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Does an equity holding tax help to stabilize a VaR regulated financial market?*

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Abstract

We investigate the capability of an equity holding tax to stabilize a VaR regulated financial market. We show that a VaR constraint induces high volatility in a distressed financial market, the phenomenon is not observed in a market with risk averse unregulated traders. A tax on equity holding smoothes the peak of volatility and stabilizes the market at the cost of a generalized higher volatility.

Keywords: equity holding tax, VaR, financial stability, volatility.
JEL codes: G11, G18.

1 Introduction

Large part of the recent financial regulation has been inspired by the idea that market and financial deregulation are beneficial to the financial system and to the economy. The argument is based on a classical result from welfare theory showing that perfect competition and complete markets allow to reach a Pareto optimal allocation of goods and of risk. In this framework financial stability is guaranteed by decentralized controls: traders are allowed to

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trade in all markets subject to a constraint on their activity, i.e, they have to evaluate the risk of their portfolio and to satisfy a constraint in terms of risk weighted assets and capital. The idea is that decentralized controls on traders' activity in isolation guarantee financial stability of the system as a whole.

The financial crisis has cast doubts on this "Theorem". On one hand credit risk has been transformed in market risk and has not been evaluated correctly by financial intermediaries and traders, on the other there is evidence that regulation may have a destabilizing effect either allowing a poor evaluation of risk and regulatory arbitrage or creating the conditions for an endogenous destabilizing response from market participants.

The relation between regulation and market behavior may be perverse, rules may induce a behavior that destabilizes the market. As a matter of fact, decentralized regulation is mainly developed in a partial equilibrium setting, it doesn't consider the equilibrium framework and the endogenous response of agents to regulation and to the behavior of other agents. Considering actual regulation and in particular risk constraints based on VaR, this point has been made clear by [Adrian and Shin (2008), Adrian and Shin (2010), Danielsson, et al. (2009)]: a VaR constraint leads traders to a demand of risky assets increasing in their price, the outcome is that in bad times traders are forced to sell assets inducing a negative spiral in the market with a destabilizing effect. [Danielsson, et al. (2009)] analyze the feedback effect of a VaR regulation on financial market volatility with heterogeneous traders. They show that in a market with VaR constrained rational traders and passive traders (traders providing a downward sloping demand function) there is a strong feedback effect: traders constrained by a VaR limit play a destabilizing role as they detain wealth in the risky asset equal to a fraction of their wealth. In this setting, volatility becomes endogenous with a hump-shape in the wealth of rational traders: volatility is high when financial wealth of VaR constrained traders is limited and then decreases as their wealth goes up. According to this result, VaR based regulation is not efficient: it renders a quiet market only when financial wealth is high with no fear of a crisis.

In this paper we investigate the effect of equity holding taxation in a VaR regulated market. In the post financial crisis debate, a number of economists, regulators and politicians have claimed that a tax on risky financial assets (transactions or holdings) may limit speculation with positive relapses on the economy and a lower volatility, e.g. see [International Monetary Fund (2010)]. This argument has not been fully proved. For example the literature has shown that the effect of a tax on financial transactions is ambiguous on a theoretical and an empirical ground: we have less noise and manipulation in the market but also less dissemination of information and activity equilibrating the market when the price differs from the no arbitrage solution. As a consequence, the effect may be perverse, see [Heaton and Lo (1995),Schwert and Seguin (1993),Song and Zhang (2005),Shi and Xu (2009),Dow and Rahi (2000)]. Empirically the effect on volatility is ambiguous, in many cases the analysis has shown an increase of volatility after the introduction of a tax on financial transactions, e.g. see [Roll (1989),Umlauf (1993)]. A tax on holding risky activities seems to be more efficient because it changes the asset relative prices affecting directly their demand. The claim is that a tax on financial holding should smooth the volatility peak induced by a VaR regulation. We prove that the claim holds true in a VaR regulated market at the cost of a generalized higher volatility.

There is little research on this issue. To evaluate the role of a tax on financial holding we investigate a VaR regulated financial market in two steps. First, we compare the VaR constrained market model with a market populated by risk averse traders and no risk constraint. [Danielsson, et al. (2009)] claim that a VaR constraint induces the agents to behave as risk averse traders. This claim contrasts with theoretical evidence showing that a VaR constraint leads agents to detain a portfolio riskier than that of risk averse agents, see [Basak and Shapiro (2001),Fusai and Luciano (2001),Leippold et al. (2006),Barucci and Cosso (2011)]. We show that the portfolio of risk neutral-VaR constrained traders looks different from the portfolio of unregulated risk averse traders and that VaR regulation plays a crucial role to generate the volatility peak. With a VaR constraint we observe a volatility peak which is not observed in a market with risk averse traders. As the VaR confidence level decreases (probability of loss on the right hand side), the maximum of the volatility is observed for a higher level of wealth and the peak level first decreases and then increases, i.e., there is an optimal VaR constraint level minimizing the volatility peak. For a large set of parameter values the volatility peak turns out to be higher than the volatility level observed with risk averse traders, but for a large enough wealth the reverse holds true. A weakly binding constraint (large VaR confidence level) renders volatility higher than the risk averse trader market for a low level of wealth, a strongly binding constraint (small VaR confidence level) renders volatility higher than that observed in a risk averse trader model for a large wealth.

The destabilizing effect of a VaR constraint comes from the different shape of the portfolio of rational traders: the risk averse portfolio increases linearly in wealth but it is almost constant, instead the VaR constrained portfolio is increasing with a convex shape. We show that the fraction of rational VaR constrained traders plays an asymmetric effect: a large fraction destabilizes the market for a low wealth, a small fraction destabilizes the market for a large wealth. This confirms that a VaR constraint doesn't play a positive effect to prevent a financial crisis.

Then, we analyze the effect of a tax on equity holding considering heterogeneous traders. We abstract from information matters and we simply consider the interaction of heterogeneous agents: rational traders, liquidity/passive traders. Rational traders are subject to a VaR constraint. The result is in favor of risky asset taxation. Taxation of risky financial assets smooths significantly the hump-shaped pattern of volatility, as a matter of fact the equilibrium volatility pattern is similar to that in the risk averse setting. The volatility peak for low values of wealth induced by the VaR regulation disappears in the presence of taxation on risky asset holding at the cost of a generalized higher volatility. We also show that to stabilize the market it is enough to tax rational traders and not passive/liquidity traders.

Summing up our analysis shows two main results. A VaR regulation destabilizes the market for a low wealth with a volatility larger than that observed in a market with risk averse traders, only a strong VaR constraint renders a low volatility for a small wealth and a large peak for a high wealth. A tax on financial asset holding reduces significantly the phenomenon smoothing the volatility pattern with a generalized higher volatility.

The paper is organized as follows. In Section 2 we present the market model introduced in [Danielsson, et al. (2009)] with VaR constrained rational traders. In Section 3 we consider the model with risk averse agents and we compare the results to those presented in [Danielsson, et al. (2009)]. In Section 4 we analyze the effect of equity holding taxation on market volatility. In Section 5 we consider the case of selective taxation only on rational traders.

2 VaR constrained market model

We start from the model of [Danielsson, et al. (2009)] with risk neutral traders subject to a VaR constraint.

There are two classes of traders in the economy: rational traders or arbitrageurs (denoted by R) and passive traders (denoted by P). Rational traders negotiate in the market maximizing the expected value of the instantaneous rate of return of their wealth under a VaR constraint on their portfolio, instead passive traders are characterized by a log demand decreasing in the asset price.

There are two assets in the economy: a risk free asset and a risky asset. The risk free asset doesn't pay dividends, its price at time t is $B(t),\ t \in [0, \infty)$, and satisfies the differential equation

$$\mathrm{d}B(t) = rB(t)\mathrm{d}t, \qquad B(0) = 1,$$

where r is the constant risk free rate. The dynamics of the price of the risky asset is determined imposing the equilibrium condition in the market. We look for an equilibrium in which the price process evolves according to the following stochastic differential equation

$$\mathrm{d}S(t) = \mu(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}W(t), \qquad S(0) = 1, \tag{1}$$

where $W(t),\ t \in [0, \infty)$, is a Brownian motion. The drift $\mu(t)$ and the volatility $\sigma(t)$ are to be determined in equilibrium.

Passive traders act against the market, i.e., they buy the risky security when market falls and vice versa. In a sense passive traders are fundamentalists with a downward sloping demand. As far as rational traders is concerned we assume that they are fully rational, i.e., they know that the evolution of the asset price is given by (1) with the functions $\mu(t)$ and $\sigma(t)$ to be determined in equilibrium with rational expectations, i.e., actual drift and volatility confirm traders' conjectures.

Let $\vartheta(t)$ and $b(t)$ be the number of shares of the risky asset and of the risk free asset held at time t by rational traders and $D^R(t) = \vartheta(t)S(t)$ the amount of money invested in the risky asset. The wealth of rational traders is $V(t) = b(t)B(t) + \vartheta(t)S(t)$. As the portfolio is self-financing, the wealth satisfies the following stochastic differential equation

$$\mathrm{d}V(t) = \Big(rV(t) + D^R(t)(\mu(t) - r)\Big)\mathrm{d}t + D^R(t)\sigma(t)\mathrm{d}W(t). \tag{2}$$

The VaR of rational traders detaining an amount of wealth $D^R(t)$ invested in the risky asset at time t is α times the volatility of the financial wealth $V(t)$, namely

$$\mathrm{VaR}(t) = \alpha\sqrt{\mathrm{Var}(\mathrm{d}V(t))} = \alpha D^R(t)\sigma(t)$$

where α depends on the confidence level of the VaR: it is high for a small probability on the right hand side of the loss, i.e., a low VaR confidence level.

