12/12/2006

EULER EQUATIONS FOR BLAKE & ZISSERMAN FUNCTIONAL

Michele Carriero* Antonio Leaci* Franco Tomarelli**

Abstract - We derive many necessary conditions for minimizers of a functional depending on free discontinuities, free gradient discontinuities and second derivatives, which is related to image segmentation.
A candidate for minimality of main part of the functional is explicitly exhibited.

* Dipartimento di Matematica "Ennio De Giorgi" – Via Arnesano – 73100 – Lecce – Italia
** Dipartimento di Matematica "Francesco Brioschi" – Politecnico – Piazza Leonardo da Vinci 32 – 20133 – Milano – Italia

12/12/2006

1. Introduction.

Calculus of variations is the framework where energy minimization and equilibrium notions find a precise language and formalization by means of variational principles. Image segmentation is a relevant problem both in digital image processing and in the understanding of biological vision.
Roughly speaking, *segmenting an image* means to find regions of interest in a picture, so that these regions can be parcelled out for further analysis: the process cuts the picture into the simplest shaped pieces possible while keeping the color or luminance of each piece as slowly varying as possible.

In the book [BZ] a variational principle for image segmentation was introduced in the context of visual reconstruction: the Blake & Zisserman functional which depends on second derivatives, free discontinuities and free gradient discontinuities of the intensity levels.

The Blake-Zisserman model faces the segmentation as an energy minimization problem. It takes an image and produces two outputs: a boundary process map which indicates the location of boundaries (jump and creases of luminance), and a surface attribute map which indicates the smoothed (interpolated) luminance values on the surface of objects in the field.

We introduced the formalized weak version of this principle and proved the existence of weak minimizers and the corresponding optimal segmentation in [CLT3], [CLT2], [CLT9]. Then we showed regularity properties, energy and density estimates for optimal segmentation in [CLT3], [CLT4], [CLT5], [CLT8], [CLT10].
Here we derive many necessary conditions about extremals by performing various kind of first variations: these delicate computations are performed by taking into account the differential geometry of free discontinuity set in several dimensions. Some of the results were announced in [CLT7]. In particular we develop here the full analysis of crack-tip and crease-tip (boundaries of free discontinuity set), whose properties were stated in [CLT7] only for the flat case.
We recall the strong formulation F of Blake & Zisserman functional ([CLT4]), say

$$\begin{aligned} F(K_0,K_1,u) \ :=\ & \int_{\Omega\setminus(K_0\cup K_1)} \left(|D^2u|^2+\mu|u-g|^q\right)dy \\ & + \alpha\mathcal{H}^{n-1}(K_0\cap\Omega)+\beta\mathcal{H}^{n-1}((K_1\setminus K_0)\cap\Omega)\ , \end{aligned} \tag{1.1}$$

to be minimized over triplets(K_0,K_1,u) in order to achieve an optimal segmentation, and label the main part E of functional F as follows

$$E(K_0,K_1,u) := \int_{\Omega\setminus(K_0\cup K_1)} |D^2u|^2\,dy+\alpha\mathcal{H}^{n-1}(K_0\cap\Omega)+\beta\mathcal{H}^{n-1}((K_1\setminus K_0)\cap\Omega)\,, \tag{1.2}$$

where $\Omega \subset \mathbf{R}^n$ is an open set, $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, and $\alpha, \beta, \mu, q \in \mathbf{R}$, with

$$q \geq 1 \ , \ \mu > 0 \ , \ 0 < \beta \leq \alpha \leq 2\beta \ , \ g \in L^q(\Omega) \ , \tag{1.3}$$

are given; while $K_0, \ K_1 \subset \mathbf{R}^n$ are Borel sets (a priori unknown) with $K_0 \cup K_1$ closed, $u \in C^2(\Omega \setminus (K_0 \cup K_1))$ and it is approximately continuous on $\Omega \setminus K_0$.
If K_0, K_1, u is a minimizing triplet of F and $n = 2, 3$ then $K_0 \cup K_1$ can be interpreted as an optimal segmentations of the monochromatic image of brightness intensity g.
Existence of minimizers of (1.1) was proved by regularization of solution of the weak formulation (1.4) for $n = 2$, provided the additional assumption $g \in L^{2q}_{loc}(\Omega)$ is satisfied.
When $n \geq 2$ and $g \notin L^{nq}_{loc}(\Omega)$ then the infimum cannot be achieved in general (see [CLT5], section 5).

We recall that the weak functional $\mathcal{F}$ is defined by ([CLT3])

$$\mathcal{F}(v) \ := \ \int_\Omega (|\nabla^2 v|^2 + \mu |v - g|^q)\, dy + \alpha \mathcal{H}^{n-1}(S_v) + \beta \mathcal{H}^{n-1}(S_{\nabla v} \setminus S_v)\,, \tag{1.4}$$

for any $v \in L^q(\Omega) \cap GSBV(\Omega)$ with $\nabla v \in (GSBV(\Omega))^n$ (for the precise setting of the functional framework we refer to Definition 2.1). The main part $\mathcal{E}$ of the functional $\mathcal{F}$ will be denoted by

$$\mathcal{E}(v) \ := \ \int_\Omega |\nabla^2 v|^2\, dy + \alpha \mathcal{H}^{n-1}(S_v) + \beta \mathcal{H}^{n-1}(S_{\nabla v} \setminus S_v)\,. \tag{1.5}$$

Due to the dependence on second derivatives $D^2 u$ the Blake & Zisserman functional detects both jump and crease sets. Moreover second order functionals avoid the inconvenient of ramp effect due to over-segmentation of steep gradients: that is the appearing of one or more spurious discontinuities in the output image u determined by Mumford & Shah model ([MSh],[MoSo],[Ma]), when the datum g is a continuous ramp with gradient steep enough.
The Blake & Zisserman functional (1.1) depends both on bulk energy and a lineic (or surfacic) discontinuity energy; their coupling is rather intriguing. Moreover the discontinuities of u and of Du take place respectively on the sets $K_0, \ K_1$ which are “a priori” unknown, hence the associated minimization problem turns out to be essentially non-convex, and non uniqueness of minimizers may develop for some choice of data.
Notice that uniqueness fails due to lack of convexity of functional F (see [BoT] for explicit examples of multiplicity), nevertheless generic uniqueness of minimizers (with respect to data α, β, g) is proven in the 1 dimensional case in [BoT].

Another difficulty in the mathematical analysis of the BZ functional is the fact that (1.1) does not control the intermediate (first) derivatives, moreover truncation of competing functions does not reduce the energy, while in case of functional MS truncation reduces energy.

Here a deep analysis of first variation is done.
First (by performing suitable smooth variations of a function u minimizing $\mathcal{F}$) we find the Euler partial differential equation satisfied by u in $\Omega \setminus \overline{(S_u \cup S_{\nabla u})}$ (see Theorem 3.4) and jump conditions for natural boundary operator evaluated on u in $S_u \cup S_{\nabla u}$ (see Theorems 4.3 and 4.4).
Second (by performing smooth variations of the sets S_u and $S_{\nabla u}$ around a minimizer u for $\mathcal{F}$) we find integral and geometric conditions on optimal segmentation sets (Theorems 5.1, 5.3, 5.4, 5.5 and 5.6). More precisely we evaluate the first variation of the energy functional (1.4) around a local minimizer u, under compactly supported smooth deformation of S_u and $S_{\nabla u}$ and we get the complete integral Euler equation (5.1); in order to obtain additional information on local minimizers, it is useful to perform a careful integration by parts of the volume integral of Euler equation (5.1) independent of the forcing term, hence deduce a relationship between the curvature of S_u and the square hessian jump (Theorem 5.3) and a relationship between the curvature of $S_{\nabla u}$ and the square hessian jump (Theorem 5.4). Then we perform a qualitative analysis of the "boundary" of the singular set, by assuming it is manifold as smooth as required by the computation of boundary operators: the strategy is a new choice of the test functions in Euler equation (5.1.): a vector field η tangential to S_u (or $S_{\nabla u}$) and we obtain quantitative information about crack-tip and crease-tip (Theorems 5.5, 5.6).
All the results listed above hold true also for the strong functionals F and E, as stated in Remarks 5.7 and 6.7.
A Caccioppoli inequality holds (Theorem 6.2): as a consequence local minimizers of $\mathcal{E}$ in $\mathbf{R}^n$ cannot have both nonempty compact segmentation set $K_0 \cup K_1$ and finite energy (Theorem 6.3). We recall that neither an infinite wedge nor a 1–dimensional uniform jump are local minimizers of $\mathcal{E}$ in $\mathbf{R}^2$ ([CLT8]).
We extend here a Liouville type property (proven in [CLT8] for $n = 2$) for Blake & Zisserman local minimizers to any dimension n (Theorem 6.4): bi-harmonic functions in $\mathbf{R}^n$ are local minimizers of $\mathcal{E}$ if and only if they are affine.
We prove an Almansi decomposition property in 2 dimensional ball $B_\varrho(0)$ with a cut up to the origin (Theorem 7.2) and analyze asymptotic expansion of a bi-harmonic function in a disk with a cut (more precisely functions in the space V): see Definitions 7.3, 7.6 and Lemmas 7.4, 7.7.
The huge amount of information around a crack-tip or a crease-tip leads us to restrict severely qualitative and quantitative behavior allowed to extremals: in Theorem 7.9 the generic asymptotic expansion of any local minimizer of $\mathcal{E}$ with jump discontinuity along the negative real axis is exhibited, together with the fact that the main part

