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Barucci, E.; Cosso, A.

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Portfolio choices and VaR constraint with a defaultable asset*

Emilio Barucci, Andrea Cosso

Politecnico di Milano, Department of Mathematics

Via Bonardi, 9 - 20133 Milano, Italy

Abstract

Assuming a Constant Elasticity of Variance (CEV) model for the asset price, that is a defaultable asset showing the so called leverage effect (high volatility when the asset price is low), a VaR constraint reevaluated over time induces an agent more risk averse than a logarithmic utility to take more risk than in the unconstrained setting.

Keywords: VaR, Optimal portfolio, regulation, CEV.

JEL Code: G11, G21, G28.

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1 Introduction

In this paper we show that assuming a Constant Elasticity of Variance (CEV) model for the asset price, a VaR constraint reevaluated over time induces an agent to take more risk than in the unconstrained setting.

This result contributes to the literature on optimal investment and regulation that has investigated the effect of a VaR constraint on the optimal portfolio. Assuming a defaultable asset - as in a CEV model - we show that an agent more risk averse than a logarithmic utility detains a position in the risky asset larger than the one of an uncostrained agent.

The literature on this issue is not conclusive as the effect of a VaR constraint depends on the model for the asset price and on the definition of the risk constraint. According to [Basak and Shapiro (2001)], imposing a static VaR constraint, i.e., the loss refers to the difference between the initial and the terminal wealth, and assuming a constant opportunity set, i.e., a lognormal process for the asset price, we have a portfolio riskier than the one obtained without constraint. Because of the VaR constraint, the agent optimally chooses to insure against intermediate loss states and to incur losses in the worst states of the world. As a matter of fact, under a VaR constraint uninsured states are the worst states. This undesired effect is due to non coherency of VaR, assuming a coherent risk measure, e.g. the Expected Shortfall, the effect disappears and the agent chooses a less risky portfolio. [Cuoco et al. (2008), Leippold et al. (2006)] point out that the excess risk taking is also due to the static nature of the VaR constraint considered in [Basak and Shapiro (2001)]: the VaR constraint is placed in $t = 0$ and concerns the final wealth. This approach has two main drawbacks: the policy is dynamically inconsistent, i.e., the constraint is only placed at the beginning of the optimization horizon and the trader may have the incentive to change the investment policy later on; the probability of the portfolio loss is not updated as time goes, and this is different from what happens in practice, as a matter of fact financial institutions reevaluate the VaR on a daily or weekly basis. To address these problems they consider a dynamic VaR constraint: the constraint is posed $\forall t \geq 0$ for a short horizon $\tau > 0$, in the interval $[t, t+\tau]$ the portfolio is kept constant. Assuming that the agent has to satisfy the dynamic VaR constraint, its effect on the optimal investment is ambiguous. [Cuoco et al. (2008)] consider a lognormal process for the asset price and prove that a dynamic VaR constraint leads the agent to take less risk, i.e., the expected value of losses and the portfolio are lower under a VaR constraint than without. [Leippold et al. (2006)] consider a more general incomplete market model with a single risky asset whose dynamics depend on a state variable in such a way that both drift and volatility are stochastic. The complexity of the model forces the authors to consider a utility function nearly logarithmic. The effect of a VaR constraint on the portfolio strategy depends on the opportunity set dynamics. In general they cannot say that the constraint induces the agent to take less risk, they provide some examples in which it induces banks to increase their exposure in high volatility states.

In this paper we analyze the optimal investment problem with a dynamic VaR constraint as in [Cuoco et al. (2008), Leippold et al. (2006)] assuming a CEV model for the asset price. We consider a CEV model as a good choice looking for a realistic market model. We maintain market completeness and tractability removing the constant opportunity set assumption and allowing for a negative correlation between asset price and volatility. Moreover, differently from the lognormal and stochastic volatility cases, an asset following a CEV model may default (when the asset price touches the zero barrier). We derive a clear cut analysis on the effect of a VaR constraint on the investment policy: for a wide set of parameters, a VaR constraint induces an agent more (less) risk averse than a logarithmic utility to take more (less) risk than a risk unconstrained agent. As in [Leippold et al. (2006)] we provide an approximation analysis for a utility function in a neighborhood of a logarithmic utility, as optimal solution we consider the one obtained when the constraint is not binding, i.e., we evaluate the effect on the optimal portfolio of the possibility that the VaR might become binding.

The perverse effect of a VaR constraint is strong when the asset is risky or the risk premium is high. A stronger VaR constraint (low α) induces a stronger perverse effect, the only way to limit the phenomenon is to increase the constraint on the VaR in terms of the fraction of wealth, i.e., increase the capital requirement for a bank. The undesired effect disappears (a VaR constrained agent takes less risk) when the risk of default is very high, i.e., the price is small enough and the asset return-volatility correlation is strongly negative.

These results contribute to the recent debate on the destabilizing role of VaR and in particular on its role in generating the recent subprime financial crisis, see [Adrian and Shin

(2008), Adrian and Shin (2010), Danielsson, et al. (2009), Barucci and Cosso (2010)]. As a matter of fact, there are theoretical and empirical results showing that a VaR constraint is procyclical and leads to a positive correlation between asset and leverage of financial intermediaries. These features may contribute to destabilize the financial market. Showing that a VaR constraint exacerbates risky bets when the asset may default, our analysis contributes to explain why banks before the crisis detained asset backed securities that looked as catastrophe bonds, see [Coval et al. (2009), Coval et al. (2009a)].

The paper is organized as follows. In Section 2 we introduce our setting and the optimization problem. In Section 3 we derive the optimal solution for the VaR constrained problem. In Section 4 we provide a comparative statics analysis on the effect of a VaR constraint on the optimal portfolio. In Appendix A we provide the proofs of the main results.

2 The model

We consider a finite horizon $[0,T]$ model with two assets: a risk free asset and a risky asset. The peculiarity of our setting is that the risky asset is defaultable, i.e., the asset price can attain the point 0.

The risk-free asset is a bond, its price evolves according to an ordinary differential equation:

$$dB(t) = rB(t), \quad B(0) = 1,$$

where the risk-free interest rate r is a positive constant. The price of the risky asset evolves according to a CEV model, see [Cox (1975)]:

$$dS(t) = (\xi + r)S(t)dt + \sigma S(t)^{1+\beta} dW(t), \quad S(0) = s, \tag{1}$$

where the initial price s, the excess return ξ and σ are all positive constants. We assume $\beta \in (-1, 0)$ which implies that the point 0 is an attainable state for the asset price S. To guarantee uniqueness we assume that after reaching zero, the asset price remains at zero, on this point see [Delbaen and Shirakawa (2002)]. The filtered probability space governing the model is $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the natural filtration generated by a continuous unidimensional Brownian motion $\{W(t)\}_{t\in[0,T]}$.

We remark two important features of the CEV model. First of all, if the asset price evolves as in (1) then we observe the so called leverage effect, i.e., volatility is negatively correlated with asset returns and is high when the asset price is low. Indeed assuming $\beta < 0$, the diffusion coefficient $\sigma S(t)^\beta$ is inversely proportional to the price $S(t)$. Furthermore, a process like (1) is suitable to describe a default event. As a matter of fact, there is a positive probability that the asset price will reach the zero barrier and therefore that the default event will occur, see [Campi and Sbuelz (2005)].

The agent chooses the portfolio of financial assets. The portfolio weight at time t is denoted by $\boldsymbol{\pi}(t) = (1 - \pi(t), \pi(t))$, where $\pi(t)$ represents the fraction of wealth invested in the risky asset. Since the portfolio is self-financing, the wealth $V(t)$ evolves according to the following stochastic differential equation

$$dV(t) = (\pi(t)\xi + r)V(t)dt + \pi(t)\sigma S(t)^\beta V(t)dW(t), \quad V(0) = v, \tag{2}$$

where $v > 0$ is the initial wealth. The process $\pi(t)$ is admissible if $\int_0^T |\pi(s)|ds < \infty$ and the resulting wealth process $V(t)$ is such that $V(t) \geq 0 \ \forall t \in [0, T]$. If the portfolio $\pi(t)$ is admissible then we write $\pi \in \mathcal{A}$.