The VaR constraint requires the wealth be larger than the VaR:

$$\mathrm{VaR}(t) \leqslant V(t),$$

The objective of rational traders is to maximize the instantaneous expected return of their portfolio subject to the VaR constraint. The portfolio choice problem then becomes

$$\sup_{D^R(t)} rV(t) + D^R(t)(\mu(t) - r)$$

subject to

$$\alpha D^R(t)\sigma(t) \leqslant V(t). \tag{3}$$

As shown in [Danielsson, et al. (2009)], the optimal risk constrained solution is obtained when the constraint is binding

$$D^R_{\mathrm{VaR}}(t) = \frac{V(t)}{\alpha\sigma(t)}. \tag{4}$$

Note that as α goes up the VaR constraint becomes more binding and the demand of the risky asset becomes a smaller fraction of the wealth.

As far as passive traders is concerned, we follow [Danielsson, et al. (2009)] assuming that they have a demand curve for the risky asset with a negative slope. We consider a log demand, i.e., they compare the logarithm of the bond price and the logarithm of the asset price. We assume that the demand is inversely proportional to the squared volatility of the asset

$$D^P(t) = \frac{\delta}{\sigma(t)^2}\Big(rt - \ln S(t) + \eta z(t)\Big), \qquad \delta,\, \eta > 0.$$

Passive trader demand also depends on $z(t)$, which is a positive demand shock. We may interpret $z(t)$ as noise trader demand, i.e., agents whose behavior is purely noise.

In the sequel we suppose that $z(t)$ satisfies the following stochastic differential equation

$$\mathrm{d}z(t) = \sigma_z \mathrm{d}W(t), \tag{5}$$

where σ_z is a positive constant. Note that there is only one source of risk, the Brownian motion $W(t)$ which affects both the asset price and the demand by passive traders. σ_z is a measure of the relevance of noise in the market and can be interpreted as the magnitude of exogenous risk.

Our goal is to define the evolution of the asset price (1) as a rational expectations equilibrium investigating the relationship between σ_z and $\sigma(t)$, i.e., between exogenous risk and market volatility. We assume that volatility and drift are functions of financial wealth of rational traders, i.e., $\sigma(V(t))$ and $\mu(V(t))$.

The equilibrium condition of the risky asset market becomes

$$D^R_{\mathrm{VaR}}(t) + D^P(t) = 0$$

and, therefore, the security price in equilibrium is given by

$$S_{\mathrm{VaR}}(t) = \exp\left\{\frac{\sigma_{\mathrm{VaR}}(t)V(t)}{\alpha\delta} + rt + \eta z(t)\right\}. \tag{6}$$

In this context, rational expectations equilibrium parameters μ_{VaR} and σ_{VaR} satisfy the following differential equations, see [Danielsson, et al. (2009)]:

$$\mu_{\mathrm{VaR}}(V(t)) = r + \frac{\sigma_{\mathrm{VaR}}(V(t))}{2\alpha\eta\sigma_z}\Big\{\alpha\sigma_{\mathrm{VaR}}(V(t))^2 - \eta\sigma_z + \tag{7}$$

$$+ (\sigma_{\text{VaR}}(V(t)) - \eta\sigma_z)\left[2\alpha^2 r + \frac{\alpha^2\delta}{V(t)} - 2\right]\Bigg\},$$

$$\frac{\mathrm{d}\sigma_{\text{VaR}}(V(t))}{\mathrm{d}V} = \frac{\alpha^2\delta - V(t)}{V(t)^2}\sigma_{\text{VaR}}(V(t)) - \frac{\alpha^2\delta\eta\sigma_z}{V(t)^2}. \tag{8}$$

Equations (7) and (8) can be found by equating the stochastic differential of $\ln S_{\text{VaR}}(t)$ obtained from (6) to that obtained from (1).

3 Risk averse rational traders

In this section we assume that rational traders are risk averse and that they have not to satisfy a VaR constraint. Our goal is to evaluate the role of VaR regulation on asset market volatility. We address this issue comparing the model with VaR constrained risk neutral agents described in the previous section to a model with unconstrained risk averse traders.

Rational traders solve the mean-variance optimization problem

$$\sup_{D^R(t)} rV(t) + D^R(t)(\mu(t) - r) - \tfrac{\beta}{2}D^R(t)^2\sigma(t)^2,$$

where β is a positive constant defining the coefficient of absolute risk aversion for an exponential utility. The maximization problem leads to the following demand by rational traders

$$D^R_{\text{MV}}(t) = \frac{\mu(t) - r}{\beta\sigma(t)^2}. \tag{9}$$

In equilibrium we have

$$D^R_{\text{MV}}(t) + D^P(t) = 0,$$

as a consequence we have the following expression for the security price:

$$S_{\text{MV}}(t) = \exp\left\{\frac{\mu_{\text{MV}}(V(t)) - r}{\beta\delta} + rt + \eta z(t)\right\}. \tag{10}$$

In a rational expectations equilibrium the drift and the diffusion coefficient of the asset price in (1) behave like in the following proposition.

Proposition 1. *In equilibrium, when rational traders solve a mean-variance optimization problem, the drift and the volatility of the risky asset price, namely $\mu_{\mathrm{MV}}(V(t))$ and $\sigma_{\mathrm{MV}}(V(t))$, satisfy the following differential equations*

$$\mu_{\mathrm{MV}}(V(t)) - \frac{1}{2}\sigma_{\mathrm{MV}}(V(t))^2 = r + \frac{1}{\beta\delta}\left[\frac{\mathrm{d}\mu_{\mathrm{MV}}(V(t))}{\mathrm{d}V}\left(rV(t) + \frac{(\mu_{\mathrm{MV}}(V(t)) - r)^2}{\beta\sigma_{\mathrm{MV}}(V(t))^2}\right) + \right. \tag{11}$$

$$\left. + \frac{1}{2}\frac{\mathrm{d}^2\mu_{\mathrm{MV}}(V(t))}{\mathrm{d}V^2}\frac{(\mu_{\mathrm{MV}}(V(t)) - r)^2}{\beta^2\sigma_{\mathrm{MV}}(V(t))^2}\right]$$

$$\sigma_{\mathrm{MV}}(V(t)) = \eta\sigma_z + \frac{1}{\beta\delta}\frac{\mathrm{d}\mu_{\mathrm{MV}}(V(t))}{\mathrm{d}V}\frac{\mu_{\mathrm{MV}}(V(t)) - r}{\beta\sigma_{\mathrm{MV}}(V(t))}. \tag{12}$$

Proof. The strategy to find out the drift and the volatility of the asset price in equilibrium is to determine the drift and the diffusion coefficient of the logarithm of the asset price as they result from market clearing in (10) and to equate the coefficients to those obtained from (1), namely

$$\mathrm{d}(\ln S(t)) = \left(\mu(V(t)) - \tfrac{1}{2}\sigma(V(t))^2\right)\mathrm{d}t + \sigma(V(t))\mathrm{d}W(t). \tag{13}$$

From (10) we have that in equilibrium

$$\mathrm{d}(\ln S(t)) = r\mathrm{d}t + \eta\mathrm{d}z(t) + \frac{1}{\beta\delta}\mathrm{d}\mu(V(t)). \tag{14}$$

The stochastic differential of $\mu(V(t))$ is obtained applying the Itô formula. To this end we have to evaluate the stochastic differential of $V(t)$, from (2) and (9) we have

$$\mathrm{d}V(t) = \left[rV(t) + \frac{(\mu(V(t)) - r)^2}{\beta\sigma(V(t))^2}\right]\mathrm{d}t + \frac{\mu(V(t)) - r}{\beta\sigma(V(t))}\mathrm{d}W(t). \tag{15}$$

Hence, we get

$$\begin{aligned}\mathrm{d}\mu(V(t)) &= \frac{\mathrm{d}\mu(V(t))}{\mathrm{d}V}\mathrm{d}V(t) + \frac{1}{2}\frac{\mathrm{d}^2\mu(V(t))}{\mathrm{d}V^2}(\mathrm{d}V(t))^2 \\ &= \left[\frac{\mathrm{d}\mu(V(t))}{\mathrm{d}V}\left(rV(t) + \frac{(\mu(V(t)) - r)^2}{\beta\sigma(V(t))^2}\right) + \frac{1}{2}\frac{\mathrm{d}^2\mu(V(t))}{\mathrm{d}V^2}\frac{(\mu(V(t)) - r)^2}{\beta^2\sigma(V(t))^2}\right]\mathrm{d}t + \\ &\quad + \frac{\mathrm{d}\mu(V(t))}{\mathrm{d}V}\frac{\mu(V(t)) - r}{\beta\sigma(V(t))}\mathrm{d}W(t).\end{aligned}$$

Therefore, the equilibrium price differential equation (14) becomes

$$\mathrm{d}(\ln S(t)) = \left\{ r + \frac{1}{\beta\delta}\left[\frac{\mathrm{d}\mu(V(t))}{\mathrm{d}V}\left(rV(t) + \frac{(\mu(V(t))-r)^2}{\beta\sigma(V(t))^2} \right) + \right.\right.$$
$$\left.\left. + \frac{1}{2}\frac{\mathrm{d}^2\mu(V(t))}{\mathrm{d}V^2}\frac{(\mu(V(t))-r)^2}{\beta^2\sigma(V(t))^2} \right] \right\}\mathrm{d}t + \left\{ \eta\sigma_z + \frac{1}{\beta\delta}\frac{\mathrm{d}\mu(V(t))}{\mathrm{d}V}\frac{\mu(V(t))-r}{\beta\sigma(V(t))} \right\}\mathrm{d}W(t).$$