of the expansion around the origin has homogeneity $3/2$ in r (Lemma 7.8).
If in addition equipartition of energy around the origin (among the volume integral and the segmentation length) is imposed, then the coefficients of the main part of a local minimizer are fixed and we can evaluate them explicitly (Theorem 7.11).
Eventually, in Section 8 we can show a nontrivial function, with jump discontinuity along the negative real axis,

$$\pm\sqrt{\frac{\alpha}{193\,\pi}}\,r^{3/2}\,\left(\sqrt{21}\,\omega(\vartheta)\,\pm\,w(\vartheta)\right) \tag{1.6}$$

explicitly

$$\pm\sqrt{\frac{\alpha}{193\,\pi}}\;r^{3/2}\,\left(\sqrt{21}\left(\,\sin\frac{\theta}{2}\;-\;\frac{5}{3}\,\sin\left(\frac{3}{2}\theta\right)\right)\,\pm\,\left(\,\cos\frac{\theta}{2}\;-\;\frac{7}{3}\,\cos\left(\frac{3}{2}\theta\right)\right)\right)$$

satisfying in $\mathbf{R}^2$ all the extremal conditions proved for functional $\mathcal{E}$: hence such function is a natural candidate to be a local minimizer.
Such function has jump set on the negative real axis and empty jump discontinuity set of the gradient.
All these facts lead us to formulate the following statement.

Conjecture - The candidate *(1.6) is a local minimizer of $\mathcal{E}$ in $\mathbf{R}^2$, and there are no other nontrivial local minimizers, up to (possibly independent in each mode ω and w) sign change, rigid motions of $\mathbf{R}^2$ co-ordinates and/or addition of affine functions.*

In this paper we prove that the presence of a non-vanishing $3/2$ homogeneous term is necessary and has prescribed modes and coefficients in any asymptotic development of a local minimizer (see (7.26) in Lemma 7.8): such term is then the archetype of the admissible candidate.
Moreover we emphasize that the admissible candidate (1.6) fulfills equipartition of absolutely continuous and free discontinuity lineic energy in a strong form: say they coincide on each ball centered at the origin.

Notice that the leading coefficient (S.I.F. : Stress Intensity Factor) $\sqrt{\dfrac{\alpha}{193\,\pi}}$ in the candidate (1.6) is uniquely defined (up to sign change) as soon as crack-tip extremal condition (of Theorem 5.6) and equipartition of energy (Definition 7.10) are fulfilled.
We emphasize that the above value of S.I.F. is also the only admissible value of leading coefficient in the expansion of any local minimizer of $\mathcal{E}$ (Theorem 7.11).

Some of the very long computation of Section 7 (Lemma 7.7, Theorems 7.9, 7.11) were checked also by symbolic computation routines with software MATHEMATICA 5.0 © : the Notebook with essential labelled instructions about computations and plots of the candidate are contained in the Appendix (Section 10: whose formulas are labelled by (10.xx) in the paper).

Referring to a forthcoming paper we claim the non existence of minimizers of $\mathcal{E}$ in $\mathbf{R}^2$ whose singular set is a crease along the negative real axis.

The outline of the paper is the following.

1. Introduction.
2. Notation and preliminary results.
3. Euler equations I : smooth variations for $\mathcal{F}$ $(n \geq 2)$.
4. Euler equations II : boundary-type conditions on the singular sets for extremals of $\mathcal{F}$ $(n \geq 2)$.
5. Euler equations III : singular set variations for $\mathcal{F}$ $(n \geq 2)$.
6. Local minimizers of $\mathcal{E}$ in $\mathbf{R}^n$ $(n \geq 2)$: Caccioppoli inequality and Liouville property.
7. Asymptotic expansions of bi-harmonic functions in a disk with a cut and non trivial local minimizers of $\mathcal{E}$ in $\mathbf{R}^2$.
8. A candidate for minimality of $\mathcal{E}$ in $\mathbf{R}^2$.
9. References.
10. Appendix - Notebook BZEE.NB (MATHEMATICA 5.0 ©)

We refer to the enclosed bibliography and to the web-page of the Research Group in Calculus of Variations and Geometric Measure Theory http://cvgmt.sns.it where several additional references are available.

2. Notation and preliminary results.

From now on we denote by Ω an open set in $\mathbf{R}^n$, $n \geq 2$. Given two vectors $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n)$, we set $a \cdot b = \sum_i a_i b_i$.
Given two matrices $A = (A_{ij}), B = (B_{ij})$, we set $(AB)_{ik} = \sum_j A_{ij} B_{jk}$ and $A : B = \sum_{ij} A_{ij} B_{ij}$. By A^t we denote the transposed matrix.
For a given set $U \subset \mathbf{R}^n$ we denote by ∂U its topological boundary, by $\overline{U}$ its topological closure, by $\mathcal{H}^{n-1}(U)$ its $(n-1)$-dimensional Hausdorff measure and by $|U|$ its n-dimensional Lebesgue outer measure; χ_U is the characteristic function of U. We indicate by $B_\rho(x)$ the open ball $\{y \in \mathbf{R}^n ; |y - x| < \rho\}$, $B_\rho = B_\rho(0)$ and we set $S^{n-1} = \partial B_1(0)$ and $\omega_n = |B_1(0)|$. If Ω, Ω' are open subsets in $\mathbf{R}^n$, by $\Omega \subset\subset \Omega'$ we mean that $\overline{\Omega}$ is compact and $\overline{\Omega} \subset \Omega'$.
We say that a subset E of $\mathbf{R}^n$ is *countably* $(\mathcal{H}^{n-1}, n-1)$ *rectifiable* if it is $\mathcal{H}^{n-1}$ measurable and E (up to a set of vanishing $\mathcal{H}^{n-1}$ measure) is the countable union of C^1 images of bounded subsets of $\mathbf{R}^{n-1}$; if in addition $\mathcal{H}^{n-1}(E) < +\infty$ then we say that E is $(\mathcal{H}^{n-1}, n-1)$ *rectifiable.*

For any Borel function $v : \Omega \to \mathbf{R}$ and for $x \in \Omega, z \in \bar{\mathbf{R}} := \mathbf{R} \cup \{-\infty, +\infty\}$, we set $z = \operatorname{ap}\lim_{y \to x} v(y)$ (approximate limit of v at x, denoted by $\tilde{v}(x)$) if

$$g(z) = \lim_{\rho \to 0} \fint_{B_\rho(0)} g(v(x+\xi))d\xi$$

for every $g \in C^0(\bar{\mathbf{R}})$; if $z \in \mathbf{R}$ this definition is equivalent to 2.9.12 in [F]. If there exist $\nu \in S^{n-1}$ and $z \in \bar{\mathbf{R}}$ such that

$$g(z) = \lim_{\rho \to 0} \fint_{B_\rho(0) \cap \{\xi \cdot \nu > 0\}} g(v(x+\xi))d\xi \qquad \forall g \in C^0(\bar{\mathbf{R}}),$$

we set $z = tr^+(x, v, \nu)$. Analogously we define $z = tr^-(x, v, \nu)$ if $z = tr^+(x, v, -\nu)$. Let $x \in \Omega$ such that $\tilde{v}(x) \in \mathbf{R}$; we say that v is approximately differentiable at x if there exists a vector $\nabla v(x)$ (approximate gradient of v at x) such that

$$\operatorname{ap}\lim_{y \to x} \frac{|v(y) - \widetilde{v}(x) - \nabla v(x) \cdot (y - x)|}{|y - x|} = 0.$$