The agent maximizes the expected utility of the final wealth

$$E[u(V(T))].$$

In the sequel we consider a CRRA utility function:

$$u(x) = \begin{cases} \dfrac{x^\gamma - 1}{\gamma}, & \gamma < 0 \text{ and } 0 < \gamma < 1, \\ \ln x, & \gamma = 0. \end{cases}$$

The agent maximizes the expected utility subject to a dynamic VaR constraint as in [Fusai and Luciano (2001), Yiu (2004), Cuoco et al. (2008), Leippold et al. (2006)]. Given time horizon $\tau > 0$ and a confidence level $1-\alpha$, the VaR at time t with a constant portfolio over the time interval $[t, t+\tau]$ is defined as

$$\mathrm{VaR}_t^{\alpha,\tau} = \inf\{\ell \in \mathbb{R}^+ : \mathbb{P}(V(t) - \mathcal{V}(t+\tau) > \ell) \leqslant \alpha\}, \tag{3}$$

where $\mathcal{V}(t+\tau)$ is the wealth value at time $t+\tau$ assuming a constant portfolio $\boldsymbol{\pi}(t)$ in the time interval $[t, t+\tau]$. Indeed, we evaluate the VaR considering the frozen portfolio $\boldsymbol{\pi}(t)$ for the interval of time $[t, t+\tau]$.

According to the financial regulation, the VaR should be smaller than a fraction of the assets (wealth). In our analysis we follow [Cuoco et al. (2008),Leippold et al. (2006)] assuming that

$$\mathrm{VaR}_t^{\alpha,\tau} \leqslant \zeta V(t), \qquad \forall t \in [0,T], \tag{4}$$

where $\zeta \in (0,1)$. Hence, the maximum loss with probability $1-\alpha$ is smaller than a fraction of the portfolio value.

We now express the constraint (4) as a constraint on $\pi(t)$. To this end, we follow [Leippold et al. (2006)] performing an Itô-Taylor expansion of $\mathcal{V}(t+\tau)$ centered in $V(t)$. The approximation is provided in Appendix A.1 with a discussion of the order of the approximation error. Thanks to the Itô-Taylor expansion of $\mathcal{V}(t+\tau)$, the following result holds on the VaR constraint with respect to $\pi(t)$.

Proposition 1. *Using the Itô-Taylor expansion (23) of $\mathcal{V}(t+\tau)$, the VaR constraint (4) can be expressed as*

$$\pi^-(S(t)) \leqslant \pi(t) \leqslant \pi^+(S(t)), \qquad t \in [0,T], \tag{5}$$

where

$$\begin{aligned}\pi^{\pm}(S(t)) =& \frac{\xi\tau + \Phi^{-1}(\alpha)\sigma S(t)^{\beta}\sqrt{\tau}}{\sigma^2 S(t)^{2\beta}\tau} \\ &\pm \frac{\sqrt{(\xi\tau + \Phi^{-1}(\alpha)\sigma S(t)^{\beta}\sqrt{\tau})^2 + 2r\sigma^2 S(t)^{2\beta}\tau^2 - 2\ln(1-\zeta)\sigma^2 S(t)^{2\beta}\tau}}{\sigma^2 S(t)^{2\beta}\tau},\end{aligned} \tag{6}$$

and Φ is the cumulative distribution function of the standard normal distribution.

Proof. See Appendix A.1. □

Assuming that the agent has to satisfy the VaR constraint (4) $\forall t \in [0,T]$, the set of viable portfolios in t is provided by $\Pi_{ad}(t)$, i.e., portfolios that are admissible and satisfy the constraint (5) $\forall s \in [t,T]$.

Let J be the value function:

$$J(v,s,t) := \sup_{\pi \in \boldsymbol{\Pi}_{ad}(t)} \mathbb{E}[u(V(T))|V(t) = v,\, S(t) = s], \tag{7}$$

for every $(v,s,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \times [0,T]$.

An admissible portfolio $\boldsymbol{\pi}$ which maximizes the expected value (7) in $t = 0$ is called the optimal portfolio and is denoted by $\boldsymbol{\pi}^* = (1 - \pi^*, \pi^*)$. The value function and the optimal portfolio are fully characterized by an Hamilton-Jacobi-Bellman equation.

Theorem 2. *The Hamilton-Jacobi-Bellman equation for the value function J is*

$$\begin{cases} \dfrac{\partial J}{\partial t} + \displaystyle\sup_{\pi^-(s)\leqslant \pi(t) \leqslant \pi^+(s)} \Bigg\{ (\pi\xi + r)v\dfrac{\partial J}{\partial v} + (\xi + r)s\dfrac{\partial J}{\partial s} + \tfrac{1}{2}\pi^2\sigma^2 s^{2\beta} v^2 \dfrac{\partial^2 J}{\partial v^2} + \\ \qquad + \pi\sigma^2 s^{1+2\beta} v \dfrac{\partial^2 J}{\partial v \partial s} + \tfrac{1}{2}\sigma^2 s^{2+2\beta}\dfrac{\partial^2 J}{\partial s^2} \Bigg\} = 0, \qquad \forall\,(v,s,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \times (0,T), \\ J(v,s,T) = u(v), \qquad\qquad \forall\,(v,s) \in \mathbb{R}^+ \times \mathbb{R}^+. \end{cases}$$

The optimal portfolio of the risky asset has the following expression

$$\pi^*(t) = \begin{cases} \pi^-(S(t)), & \widetilde{\pi}(t) \leqslant \pi^-(S(t)), \\ \widetilde{\pi}(t), & \pi^-(S(t)) < \widetilde{\pi}(t) < \pi^+(S(t)), \\ \pi^+(S(t)), & \widetilde{\pi}(t) \geqslant \pi^+(S(t)), \end{cases} \tag{8}$$

where

$$\widetilde{\pi}(t) = -\frac{\dfrac{\partial J}{\partial v}}{V(t)\dfrac{\partial^2 J}{\partial v^2}} \frac{\xi}{\sigma^2 S(t)^{2\beta}} - \frac{S(t)\dfrac{\partial^2 J}{\partial v \partial s}}{V(t)\dfrac{\partial^2 J}{\partial v^2}}. \tag{9}$$

Since u is a CRRA utility function, the value function J can be written as follows

$$J(v,s,t)=\begin{cases}\dfrac{e^{\gamma g_\gamma(s,t)}v^\gamma-1}{\gamma}, & \gamma<0 \ \textit{and}\ 0<\gamma<1,\\ g_0(s,t)+\ln v, & \gamma=0,\end{cases} \tag{10}$$

where g_γ is a function of γ, s and t, but it doesn't depend on v and g_0 is defined as

$$g_0(s,t):=\lim_{\gamma\to 0} g_\gamma(s,t), \qquad \forall\,(s,t)\in\mathbb{R}^+\times[0,T]. \tag{11}$$

Proof. The derivation of the Hamilton-Jacobi-Bellman equation from the optimization problem follows from classical dynamic programming techniques. As far as the optimal portfolio π^* is concerned, its expression can be obtained considering the second-degree polynomial in π of the Hamilton-Jacobi-Bellman equation. As the second derivative of J with respect to v is negative, the supremum of the polynomial is attained at π^*. Indeed, for a CRRA utility J is homogeneous and has the expression given in (10). Hence, J is a concave function with respect to v, therefore the second derivative of J with respect to v is negative. □

Substituting the expression of J provided in (10) in (9) we find

$$\widetilde{\pi}(t)=\begin{cases}\dfrac{1}{1-\gamma}\dfrac{\xi}{\sigma^2 S(t)^{2\beta}}+\dfrac{\gamma S(t)}{1-\gamma}\dfrac{\partial g_\gamma}{\partial s}, & \gamma<0 \text{ and } 0<\gamma<1,\\ \dfrac{\xi}{\sigma^2 S(t)^{2\beta}}, & \gamma=0.\end{cases} \tag{12}$$

Our interest is now to compare this portfolio with that obtained without the VaR constraint and therefore to analyze the consequences of a VaR constraint on the optimal investment problem.

3 The portfolio strategy

We compare the optimal portfolio derived in the presence of the VaR constraint with the one obtained in its absence. In particular we are interested in analyzing the optimal portfolio in the absence of the VaR constraint and the portfolio $\widetilde{\pi}$, i.e., the optimal portfolio when there is a VaR constraint but is not binding.

The optimal portfolio for the optimal investment problem without VaR constraint (π^f) has been computed in [Battauz and Sbuelz (2010)]:

$$\pi^f(t) = \begin{cases} \dfrac{1}{1-\gamma}\dfrac{\xi}{\sigma^2 S(t)^{2\beta}} + \dfrac{\gamma S(t)}{1-\gamma}\dfrac{\partial g_\gamma^f}{\partial s}, & \gamma < 0 \text{ and } 0 < \gamma < 1, \\ \dfrac{\xi}{\sigma^2 S(t)^{2\beta}}, & \gamma = 0, \end{cases} \tag{13}$$

where the function g_γ^f has the following expression

$$g_\gamma^f(s,t) = \frac{1}{1-\gamma}\frac{\xi^2}{\sigma^2 s^{2\beta}}\frac{1-e^{-q(T-t)}}{2q - \left(q - 2\beta\frac{\xi + r(1-\gamma)}{1-\gamma}\right)\left(1 - e^{-q(T-t)}\right)}, \qquad \forall\,(s,t) \in \mathbb{R}^+ \times [0,T],$$

and q given by

$$q = \sqrt{4\beta^2\left(r^2 + \frac{1}{1-\gamma}\left((r+\xi)^2 - r^2\right)\right)}.$$

The first component of π^f is the myopic demand of the defaultable asset (when the optimization horizon shrinks to 0). The second component is the intertemporal non myopic demand.