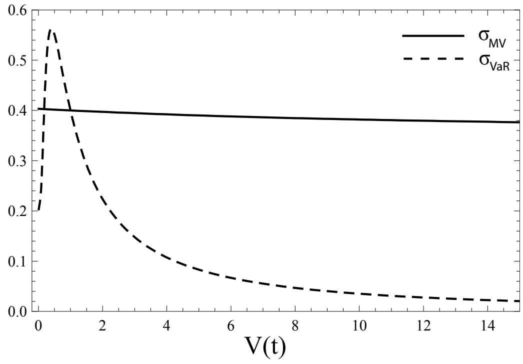
Equating the drift and the diffusion coefficient to those in (13) we determine the restrictions on the coefficients of the asset price process yielding equations (11) and (12). □

In Figure 1 we plot σ_{MV}, namely the volatility obtained from Proposition 1 with risk averse rational traders, and the volatility σ_{VaR}, obtained when rational traders maximize the expected instantaneous rate of return of their wealth subject to a VaR constraint, which is the solution of equation (8). The three pictures differ only for the value of α. While [Danielsson, et al. (2009)] only fix the initial value of the volatility σ_{VaR} and the initial value of μ_{VaR} is obtained from (7), in the mean-variance setting we have to set both a drift and a volatility starting point. In Figure 1 we set $\sigma_{\mathrm{MV}}(1) = \sigma_{\mathrm{VaR}}(1) = 0.4$, from (7) we have $\mu_{\mathrm{VaR}}(1) = 0.14$ for $\alpha = 5$.

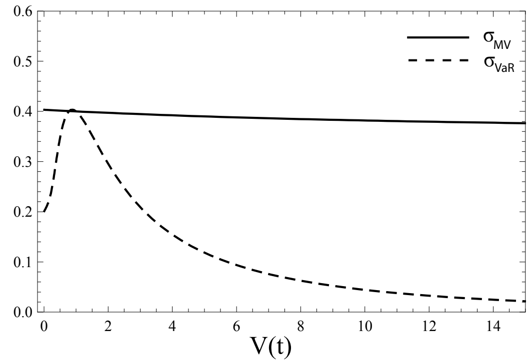
Volatility shows a different pattern in the two models. In the unconstrained risk averse agents setting, for a small coefficient of risk aversion we have that volatility is a monotone decreasing function of the wealth with a reduction that is of little significance; as the coefficient of risk aversion goes up we may observe a U-shaped volatility decreasing and then increasing. Instead, in the risk constrained case we have a hump-shaped volatility first increasing and then decreasing.

The different pattern of volatility originates from the portfolio holdings of rational traders. In Figure 2 we plot the portfolio holdings of rational traders in the two settings, namely equity holdings $D^R(t)$ and bond holdings $V(t) - D^R(t)$ for risk averse traders and VaR constrained traders. The difference is striking. Portfolio holding of risk averse traders is almost constant, it increases linearly. Portfolio holding of VaR constrained traders is increasing with a positive second derivative. The elasticity of equity portfolio with respect to wealth in case of VaR constrained traders is much higher than that of risk averse traders.

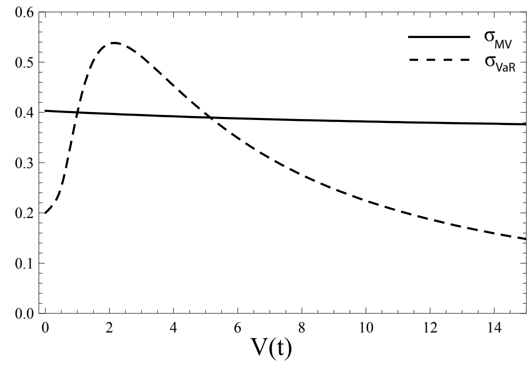
We can conclude that VaR regulation plays a crucial role to induce a destabilizing effect.



(a) $\alpha = 3$



(b) $\alpha = 5$



(c) $\alpha = 7$

Figure 1: Market volatility in the mean-variance model (bold line) and in the model of [Danielsson, et al. (2009)] with a VaR constraint (dashed line), $\beta = 0.05$, $\delta = 0.07$, $\eta = 1$, $\sigma_z = 0.2$, $r = 0.01$, $\mu_{\mathrm{MV}}(1) = 0.14$ and $\sigma_{\mathrm{MV}}(1) = \sigma_{\mathrm{VaR}}(1) = 0.4$.

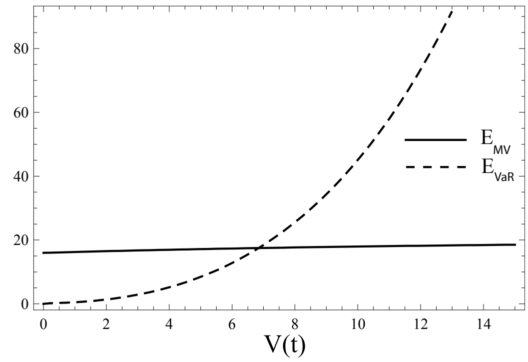
As α increases the VaR confidence level also decreases, the constraint becomes more binding, equity holding decreases and the maximum of the volatility is observed for a higher level of wealth. The maximum level of the volatility first decreases and then increases in α: for the parameters in Figure 1 the maximum decreases when α varies from 3 to 5 and increases when α goes from 5 to 7. In general, it can be shown numerically that a weakly binding constraint (large VaR confidence level), e.g. $\alpha = 3$, renders a volatility higher than that observed in the risk averse trader market for a low level of wealth; on the other hand, a strongly binding constraint (small VaR confidence level), e.g. $\alpha = 7$, renders a volatility higher than the risk averse trader market for a high level of wealth (and smaller for a small wealth). The rationale of this behavior is that as α goes up the demand of risk constrained traders shifts downward and therefore the volatility peak is observed for a higher volatility (when the demand of risk constrained traders is high enough compared to that of risk averse traders).

From Figure 1 we observe that when the VaR constraint becomes more binding the interval of wealth for which $\sigma_{\text{VaR}}(V(t)) \geq \sigma_{\text{MV}}(V(t))$ enlarges. Indeed, it can be shown numerically, that the second derivative of the volatility at the maximum point tends to zero as α increases, that is the graph of the volatility near the maximum becomes more and more flat. Consequently, as α goes up (more binding VaR constraint) we observe an expansion of the neighborhood of the maximum wealth for which the volatility in the VaR constrained market is higher than the volatility in the mean-variance model. Summing up we face a tradeoff: a strongly binding constraint leads to a high volatility for a large interval of wealth far away from zero, a weakly binding constraint leads to a peak of volatility for a small level of wealth.

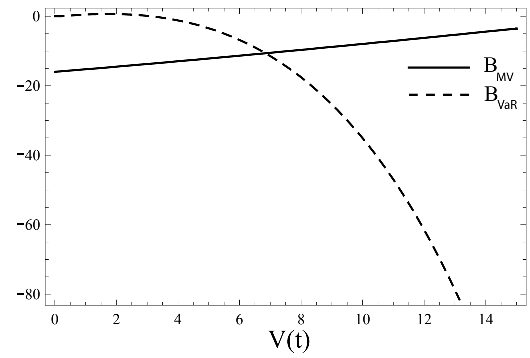
The comparison of volatility in the two settings shows that the VaR constraint does not help to stabilize the market. As a matter of fact, a VaR constraint renders a smaller volatility than in the unregulated market for a large wealth and not in case of financial distress. As far as a low wealth level is concerned, we may have a volatility lower than in the risk averse case only with a fine tuning of the VaR constraint ($\alpha = 5$ in our case).

We can also compute the volatility of volatility in equilibrium. Its evolution is derived from the stochastic differential of $\sigma(V(t))$

$$\mathrm{d}\sigma(V(t)) = \mu^{\sigma}(t)\mathrm{d}t + \widetilde{\sigma}^{\sigma}(t)\mathrm{d}W(t).$$



(a) Equity holdings



(b) Bond holdings

Figure 2: Equity and bond holding in the mean-variance model (bold line) and in the presence of the VaR constraint (dashed line), $\beta = 0.05$, $\alpha = 5$, $\delta = 0.07$, $\eta = 1$, $\sigma_z = 0.2$, $r = 0.01$, $\mu_{\text{MV}}(1) = 0.14$, $\sigma_{\text{MV}}(1) = \sigma_{\text{VaR}}(1) = 0.4$.

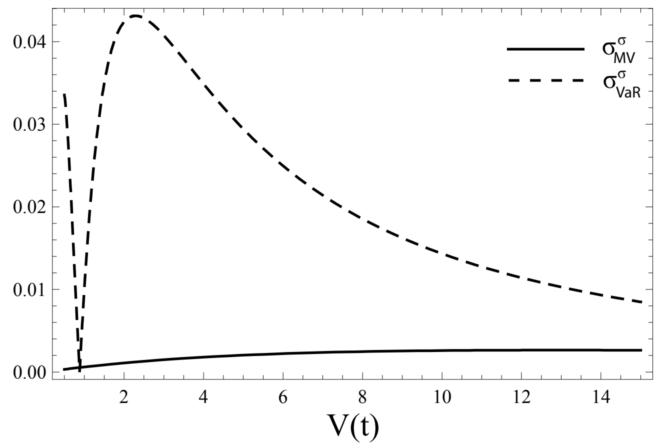


Figure 3: Volatility of volatility in the mean-variance model (bold line) and vol of vol in the model presented in [Danielsson, et al. (2009)], with a VaR constraint (dashed line). Parameters: $\beta = 0.05$, $\alpha = 5$, $\delta = 0.07$, $\eta = 1$, $\sigma_z = 0.2$, $r = 0.01$, $\mu_{\text{MV}}(1) = 0.14$ and $\sigma_{\text{MV}}(1) = \sigma_{\text{VaR}}(1) = 0.4$.