The set $S_v = \{x \in \Omega : \operatorname{ap}\lim_{y \to x} v(y) \text{ does not exist}\}$ will be called the singular set of v; S_v is a Borel set and it has null Lebesgue measure (see e.g. [F], 2.9.13).
In the following with the notation $|\nabla v|$ we mean the euclidean norm of ∇v and we set $\nabla_i v = (\mathbf{e}_i \cdot \nabla)v$, where $\{\mathbf{e}_i\}$ denotes the canonical basis of $\mathbf{R}^n$. When the right hand side is meaningful, we set $\nabla^2_{ij} v = \nabla_i(\nabla_j v)$; moreover we set $\Delta v = \sum_{j=1}^n D_j(D_j v)$ and $\Delta^2 v = \Delta(\Delta v)$ (distributional laplacian and distributional bilaplacian respectively).
If A is an open set and $k \in \mathbf{N}$ then $W^{k,s}(A)$ denotes the Sobolev space of distributions $v \in L^s(A)$ such that all weak derivatives of v up to order k are in $L^s(A)$.
We recall the definition of the space of real valued functions with bounded variation in Ω : $BV(\Omega) = \{v \in L^1(\Omega); Dv \in \mathcal{M}(\Omega)\}$, where Dv denotes the distributional gradient of v and $\mathcal{M}(\Omega)$ denotes the space of vector-valued Radon measure with finite total variation. We denote by $\int_\Omega |Dv|$ the total variation of the measure Dv in Ω.
For every $v \in BV(\Omega)$ the following properties hold ([F]):

1) S_v is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable and $\mathcal{H}^{n-1}$-a.e. an approximate unit normal vector ν is uniquely defined (up to the orientation);
2) ∇v exists a.e. in Ω and coincides with the Radon–Nikodym derivative of Dv with respect to the Lebesgue measure;
3) for $\mathcal{H}^{n-1}$-a.a. $x \in S_v$ there exists a unique (up to the sign) $\nu(x) \in S^{n-1}$ such that $tr^\pm(x, v, \nu)$ exist (finite and not equal); once $\nu(x)$ is fixed we put $v^\pm(x) = tr^\pm(x, v, \nu(x))$.

We list the definitions of functional classes related to first derivatives which are De Giorgi special measures, and we refer to [DA],[AFP], [CLT1,2,3,4] [P] for their properties.

Definition 2.1. *$SBV(\Omega)$ denotes the class of functions $v \in BV(\Omega)$ such that*

$$\int_\Omega |Dv| = \int_\Omega |\nabla v|\, dy + \int_{S_v} |v^+ - v^-|\, d\mathcal{H}^{n-1}.$$

$$SBV_{loc}(\Omega) := \{v \in SBV(\Omega') : \ \forall \Omega' \subset\subset \Omega\},$$

$$GSBV(\Omega) := \left\{v : \Omega \to \mathbf{R} \text{ Borel function}; -k \vee v \wedge k \in SBV_{loc}(\Omega)\ \forall k \in \mathbf{N}\right\}.$$

$$GSBV^2(\Omega) := \left\{v \in GSBV(\Omega),\ \nabla v \in \left(GSBV(\Omega)\right)^n\right\}.$$

We emphasize that $GSBV(\Omega)$, $GSBV^2(\Omega)$ are neither vector spaces nor subsets of distributions in Ω ([AFM],[FLP]). Nevertheless smooth variations of a function in $GSBV^2(\Omega)$ still belong to the same class $GSBV^2(\Omega)$.
Notice that, if $v \in GSBV(\Omega)$, then S_v is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable and ∇v exists a.e. in Ω. Moreover we can define the tangential derivatives of a C^1 function along the singular set of a function in $GSBV$ as follows.
Notice that $Dv \neq \nabla v$ in $GSBV^2(\Omega)$; moreover we set $S_{\nabla v} = \bigcup_{i=1}^n S_{\nabla_i v}$.
For simplicity of notation we set

$$K_v = \overline{S_v \cup S_{\nabla v}}. \tag{2.1}$$

We remark that the set $S_v \cup S_{\nabla v}$ is not closed in general and its closure may be the whole set Ω.

Definition 2.2. *If $v \in GSBV(\Omega)$, $\psi \in C^1(\Omega)$ and ν is the approximate normal vector to S_v we set for $\mathcal{H}^{n-1}$ a.e. $x \in S_v$, $i = 1 \dots, n$,*

$$\delta_i \psi = D_i \psi - (D\psi \cdot \nu)\, \nu_i\ ,$$

$\delta_i \psi$ are the tangential derivatives, say the components of the tangential gradient $\delta\psi$. Moreover if $\eta \in C^1(\Omega, \mathbf{R}^n)$, we define the tangential gradient $\delta\psi = (\delta_1\psi, \dots, \delta_n\psi)$ and the tangential divergence of η $\mathcal{H}^{n-1}$ a.e. on S_v by

$$\mathrm{div}^\tau_{\mathrm{S_v}}\, \eta = \sum_{i=1}^n \delta_i \eta^i.$$

We will write shortly $\mathrm{div}^\tau\, \eta$ whenever there is no risk of confusion.

We recall the following definition (see [Giu], Remark 10.6):

Definition 2.3. *Let M be a C^2 hypersurphace in Ω and $\eta \in C^1_0(\Omega, \mathbf{R}^n)$ a vector field such that $\eta(x) = \nu_M(x)$ on M, where $\nu_M(x) \in \partial B_1$ is a normal vector to M at x. The mean curvature of M at x is defined by*

$$\mathcal{K}(M)(x) \;=\; \frac{1}{n-1}\,\mathrm{div}^\tau_{\mathrm{M}}\,\eta(x) \qquad x \in M.$$

Again we will write shortly $\mathrm{div}^\tau\,\eta$ *whenever there is no risk of confusion. We notice that, if $n = 2$, then M is a C^2 arc and, by setting $\nu = (\nu_1, \nu_2)$, $\tau = (\nu_2, -\nu_1)$, we get for every $x \in M$,*

$$\tau \cdot D\eta\,\tau \;=\; \delta_1\eta^1 + \delta_2\eta^2 \;=\; \mathcal{K}(M).$$

We recall the precise statement of the weak minimization.

Definition 2.4. (**Weak formulation of Blake & Zisserman functional**)
For $\Omega \subset \mathbf{R}^n$ open set, under the assumption (1.3), we define $\mathcal{F} : X(\Omega) \to [0, +\infty]$ by

$$\mathcal{F}(v) \;:=\; \int_\Omega (|\nabla^2 v|^2 + \mu|v - g|^q)\,dy + \alpha\mathcal{H}^{n-1}(S_v) + \beta\mathcal{H}^{n-1}(S_{\nabla v} \setminus S_v)\,. \tag{2.2}$$

where $X(\Omega) := GSBV^2(\Omega) \cap L^q(\Omega)$. We will use also localization of functional $\mathcal{F}$

$$\begin{aligned}\mathcal{F}(v, A) = \int_A \big(|\nabla^2 v|^2 + \mu|v - g|^q\big)\,dy \\ + \alpha\mathcal{H}^{n-1}(S_v \cap A) + \beta\mathcal{H}^{n-1}\big((S_{\nabla v} \setminus S_v) \cap A\big)\end{aligned} \tag{2.3}$$

for every Borel set $A \subseteq \Omega$.

We notice that the subset of $GSBV^2(\Omega)$ where $\mathcal{F}$ is finite is a vector space (see Corollary 4.5 of [AFM]) while $GSBV^2(\Omega)$ is not a vector space.

We proved the following results in [CLT3], [CLT4] and [CLT5].

Theorem 2.5. (Existence of weak solutions)
Let $\Omega \subset \mathbf{R}^n$ be an open set and assume (1.3). Then there is $v_0 \in X(\Omega)$ such that

$$\mathcal{F}(v_0) \le \mathcal{F}(v) \qquad \forall v \in X(\Omega).$$

We recall that assumption $\beta \le \alpha \le 2\beta$ is necessary for lower semicontinuity of $\mathcal{F}$.

Definition 2.6. (**Strong formulation of Blake & Zisserman functional**)

For $\Omega \subset \mathbf{R}^n$ open set, $K_0, K_1 \subset \mathbf{R}^n$ Borel sets with $K_0 \cup K_1$ closed, u approximately continuous on $\Omega \setminus K_0$ and $u \in C^2(\Omega \setminus (K_0 \cup K_1))$, under the assumption (1.3), we set

$$
\begin{aligned}
F(K_0, K_1, u) \ :=\ & \int_{\Omega \setminus (K_0 \cup K_1)} \left(|D^2 u|^2 + \mu |u - g|^q \right) dy \\
& + \alpha \mathcal{H}^{n-1}(K_0 \cap \Omega) + \beta \mathcal{H}^{n-1}\left((K_1 \setminus K_0) \cap \Omega \right).
\end{aligned}
\tag{2.4}
$$

We write $F(K_0, K_1, u, A)$ when the previous functional is localized on a Borel set $A \subset \Omega$.