The portfolio π^f is analogous to that of $\tilde{\pi}$ in (12), the difference is provided by the hedging demand. Note that considering a log-investor, that is when $\gamma = 0$, the two strategies coincide.

As in [Leippold et al. (2006)], we cannot compare $\tilde{\pi}$ and π^f for a generic CRRA utility function. We restrict our attention to a γ in a neighborhood of 0, i.e., utility in a neighborhood of a logarithmic utility. Taking the difference between $\tilde{\pi}$ and π^f when $\gamma \neq 0$, we get

$$\tilde{\pi} - \pi^f = \frac{\gamma S(t)}{1-\gamma}\left(\frac{\partial g_\gamma}{\partial s} - \frac{\partial g_\gamma^f}{\partial s}\right), \tag{14}$$

therefore the difference takes only into account the two hedging demands, namely:

$$\pi_h = \frac{\gamma S(t)}{1-\gamma}\frac{\partial g_\gamma}{\partial s} \qquad \text{and} \qquad \pi_h^f = \frac{\gamma S(t)}{1-\gamma}\frac{\partial g_\gamma^f}{\partial s}.$$

Since γ is in a neighborhood of 0, we expand the functions g_γ and g_γ^f around $\gamma = 0$. When

$\gamma = 0$ the functions g_γ and g_γ^f become g_0 and g_0^f, respectively. Hence we have:

$$g_\gamma = g_0 + O(\gamma) \qquad \text{and} \qquad g_\gamma^f = g_0^f + O(\gamma), \qquad \text{as } \gamma \to 0.$$

Consequently from (14) we get

$$\pi_h - \pi_h^f = \frac{\gamma S(t)}{1-\gamma}\left(\frac{\partial g_0}{\partial s} - \frac{\partial g_0^f}{\partial s}\right) + O(\gamma^2), \qquad \text{as } \gamma \to 0. \tag{15}$$

Therefore, the sign of the difference between π_h and π_h^f is determined by the difference between $\frac{\partial g_0}{\partial s}$ and $\frac{\partial g_0^f}{\partial s}$: if $0 < \gamma < 1$ then it's equal to the sign of $\frac{\partial g_0}{\partial s} - \frac{\partial g_0^f}{\partial s}$, otherwise, if γ is negative, it is the opposite.

To determine the sign of the difference in (15), we have to compute the difference between g_0 and g_0^f, then we can take the derivative with respect to s. g_0 and g_0^f appear in the value functions of the two problems, exploiting the expression of the value function J, we can compute the difference between g_0 and g_0^f, as shown in the following Lemma.

Lemma 3. *The difference between the functions g_0 and g_0^f is given by*

$$g_0(s,t) - g_0^f(s,t) = -\frac{1}{2}\int_t^T \mathbb{E}\Big[1_{\{\hat{\pi}(S(u))<0\}}(\sigma S(u)^\beta \hat{\pi}(S(u)))^2 \Big| S(t) = s\Big]\,\mathrm{d}u, \tag{16}$$

where

$$\hat{\pi}(S(t)) = \pi^+(S(t)) - \frac{\xi}{\sigma^2 S(t)^{2\beta}}. \tag{17}$$

Proof. See Appendix A.2. □

We can now derive the expression for $g_0(s,t) - g_0^f(s,t)$. The difference between the two portfolios when γ is in a neighborhood of 0 is provided in the following Theorem.

Theorem 4. *The difference between the two hedging demands π_h and π_h^f when γ is in a neighborhood of* 0 *is given by*

$$\pi_h - \pi_h^f = -\frac{\gamma s}{1-\gamma}\int_t^T C(u)\Big\{\int_{a(u)}^{b(u)} \Big(\xi\sqrt{\tau}\sigma^{-1}e^{-\beta(\xi+r)(u-t)}s^\beta \eta(u) y + \Phi^{-1}(\alpha)\Big)\cdot \tag{18}$$

$$\cdot \left(\frac{\Phi^{-1}(\alpha)}{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}e^{-\beta(\xi+r)(u-t)}s^{\beta}\eta(u)y+\Phi^{-1}(\alpha))^2+2r\tau-2\ln(1-\zeta)}} + \right.$$

$$\left. + \sigma \right) e^{-\frac{\eta(u)(1+\beta)^2}{2}y^2} y^{2+\frac{1}{2\beta}} I_{\frac{1}{2\beta}}(y)\, dy \Bigg\} \, du + O(\gamma^2), \qquad as\ \gamma \to 0.$$

where $I_{\frac{1}{2\beta}}$ *is the modified Bessel function of the first kind and* $C(u)$, $a(u)$, $b(u)$ *and* $\eta(u)$ *are positive functions of* u, *independent of* y, *which have the following expressions:*

$$C(u) = s^{1+\beta}(1+\beta)^{4-\frac{1}{\beta}}\eta(u)^{3+\frac{1}{4\beta}}\frac{\xi(-\beta)}{\sqrt{\tau}}e^{-\frac{(1+\beta)^{-2}+2s^{-2\beta}}{2\eta(u)}}e^{-\beta(\xi+r)(u-t)} \tag{19}$$

$$a(u) = \frac{-\Phi^{-1}(\alpha)-\sqrt{\Phi^{-1}(\alpha)^2-2r\tau+2\ln(1-\zeta)}}{\xi\sqrt{\tau}\sigma^{-1}\eta(u)}e^{\beta(\xi+r)(u-t)}s^{-\beta} \tag{20}$$

$$b(u) = \frac{-\Phi^{-1}(\alpha)+\sqrt{\Phi^{-1}(\alpha)^2-2r\tau+2\ln(1-\zeta)}}{\xi\sqrt{\tau}\sigma^{-1}\eta(u)}e^{\beta(\xi+r)(u-t)}s^{-\beta} \tag{21}$$

$$\eta(u) = \frac{\beta\sigma^2}{2(\xi+r)}\left(e^{2\beta(\xi+r)(u-t)}-1\right) \tag{22}$$

Proof. See Appendix A.2. □

4 Comparative statics

We are now in the position to analyze the effect of the VaR regulation on the optimal portfolio comparing the optimal investment policy before the VaR constraint becomes binding ($\tilde{\pi}$) with the unconstrained solution (π^f). This exercise allows us to evaluate how the anticipation that the VaR constraint might become binding in the future affects the optimal investment policy now.

To this end we determine the sign of the difference $\pi_h - \pi_h^f$ in a neighborhood of $\gamma = 0$ evaluating numerically the integral in (18). In our analysis we assume $\xi = 0.03,\ r = 0.02, \sigma = 0.15, T = 1, \tau = 10/250$ and a confidence level of the VaR $1-\alpha$ at 99% (so that $\Phi^{-1}(\alpha) = -2.32635$). The time horizon T corresponds to one year and τ to ten days. The fraction ζ of the portfolio value that appears in the VaR constraint (4) is set equal to 5%.

We can determine numerically the sign of the difference $\pi_h - \pi_h^f$ as a function of the today price s and the exponent of the CEV model β. Results are reported in Table 1, the magnitude

β \ s	0.1	1	10	100
-0.1	+	+	+	+
-0.2	+	+	+	+
-0.3	+	+	+	+
-0.4	+	+	+	+
-0.5	+	+	+	+
-0.6	+	+	+	+
-0.7	−	+	+	+
-0.8	−	+	+	+
-0.9	−	−	+	+

β \ s	0.1	1	10	100
-0.1	−	−	−	−
-0.2	−	−	−	−
-0.3	−	−	−	−
-0.4	−	−	−	−
-0.5	−	−	−	−
-0.6	−	−	−	−
-0.7	+	−	−	−
-0.8	+	−	−	−
-0.9	+	+	−	−

Table 1: Sign of the difference $\pi_h - \pi_h^f$ when $1-\gamma > 1$ (table in the left) and when $0 < 1-\gamma < 1$ (table in the right). Parameters: $\xi = 0.03$, $r = 0.02$, $\sigma = 0.15$, $T = 1$, $\tau = 10/250$, $\alpha = 0.01$ and $\zeta = 0.05$.

is small as we are considering a utility function nearly logarithmic and for the logarithmic case the difference is 0. From Table 1 we can conclude that in case of an agent more risk averse than a log-utility investor (table on the left) a VaR constraint leads to increase the holding of the risky asset, i.e., the VaR constraint induces a riskier portfolio strategy. The opposite holds true for an agent less risk averse than a log-utility investor (table on the right).