The volatility of volatility, denoted by σ^σ, is the absolute value of the diffusion coefficient in the above expression, i.e., $\sigma^\sigma = |\widetilde{\sigma}^\sigma|$. In our setting, the volatility of volatility is given by

$$\sigma^\sigma(V(t)) = \frac{V(t)}{\alpha}\left|\frac{\mathrm{d}\sigma(V(t))}{\mathrm{d}V}\right|. \tag{16}$$

In Figure 3 we plot the volatility of volatility in the mean-variance model and in the presence of the VaR constraint for $\alpha = 5$ (the graph is similar for other parameter choices). The volatility of volatility in the unconstrained case is much smaller than the volatility of volatility in the VaR constrained case.

If a VaR constraint plays a stabilizing role then a higher fraction of VaR constrained traders will induce a lower volatility. Otherwise, we expect that the effect of regulation as an amplifier of endogenous risk will be more remarkable. To test this hypothesis we consider an economy with a proportion $\pi \in (0,1)$ of rational traders and a proportion $1-\pi$ of passive traders. In the sequel we denote by $\sigma_{\mathrm{MV},p}$ and $\sigma_{\mathrm{VaR},p}$ the market volatility obtained in the risk averse trader market and in the VaR constrained trader market. Market clearing requires $\pi D^R(t) + (1-\pi)D^P(t) = 0$.

If rational traders solve a mean-variance optimization problem to determine their strategy, then the equilibrium risky asset price is given by

$$S_{\mathrm{MV,p}}(t) = \exp\left\{\frac{\pi}{1-\pi}\frac{\mu_{\mathrm{MV,p}}(V(t)) - r}{\beta\delta} + rt + \eta z(t)\right\}.$$

Instead, if rational traders are subject to a VaR constraint, then the security price has the following expression

$$S_{\mathrm{VaR,p}}(t) = \exp\left\{\frac{\pi}{1-\pi}\frac{\sigma_{\mathrm{VaR,p}}(V(t))V(t)}{\alpha\delta} + rt + \eta z(t)\right\}.$$

As in the previous section, we can now proceed to find the equilibrium market volatility. The following proposition holds true.

Proposition 2. *When rational traders solve a mean-variance optimization problem, the equi-*

librium market volatility $\sigma_{\mathrm{MV,p}}(V(t))$ and the drift $\mu_{\mathrm{MV,p}}(V(t))$ satisfy the following equations

$$\begin{aligned}\mu_{\mathrm{MV,p}}(V(t)) - \frac{1}{2}\sigma_{\mathrm{MV,p}}(V(t))^2 = r+ \\ + \frac{\pi}{1-\pi}\frac{1}{\beta\delta}\left[\frac{\mathrm{d}\mu_{\mathrm{MV,p}}(V(t))}{\mathrm{d}V}\left(rV(t) + \frac{(\mu_{\mathrm{MV,p}}(V(t)) - r)^2}{\beta\sigma_{\mathrm{MV,p}}(V(t))^2}\right)+\right. \\ \left. + \frac{1}{2}\frac{\mathrm{d}^2\mu_{\mathrm{MV,p}}(V(t))}{\mathrm{d}V^2}\frac{(\mu_{\mathrm{MV,p}}(V(t)) - r)^2}{\beta^2\sigma_{\mathrm{MV,p}}(V(t))^2}\right]\end{aligned} \tag{17}$$

$$\sigma_{\mathrm{MV,p}}(V(t)) = \frac{1-\pi}{\pi}\frac{1}{\beta\delta}\frac{\mathrm{d}\mu_{\mathrm{VaR,p}}(V(t))}{\mathrm{d}V}\frac{\mu_{\mathrm{MV,p}}(V(t)) - r}{\beta\sigma_{\mathrm{MV,p}}(V(t))} + \eta\sigma_z. \tag{18}$$

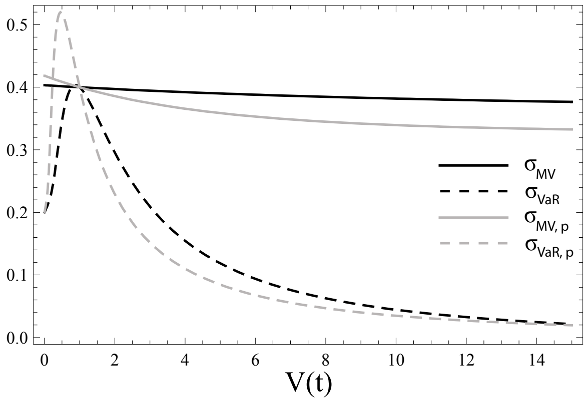
When rational traders are subject to a VaR constraint, the equilibrium market volatility satisfies the following equation

$$\frac{\mathrm{d}\sigma_{\mathrm{VaR,p}}(V(t))}{\mathrm{d}V} = \frac{\frac{1-\pi}{\pi}\alpha^2\delta - V(t)}{V(t)^2}\sigma_{\mathrm{VaR,p}}(V(t)) - \frac{1-\pi}{\pi}\frac{\alpha^2\delta\eta\sigma_z}{V(t)^2}. \tag{19}$$

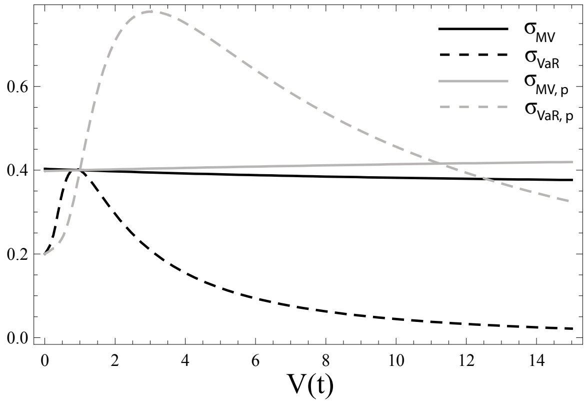
In Figure 4 we plot the market volatility in the mean-variance model and in the presence of the VaR constraint, when the proportion of rational traders is equal to 70% (Figure 4a) and to 30% (Figure 4b). Note that $\mu_{\mathrm{VaR,p}}(1)$ is obtained from $\sigma_{\mathrm{VaR,p}}(1)$ by

$$\begin{aligned}\mu_{\mathrm{VaR,p}}(V(t)) = r + \frac{\sigma_{\mathrm{VaR,p}}(V(t))}{2\alpha\eta\sigma_z}\Big\{\alpha\sigma_{\mathrm{VaR,p}}(V(t))^2 - \eta\sigma_z+ \\ + (\sigma_{\mathrm{VaR,p}}(V(t)) - \eta\sigma_z)\left[2\alpha^2 r + \frac{1-\pi}{\pi}\frac{\alpha^2\delta}{V(t)} - 2\right]\Big\}.\end{aligned}$$

As far as the VaR constrained economy is concerned, a large fraction of rational traders renders a high peak of volatility for a small level of financial wealth. When the fraction of rational traders decreases, we observe a high volatility for a large wealth and a small stabilizing effect for a small level of wealth. This analysis shows that the effect of a VaR constraint is ambiguous: a large fraction of VaR constrained traders destabilizes the market for a small wealth, a small fraction of VaR constrained traders destabilizes the market for a large wealth with little gain for a low wealth. If the goal of the regulation is to prevent financial crises then the policy implication is to reduce the space for VaR regulation.



(a) Proportion of rational traders: $\pi = 70\%$



(b) Proportion of rational traders: $\pi = 30\%$

Figure 4: Market volatility in the mean-variance model (bold line) and in the presence of the VaR constraint (dashed line), with a different proportion of rational traders with respect to passive traders (gray lines). Black lines are referred to the case of equal proportion, $\beta = 0.05$, $\alpha = 5$, $\delta = 0.07$, $\eta = 1$, $\sigma_z = 0.2$, $r = 0.01$, $\mu_{\mathrm{MV}}(1) = 0.14$, $\sigma_{\mathrm{MV}}(1) = \sigma_{\mathrm{VaR}}(1) = \sigma_{\mathrm{MV,p}}(1) = \sigma_{\mathrm{VaR,p}}(1) = 0.4$, $\mu_{\mathrm{MV,p}}(1) = 0.1$ for $\pi = 70\%$ and $\mu_{\mathrm{MV,p}}(1) = 0.23$ for $\pi = 30\%$.

4 Equity holding taxation

In this Section we analyze the market with taxation on equity holdings. Our goal is to investigate the capability of taxation to dampen the amplification effect of VaR regulation. First we consider taxation on rational and passive traders, then in the next section we will consider taxation only on rational traders.