Theorem 2.7. (Existence of strong solutions)
Let $n = 2$, $\Omega \subset \mathbf{R}^2$ be an open set. Assume (1.3) and $g \in L^{2q}_{loc}(\Omega)$. Then there is at least one triplet among $K_0, K_1 \subset \mathbf{R}^2$ Borel sets with $K_0 \cup K_1$ closed and $u \in C^2(\Omega \setminus (K_0 \cup K_1))$ approximately continuous on $\Omega \setminus K_0$ minimizing the functional (2.4) with finite energy. Moreover the sets $K_0 \cap \Omega$ and $K_1 \cap \Omega$ are $(\mathcal{H}^1, 1)$ rectifiable.

Theorem 2.8. *Let $n = 2$, $\Omega \subset \mathbf{R}^2$ be an open set. Assume (1.3), $g \in L^{2q}_{loc}(\Omega)$ and $\alpha = \beta$. Then there is at least one pair among $K \subset \mathbf{R}^2$ closed set and $u \in C^2(\Omega \setminus K)$ minimizing the functional*

$$
\int_{\Omega \setminus K} \left(|D^2 u|^2 + \mu |u - g|^q \right) \, dy + \alpha \mathcal{H}^1(K \cap \Omega)
$$

with finite energy. Moreover the set $K \cap \Omega$ is $(\mathcal{H}^1, 1)$ rectifiable.

Definition 2.9. (Strong minimizing triplet of F)
A triplet (T_0, T_1, u) such that, $T_0, T_1 \subset \mathbf{R}^n$ are Borel sets, $T_0 \cup T_1$ is a closed set, $u \in C^2\big(\Omega \setminus (T_0 \cup T_1)\big)$ and approximately continuous in $\Omega \setminus T_0$, is a strong minimizing triplet of the functional (2.4) if

$$
(T_0, T_1, u) \ \in \ \operatorname{argmin} F \ .
$$

Remark 2.10. If $n = 2$, $g \in L^{2q}_{loc}$ and (T_0, T_1, u) is a strong minimizing triplet of F then u is a weak minimizer of $\mathcal{F}$ and $F(T_0, T_1, u) = \mathcal{F}(u) = \min \mathcal{F}$.

Definition 2.11. (Essential minimizing triplet of F)
Given a strong minimizing triplet (T_0, T_1, v) of the functional (2.4), there is another triplet (K_0, K_1, u) , called essential minimizing triplet, uniquely defined by

$$
\begin{aligned}
K_0 &= \overline{T_0 \cap K} \setminus (S_{\nabla v} \setminus S_v) \\
K_1 &= \overline{T_1 \cap K} \setminus S_v \\
u &= \tilde{v}
\end{aligned}
$$

where K is the smallest closed subset of $T_0 \cup T_1$ such that $\tilde{v} \in C^2(\Omega \setminus K)$.

Theorem 2.12. (Density upper bound for the functional F)
Let (K_0, K_1, u) be a strong minimizing triplet for the functional (2.4) under assumptions (1.3). Then for every $0 < \rho \leq 1$ and for every $x \in \Omega$ such that $\overline{B}_\rho(x) \subset \Omega$ we have

$$
F(K_0, K_1, u, \overline{B}_\rho(x)) \leq c_0 \rho^{n-1}. \tag{2.5}
$$

where $c_0 = \omega_n^{\frac{n-1}{n}} \mu \|g\|^q_{L^{nq}(B_\rho(x))} + \alpha n \omega_n$.

If $q = 2$ and $g \in L^\infty(\Omega)$, then $c_0 = \omega_n \mu \|g\|^2_{L^\infty(B_\rho(x))} + \alpha n \omega_n$.

Remark 2.13. We notice that the density upper bound of Theorem 2.12 holds true, by substituting F with $\mathcal{F}$, also for the minimizers of the weak formulation.

Additional and more precise informations are available in the two dimensional case: we list below some quantitative geometric properties about optimal segmentation which were used in the approximation of the 2 dimensional Blake & Zisserman energy by elliptic functionals ([AFM], [CFS]).

Theorem 2.14. (Density lower bound for the functional F and for the segmentation lenght) ([CLT4],[CLT5])
Let $n = 2$ and (K_0, K_1, u) be an essential minimizing triplet for the functional (2.4) with $g \in L^{2q}_{loc}(\Omega)$. Then there exist $\varepsilon_0 > 0, \varrho_0 > 0$ and $\varepsilon_1 > 0, \varrho_1 > 0$ such that

$$
\begin{aligned}
F(K_0, K_1, u, B_\varrho(x)) &\geq \varepsilon_0 \varrho \qquad \forall x \in K_0 \cup K_1, \quad \forall \varrho \leq \varrho_0, \\
\mathcal{H}^1\left((K_0 \cup K_1) \cap B_\varrho(x)\right) &\geq \varepsilon_1 \varrho \qquad \forall x \in K_0 \cup K_1, \quad \forall \varrho \leq \varrho_1.
\end{aligned}
$$

Theorem 2.15. (Elimination Property) ([CLT4],[CLT5])

Let $n = 2$ and let (K_0, K_1, u) be an essential minimizing triplet for the functional (2.4) with $g \in L^{2q}_{loc}(\Omega)$ and let $\varepsilon_1 > 0, \varrho_1 > 0$ as in Theorem 2.14 and $\rho \leq \rho_1$. If $x \in \Omega$ and

$$\mathcal{H}^1\left((K_0 \cup K_1) \cap B_\varrho(x)\right) \quad < \quad \frac{\varepsilon_1}{2}\rho$$

then

$$(K_0 \cup K_1) \cap B_{\rho/2}(x) = \emptyset.$$

The elimination property states that, when an optimal segmentation has length, in a small ball, less than an absolute constant times the radius of the ball, then such segmentation does not intersect the ball with half the radius.

Theorem 2.16. **(Minkowski content of the segmentation)** ([CLT5])
Let $n = 2$ and let (K_0, K_1, u) be an essential minimizing triplet for the functional (2.4) with $g \in L^{2q}_{loc}(\Omega)$. Then the following equality holds for every $\Omega' \subset\subset \Omega$

$$\lim_{\rho \to 0} \frac{|\{x \in \Omega\,;\ dist(x, (K_0 \cup K_1) \cap \Omega') < \rho\ \}|}{2\rho} = \mathcal{H}^1\left((K_0 \cup K_1) \cap \Omega'\right)\ .$$

Roughly speaking, the above theorem says that a uniform fattening of an optimal segmentation is a reasonable approximation of the segmentation itself.

We notice that the various constants $c_0, \varepsilon_0, \varepsilon_1, \rho_0, \rho_1$ depend on the data n, α, β, μ, g.

3. Euler equations I : smooth variations for $\mathcal{F}$ $(n \geq 2)$.

In this section we obtain some regularity properties of the minimizers of the functional $\mathcal{F}$ outside of the closure of the singular sets. First we recall the definition of local minimizer of $\mathcal{F}$.

Definition 3.1. **(Local minimizer of $\mathcal{F}$)**
We say that u is a local minimizer of the functional $\mathcal{F}$ if

$$u \in GSBV^2(A), \qquad\qquad \mathcal{F}(u, A) < +\infty$$

and

$$\mathcal{F}(u, A) \leq \mathcal{F}(u + \varphi, A)$$

for every open subset $A \subset\subset \Omega$ and for every $\varphi \in GSBV^2(\Omega)$ with compact support in A.

We say that u is a local minimizer of the functional $\mathcal{E}$ in Ω

$$\mathcal{E}(v) \;=\; \int_\Omega |\nabla^2 v|^2\,dx + \alpha\mathcal{H}^{n-1}(S_v) + \beta\mathcal{H}^{n-1}(S_{\nabla v} \setminus S_v)$$

if, by denoting $\mathcal{E}(\cdot, A)$ the localization of $\mathcal{E}$,

$$u \in GSBV^2(A), \qquad\qquad \mathcal{E}(u,A) < +\infty, \qquad \mathcal{E}(u,A) \le \mathcal{E}(u+\varphi, A)$$

for every open subset $A \subset\subset \Omega$ and for every $\varphi \in GSBV^2(\Omega)$ with compact support in A.