According to these results, the VaR constraint instead of preventing the agent from taking risk, encourages him. The effect is stronger as the price decreases and the β goes up in absolute value. This result shows that indeed a VaR constraint has a perverse and strong effect when the default probability is high (low asset price) and the leverage effect is strong (low β). However, for very low price and low β the effect is reversed (the effect is confirmed considering a finer grid). As a consequence, the VaR constraint works against risk only when the probability of default is very high.

As far as the other parameters are concerned, we observe that the effect of a VaR constraint goes up with the risk premium and the volatility (ξ and σ go up), as shown in Table 2 and in Table 3. The effect of a VaR constraint instead decreases with the VaR quantile α (Table 4) and with the regulatory parameter ζ (Table 5). Again, the effect is confirmed considering a finer grid.

From the above analysis we have two interesting insights:

β \ s	0.1	1	10	100
-0.1	$+$	$+$	$+$	$+$
-0.2	$+$	$+$	$+$	$+$
-0.3	$-$	$+$	$+$	$+$
-0.4	$-$	$+$	$+$	$+$
-0.5	$+$	$-$	$+$	$+$
-0.6	$-$	$-$	$+$	$+$
-0.7	$-$	$-$	$+$	$+$
-0.8	$+$	$-$	$-$	$+$
-0.9	$-$	$+$	$-$	$-$

β \ s	0.1	1	10	100
-0.1	$-$	$-$	$-$	$-$
-0.2	$-$	$-$	$-$	$-$
-0.3	$-$	$-$	$-$	$-$
-0.4	$-$	$-$	$-$	$-$
-0.5	$-$	$-$	$-$	$-$
-0.6	$+$	$-$	$-$	$-$
-0.7	$+$	$-$	$-$	$-$
-0.8	$+$	$-$	$-$	$-$
-0.9	$-$	$+$	$-$	$-$

Table 2: On the Table in the left $\xi = 0.1$, on the Table in the right $\xi = 0.005$. The plus (minus) sign represents an increase (decrease) in the difference $\pi_h - \pi_h^f$ (when $1-\gamma > 1$) with respect to the case with $\xi = 0.03$ (Table 1). The other parameters remain unchanged.

β \ s	0.1	1	10	100
-0.1	$+$	$+$	$+$	$+$
-0.2	$+$	$+$	$+$	$+$
-0.3	$+$	$+$	$+$	$+$
-0.4	$+$	$+$	$+$	$+$
-0.5	$+$	$+$	$+$	$+$
-0.6	$+$	$+$	$+$	$+$
-0.7	$+$	$+$	$+$	$+$
-0.8	$-$	$+$	$+$	$+$
-0.9	$-$	$+$	$+$	$+$

β \ s	0.1	1	10	100
-0.1	$-$	$-$	$-$	$-$
-0.2	$-$	$-$	$-$	$-$
-0.3	$-$	$-$	$-$	$-$
-0.4	$-$	$-$	$-$	$-$
-0.5	$-$	$-$	$-$	$-$
-0.6	$-$	$-$	$-$	$-$
-0.7	$-$	$-$	$-$	$-$
-0.8	$+$	$-$	$+$	$-$
-0.9	$+$	$+$	$-$	$-$

Table 3: On the Table in the left $\sigma = 0.3$, on the Table in the right $\sigma = 0.01$. The plus (minus) sign represents an increase (decrease) in the difference $\pi_h - \pi_h^f$ (when $1 - \gamma > 1$) with respect to the case with $\sigma = 0.15$ (Table 1). The other parameters remain unchanged.

β \ s	0.1	1	10	100
-0.1	$+$	$+$	$+$	$+$
-0.2	$+$	$+$	$+$	$+$
-0.3	$+$	$+$	$+$	$+$
-0.4	$+$	$+$	$+$	$+$
-0.5	$+$	$+$	$+$	$+$
-0.6	$+$	$+$	$+$	$+$
-0.7	$+$	$+$	$+$	$+$
-0.8	$-$	$+$	$+$	$+$
-0.9	$-$	$-$	$+$	$+$

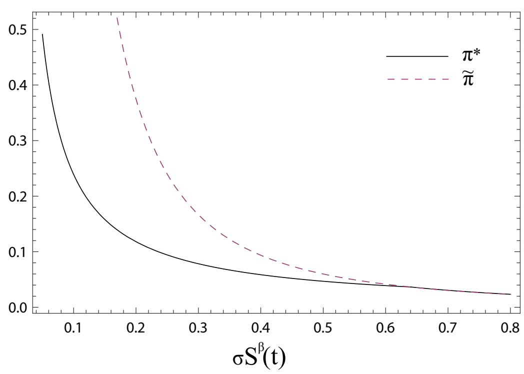
β \ s	0.1	1	10	100
-0.1	$-$	$-$	$-$	$-$
-0.2	$-$	$-$	$-$	$-$
-0.3	$-$	$-$	$-$	$-$
-0.4	$-$	$-$	$-$	$-$
-0.5	$-$	$-$	$-$	$-$
-0.6	$-$	$-$	$-$	$-$
-0.7	$-$	$-$	$-$	$-$
-0.8	$+$	$-$	$+$	$-$
-0.9	$+$	$+$	$-$	$-$

Table 4: On the Table in the left $\alpha = 0.005$, on the Table in the right $\alpha = 0.05$. The plus (minus) sign represents an increase (decrease) in the difference $\pi_h - \pi_h^f$ (when $1 - \gamma > 1$) with respect to the case with $\alpha = 0.01$ (Table 1). The other parameters remain unchanged.

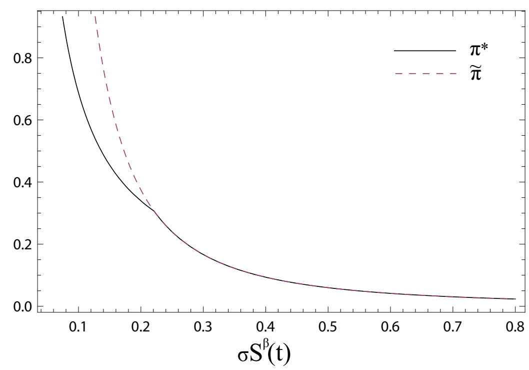
β \ s	0.1	1	10	100
-0.1	$-$	$+$	$+$	$+$
-0.2	$+$	$+$	$+$	$+$
-0.3	$+$	$+$	$+$	$+$
-0.4	$+$	$-$	$+$	$+$
-0.5	$-$	$+$	$+$	$+$
-0.6	$+$	$+$	$+$	$+$
-0.7	$-$	$+$	$-$	$+$
-0.8	$-$	$+$	$-$	$+$
-0.9	$-$	$-$	$+$	$+$

β \ s	0.1	1	10	100
-0.1	$-$	$-$	$-$	$-$
-0.2	$-$	$-$	$-$	$-$
-0.3	$-$	$-$	$-$	$-$
-0.4	$-$	$-$	$-$	$-$
-0.5	$-$	$-$	$-$	$-$
-0.6	$-$	$-$	$-$	$+$
-0.7	$-$	$-$	$-$	$-$
-0.8	$+$	$-$	$+$	$-$
-0.9	$+$	$+$	$-$	$-$

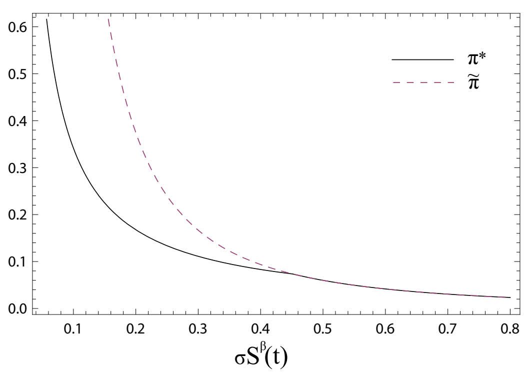
Table 5: On the Table in the left $\zeta = 0.01$, on the Table in the right $\zeta = 0.2$. The plus (minus) sign represents an increase (decrease) in the difference $\pi_h - \pi_h^f$ (when $1 - \gamma > 1$) with respect to the case with $\zeta = 0.05$ (Table 1). The other parameters remain unchanged.