We assume that rational traders are subject to a tax on equity holdings, which is expressed as the square of equity holdings times $\tau/2$, where τ is a positive constant, i.e., $\tau D^R(t)^2/2$, see [Dow and Rahi (2000)] for a similar hypothesis on transaction costs. As a consequence, the wealth of rational traders $V(t) = b(t)B(t) + \vartheta(t)S(t)$ is no more self-financing, its dynamics becomes

$$\mathrm{d}V(t) = \Big(rV(t) + D^R(t)(\mu(t) - r) - \tfrac{\tau}{2}D^R(t)^2\Big)\mathrm{d}t + D^R(t)\sigma(t)\mathrm{d}W(t). \qquad (20)$$

Passive traders are subject to the same kind of tax, their risky asset demand is inversely proportional to the taxation coefficient τ.

Our goal is to compare the effect of financial asset tax on market volatility when rational traders are subject to a risk constraint and when they solve a mean-variance optimization problem.

Firstly we study the mean-variance optimization problem. The portfolio choice problem of rational traders becomes

$$\sup_{D^R(t)} rV(t) + D^R(t)(\mu(t) - r) - \tfrac{\beta}{2}D^R(t)^2\sigma(t)^2 - \tfrac{\tau}{2}D^R(t)^2.$$

First order condition leads to the following demand

$$D^R_{\mathrm{MV},\tau}(t) = \frac{\mu(t) - r}{\beta\sigma(t)^2 + \tau}. \qquad (21)$$

As far as passive traders is concerned, their demand of the risky asset becomes

$$D^P_\tau(t) = \frac{\delta}{\sigma(t)^2 + \tau}\Big(rt - \ln S(t) + \eta z(t)\Big).$$

In equilibrium we obtain the following expression for the security price

$$S_{\mathrm{MV},\tau}(t) = \exp\left\{\frac{\sigma_{\mathrm{MV},\tau}(t)^2+\tau}{\beta\sigma_{\mathrm{MV},\tau}(t)^2+\tau}\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\delta}+rt+\eta z(t)\right\}. \tag{22}$$

The drift and the volatility of the risky asset price in equilibrium are given in the following proposition.

Proposition 3. *In equilibrium, the volatility* $\sigma_{\mathrm{MV},\tau}(V(t))$ *and the drift* $\mu_{\mathrm{MV},\tau}(V(t))$ *satisfy the following system of two ordinary differential equations of second order*

$$\begin{aligned}
\mu_{\mathrm{MV},\tau}(V(t)) - \frac{1}{2}\sigma_{\mathrm{MV},\tau}(V(t))^2 &= r + \frac{\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\frac{1}{\delta}\left[\frac{\mathrm{d}\mu_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\left(rV(t)+\right.\right.\\
&+\left.\frac{(\mu_{\mathrm{MV},\tau}(V(t))-r)^2}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau} - \frac{\tau(\mu_{\mathrm{MV},\tau}(V(t))-r)^2}{2(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}\right) + \frac{1}{2}\frac{\mathrm{d}^2\mu_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V^2}\cdot\\
&\cdot\left.\left(\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau}(V(t))\right)^2\right] + \frac{2\tau(1-\beta)\sigma_{\mathrm{MV},\tau}(V(t))}{(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}\cdot\\
&\cdot\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\delta}\left[\frac{\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\left(rV(t)+\frac{(\mu_{\mathrm{MV},\tau}(V(t))-r)^2}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\right.\right.\\
&-\left.\frac{\tau(\mu_{\mathrm{MV},\tau}(V(t))-r)^2}{2(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}\right) + \frac{1}{2}\frac{\mathrm{d}^2\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V^2}\cdot\\
&\cdot\left.\left(\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau}(V(t))\right)^2\right] + \tau(1-\beta)\frac{\tau-3\beta\sigma_{\mathrm{MV},\tau}(V(t))^2}{(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^3}\cdot\\
&\cdot\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\delta}\left(\frac{\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\right)^2\left(\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau}(V(t))\right)^2+\\
&+\frac{2\tau(1-\beta)\sigma_{\mathrm{MV},\tau}(V(t))}{(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}\frac{1}{\delta}\frac{\mathrm{d}\mu_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\frac{\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\cdot\\
&\cdot\left(\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau}(V(t))\right)^2
\end{aligned} \tag{23}$$

$$\begin{aligned}
\sigma_{\mathrm{MV},\tau}(V(t)) &= \frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\frac{\sigma_{\mathrm{MV},\tau}(V(t))}{\delta}\left(\frac{\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\frac{\mathrm{d}\mu_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}+\right.\\
&+\left.\frac{2\tau(1-\beta)\sigma_{\mathrm{MV},\tau}(V(t))}{(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}(\mu_{\mathrm{MV},\tau}(V(t))-r)\frac{\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\right)+\eta\sigma_z.
\end{aligned} \tag{24}$$

Proof. The strategy to find out the drift and the volatility of the asset price in equilibrium is to determine the drift and the diffusion coefficient of the logarithm of the asset price in (22) and to equate them to those obtained from (1).

We consider the logarithmic price, from (1) we obtain

$$\mathrm{d}(\ln S_{\mathrm{MV},\tau}(t)) = \Big(\mu_{\mathrm{MV},\tau}(V(t)) - \tfrac{1}{2}\sigma_{\mathrm{MV},\tau}(V(t))^2\Big)\mathrm{d}t + \sigma_{\mathrm{MV},\tau}(V(t))\mathrm{d}W(t) \tag{25}$$

from (22) we have that in equilibrium

$$\begin{aligned}
\mathrm{d}(\ln S_{\mathrm{MV},\tau}(t)) = {} & r\mathrm{d}t + \eta\mathrm{d}z(t) + \frac{\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\frac{1}{\delta}\mathrm{d}\mu_{\mathrm{MV},\tau}(V(t)) + \\
& + \frac{2\tau(1-\beta)\sigma_{\mathrm{MV},\tau}(V(t))}{(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\delta}\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t)) + \\
& + \tau(1-\beta)\frac{\tau-3\beta\sigma_{\mathrm{MV},\tau}(V(t))^2}{(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^3}\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\delta}(\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t)))^2 + \\
& + \frac{2\tau(1-\beta)\sigma_{\mathrm{MV},\tau}(V(t))}{(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}\frac{1}{\delta}\mathrm{d}\mu_{\mathrm{MV},\tau}(V(t))\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t)).
\end{aligned} \tag{26}$$

The stochastic differentials of $\sigma_{\mathrm{MV},\tau}(V(t))$ and $\mu_{\mathrm{MV},\tau}(V(t))$ are obtained applying the Itô formula. To this end we have to evaluate the stochastic differential of $V(t)$, from (20) and (21) we find

$$\begin{aligned}
\mathrm{d}V(t) = {} & \left[rV(t) + \frac{(\mu_{\mathrm{MV},\tau}(V(t))-r)^2}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau} - \frac{\tau(\mu_{\mathrm{MV},\tau}(V(t))-r)^2}{2(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}\right]\mathrm{d}t + \\
& + \frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau}(V(t))\mathrm{d}W(t).
\end{aligned} \tag{27}$$

Hence, we get

$$\begin{aligned}
\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t)) & = \frac{\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\mathrm{d}V(t) + \frac{1}{2}\frac{\mathrm{d}^2\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V^2}(\mathrm{d}V(t))^2 \\
& = \left[\frac{\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\left(rV(t) + \frac{(\mu_{\mathrm{MV},\tau}(V(t))-r)^2}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\right.\right. \\
& \quad \left. - \frac{\tau(\mu_{\mathrm{MV},\tau}(V(t))-r)^2}{2(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}\right) + \frac{1}{2}\frac{\mathrm{d}^2\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V^2}\cdot \\
& \quad \left.\cdot\left(\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau}(V(t))\right)^2\right]\mathrm{d}t +
\end{aligned} \tag{28}$$

$$+\frac{\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau}(V(t))\mathrm{d}W(t)$$

and

$$\begin{aligned}\mathrm{d}\mu_{\mathrm{MV},\tau}(V(t)) &= \frac{\mathrm{d}\mu_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\mathrm{d}V(t)+\frac{1}{2}\frac{\mathrm{d}^2\mu_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V^2}(\mathrm{d}V(t))^2 \\ &= \left[\frac{\mathrm{d}\mu_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\left(rV(t)+\frac{(\mu_{\mathrm{MV},\tau}(V(t))-r)^2}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\right.\right. \\ &\quad \left.-\frac{\tau(\mu_{\mathrm{MV},\tau}(V(t))-r)^2}{2(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}\right)+\frac{1}{2}\frac{\mathrm{d}^2\mu_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V^2}\cdot \\ &\quad \left.\cdot\left(\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau}(V(t))\right)^2\right]\mathrm{d}t+ \\ &\quad +\frac{\mathrm{d}\mu_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau}(V(t))\mathrm{d}W(t).\end{aligned} \tag{29}$$