Remark 3.2. If u is a local minimizer of $\mathcal{E}$ in Ω then also the function $u(x)+a\cdot x+b$ is a local minimizer in Ω for every $a \in \mathbf{R}^n, b \in \mathbf{R}$. Moreover, if $B_\rho(x_0) \subset \Omega$, then the re-scaling

$$u_\rho(x) = \rho^{-3/2} u(x_0 + \rho x)$$

defines a local minimizer of $\mathcal{E}$ in $B_1(0)$ and we have

$$\mathcal{E}(u, B_\rho(x_0)) = \rho^{n-1}\mathcal{E}(u_\rho, B_1(0)).$$

We recall that the subset of $GSBV^2(\Omega)$ where $\mathcal{F}$ is finite is a vector space (see [AFM], [AFP]) and moreover the following property holds:

Proposition. 3.3. *If $v \in GSBV^2(\Omega)$, $B \subset \Omega$ is an open ball, $\mathcal{F}(v,B) < +\infty$ and $\mathcal{H}^{n-1}\big((S_v \cup S_{\nabla v}) \cap B\big) = 0$, then $v \in W^{2,2}(B)$.*

Proof - $\nabla^2 v \in L^2$ and $\mathcal{H}^{n-1}(S_{\nabla v} \cap B) = 0$ entail (in B) $\nabla^2 v = D\nabla v$, hence

$$D\nabla v \in L^2\,, \qquad \nabla v \in L^2\,, \qquad \mathcal{H}^{n-1}\,(\,S_v \cap B) = 0\,.$$

Then we have $Dv = \nabla v$ and $D^2 v = \nabla^2 v$. Moreover

$$\|D^2 v\|^2_{L^2(B)} + \|v\|^q_{L^q(B)} \;\le\; (1 + 2^q\mu^{-1})\,\mathcal{F}(v,B) + 2^q\|g\|^q_{L^q(B)}\,. \blacksquare$$

We now show that each local minimizer u solves a fourth order elliptic equation in the interior of $\Omega \setminus K_u$ and has (internal uniform) $\frac{1}{2}$-Hölder continuous derivatives in every ball contained in $\Omega \setminus K_u$ (see (2.1) for the definition of K_u).

Theorem 3.4. *If $u \in GSBV^2(\Omega)$ is a local minimizer of $\mathcal{F}$ in $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$, $g \in L^q(\Omega)$ $(q > 1)$, then*

(i) $\Delta^2 u = -\frac{q}{2}\mu|u-g|^{q-2}(u-g)$ *in* $\Omega \setminus K_u$;

(ii) $u \in W^{4,q/(q-1)}_{loc}(\Omega \setminus K_u)$;

(iii) there exists a constant $c > 0$ (depending on n, α, β, μ, g) such that for every open ball $B \subset \Omega \setminus K_u$

$$\sup_{\substack{x,y\in B\\ x\neq y}} \frac{|Du(x) - Du(y)|}{|x-y|^{1/2}} \leq c,$$

say $u \in C^{1,1/2}_{loc}(\Omega \setminus K_u)$.

Here $c = c_0\, c_\Omega$ where c_0 is the constant in density upper bound (2.5) and c_Ω is the Poincaré inequality constant:

$$\fint_B \left| v - \fint_B v \right|^2 dx \ \leq \ c_\Omega \fint_B |Dv|^2 \qquad \forall v \in W^{1,2}(\Omega),\ \forall B \subset \Omega\,.$$

Proof. (i) For every open set $A \subset\subset \Omega \setminus K_u$, for every $\varepsilon \in \mathbf{R}$ and for every $\varphi \in C_0^\infty(A)$ we have

$$0 \leq \mathcal{F}(u + \varepsilon\varphi, A) - \mathcal{F}(u, A)$$

$$= 2\varepsilon \left(\int_A (D^2 u) : (D^2\varphi)\, dx + \frac{q}{2}\mu \int_A |u-g|^{q-2}(u-g)\varphi\, dx \right) + o(\varepsilon)\ ,$$

where $o(\varepsilon)$ is an infinitesimal of order greater than ε. Hence

$$\int_A (D^2u) : (D^2\varphi)\, dx = -\frac{q}{2}\mu \int_A |u-g|^{q-2}(u-g)\varphi\, dx$$

for every $\varphi \in C_0^\infty(A)$. The thesis follows integrating by parts.
(ii) Since u is a local minimizer, then $u \in L^q_{loc}(\Omega)$. By (i)

$$\Delta^2 u = -\frac{q}{2}\mu|u-g|^{q-2}(u-g) \in L^{\frac{q}{q-1}}_{loc} \qquad \text{in } \Omega \setminus K_u.$$

Fix a ball $B \subset \Omega \setminus K_u$, then by elliptic regularity

$$u \in W^{4,\frac{q}{q-1}}(B).$$

(iii) Let $B \subset \Omega \setminus K_u$ be a ball; then $u \in W^{2,2}(B)$ by Proposition 3.3 and, for every $x_0 \in B$, by Poincaré inequality, density upper bound (Theorem 2.12) and Remark 2.13 we have

$$\int_{A(x_0,\rho)} |Du - (Du)_{x_0,\rho}|^2 dx \;\leq\; c_\Omega\, \rho^2 \int_{A(x_0,\rho)} |D^2 u|^2 dx \;\leq\; c_\Omega\, c_0\, \rho^{n+1},$$

where $A(x_0,\rho) = B \cap B_\rho(x_0)$ and $(Du)_{x_0,\rho}$ is the mean value of Du in $A(x_0,\rho)$. Since $|A(x_0,\rho)| \geq \left(\frac{\rho}{2}\right)^n$, the thesis follows by a well known characterization of Hölder spaces (see for instance [Gia],Th. 1.3, Ch. 3). ■

Remark 3.5. The fact that increasing q (integrability of g) implies decreasing regularity of the solution u is a paradox only at a first glance: actually this is due to the fact that q is also the exponent in the forcing term of the functional, hence the right-hand side of the equation belongs to $L^{q/(q-1)}(\Omega)$, say it has a summability which is decreasing in q.
To avoid this ambiguity and to focus the consequences of higher integrability of g, we introduce the parameter $s \geq q$ to denote the summability of g, and we get better regularity results under the restriction $s \geq n(q-1)$.

Theorem 3.6. (Further regularity)
Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, $q > 1$, $u \in GSBV^2(\Omega)$ is a local minimizer of $\mathcal{F}$ in Ω, and $g \in L^s(\Omega)$ with $s \geq \max\{q, n(q-1)\}$, then

$$u \;\in\; W^{4,s/(q-1)}_{loc}(\Omega \setminus K_u) \;\subset\; C^{3,\gamma}_{loc}(\Omega \setminus K_u),$$

where $\gamma = 1 - \frac{n(q-1)}{s}$.

Proof - By Theorem 3.4(iii) the function u is locally bounded in $\Omega \setminus K_u$ and

$$\Delta^2 u \;=\; -\frac{q}{2}\,\mu\,|u-g|^{q-2}(u-g) \in L^{\frac{s}{q-1}}_{loc} \qquad \text{in } \Omega \setminus K_u.$$

Hence for every ball $B \subset \Omega \setminus K_u$ we have $u \in W^{4,\frac{s}{q-1}}(B)$. By Sobolev embedding theorem we get the thesis. ■

4. Euler equations II : boundary-type conditions on the singular sets for extremals of $\mathcal{F}$ ($n \geq 2$).

We recall a Green formula for the bi-harmonic operator Δ^2. Here and in the following we assume the involved functions regular enough to have all the traces that are needed.

Let A be an open subset of $\Omega \subset \mathbf{R}^n$. For every $u, \varphi \in W^{2,2}(A)$ we set

$$a_A(u,\varphi) := \int_A (D^2 u) : (D^2 \varphi)\, dx \ . \tag{4.1}$$

The form a_A is bilinear and symmetric on $W^{2,2}(A)$.