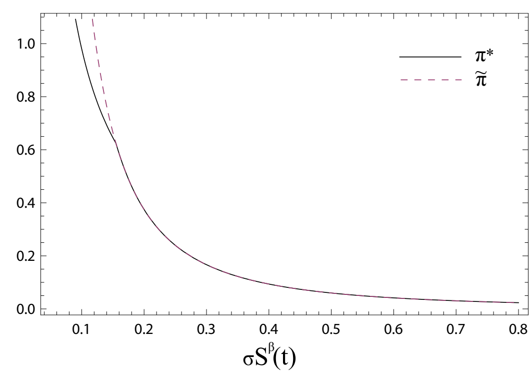
(a) $\alpha = 0.01$, $\zeta = 0.01$ and $1 - \gamma = 2$



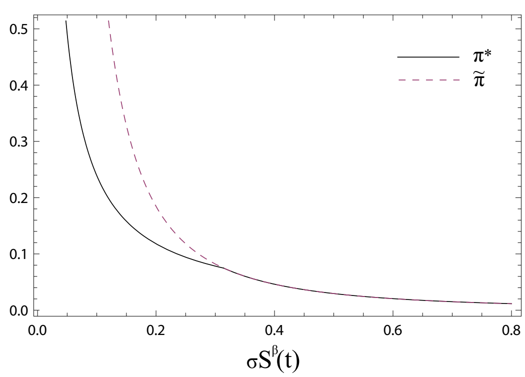
(b) $\alpha = 0.01$, $\zeta = 0.03$ and $1 - \gamma = 2$



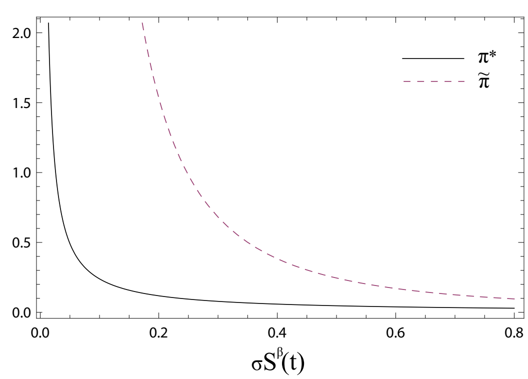
(c) $\alpha = 0.05$, $\zeta = 0.01$ and $1 - \gamma = 2$



(d) $\alpha = 0.05$, $\zeta = 0.03$ and $1 - \gamma = 2$



(e) $\alpha = 0.01$, $\zeta = 0.01$ and $1 - \gamma = 4$



(f) $\alpha = 0.01$, $\zeta = 0.01$ and $1 - \gamma = 0.5$

Figure 1: Plot of π^* (solid line) and $\widetilde{\pi}$ (dashed line). Parameters: $S(t) = 1$, $\beta = -0.7$, $\xi = 0.03$, $r = 0.02$, $T = 1$ and $\tau = 10/250$.

a) the perverse effect of a VaR constraint is strong when the asset is risky or the premium is high;

b) a stronger VaR constraint (low α) induces a strong perverse effect, the only way to limit the effect is to strengthen the constraint on the VaR in terms of fraction of wealth (higher capital requirement).

The above analysis compares the VaR constrained solution when the constraint is not binding ($\widetilde{\pi}$) with the unconstrained solution. We now compare the optimal strategy π^* with the strategy $\widetilde{\pi}$, which coincides with π^* when the VaR constraint is not binding. We want to decipher when the VaR constraint becomes binding as the volatility changes for different values of the parameters. In Figure 1 we plot π^* and $\tilde{\pi}$ when α is equal to 0.01 or 0.05, ζ is equal to 0.01 or 0.03 and $1-\gamma$ is equal to 0.5, 2 or 4. The VaR constraint and risk aversion allow to invest more in the risky asset when the volatility is low: both π^* and $\widetilde{\pi}$ are decreasing functions of $\sigma S(t)^\beta$. When $\alpha = 0.01$ and $\zeta = 0.01$ (Figure 1(a)) the VaR constraint is at the strongest level and the optimal portfolio π^* is almost always smaller than $\widetilde{\pi}$, i.e., the VaR constraint is binding. From Figure 1 we note that the constraint becomes less binding as α goes from 0.01 to 0.05 and ζ from 0.01 to 0.03. Comparing Figure 1(a), Figure 1(e) and Figure 1(f) we see that when risk aversion is high $\widetilde{\pi}$ coincides with π^* for a larger interval of values of volatility: risk aversion induces the unconstrained solution to satisfy the VaR constraint when volatility is high and the interval of volatility with coincidence of the two solutions enlarges as risk aversion increases. In all figures, the difference between the two strategies is large when the volatility is low. It seems that a high risk aversion or a high volatility renders the VaR constraint less binding in a sense that the optimal solution before the VaR constraint becomes binding turns out to be optimal. The rationale is that a high volatility or a high risk aversion renders the agent more prudent.

5 Conclusions

The recent subprime financial crisis has shown that VaR limits in banking activity may have a perverse effect generating feedback effects destabilizing the market. We have shown that indeed a VaR constraint when the asset is defaultable may induce the agent to take more risk.

This result contributes to the literature on banking regulation showing a clear cut result on VaR effects: a constraint reevaluated over time as in [Cuoco et al. (2008), Leippold et al. (2006)] may induce a risky strategy when the financial asset may default. In a way we have shown that a VaR constraint induces the agent to take a risky bet. The agent is conscious that the asset may default, a VaR limit induces him to bet until when default becomes extremely likely, exactly what happened before the financial crisis. The only way to limit the phenomenon is to strengthen capital requirements.

A Proofs

A.1 Itô-Taylor expansion of $\mathcal{V}(t+\tau)$ and proof of Proposition 1

The Itô-Taylor expansion of $\mathcal{V}(t+\tau)$ is equivalent to the Euler discretization of a stochastic differential equation with time step τ. Since $\ln \mathcal{V}(s)$ solves the following stochastic differential equation

$$\begin{cases} \mathrm{d}\ln \mathcal{V}(s) = (\pi(t)\xi + r)\mathrm{d}s - \frac{1}{2}\pi(t)^2\sigma^2 S(s)^{2\beta}\mathrm{d}s + \pi(t)\sigma S(s)^\beta \mathrm{d}W(s), & s > t, \\ \ln \mathcal{V}(t) \;\; = \ln V(t), \end{cases}$$

the Euler discretization with time step τ gives us

$$\ln \mathcal{V}(t+\tau) \approx \ln V(t) + (\pi(t)\xi + r)\tau - \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau + \pi(t)\sigma S(t)^\beta (W(t+\tau) - W(t)).$$

Consequently, $\mathcal{V}(t+\tau)$ is approximated as

$$\mathcal{V}(t+\tau) \approx \widetilde{\mathcal{V}}(t+\tau) = V(t)e^{(\pi(t)\xi+r)\tau - \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau + \pi(t)\sigma S(t)^\beta (W(t+\tau)-W(t))}. \tag{23}$$

The Euler discretization has an absolute error of order $\sqrt{\tau}$, for a proof see [Kloeden and Platen (1992)], as a consequence, the absolute error of the Itô-Taylor expansion of $\mathcal{V}(t+\tau)$ in (23) is of order $\sqrt{\tau}$:

$$\varepsilon(\tau) := \mathbb{E}[|\mathcal{V}(t+\tau) - \widetilde{\mathcal{V}}(t+\tau)|] = O(\sqrt{\tau}). \tag{24}$$

Proof of Theorem 1. From the definition of VaR in (3), we have to evaluate $\mathbb{P}(V(t) - \mathcal{V}(t+\tau) > \ell)$, where ℓ is a nonnegative real number.