Therefore, the equilibrium price differential equation (26) becomes

$$\begin{aligned}\mathrm{d}(\ln S_{\mathrm{MV},\tau}(t)) = &\left\{r+\frac{\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\frac{1}{\delta}\left[\frac{\mathrm{d}\mu_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\left(rV(t)+\right.\right.\right. \\ &+\frac{(\mu_{\mathrm{MV},\tau}(V(t))-r)^2}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}-\left.\frac{\tau(\mu_{\mathrm{MV},\tau}(V(t))-r)^2}{2(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}\right)+\frac{1}{2}\frac{\mathrm{d}^2\mu_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V^2}\cdot \\ &\left.\cdot\left(\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau}(V(t))\right)^2\right]+\frac{2\tau(1-\beta)\sigma_{\mathrm{MV},\tau}(V(t))}{(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}\cdot \\ &\cdot\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\delta}\left[\frac{\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\left(rV(t)+\frac{(\mu_{\mathrm{MV},\tau}(V(t))-r)^2}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\right.\right. \\ &\left.-\frac{\tau(\mu_{\mathrm{MV},\tau}(V(t))-r)^2}{2(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}\right)+\frac{1}{2}\frac{\mathrm{d}^2\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V^2}\cdot \\ &\left.\cdot\left(\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau}(V(t))\right)^2\right]+\tau(1-\beta)\frac{\tau-3\beta\sigma_{\mathrm{MV},\tau}(V(t))^2}{(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^3}\cdot \\ &\cdot\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\delta}\left(\frac{\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\right)^2\left(\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau}(V(t))\right)^2+ \\ &+\frac{2\tau(1-\beta)\sigma_{\mathrm{MV},\tau}(V(t))}{(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}\frac{1}{\delta}\frac{\mathrm{d}\mu_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\frac{\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\cdot \\ &\left.\cdot\left(\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau}(V(t))\right)^2\right\}\mathrm{d}t+ \\ &+\left\{\frac{\mu_{\mathrm{MV},\tau}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\frac{\sigma_{\mathrm{MV},\tau}(V(t))}{\delta}\left(\frac{\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}{\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau}\frac{\mathrm{d}\mu_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}+\right.\right.\end{aligned}$$

$$+\frac{2\tau(1-\beta)\sigma_{\mathrm{MV},\tau}(V(t))}{(\beta\sigma_{\mathrm{MV},\tau}(V(t))^2+\tau)^2}(\mu_{\mathrm{MV},\tau}(V(t))-r)\frac{\mathrm{d}\sigma_{\mathrm{MV},\tau}(V(t))}{\mathrm{d}V}\Big)+\eta\sigma_z\Big\}\mathrm{d}W(t).$$

Equating the drift and the diffusion coefficient to those in (25) we determine the restrictions on the coefficients of the asset price process yielding the differential equations (23) and (24). □

Now we study market volatility when risk neutral rational traders choose their strategy maximizing the instantaneous expected return of their portfolio subject to a VaR constraint.

The portfolio choice problem becomes

$$\sup_{D^R(t)} rV(t)+D^R(t)(\mu(t)-r)-\tfrac{\tau}{2}(D^R(t))^2$$

subject to the VaR constraint

$$\alpha D^R(t)\sigma(t)\leqslant V(t).$$

To solve this problem we introduce the Lagrangian function

$$\mathcal{L}(D^R(t),\lambda)=rV(t)+D^R(t)(\mu(t)-r)-\tfrac{\tau}{2}(D^R(t))^2-\lambda\Big(\alpha D^R(t)\sigma(t)-V(t)\Big).$$

Kuhn-Tucker conditions are given by

$$\begin{cases}\dfrac{\partial\mathcal{L}}{\partial D^R}=\mu(t)-r-\tau D^R(t)-\lambda\alpha\sigma(t)=0,\\ \dfrac{\partial\mathcal{L}}{\partial\lambda}=-\Big(\alpha D^R(t)\sigma(t)-V(t)\Big)\geqslant 0,\\ \lambda\geqslant 0 \qquad \text{and} \qquad \lambda\Big(\alpha D^R(t)\sigma(t)-V(t)\Big)=0.\end{cases}$$

Hence we get the following optimal solution

$$D^R_{\mathrm{Opt}}(t)=\begin{cases}D^R_{\mathrm{Free}}(t), & \text{if } U_{\mathrm{Free}}(t)\geqslant U_{\mathrm{VaR},\tau}(t) \text{ and } \alpha D^R_{\mathrm{Free}}(t)\sigma(t)\leqslant V(t),\\ D^R_{\mathrm{VaR},\tau}(t), & \text{otherwise,}\end{cases} \tag{30}$$

where

$$U_{\mathrm{Free}}(t)=rV(t)+D^R_{\mathrm{Free}}(t)(\mu(t)-r)-\tfrac{\tau}{2}(D^R_{\mathrm{Free}}(t))^2,$$

$$U_{\mathrm{VaR},\tau}(t) = rV(t) + D^R_{\mathrm{VaR},\tau}(t)(\mu(t) - r) - \tfrac{\tau}{2}(D^R_{\mathrm{VaR},\tau}(t))^2$$

and

$$D^R_{\mathrm{Free}}(t) = \frac{\mu(V(t)) - r}{\tau}, \qquad D^R_{\mathrm{VaR},\tau}(t) = \frac{V(t)}{\alpha\sigma(V(t))}. \tag{31}$$

To determine $D^R_{\mathrm{Opt}}(t)$, we have to compute $D^R_{\mathrm{Free}}(t)$ (the solution when the VaR constraint is not binding) and $D^R_{\mathrm{VaR},\tau}(t)$ (the solution when the VaR constraint is binding). To this end we have to evaluate the drift and volatility. The problem is complex because the drift and the volatility for a certain level of wealth depends on the portfolio chosen for a smaller level of wealth, i.e., they are path dependent.

We start considering the case in which $D^R_{\mathrm{Free}}(t)$ or $D^R_{\mathrm{VaR},\tau}(t)$ is adopted for all drift-volatility-wealth values. Assuming that $D^R_{\mathrm{Free}}(t)$ is adopted for all drift-volatility-wealth values, the drift and the volatility coincide with the solution of the mean-variance model for a risk neutral agent, i.e., $\beta = 0$, for all levels of wealth. In this case we obtain $\mu_{\mathrm{Free}}(t)$ and $\sigma_{\mathrm{Free}}(t)$ from equations (23) and (24) with $\beta = 0$. Given this solution we can verify whether the demand satisfies the VaR constraint, i.e.:

$$\alpha D^R_{\mathrm{Free}}(t)\sigma_{\mathrm{Free}}(t) = \alpha\frac{\mu_{\mathrm{Free}}(V(t)) - r}{\tau}\sigma_{\mathrm{Free}}(t) \leqslant V(t).$$

When $D^R_{\mathrm{VaR},\tau}(t)$ is adopted for all drift-volatility-wealth values we are able to determine $\mu_{\mathrm{VaR},\tau}(t)$ and $\sigma_{\mathrm{VaR},\tau}(t)$. The security price in equilibrium is given by

$$S_{\mathrm{VaR},\tau}(t) = \exp\left\{\frac{(\sigma_{\mathrm{VaR},\tau}(t)^2 + \tau)V(t)}{\alpha\delta\sigma_{\mathrm{VaR},\tau}(t)} + rt + \eta z(t)\right\}.$$

In this context, the rational expectations equilibrium is described in the following proposition.

Proposition 4. *In equilibrium, the market volatility when rational traders are subject to a* VaR *constraint satisfies the following ordinary differential equation*

$$\begin{aligned}\frac{\mathrm{d}\sigma_{\mathrm{VaR},\tau}(V(t))}{\mathrm{d}V} = \frac{\alpha^2\delta - V(t)}{V(t)^2}\frac{\sigma_{\mathrm{VaR},\tau}(V(t))^3}{\sigma_{\mathrm{VaR},\tau}(V(t))^2 - \tau} - \frac{\alpha^2\delta\eta\sigma_z}{V(t)^2}\frac{\sigma_{\mathrm{VaR},\tau}(V(t))^2}{\sigma_{\mathrm{VaR},\tau}(V(t))^2 - \tau}\\ - \frac{\tau}{V(t)}\frac{\sigma_{\mathrm{VaR},\tau}(V(t))}{\sigma_{\mathrm{VaR},\tau}(V(t))^2 - \tau}.\end{aligned} \tag{32}$$

The proof is similar to that of Proposition 3.

We can also determine the value of $\mu_{\text{VaR},\tau}(V(t))$ from the following condition (which can be derived similarly to (23)):

$$\mu_{\text{VaR},\tau}(V(t)) = \frac{1}{1 - \frac{\sigma_{\text{VaR},\tau}(V(t))^2 - \tau}{\alpha^2 \delta \sigma_{\text{VaR},\tau}(V(t))^3} \frac{\mathrm{d}\sigma_{\text{VaR},\tau}(V(t))}{\mathrm{d}V} V(t) - \frac{\sigma_{\text{VaR},\tau}(V(t))^2 + \tau}{\alpha^2 \delta \sigma_{\text{VaR},\tau}(V(t))^2} V(t)} \Bigg\{ r + \\ + \frac{1}{2}\sigma_{\text{VaR},\tau}(V(t))^2 + \frac{\sigma_{\text{VaR},\tau}(V(t))^2 - \tau}{\alpha \delta \sigma_{\text{VaR},\tau}(V(t))^2} V(t) \left[\frac{\mathrm{d}\sigma_{\text{VaR},\tau}(V(t))}{\mathrm{d}V} \left(rV(t) \right.\right. \\ \left.\left. - \frac{rV(t)}{\alpha \sigma_{\text{VaR},\tau}(V(t))} - \frac{\tau}{2} \frac{V(t)^2}{\alpha^2 \sigma_{\text{VaR},\tau}(V(t))^2} \right) + \frac{1}{2} \left(\frac{V(t)}{\alpha} \right)^2 \frac{\mathrm{d}^2 \sigma_{\text{VaR},\tau}(V(t))}{\mathrm{d}V(t)^2} \right] + \\ + \frac{\sigma_{\text{VaR},\tau}(V(t))^2 + \tau}{\alpha \delta \sigma_{\text{VaR},\tau}(V(t))} \left(rV(t) - \frac{rV(t)}{\alpha \sigma_{\text{VaR},\tau}(V(t))} - \frac{\tau}{2} \frac{V(t)^2}{\alpha^2 \sigma_{\text{VaR},\tau}(V(t))^2} \right) + \\ + \frac{\tau}{\sigma_{\text{VaR},\tau}(V(t))^3} V(t) \left(\frac{\mathrm{d}\sigma_{\text{VaR},\tau}(V(t))}{\mathrm{d}V} \right)^2 \left(\frac{V(t)}{\alpha} \right)^2 + \\ + \frac{\sigma_{\text{VaR},\tau}(V(t))^2 - \tau}{\alpha \delta \sigma_{\text{VaR},\tau}(V(t))^2} \frac{\mathrm{d}\sigma_{\text{VaR},\tau}(V(t))}{\mathrm{d}V} \left(\frac{V(t)}{\alpha} \right)^2 \Bigg\}. \tag{33}$$

If the optimal demand is given by $D_{\text{VaR},\tau}(t)$ for all levels of volatility and drift, then the corresponding market volatility $\sigma_{\text{VaR},\tau}(V(t))$ looks similar to $\sigma_{\text{VaR}}(V(t))$ (the volatility without taxation), i.e., the volatility is hump-shaped.