Lemma 4.1. (**Green formula.**) *Assume A is a C^2 uniformly regular open set and (4.1). Denote by $N = (N_1 \dots, N_n)$ the outward unit normal to ∂A and denote by $\{t^k = t^k(x)\ ;\ k = 1, \dots, n{-}1, x \in \partial A\ \}$ a system of local tangential coordinates. Then for every $\varphi \in W^{2,2}(A)$, $r > 1$, $r \geq (2n)/(n{+}2)$ and $u \in W^{2,2}(A) \cap \{u;\ \Delta^2 u \in L^r(A)\}$ the following Green formula holds true:*

$$\begin{aligned} a_A(u,\varphi) \ &= \\ &= \int_A (\Delta^2 u)\varphi\, dx + \int_{\partial A} \Big(S(u) - \frac{\partial}{\partial N}\Delta u \Big)\varphi\, d\mathcal{H}^{\,n-1} + \int_{\partial A} T(u) \frac{\partial \varphi}{\partial N}\, d\mathcal{H}^{\,n-1} \end{aligned} \tag{4.2}$$

where the natural boundary operators $T(u)$ and $S(u)$ are defined by

$$T(u) \ := \ \sum_{i,j=1}^{n} \nabla^2_{ij} u\ N_i N_j \ = N \cdot (\nabla^2 u\, N)\ ,$$

$$S(u) \ := \ -\sum_{i,j=1}^{n} \sum_{k=1}^{n-1} \frac{\partial}{\partial t^k} \left(\nabla^2_{ij} u\ N_j \frac{\partial t^k}{\partial x_i} \right)\ .$$

Proof - See [Li], pp. 75–76 for $r = 2$. The same proof works in the general case too.

Remark 4.2. If $n = 2$, in Lemma 4.1 we can choose $\tau = (\tau_1, \tau_2)$ the unit tangent vector to ∂A which orients ∂A counter-clockwise defined by $\tau_1 = -N_2,\ \tau_2 = N_1$ (where $N = (N_1, N_2)$ is the outward unit normal vector to ∂A) and we get (4.2) where

$$T(u) := N \cdot (D^2 u\, N)\ , \qquad S(u) := -\frac{\partial}{\partial \tau}\left(\tau \cdot D^2 u\, N\right);$$

moreover in a flat portion of ∂A parallel to the x_1 axis, we get the identities:

$$T(u) = \frac{\partial^2 u}{\partial N^2} = D_{22} u, \qquad S(u) = -\frac{\partial}{\partial N}\left(\frac{\partial^2 u}{\partial \tau^2}\right) = -D_{112} u.$$

Now we want to evaluate the first variation of the energy functional (2.2) around a local minimizer u under compactly supported deformations of u, which are smooth outside K_u.

Theorem 4.3. (**Necessary conditions on S_u for natural boundary operators**) *Assume (1.3), $n \geq 2$, $q > 1$ and u is a local minimizer of $\mathcal{F}$, $B \subset\subset \Omega$ an open ball such that $S_u \cap B$ is the graph of a C^4 function and $(S_{\nabla u} \setminus S_u) \cap B = \emptyset$. Denote by B^+, B^- the two connected components of $B \setminus S_u$ and by N the unit normal to S_u pointing toward B^+. Assume that $u \in C^4(\overline{B^+}) \cap C^4(\overline{B^-})$. Then, by defining v^+, v^- the traces of any v on S_u respectively from B^+ and B^-, we have*

$$\Big(T(u)\Big)^{\pm} = 0 \qquad \text{on } S_u \cap B\ . \tag{4.3}$$

$$\Big(S(u) - \frac{\partial}{\partial N}\Delta u\Big)^{\pm} = 0 \qquad \text{on } S_u \cap B\ , \tag{4.4}$$

More explicitly, if $n = 2$ and $S_u \cap B$ is a segment, (4.3) becomes

$$\left(\frac{\partial^2 u}{\partial N^2}\right)^{\pm} \ = \ 0, \tag{4.5}$$

and (4.4) becomes

$$\left(\frac{\partial^3 u}{\partial N^3} \ + \ 2\,\frac{\partial}{\partial N}\left(\frac{\partial^2 u}{\partial \tau^2}\right)\right)^{\pm} \ = \ 0. \tag{4.6}$$

Proof - Let $\varphi \in C^2(\overline{B^+}) \cap C^2(\overline{B^-})$ be a function such that $\operatorname{spt}\varphi \subset B$. Then $\varphi \in GSBV^2(B)$ and for every $\varepsilon \in \mathbf{R}$ we have

$$\big(S_{u+\varepsilon\varphi} \cup S_{\nabla(u+\varepsilon\varphi)}\big) \cap B \ \ \subset \ \ S_u \cap B\ .$$

By (4.2) we have:

$$\begin{aligned}
0 &\leq \mathcal{F}(u+\varepsilon\varphi, B) - \mathcal{F}(u, B) \\
&= \alpha\left(\mathcal{H}^{n-1}(S_{u+\varepsilon\varphi} \cap B) - \mathcal{H}^{n-1}(S_u \cap B)\right) + \beta\mathcal{H}^{n-1}\left((S_{\nabla(u+\varepsilon\varphi)} \setminus S_{u+\varepsilon\varphi}) \cap B\right) + \\
&\qquad 2\varepsilon\left(a_{B^+}(u,\varphi) + a_{B^-}(u,\varphi) + \frac{q}{2}\mu\int_B |u-g|^{q-2}(u-g)\varphi\,dx\right) + o(\varepsilon) \\
&\leq \beta\mathcal{H}^{n-1}\left((S_{\nabla(u+\varepsilon\varphi)} \setminus S_{u+\varepsilon\varphi}) \cap B\right) + \\
&\qquad 2\varepsilon\left(\int_{B^+\cup B^-}(\Delta^2 u)\varphi\,dx + \frac{q}{2}\mu\int_B |u-g|^{q-2}(u-g)\varphi\,dx + \right. \\
&\qquad \left.\int_{S_u\cap B}\left[\!\!\left[\left(S(u) - \frac{\partial}{\partial N}\Delta u\right)\varphi\right]\!\!\right] d\mathcal{H}^{n-1} + \int_{S_u\cap B}\left[\!\!\left[T(u)\frac{\partial\varphi}{\partial N}\right]\!\!\right] d\mathcal{H}^{n-1}\right) + o(\varepsilon),
\end{aligned}$$

where for a function w we have set $[\![w]\!] = w^+ - w^-$.
Up to a countable set of values of ε, we have $\mathcal{H}^{n-1}(S_{u+\varepsilon\varphi} \cap B) = \mathcal{H}^{n-1}(S_u \cap B)$ so that (by taking into account of $S_{u+\varepsilon\varphi} \subset S_u$) we can choose arbitrarily small ε satisfying also

$$\mathcal{H}^{n-1}((S_{\nabla(u+\varepsilon\varphi)} \setminus S_{u+\varepsilon\varphi}) \cap B) = 0.$$

Taking into account Theorem 3.4(i), for small ε, by the arbitrariness of the two traces of φ and $\frac{\partial\varphi}{\partial N}$ on the two sides of S_u, we can choose φ with $\varphi^{\pm} = 0$, $\left(\frac{\partial\varphi}{\partial N}\right)^+ = 0$ and $\left(\frac{\partial\varphi}{\partial N}\right)^-$ arbitrary or viceversa to get (4.3). Similarly, choosing $\left(\frac{\partial\varphi}{\partial N}\right)^{\pm} = 0$, $\varphi^+ = 0$ and φ^- arbitrary or viceversa, we obtain (4.4). ■

Theorem 4.4. (**Necessary conditions on $S_{\nabla u}$ for jumps of natural boundary operators**) *Assume (1.3), $n \geq 2$, $q > 1$ and u is a local minimizer of $\mathcal{F}$, $B \subset\subset \Omega$ an open ball such that $S_{\nabla u} \cap B$ is the graph of a C^4 function and $S_u \cap B = \emptyset$. Denote by B^+, B^- the two connected components of $B \setminus S_{\nabla u}$ and by N the unit normal vector to $S_{\nabla u}$ pointing toward B^+. Assume that $u \in C^4(\overline{B^+}) \cap C^4(\overline{B^-})$. Then, by defining v^+, v^- the traces of any v on $S_{\nabla u}$ respectively from B^+ and B^-, and $[\![v]\!] = v^+ - v^-$, we have*

$$\left(T(u)\right)^{\pm} = 0 \qquad \text{on } S_{\nabla u} \cap B\ , \tag{4.7}$$

$$\left[\!\!\left[S(u) - \frac{\partial}{\partial N}\Delta u \right]\!\!\right] = 0 \qquad \text{on } S_{\nabla u} \cap B\ . \tag{4.8}$$

If $\alpha = \beta$ we have also

$$\left(S(u) - \frac{\partial}{\partial N}\Delta u \right)^{\pm} = 0 \qquad \text{on } S_{\nabla u} \cap B\ . \tag{4.9}$$

