rho(n)t^n =: F(t)$. Clearly, any fixed point z of G must satisfy the inequality $z(x) \geq \bar{q}(x) = \bar{q}$. Since $F(t) < t$ for all $t \in (\bar{q}, 1)$ then

$$z(x) = G(z|x) = F(Pz(x)) \leq Pz(x),$$

hence z is a bounded subharmonic function. It is well known that the existence of non-constant subharmonic functions which are bounded from above is equivalent to transience, thus, in the recurrent case we have necessarily $z = t\mathbf{1}$ which implies that $t = F(t)$ and $t \in \{\bar{q}, 1\}$. Suppose that $\bar{q} < 1$, since the random walk is recurrent, then $q_0(\cdot, A) \leq \bar{q}\mathbf{1}$ (for all $A \subseteq X$), hence by Remark 4.1 $q(\cdot, A) = \bar{q}\mathbf{1}$ which is equivalent to strong local survival in A. $\square$

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