Exploiting the Itô-Taylor expansion (23) of $\mathcal{V}(t+\tau)$, we have that

$$\begin{aligned} \mathbb{P}(V(t) - \widetilde{\mathcal{V}}(t+\tau) > \ell) &= \mathbb{P}\Big(V(t) - V(t)e^{(\pi(t)\xi+r)\tau - \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau + \pi(t)\sigma S(t)^\beta (W(t+\tau)-W(t))} > \ell\Big) \\ &= \mathbb{P}\Big(e^{(\pi(t)\xi+r)\tau - \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau + \pi(t)\sigma S(t)^\beta (W(t+\tau)-W(t))} < 1 - \frac{\ell}{V(t)}\Big) \\ &= \mathbb{P}\Big((\pi(t)\xi + r)\tau - \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau + \pi(t)\sigma S(t)^\beta (W(t+\tau) \end{aligned}$$

$$
\begin{aligned}
&\qquad - W(t)) < \ln\left(1 - \frac{\ell}{V(t)}\right)\Bigg) \\
&= \mathbb{P}\Bigg(W(t+\tau) - W(t) < \frac{\ln(1 - \frac{\ell}{V(t)}) - (\pi(t)\xi + r)\tau + \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau}{\pi(t)\sigma S(t)^{\beta}}\Bigg).
\end{aligned}
$$

Since $W(t+\tau) - W(t)$ is distributed as $\sqrt{\tau}Z$, where Z is a standard normal random variable, we obtain

$$
\mathbb{P}(V(t) - \widetilde{\mathcal{V}}(t+\tau) > \ell) = \mathbb{P}\Bigg(Z < \frac{\ln(1 - \frac{\ell}{V(t)}) - (\pi(t)\xi + r)\tau + \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau}{\pi(t)\sigma S(t)^{\beta}\sqrt{\tau}}\Bigg).
$$

Set $\mathbb{P}(V(t) - \widetilde{\mathcal{V}}(t+\tau) > \ell) = \alpha$, we can find the value of ℓ:

$$
\frac{\ln(1 - \frac{\ell}{V(t)}) - (\pi(t)\xi + r)\tau + \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau}{\pi(t)\sigma S(t)^{\beta}\sqrt{\tau}} = \Phi^{-1}(\alpha),
$$

and therefore

$$
\mathrm{VaR}_t^{\alpha,\tau} = V(t)\Big(1 - e^{\Phi^{-1}(\alpha)\pi(t)\sigma S(t)^{\beta}\sqrt{\tau} + (\pi(t)\xi + r)\tau - \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau}\Big).
$$

As a consequence, the VaR constraint (4) becomes

$$
1 - e^{\Phi^{-1}(\alpha)\pi(t)\sigma S(t)^{\beta}\sqrt{\tau} + (\pi(t)\xi + r)\tau - \frac{1}{2}\pi(t)^2\sigma^2 S(t)^{2\beta}\tau} \leqslant \zeta
$$

which yields

$$
\pi(t)^2\sigma^2 S(t)^{2\beta}\tau - 2\pi(t)(\Phi^{-1}(\alpha)\sigma S(t)^{\beta}\sqrt{\tau} + \xi\tau) + 2\ln(1-\zeta) - 2r\tau \leqslant 0
$$

and therefore we obtain

$$
\pi^{-}(S(t)) \leqslant \pi(t) \leqslant \pi^{+}(S(t)),
$$

where

$$
\pi^{\pm}(S(t)) = \frac{\xi\tau + \Phi^{-1}(\alpha)\sigma S(t)^{\beta}\sqrt{\tau}}{\sigma^2 S(t)^{2\beta}\tau}
$$

$$\pm \frac{\sqrt{(\xi\tau + \Phi^{-1}(\alpha)\sigma S(t)^{\beta}\sqrt{\tau})^2 + 2r\sigma^2 S(t)^{2\beta}\tau^2 - 2\ln(1-\zeta)\sigma^2 S(t)^{2\beta}\tau}}{\sigma^2 S(t)^{2\beta}\tau}.$$

□

A.2 Proof of Lemma 3 and of Theorem 4

Proof of Lemma 3. As shown in (10), the value function of the log-investor ($\gamma = 0$) is given by

$$J(v, s, t) = g_0(s, t) + \ln v.$$

We can prove that a similar expression holds true in the absence of the VaR constraint:

$$J^f(v, s, t) = g_0^f(s, t) + \ln v,$$

where J^f represents the value function for the optimal investment problem without the VaR constraint.

By definition, we have

$$J(v, s, t) = \mathbb{E}[\ln(V^*(T))|V^*(t) = v,\ S(t) = s]$$

and

$$J^f(v, s, t) = \mathbb{E}[\ln(V^f(T))|V^f(t) = v,\ S(t) = s]$$

where $V^*(T)$ is the wealth at time T in the presence of the VaR constraint, and $V^f(T)$ is the wealth at time T in the absence of the VaR constraint. From (2) we have that $V^*(T)$ and $V^f(T)$ are given by:

$$V^*(T) = ve^{\int_t^T \left(\pi^*(S(u))\xi + r - \frac{1}{2}\pi^*(S(u))^2\sigma^2 S^{2\beta}(u)\right)\mathrm{d}u + \int_t^T \pi^*(S(u))\sigma S^{\beta}(u)\,\mathrm{d}W(u)}$$

and

$$V^f(T) = ve^{\int_t^T \left(\pi^f(S(u))\xi + r - \frac{1}{2}\pi^f(S(u))^2\sigma^2 S^{2\beta}(u)\right)\mathrm{d}u + \int_t^T \pi^f(S(u))\sigma S^{\beta}(u)\,\mathrm{d}W(u)}$$

where, from (8) and (12), we have

$$\pi^*(S(t)) = \begin{cases} \dfrac{\xi}{\sigma^2 S(t)^{2\beta}}, & \dfrac{\xi}{\sigma^2 S(t)^{2\beta}} < \pi^+(S(t)), \\ \pi^+(S(t)), & \dfrac{\xi}{\sigma^2 S(t)^{2\beta}} \geqslant \pi^+(S(t)), \end{cases}$$

as $\dfrac{\xi}{\sigma^2 S(t)^{2\beta}}$ is always greater than $\pi^-(S(t))$. From (13), we get

$$\pi^f(S(t)) = \frac{\xi}{\sigma^2 S(t)^{2\beta}}.$$

By inserting the expression of $V^*(T)$ in J, we obtain

$$J(v,s,t) = \mathbb{E}[\ln(V^*(T))|V^*(t) = v,\, S(t) = s] = \ln v + \\ + \int_t^T \mathbb{E}[\pi^*(S(u))\xi + r - \tfrac{1}{2}\pi^*(S(u))^2\sigma^2 S^{2\beta}(u)|S(t) = s]\,\mathrm{d}u.$$

From the expression of J given at the beginning of the proof, we deduce that

$$g_0(s,t) = \int_t^T \mathbb{E}[\pi^*(S(u))\xi + r - \tfrac{1}{2}\pi^*(S(u))^2\sigma^2 S^{2\beta}(u)|S(t) = s]\,\mathrm{d}u. \tag{25}$$

Analogously it can be proved that g_0^f is given by

$$g_0^f(s,t) = \int_t^T \mathbb{E}[\pi^f(S(u))\xi + r - \tfrac{1}{2}\pi^f(S(u))^2\sigma^2 S^{2\beta}(u)|S(t) = s]\,\mathrm{d}u. \tag{26}$$

Taking the difference between g_0 and g_0^f, we get

$$g_0(s,t) - g_0^f(s,t) =$$

$$\int_t^T \mathbb{E}\Big[\Big(\pi^*(S(u)) - \pi^f(S(u))\Big)\xi - \tfrac{1}{2}\Big(\pi^*(S(u))^2 - \pi^f(S(u))^2\Big)\sigma^2 S^{2\beta}(u)\Big|S(t) = s\Big]\,\mathrm{d}u =$$

$$= \int_t^T \mathbb{E}\Big[\Big(\pi^*(S(u)) - \pi^f(S(u))\Big)\Big(\xi - \tfrac{1}{2}\Big(\pi^*(S(u)) + \pi^f(S(u))\Big)\sigma^2 S^{2\beta}(u)\Big)\Big|S(t) = s\Big]\,\mathrm{d}u.$$

We define $\hat{\pi}(S(t)) := \pi^+(S(t)) - \pi^f(S(t))$. Consequently, we have

$$\pi^*(S(t)) - \pi^f(S(t)) = 1_{\{\hat{\pi}(S(t))<0\}}\hat{\pi}(S(t))$$

and

$$\pi^*(S(t)) + \pi^f(S(t)) = 1_{\{\hat{\pi}(S(t))<0\}}\hat{\pi}(S(t)) + 2\pi^f(S(t)) = 1_{\{\hat{\pi}(S(t))<0\}}\hat{\pi}(S(t)) + 2\frac{\xi}{\sigma^2 S(t)^{2\beta}}.$$