More explicitly, if $n = 2$ and $S_{\nabla u} \cap B$ is a segment, (4.7) and (4.8) become respectively

$$\left(\frac{\partial^2 u}{\partial N^2} \right)^{\pm} = 0, \tag{4.10}$$

$$\left[\!\!\left[\frac{\partial^3 u}{\partial N^3} + 2\frac{\partial}{\partial N}\left(\frac{\partial^2 u}{\partial \tau^2} \right) \right]\!\!\right] = 0. \tag{4.11}$$

If, in addition, $\alpha = \beta$, then(4.9) becomes

$$\left(\frac{\partial^3 u}{\partial N^3} + 2\, \frac{\partial}{\partial N} \left(\frac{\partial^2 u}{\partial \tau^2} \right) \right)^{\pm} = 0. \tag{4.12}$$

Proof - Let $\varphi \in C^2(\overline{B^+}) \cap C^2(\overline{B^-})$ be a function such that $\operatorname{spt} \varphi \subset B$ and $S_\varphi = \emptyset$. Then $\varphi \in GSBV^2(B)$ and for every $\varepsilon \in \mathbf{R}$ we have

$$S_{u+\varepsilon\varphi} \cap B = \emptyset\,, \qquad S_{\nabla(u+\varepsilon\varphi)} \cap B \subset S_{\nabla u} \cap B\ .$$

Moreover, by (4.2):

$$\begin{aligned}
0 &\le \mathcal{F}(u+\varepsilon\varphi, B) - \mathcal{F}(u,B) \\
&= \beta \left(\mathcal{H}^{n-1} \left(S_{\nabla(u+\varepsilon\varphi)} \cap B \right) - \mathcal{H}^{n-1} \left(S_{\nabla u} \cap B \right) \right) + \\
&\quad 2\varepsilon \left(a_{B^+}(u,\varphi) + a_{B^-}(u,\varphi) + \frac{q}{2}\mu \int_B |u-g|^{q-2}(u-g)\varphi\, dx \right) + o(\varepsilon) \\
&\le 2\varepsilon \left(\int_{B^+ \cup B^-} (\Delta^2 u)\varphi\, dx + \frac{q}{2}\mu \int_B |u-g|^{q-2}(u-g)\varphi\, dx + \right. \\
&\quad \left. \int_{S_{\nabla u}\cap B} \left[\!\!\left[\left(S(u) - \frac{\partial}{\partial N}\Delta u \right) \varphi \right]\!\!\right] d\mathcal{H}^{n-1} + \int_{S_{\nabla u}\cap B} \left[\!\!\left[T(u) \frac{\partial \varphi}{\partial N} \right]\!\!\right] d\mathcal{H}^{n-1} \right) + o(\varepsilon).
\end{aligned}$$

Taking into account Theorem 3.4(i), for small ε and by the arbitrariness of φ and of the two traces of $\frac{\partial\varphi}{\partial N}$ on the two sides of $S_{\nabla u}$, we can choose φ with $\left(\frac{\partial\varphi}{\partial N}\right)^{\pm} = 0$, $\varphi^+ = \varphi^-$ arbitrary, to get (4.8). Analogously by choosing $\varphi^{\pm} = 0$, $\left(\frac{\partial\varphi}{\partial N}\right)^{+} = 0$ and $\left(\frac{\partial\varphi}{\partial N}\right)^{-}$ arbitrary or viceversa, we obtain (4.7).

Now, let $\varphi \in C^2(\overline{B^+}) \cap C^2(\overline{B^-})$ be a function such that $\operatorname{spt} \varphi \subset B$ and $\emptyset \neq S_\varphi \subset S_{\nabla u}$. Then $\varphi \in GSBV^2(B)$ and for every $\varepsilon \in \mathbf{R}$ we have

$$\left(S_{u+\varepsilon\varphi} \cup S_{\nabla(u+\varepsilon\varphi)} \right) \cap B \subset S_{\nabla u} \cap B\ .$$

By (4.2) and Theorem 3.4(i) we have

$$
\begin{aligned}
0 &\leq \mathcal{F}(u+\varepsilon\varphi, B) - \mathcal{F}(u, B) \\
&= \alpha\,\mathcal{H}^{n-1}\left(S_{u+\varepsilon\varphi} \cap B\right) + \beta\left(\mathcal{H}^{n-1}\left(\left(S_{\nabla(u+\varepsilon\varphi)} \setminus S_{u+\varepsilon\varphi}\right) \cap B\right) - \mathcal{H}^{n-1}\left(S_{\nabla u} \cap B\right)\right) \\
&\quad + \ 2\varepsilon\left(a_{B^+}(u,\varphi) + a_{B^-}(u,\varphi) + \frac{q}{2}\mu\int_B |u-g|^{q-2}(u-g)\varphi\,dx\right) + o(\varepsilon) \\
&= \alpha\,\mathcal{H}^{n-1}(S_{u+\varepsilon\varphi} \cap B) - \beta\mathcal{H}^{n-1}(S_{\nabla u} \cap B) + \beta\mathcal{H}^{n-1}\left((S_{\nabla(u+\varepsilon\varphi)} \setminus S_{u+\varepsilon\varphi}) \cap B\right) \\
&\quad + 2\varepsilon\Bigg(\int_{B^+\cup B^-} (\Delta^2 u)\varphi\,dx + \frac{q}{2}\mu\int_B |u-g|^{q-2}(u-g)\varphi\,dx \, + \\
&\quad \int_{S_{\nabla u}\cap B} \left[\!\left[\left(S(u) - \frac{\partial}{\partial N}\Delta u\right)\varphi\right]\!\right] d\mathcal{H}^{n-1} + \int_{S_{\nabla u}\cap B} \left[\!\left[T(u)\frac{\partial\varphi}{\partial N}\right]\!\right] d\mathcal{H}^{n-1}\Bigg) + o(\varepsilon). \\
&= \alpha\,\mathcal{H}^{n-1}(S_{u+\varepsilon\varphi} \cap B) - \beta\mathcal{H}^{n-1}(S_{\nabla u} \cap B) + \beta\mathcal{H}^{n-1}\left((S_{\nabla(u+\varepsilon\varphi)} \setminus S_{u+\varepsilon\varphi}) \cap B\right) \\
&\quad + 2\varepsilon\left(\int_{S_{\nabla u}\cap B} \left[\!\left[\left(S(u) - \frac{\partial}{\partial N}\Delta u\right)\varphi\right]\!\right] d\mathcal{H}^{n-1} + \int_{S_{\nabla u}\cap B} \left[\!\left[T(u)\frac{\partial\varphi}{\partial N}\right]\!\right] d\mathcal{H}^{n-1}\right) + o(\varepsilon).
\end{aligned}
$$

If $\alpha > \beta$ then the inequality is fulfilled for ε small enough, hence we do not obtain further information. On the other hand, when $\alpha = \beta$, by taking into account the inclusion $S_{u+\varepsilon\varphi} \cup \left(S_{\nabla(u+\varepsilon\varphi)} \setminus S_{u+\varepsilon\varphi}\right) \subset S_{\nabla u}$, we get

$$
\begin{aligned}
&0 \leq \mathcal{F}(u+\varepsilon\varphi, B) - \mathcal{F}(u, B) = \\
&\quad + 2\varepsilon\left(\int_{S_{\nabla u}\cap B} \left[\!\left[\left(S(u) - \frac{\partial}{\partial N}\Delta u\right)\varphi\right]\!\right] d\mathcal{H}^{n-1} + \int_{S_{\nabla u}\cap B} \left[\!\left[T(u)\frac{\partial\varphi}{\partial N}\right]\!\right] d\mathcal{H}^{n-1}\right) + o(\varepsilon)
\end{aligned}
$$

and the coefficient of 2ε must vanish, hence by (4.7) and by the arbitrariness of the two traces of φ, we get (4.9). ■

5. Euler equations III : singular set variations for $\mathcal{F}$ ($n \geq 2$).

Now we want to compute the first variation of the functional (2.2) with respect to some directions which are different from those considered in Theorems 4.3 and 4.4 .

We evaluate the first variation of the energy functional (2.2) around a local minimizer u, under compactly supported smooth deformation of S_u and $S_{\nabla u}$.