This analysis is not enough to appreciate the effect of equity holding taxation. We cannot evaluate $D_{\text{Free}}(t)$ along (23) and (24) with $\beta = 0$ and $D^R_{\text{VaR},\tau}(t)$ along (32) and (33) respectively because drift and volatility are path dependent, i.e., they depend on their values for other levels of wealth. However, we are able to prove numerically that for a large set of parameters the optimal demand is given by $D_{\text{Free}}(t)$ for all levels of wealth. Our argument proceeds as follows. We choose $V(t) = 1$ as a starting point and we set $\mu_{\text{VaR},\tau}(1) = \mu_{\text{Free}}(1)$ according to (33) for a value of the volatility in $V(t) = 1$ ($\sigma_{\text{VaR},\tau}(1) = \sigma_{\text{Free}}(1)$). We can compare $U_{\text{VaR},\tau}(t)$ and $U_{\text{Free}}(t)$ in $V(t) = 1$. Assuming $\beta = 0.05$, $\alpha = 5$, $\delta = 0.07$, $\eta = 1$, $\sigma_z = 0.2$, we verify that $D_{\text{Free}}(t)$ satisfies the VaR constraint in $V(t) = 1$ if $\sigma_{\text{VaR},\tau}(1) \leq 0.23$, namely for a reasonable initial value of volatility the demand $D_{\text{Free}}(t)$ is viable. For these values of wealth and of initial volatility $D_{\text{Free}}(t)$ is also superior to $D^R_{\text{VaR},\tau}(t)$, i.e., $U_{\text{Free}}(t) \geq U_{\text{VaR},\tau}(t)$. In what follows we assume that $\sigma_{\text{VaR},\tau}(1) = \sigma_{\text{Free}}(1) = 0.2$.

Starting form this point, admissibility and optimality of $D_{\text{Free}}(t)$ is confirmed for all values $V(t) \geqslant 1$: the demand $D_{\text{Free}}(t)$ continues to satisfy the VaR constraint along the solution of the differential equations of $\mu_{\text{Free}}(V(t))$ and $\sigma_{\text{Free}}(V(t))$, i.e., along the solution of (23) and (24) with $\beta = 0$. This result is shown in Figure 6, where we plot the difference $\alpha D^R_{\text{Free}}(t)\sigma_{\text{Free}}(V(t)) - V(t)$ along the solution of (23) and (24) with $\beta = 0$, which is negative when the VaR constraint is satisfied. In Figure 7 we provide the graph of the difference $U_{\text{Free}}(t) - U_{\text{VaR},\tau}(t)$, which is positive for all values of wealth $V(t) \geqslant 1$. Hence, the rational trader has no reason to adopt $D_{\text{VaR},\tau}(t)$ instead of $D_{\text{Free}}(t)$ for all $V(t) \geqslant 1$.

In Figure 5 we report the volatility in a VaR constrained setting with a tax on equity holding. The result obtained is striking. Taxation removes the effect of a VaR constraint, as a matter of fact volatility is almost constant with no peak, for a high wealth volatility is higher than the one observed in a VaR regulated market. There is no more a peak of volatility for low values of wealth. This result remains valid for a large set of parameters, i.e., when volatility for $V(t) = 1$ is smaller than 0.23. Moreover it holds true varying the value of α and the level of taxation τ or the magnitude of the exogenous risk σ_z. We can also change the starting point $V(t) = 1$, again we get that taxation eliminates the peak of volatility. Note that this is obtained at the cost of a generalized higher volatility.

Considering a higher level of volatility, for example at $V(t) = 1$ we start with a volatility equal to 0.4, then the peak reappears, so that VaR continues to affect volatility even in the presence of taxation. Nevertheless, this happens for high values of volatility. The rationale is simple: if volatility is high then the VaR constraint becomes binding and the optimal demand is the one obtained without taxation. Equity holding taxation reduces financial instability when the optimal solution is no more the one that fully exploits the VaR limit.

5 Taxation only on rational traders

In this Section we study the effect of a taxation imposed only on rational traders, passive traders are not taxed.

As in the previous section, rational traders are subject to a taxation on equity holdings, consequently their wealth evolves according to (20). Therefore, in a mean-variance setting the demand of rational traders is still given by (21). Instead, the demand of passive traders is as

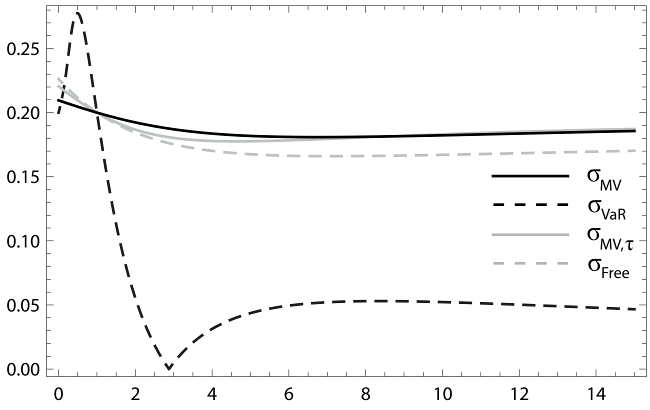


Figure 5: Market volatility in the mean-variance model (bold line) and market volatility in the presence of the VaR constraint (dashed line). Gray lines refer to the model in the presence of taxation, instead black lines refer to the model with no tax. Parameters: $\beta = 0.05$, $\alpha = 5$, $\delta = 0.07$, $\eta = 1$, $\sigma_z = 0.2$, $r = 0.01$, $\mu_{\text{MV}}(1) = 0.02$, $\mu_{\text{MV},\tau}(1) = \mu_{\text{Free}}(1) = -0.01$, $\sigma_{\text{MV}}(1) = \sigma_{\text{VaR}}(1) = \sigma_{\text{MV},\tau}(1) = \sigma_{\text{Free}}(1) = 0.2$ and $\tau = 0.005$.

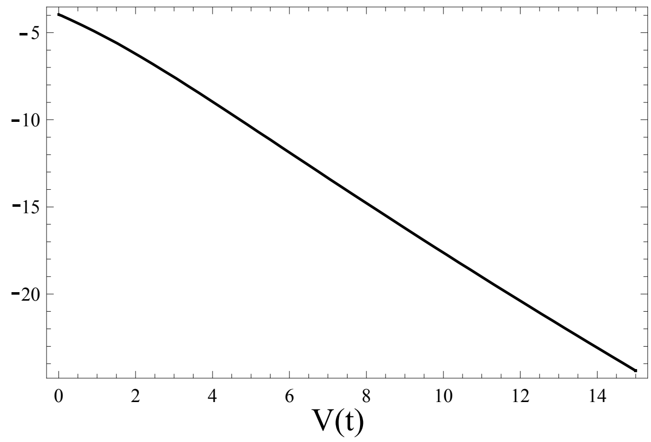


Figure 6: $\alpha D^R_{\text{Free}}(t)\sigma_{\text{Free}}(V(t)) - V(t)$ along the the solution of (23) and (24) with $\beta = 0$. Parameters: $\beta = 0.05$, $\alpha = 5$, $\delta = 0.07$, $\eta = 1$, $\sigma_z = 0.2$, $r = 0.01$, $\mu_{\text{MV}}(1) = 0.02$, $\mu_{\text{MV},\tau}(1) = \mu_{\text{Free}}(1) = -0.01$, $\sigma_{\text{MV}}(1) = \sigma_{\text{VaR}}(1) = \sigma_{\text{MV},\tau}(1) = \sigma_{\text{Free}}(1) = 0.2$ and $\tau = 0.005$.

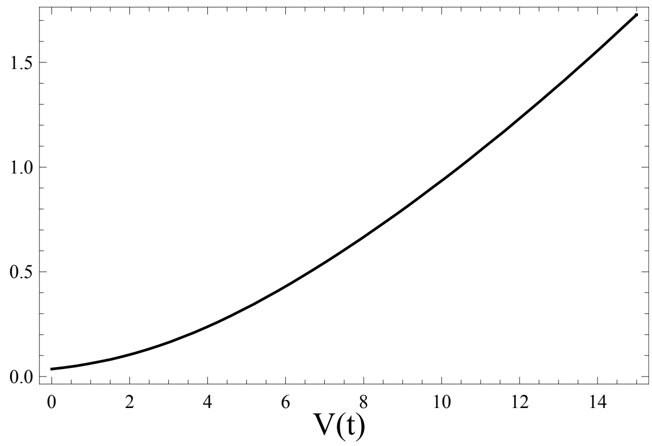


Figure 7: $U_{\text{Free}}(t) - U_{\text{VaR},\tau}(t)$ along the solution of (23) and (24) with $\beta = 0$. Parameters: $\beta = 0.05$, $\alpha = 5$, $\delta = 0.07$, $\eta = 1$, $\sigma_z = 0.2$, $r = 0.01$, $\mu_{\text{MV}}(1) = 0.02$, $\mu_{\text{MV},\tau}(1) = \mu_{\text{Free}}(1) = -0.01$, $\sigma_{\text{MV}}(1) = \sigma_{\text{VaR}}(1) = \sigma_{\text{MV},\tau}(1) = \sigma_{\text{Free}}(1) = 0.2$ and $\tau = 0.005$.