Hence, we have

$$\begin{aligned} g_0(s,t) - g_0^f(s,t) = \\ \int_t^T \mathbb{E}\Big[1_{\{\hat{\pi}(S(u))<0\}}\hat{\pi}(S(u))\Big(\xi - \tfrac{1}{2}\Big(1_{\{\hat{\pi}(S(u))<0\}}\hat{\pi}(S(u)) + 2\tfrac{\xi}{\sigma^2 S^{2\beta}(u)}\Big)\sigma^2 S^{2\beta}(u)\Big)\Big|S(t)=s\Big]\,\mathrm{d}u \\ = -\frac{1}{2}\int_t^T \mathbb{E}\Big[1_{\{\hat{\pi}(S(u))<0\}}(\sigma S^\beta(u)\hat{\pi}(S(u)))^2\Big|S(t)=s\Big]\,\mathrm{d}u, \end{aligned}$$

as stated in the Lemma. □

Proof of Theorem 4. From Lemma 3, we know the expression of the difference $g_0 - g_0^f$. We begin by taking the derivative of $S(u)^\beta\hat{\pi}(S(u))$ with respect to s. By (17) and (6) we get

$$\begin{aligned} \frac{\partial(S(u)^\beta\hat{\pi}(S(u)))}{\partial s} &= \frac{\partial\Big(S(u)^\beta\Big(\pi^+(S(u)) - \frac{\xi}{\sigma^2 S(u)^{2\beta}}\Big)\Big)}{\partial s} \\ &= \frac{\partial\Big(S(u)^\beta\Big(\frac{\Phi^{-1}(\alpha)}{\sigma S(u)^\beta\sqrt{\tau}}\Big)\Big)}{\partial s} + \\ &\quad + \frac{\partial\Big(S(u)^\beta\Big(\frac{\sqrt{(\xi\tau+\Phi^{-1}(\alpha)\sigma S(u)^\beta\sqrt{\tau})^2+2r\sigma^2 S(u)^{2\beta}\tau^2-2\ln(1-\zeta)\sigma^2 S(u)^{2\beta}\tau}}{\sigma^2 S(u)^{2\beta}\tau}\Big)\Big)}{\partial s} \\ &= \frac{\partial\Big(\frac{\Phi^{-1}(\alpha)}{\sigma\sqrt{\tau}}\Big)}{\partial s} + \\ &\quad + \frac{\partial\Big(\frac{\sqrt{(\xi\tau+\Phi^{-1}(\alpha)\sigma S(u)^\beta\sqrt{\tau})^2+2r\sigma^2 S(u)^{2\beta}\tau^2-2\ln(1-\zeta)\sigma^2 S(u)^{2\beta}\tau}}{\sigma S(u)^\beta\tau}\Big)}{\partial s} \\ &= \frac{\partial\Big(\frac{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta}+\Phi^{-1}(\alpha))^2+2r\tau-2\ln(1-\zeta)}}{\sqrt{\tau}}\Big)}{\partial s} \\ &= \frac{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta}+\Phi^{-1}(\alpha))\xi(-\beta)\sigma^{-1}S(u)^{-\beta-1}}{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta}+\Phi^{-1}(\alpha))^2+2r\tau-2\ln(1-\zeta)}}\frac{\partial S(u)}{\partial s}. \end{aligned}$$

As a consequence, the derivative of (16) with respect to s becomes

$$\frac{\partial g_0}{\partial s} - \frac{\partial g_0^f}{\partial s} = -\int_t^T \mathbb{E}\Bigg[1_{\{\hat{\pi}(S(u))<0\}}\sigma^2 S(u)^\beta \hat{\pi}(S(u)) \cdot \\ \cdot \frac{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))\xi(-\beta)\sigma^{-1}S(u)^{-\beta-1}}{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta)}}\frac{\partial S(u)}{\partial s}\Bigg| S(t) = s\Bigg]\,\mathrm{d}u. \tag{27}$$

Since

$$S(u)^\beta \hat{\pi}(S(u)) = \frac{\Phi^{-1}(\alpha)}{\sigma\sqrt{\tau}} + \frac{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta)}}{\sqrt{\tau}},$$

we get

$$\frac{\partial g_0}{\partial s} - \frac{\partial g_0^f}{\partial s} = -\int_t^T \mathbb{E}\Bigg[1_{\{\hat{\pi}(S(u))<0\}}\frac{\xi(-\beta)}{\sqrt{\tau}}S(u)^{-\beta-1}\frac{\partial S(u)}{\partial s}(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))\Bigg(\sigma + \\ + \frac{\Phi^{-1}(\alpha)}{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta)}}\Bigg)\Bigg| S(t) = s\Bigg]\,\mathrm{d}u. \tag{28}$$

Now we have to find the expression of the derivative of $S(u)$ with respect to s and its density function. We start determining the derivative of $S(u)$ with respect to s. By (1) we know that $S(u)$ solves the following stochastic integral equation

$$S(u) = s + \int_t^u (\xi + r)S(u)\,\mathrm{d}u + \int_t^u \sigma S(u)^{1+\beta}\,\mathrm{d}W(u).$$

Taking the derivative with respect to s, we get

$$\frac{\partial S(u)}{\partial s} = 1 + \int_t^u (\xi + r)\frac{\partial S(u)}{\partial s}\,\mathrm{d}u + \int_t^u \sigma(1+\beta)S(u)^\beta\frac{\partial S(u)}{\partial s}\,\mathrm{d}W(u),$$

which in differential form becomes

$$\begin{cases} \mathrm{d}\Big(\dfrac{\partial S(u)}{\partial s}\Big) = (\xi + r)\dfrac{\partial S(u)}{\partial s}\mathrm{d}u + \sigma(1+\beta)S(u)^\beta\dfrac{\partial S(u)}{\partial s}\mathrm{d}W(u), & u > t, \\ \dfrac{\partial S(t)}{\partial s} = 1. & \end{cases}$$

Considering the logarithm of the derivative, we get the following stochastic differential equation

$$\begin{cases} \mathrm{d}\ln\left(\dfrac{\partial S(u)}{\partial s}\right) = \left(\xi + r - \dfrac{1}{2}\sigma^2(1+\beta)^2 S(u)^{2\beta}\right)\mathrm{d}u + \sigma(1+\beta)S(u)^{\beta}\mathrm{d}W(u), \qquad u > t, \\ \ln\left(\dfrac{\partial S(t)}{\partial s}\right) = 0. \end{cases}$$

As a consequence, the derivative of $S(u)$ with respect to s is given by

$$\frac{\partial S(u)}{\partial s} = e^{(\xi+r)(u-t) - \frac{1}{2}\sigma^2(1+\beta)^2\int_t^u S(u)^{2\beta}\,\mathrm{d}u + \sigma(1+\beta)\int_t^u S(u)^{\beta}\,\mathrm{d}W(u)}. \tag{29}$$

This expression is very similar to that of $S(u)$. We want to prove that $\frac{\partial S(u)}{\partial s}$ is a random variable with a distribution law related to that of $S(u)$. In particular, from [Delbaen and Shirakawa (2002)] we know that $S(u)$ can be expressed as

$$S(u) = e^{(\xi+r)(u-t)}\left(X^{(\frac{1}{\beta}+2)}(\eta(u))\right)^{-\frac{1}{2\beta}}, \tag{30}$$

where $X^{(\frac{1}{\beta}+2)}$ is a $\left(\frac{1}{\beta}+2\right)$-dimensional squared Bessel process and $\eta(u)$ is given in (22). The random variable $X^{(\frac{1}{\beta}+2)}(\eta(u))$ has the following density function

$$f_X(x) = \frac{1}{2}\sqrt{s}e^{-\frac{s^{-2\beta}+x}{2\eta(u)}}x^{\frac{1}{4\beta}}I_{\frac{1}{2\beta}}\left(\frac{s^{-\beta}\sqrt{x}}{\eta(u)}\right), \tag{31}$$

where $I_{\frac{1}{2\beta}}$ is the modified Bessel function of the first kind. As mentioned above, $\frac{\partial S(u)}{\partial s}$ has an expression very similar to that of $S(u)$, indeed it can be seen as the price of an asset, whose price today is equal to 1 instead of s and whose volatility is $\sigma(1+\beta)S(u)^{\beta}$ instead of $\sigma S(u)^{\beta}$. Therefore, we have

$$\frac{\partial S(u)}{\partial s} = e^{(\xi+r)(u-t)}\left(Z^{(\frac{1}{\beta}+2)}(\eta(u)(1+\beta)^2)\right)^{-\frac{1}{2\beta}},$$

where $Z^{(\frac{1}{\beta}+2)}$ is a $\left(\frac{1}{\beta}+2\right)$-dimensional squared Bessel process as well. To get the density function of $Z^{(\frac{1}{\beta}+2)}(\eta(u)(1+\beta)^2)$ from the density function of $X^{(\frac{1}{\beta}+2)}(\eta(u))$ we need only to swap s for 1 and $\eta(u)$ for $\eta(u)(1+\beta)^2$ in (31). Hence, the density function of $Z^{(\frac{1}{\beta}+2)}(\eta(u)(1+\beta)^2)$

is given by

$$f_Z(z) = \frac{1}{2} e^{-\frac{1+z}{2\eta(u)(1+\beta)^2}} z^{\frac{1}{4\beta}} I_{\frac{1}{2\beta}}\Big(\frac{\sqrt{z}}{2\eta(u)(1+\beta)^2}\Big).$$