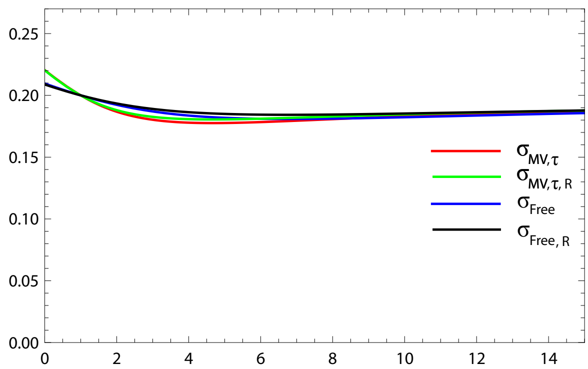


Figure 8: Market volatility in the mean-variance model with taxation for all traders (red line), market volatility in the mean-variance model with taxation only for rational traders (blue line), market volatility in the presence of the VaR constraint with taxation for all traders (green line) and market volatility in the presence of the VaR constraint with taxation only for rational traders (black line). Parameters: $\beta = 0.05$, $\alpha = 5$, $\delta = 0.07$, $\eta = 1$, $\sigma_z = 0.2$, $r = 0.01$, $\mu_{\text{MV},\tau}(1) = \mu_{\text{MV},\tau,R}(1) = \mu_{\text{Free}}(1) = \mu_{\text{Free},R}(1) = -0.01$, $\sigma_{\text{MV},\tau}(1) = \sigma_{\text{MV},\tau,R}(1) = \sigma_{\text{Free}}(1) = \sigma_{\text{Free},R}(1) = 0.2$ and $\tau = 0.005$.

in Section 2:

$$D^P(t) = \frac{\delta}{\sigma(t)^2}\Big(rt - \ln S(t) + \eta z(t)\Big).$$

In equilibrium the security price is given by

$$S_{\mathrm{MV},\tau,R}(t) = \exp\left\{\frac{\sigma_{\mathrm{MV},\tau,R}(t)^2}{\beta\sigma_{\mathrm{MV},\tau,R}(t)^2+\tau}\frac{\mu_{\mathrm{MV},\tau,R}(V(t))-r}{\delta} + rt + \eta z(t)\right\}.$$

Hence the drift and the volatility satisfy a system of two ordinary differential equations as stated in the next proposition.

Proposition 5. *In equilibrium, the volatility $\sigma_{\mathrm{MV},\tau,R}(V(t))$ and the drift $\mu_{\mathrm{MV},\tau,R}(V(t))$ satisfy the following system of two ordinary differential equations of second order*

$$\begin{aligned}
\mu_{\mathrm{MV},\tau,R}(V(t)) - \frac{1}{2}\sigma_{\mathrm{MV},\tau,R}(V(t))^2 = r + \frac{\sigma_{\mathrm{MV},\tau,R}(V(t))^2}{\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau}\frac{1}{\delta}\Bigg[\frac{\mathrm{d}\mu_{\mathrm{MV},\tau,R}(V(t))}{\mathrm{d}V}\Bigg(rV(t)+\\
+\frac{(\mu_{\mathrm{MV},\tau,R}(V(t))-r)^2}{\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau} - \frac{\tau(\mu_{\mathrm{MV},\tau,R}(V(t))-r)^2}{2(\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau)^2}\Bigg) + \frac{1}{2}\frac{\mathrm{d}^2\mu_{\mathrm{MV},\tau,R}(V(t))}{\mathrm{d}V^2}\cdot\\
\cdot\left(\frac{\mu_{\mathrm{MV},\tau,R}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau,R}(V(t))\right)^2\Bigg] + \frac{2\tau\sigma_{\mathrm{MV},\tau,R}(V(t))}{(\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau)^2}\cdot\\
\cdot\frac{\mu_{\mathrm{MV},\tau,R}(V(t))-r}{\delta}\Bigg[\frac{\mathrm{d}\sigma_{\mathrm{MV},\tau,R}(V(t))}{\mathrm{d}V}\Bigg(rV(t) + \frac{(\mu_{\mathrm{MV},\tau,R}(V(t))-r)^2}{\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau}\\
-\frac{\tau(\mu_{\mathrm{MV},\tau,R}(V(t))-r)^2}{2(\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau)^2}\Bigg) + \frac{1}{2}\frac{\mathrm{d}^2\sigma_{\mathrm{MV},\tau,R}(V(t))}{\mathrm{d}V^2}\cdot\\
\cdot\left(\frac{\mu_{\mathrm{MV},\tau,R}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau,R}(V(t))\right)^2\Bigg] + \tau\frac{\tau-\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2}{(\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau)^3}\cdot\\
\cdot\frac{\mu_{\mathrm{MV},\tau,R}(V(t))-r}{\delta}\left(\frac{\mathrm{d}\sigma_{\mathrm{MV},\tau,R}(V(t))}{\mathrm{d}V}\right)^2\left(\frac{\mu_{\mathrm{MV},\tau,R}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau,R}(V(t))\right)^2+\\
+\frac{2\tau\sigma_{\mathrm{MV},\tau,R}(V(t))}{(\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau)^2}\frac{1}{\delta}\frac{\mathrm{d}\mu_{\mathrm{MV},\tau,R}(V(t))}{\mathrm{d}V}\frac{\mathrm{d}\sigma_{\mathrm{MV},\tau,R}(V(t))}{\mathrm{d}V}\cdot\\
\cdot\left(\frac{\mu_{\mathrm{MV},\tau,R}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau}\sigma_{\mathrm{MV},\tau,R}(V(t))\right)^2 \qquad (34)
\end{aligned}$$

$$\sigma_{\mathrm{MV},\tau,R}(V(t)) = \frac{\mu_{\mathrm{MV},\tau,R}(V(t))-r}{\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau}\frac{\sigma_{\mathrm{MV},\tau,R}(V(t))}{\delta}\Bigg(\frac{\sigma_{\mathrm{MV},\tau,R}(V(t))^2}{\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau}\frac{\mathrm{d}\mu_{\mathrm{MV},\tau,R}(V(t))}{\mathrm{d}V}+$$

$$+\frac{2\tau\sigma_{\mathrm{MV},\tau,R}(V(t))}{(\beta\sigma_{\mathrm{MV},\tau,R}(V(t))^2+\tau)^2}(\mu_{\mathrm{MV},\tau,R}(V(t))-r)\frac{\mathrm{d}\sigma_{\mathrm{MV},\tau,R}(V(t))}{\mathrm{d}V}\Bigg)+\eta\sigma_z. \tag{35}$$

The proof is analogous to that of Proposition 3.

Now we consider the case when rational traders are subject to the VaR constraint. The demand is again given by (30), however D^R_{Free} and $D^R_{\mathrm{VaR},\tau}$ are different. We call them $D^R_{\mathrm{Free},R}$ and $D^R_{\mathrm{VaR},\tau,R}$. $D^R_{\mathrm{Free},R}$ is equal to the demand of rational traders in the mean-variance model with $\beta = 0$. Consequently, the drift and the volatility in equilibrium are given by Proposition 5 with $\beta = 0$. Instead, when the demand of rational traders is $D^R_{\mathrm{VaR},\tau}$ the market volatility is no more given by Proposition 4. As a matter of fact, the security price in equilibrium solves equation (6), therefore the drift and the market volatility are given respectively by equations (7) and (8).

As in the previous section for a large set of parameters D^R_{Opt} is equal to $D^R_{\mathrm{Free},R}$ for all levels of wealth. As a consequence, the resulting market volatility is $\sigma_{\mathrm{Free},R}$, i.e., $\sigma_{\mathrm{MV},\tau,R}$ with $\beta = 0$.

In Figure 8 we plot the volatility in the mean-variance model with taxation for all traders ($\sigma_{\mathrm{MV},\tau}$) and with taxation only on rational traders ($\sigma_{\mathrm{MV},\tau,R}$). Moreover we plot the volatility in the presence of the VaR-constraint in both cases (σ_{Free} and $\sigma_{\mathrm{Free},R}$). Volatility patterns coincide. The result suggests that to stabilize the market it is enough to tax rational traders and not passive/liquidity traders.

6 Conclusions

The financial crisis has shown the limits of a financial regulation based on a VaR constraint. A constraint based on a limit on the VaR of the wealth may provoke a destabilizing effect with a selling pressure when the asset prices are low. In this paper we have show that this feature is peculiar of VaR regulation, in a market with risk averse traders we do not observe the phenomenon: VaR regulation plays a destabilizing effect. To overcome the problem a tax on equity-holding may be useful. For a low level of wealth the VaR constraint is no more binding at the optimal solution eliminating the hump shape of volatility. This effect is obtained applying a tax on all traders or in a selective way only to rational/VaR constrained traders.

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