We can express f_Z in terms of f_X; indeed, set $x = z(1+\beta)^{-4}s^{2\beta}$, we get

$$\begin{aligned} f_Z(z) &= \frac{1}{2} e^{-\frac{(1+\beta)^{-2}+s^{-2\beta}(1+\beta)^2 x}{2\eta(u)}} \Big(s^{-2\beta}(1+\beta)^4\Big)^{\frac{1}{4\beta}} x^{\frac{1}{4\beta}} I_{\frac{1}{2\beta}}\Big(\frac{s^{-\beta}\sqrt{x}}{\eta(u)}\Big) \\ &= \frac{1}{s}(1+\beta)^{\frac{1}{\beta}} e^{-\frac{(1+\beta)^{-2}+s^{-2\beta}}{2\eta(u)}} e^{\frac{1-s^{-2\beta}(1+\beta)^2}{2\eta(u)} x} f_X(x) \end{aligned}$$

As a consequence, to calculate the expected value of $\frac{\partial S(u)}{\partial s}$, we can express it in terms of $S(u)$:

$$\begin{aligned} \mathbb{E}\Big[\frac{\partial S(u)}{\partial s}\Big| S(t)=s\Big] &= \int_0^\infty e^{(\xi+r)(u-t)} z^{-\frac{1}{2\beta}} f_Z(z)\,\mathrm{d}z = \int_0^\infty e^{-\frac{(1+\beta)^{-2}+s^{-2\beta}}{2\eta(u)}} \cdot \\ &\quad \cdot e^{\frac{1-s^{-2\beta}(1+\beta)^2}{2\eta(u)} x} e^{(\xi+r)(u-t)} x^{-\frac{1}{2\beta}} f_X(x) s^{-2\beta}(1+\beta)^{4-\frac{1}{\beta}}\,\mathrm{d}x \\ &= \mathbb{E}\Big[e^{-\frac{(1+\beta)^{-2}+s^{-2\beta}}{2\eta(u)}} S(u) e^{\frac{1-s^{-2\beta}(1+\beta)^2}{2\eta(u)} S(u)^{-2\beta} e^{2\beta(\xi+r)(u-t)}} \cdot \\ &\quad \cdot s^{-2\beta}(1+\beta)^{4-\frac{1}{\beta}} \Big| S(t)=s\Big]. \end{aligned}$$

Hence, the derivative of $g_0 - g_0^f$ with respect to s becomes

$$\begin{aligned} \frac{\partial g_0}{\partial s} - \frac{\partial g_0^f}{\partial s} = -\int_t^T & s^{-2\beta}(1+\beta)^{4-\frac{1}{\beta}} \frac{\xi(-\beta)}{\sqrt{\tau}} e^{-\frac{(1+\beta)^{-2}+s^{-2\beta}}{2\eta(u)}} \mathbb{E}\Big[1_{\{\hat{\pi}(S(u))<0\}} S(u)^{-\beta} \cdot \\ & \cdot e^{\frac{1-s^{-2\beta}(1+\beta)^2}{2\eta(u)} S(u)^{-2\beta} e^{2\beta(\xi+r)(u-t)}} (\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))\Bigg(\sigma + \\ & + \frac{\Phi^{-1}(\alpha)}{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta)}}\Bigg)\Bigg| S(t)=s\Bigg]\,\mathrm{d}u. \end{aligned} \tag{32}$$

Finally we find the values of $S(u)$ for which $\hat{\pi}(S(u)) < 0$:

$$\begin{aligned} \hat{\pi}(S(u)) = \pi^+(S(u)) - \frac{\xi}{\sigma^2 S(u)^{2\beta}} &= \frac{\Phi^{-1}(\alpha)}{\sigma S(u)^\beta \sqrt{\tau}} + \\ &+ \frac{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta)}}{\sigma S(u)^\beta\sqrt{\tau}} < 0, \end{aligned}$$

which yields

$$\Phi^{-1}(\alpha) + \sqrt{(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta)} < 0,$$

therefore we get

$$(\xi\sqrt{\tau}\sigma^{-1}S(u)^{-\beta} + \Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta) < \Phi^{-1}(\alpha)^2,$$

and

$$\xi^2\tau\sigma^{-2}S(u)^{-2\beta} + 2\xi\sqrt{\tau}\Phi^{-1}(\alpha)\sigma^{-1}S(u)^{-\beta} + 2r\tau - 2\ln(1-\zeta) < 0.$$

Hence, solving this inequality for the unknown $S(u)^{-\beta}$, we find the following pair of inequalities:

$$\begin{aligned} A := \frac{-\Phi^{-1}(\alpha) - \sqrt{\Phi^{-1}(\alpha)^2 - (2r\tau - 2\ln(1-\zeta))}}{\xi\sqrt{\tau}\sigma^{-1}} &< S(u)^{-\beta} < \\ &< \frac{-\Phi^{-1}(\alpha) + \sqrt{\Phi^{-1}(\alpha)^2 - (2r\tau - 2\ln(1-\zeta))}}{\xi\sqrt{\tau}\sigma^{-1}} =: B. \end{aligned} \tag{33}$$

Exploiting (30) and (31), (32) can be rewritten in the following way

$$\begin{aligned} \frac{\partial g_0}{\partial s} - \frac{\partial g_0^f}{\partial s} = -\int_t^T & s^{-2\beta}(1+\beta)^{4-\frac{1}{\beta}}\frac{\xi(-\beta)}{\sqrt{\tau}}e^{-\frac{(1+\beta)^{-2}+s^{-2\beta}}{2\eta(u)}}\Bigg\{\int_{A^2e^{2\beta(\xi+r)(u-t)}}^{B^2e^{2\beta(\xi+r)(u-t)}}\sqrt{x}\cdot \\ &\cdot\left(\sigma + \frac{\Phi^{-1}(\alpha)}{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}e^{-\beta(\xi+r)(u-t)}\sqrt{x} + \Phi^{-1}(\alpha))^2 + 2r\tau - 2\ln(1-\zeta)}}\right)\cdot \\ &\cdot e^{-\beta(\xi+r)(u-t)}e^{\frac{1-s^{-2\beta}(1+\beta)^2}{2\eta(u)}x}(\xi\sqrt{\tau}\sigma^{-1}e^{-\beta(\xi+r)(u-t)}\sqrt{x} + \Phi^{-1}(\alpha))\cdot \\ &\cdot\frac{1}{2}\sqrt{s}e^{-\frac{s^{-2\beta}+x}{2\eta(u)}}x^{\frac{1}{4\beta}}I_{\frac{1}{2\beta}}\left(\frac{s^{-\beta}\sqrt{x}}{\eta(u)}\right)\mathrm{d}x\Bigg\}\mathrm{d}u. \end{aligned} \tag{34}$$

Let y be defined as

$$y := \frac{s^{-\beta}\sqrt{x}}{\eta(u)}, \tag{35}$$

then we obtain

$$a(u) := Ae^{\beta(\xi+r)(u-t)}\frac{s^{-\beta}}{\eta(u)} < y < Be^{\beta(\xi+r)(u-t)}\frac{s^{-\beta}}{\eta(u)} =: b(u),$$

where $a(u)$ and $b(u)$ are also reported in (20) and (21), respectively.

Now we change variable inside the integral in (34), from x to y, obtaining the following expression:

$$\begin{aligned}\frac{\partial g_0}{\partial s} - \frac{\partial g_0^f}{\partial s} = -\int_t^T s^{1+\beta}(1+\beta)^{4-\frac{1}{\beta}}\eta(u)^{3+\frac{1}{4\beta}}\frac{\xi(-\beta)}{\sqrt{\tau}}e^{-\frac{(1+\beta)^{-2}+2s^{-2\beta}}{2\eta(u)}}e^{-\beta(\xi+r)(u-t)}\cdot \\ \cdot\left\{\int_{a(u)}^{b(u)}\left(\sigma + \frac{\Phi^{-1}(\alpha)}{\sqrt{(\xi\sqrt{\tau}\sigma^{-1}e^{-\beta(\xi+r)(u-t)}s^{\beta}\eta(u)y+\Phi^{-1}(\alpha))^2+2r\tau-2\ln(1-\zeta)}}\right)\cdot\right. \\ \left.\cdot\left(\xi\sqrt{\tau}\sigma^{-1}e^{-\beta(\xi+r)(u-t)}s^{\beta}\eta(u)y+\Phi^{-1}(\alpha)\right)e^{-\frac{\eta(u)(1+\beta)^2}{2}y^2}y^{2+\frac{1}{2\beta}}I_{\frac{1}{2\beta}}(y)\,\mathrm{d}y\right\}\mathrm{d}u.\end{aligned} \tag{36}$$

Introducing $C(u)$, as in (19), we get the thesis. $\square$

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