Theorem 5.1. **(Euler equation)** *Assume (1.3), $q > 1$, $g \in C^1(\Omega)$ and $u \in GSBV^2(\Omega)$ is a local minimizer of $\mathcal{F}$ in Ω with $\nabla^2 u$ symmetric. Then for every $\eta \in C_0^2(\Omega, \mathbf{R}^n)$ the following equation holds true:*

$$
\begin{aligned}
&\int_\Omega \Big(|\nabla^2 u|^2 \operatorname{div}\eta - 2\big(D\eta\, \nabla^2 u + (D\eta)^t\, \nabla^2 u + \nabla u\, D^2\eta \big) : \nabla^2 u \Big)\, dx \\
&\quad + \mu \int_\Omega \Big(|u-g|^q \operatorname{div}\eta \, - \, q|u-g|^{q-2}(u-g)\, Dg \cdot \eta \Big)\, dx \qquad\qquad (5.1)\\
&\quad + \alpha \int_{S_u} \operatorname{div}^\tau_{\mathrm{S_u}} \eta \, d\mathcal{H}^{n-1} + \beta \int_{S_{\nabla u} \setminus S_u} \operatorname{div}^\tau_{\mathrm{S_{\nabla u} \setminus S_u}} \eta \, d\mathcal{H}^{n-1} \;=\; 0 \;,
\end{aligned}
$$

where $\big(D\eta\nabla^2 u + (D\eta)^t \nabla^2 u + \nabla u D^2 \eta\big)_{ij} = \sum_k \Big(D_k\eta_i \nabla^2_{kj} u + D_i \eta_k \nabla^2_{kj} u + \nabla_k u D^2_{ij}\eta_k \Big)$ and the tangential divergence is given by Definition 2.2.

Proof - Let $\eta \in C_0^2(\Omega, \mathbf{R}^n)$ and let $\varepsilon \in \mathbf{R}$ small enough, so that the map $\tau_\varepsilon(x) = x + \varepsilon\eta(x)$ is a diffeomorphism of Ω onto itself. Set $u_\varepsilon(\tau_\varepsilon(x)) = u(x)$ i.e $u_\varepsilon = u \circ {\tau_\varepsilon}^{-1}$; $A^{-t} = (A^{-1})^t$; $(D\tau_\varepsilon)_{il} = D_l(\tau_\varepsilon)_i$; $(D\eta)_{il} = D_l \eta_i$; $\nabla^2_{ij} u = \nabla_j \nabla_i u$.
Then $(D\eta) \circ \tau_\varepsilon = D\eta + \varepsilon D^2\eta D\eta + o(\varepsilon)$ and

$$
D\tau_\varepsilon(x) = I + \varepsilon D\eta(x) \;, \quad D^2\tau_\varepsilon(x) \;=\; \varepsilon D^2\eta(x) \;, \quad \big(D\tau_\varepsilon(x)\big)^{-1} \;=\; I - \varepsilon D\eta(x) + o(\varepsilon) \;,
$$

$$
\big(D\tau_\varepsilon(x)\big)^{-t} \;=\; I - \varepsilon (D\eta(x))^t + o(\varepsilon) \;, \quad \big(D\tau_\varepsilon(x)\big)^{-2} \;=\; I - 2\varepsilon D\eta(x) + o(\varepsilon) \;,
$$

where $o(\varepsilon)$ is an infinitesimal of order greater than ε uniformly in x, we compute gradient and hessian of u_ε by the chain-rule

$$
(\nabla u_\varepsilon) \circ \tau_\varepsilon \; D\tau_\varepsilon \;=\; \nabla u
$$

$$
(\nabla u_\varepsilon) \circ \tau_\varepsilon \;=\; \nabla u \, (D\tau_\varepsilon)^{-1}
$$

$$
\big((\nabla^2 u_\varepsilon) \circ \tau_\varepsilon\big) \; D\tau_\varepsilon \; D\tau_\varepsilon \;+\; \big((\nabla u_\varepsilon) \circ \tau_\varepsilon\big) \; D^2\tau_\varepsilon \;=\; \nabla^2 u
$$

Now, by setting $(D\tau_\varepsilon)_{i\,j} = \partial(\tau_\varepsilon)_i / \partial x_j$, $(D^2_{k\,l}\, \tau_\varepsilon)_j = \partial^2 (\tau_\varepsilon)_j / \partial x_k \partial x_l$, we exploit $D^2_{ij}\eta = D^2_{ji}\eta$, $\nabla^2_{ij}\eta = \nabla^2_{ji}\eta$, $(D\tau_\varepsilon)_{il}\, (D\tau_\varepsilon)^{-1}_{lj} = \delta_{ij}$ (Kronecker delta) and the summation convention over repeated indices:

$$
(\nabla_i u_\varepsilon) \circ \tau_\varepsilon \; D_l(\tau_\varepsilon)_i \;=\; \nabla_l u
$$

$$
(\nabla_i u_\varepsilon) \circ \tau_\varepsilon \;=\; \nabla_l\, u \; (D\tau_\varepsilon)^{-1}_{l\,i}
$$

$$
\big((\nabla^2_{ij} u_\varepsilon) \circ \tau_\varepsilon\big) \; D_h(\tau_\varepsilon)_j \; D_k(\tau_\varepsilon)_i \;+\; \big((\nabla_i u_\varepsilon) \circ \tau_\varepsilon\big) \; \big(D^2_{kh}\, \tau_\varepsilon\big)_i \;=\; \nabla^2_{kh} u
$$

and, by right-multiplying the last identity times $(D\tau_\varepsilon)^{-1}_{k\,s}\,(D\tau_\varepsilon)^{-1}_{h\,r}$,

$$
\begin{aligned}
(\nabla^2_{sr} u_\varepsilon)\circ\tau_\varepsilon =& \\
&= \nabla^2_{k\,h} u (D\tau_\varepsilon)^{-1}_{ks}(D\tau_\varepsilon)^{-1}_{hr} \;-\; \nabla_l u\,(D\tau_\varepsilon)^{-1}_{li}\,\left(D^2_{kh}\,\tau_\varepsilon\right)_i\,(D\tau_\varepsilon)^{-1}_{ks}\,(D\tau_\varepsilon)^{-1}_{hr} \;= \\
&= (D\tau_\varepsilon)^{-t}_{rh}\,\left(\nabla^2_{hk} u (D\tau_\varepsilon)^{-1}_{ks} - \nabla_l u\,(D\tau_\varepsilon)^{-1}_{li}\,\left(D^2_{kh}\,\tau_\varepsilon\right)_i\,(D\tau_\varepsilon)^{-1}_{ks}\right)
\end{aligned}
$$

say

$$
\begin{aligned}
(\nabla^2 u_\varepsilon)\circ\tau_\varepsilon \;=& \\
&= (D\tau_\varepsilon)^{-t}\,\left(\nabla^2 u\,(D\tau_\varepsilon)^{-1}\right) \;-\; (D\tau_\varepsilon)^{-t}\left(\nabla u\,(D\tau_\varepsilon)^{-1}\right)\left(D^2\,\tau_\varepsilon\,(D\tau_\varepsilon)^{-1}\right) \;= \\
&= \big(I-\varepsilon(D\eta)^t\big)\nabla^2 u\big(I-\varepsilon D\eta\big) \\
&\qquad\qquad - \varepsilon\big(I-\varepsilon(D\eta)^t\big)\nabla u\big(I-\varepsilon D\eta\big)D^2\eta\big(I-\varepsilon D\eta\big) \;+\; o(\varepsilon) = \\
&= \nabla^2 u \;-\; \varepsilon\,\left((D\eta)^t\,\nabla^2 u \;+\; \nabla^2 u\,D\eta \;+\; \nabla u\,D^2\eta\right) \;+\; o(\varepsilon)
\end{aligned}
$$

$$
(\nabla^2_{sr} u_\varepsilon)\circ\tau_\varepsilon \;=\; \nabla^2_{sr} u \;-\; \varepsilon\,\left((D\eta)_{rh}\,\nabla^2_{hs} u \;+\; \nabla^2_{rk} u\,(D\eta)_{ks} \;+\; \nabla_l u\,D^2_{sr}\eta_l\right) \;+\; o(\varepsilon)
$$

which entail

$$
\begin{aligned}
\left|(\nabla^2_{sr} u_\varepsilon)\circ\tau_\varepsilon\right|^2 &= \left((\nabla^2_{sr} u_\varepsilon)\circ\tau_\varepsilon\right)\,\left((\nabla^2_{sr} u_\varepsilon)\circ\tau_\varepsilon\right) \;= \\
&= \;\nabla^2_{sr} u\;\nabla^2_{sr} u \;-\; \varepsilon\;\Big(\,(D\eta)_{rh}\,\nabla^2_{hs} u\,\nabla^2_{sr} u + \nabla^2_{sr} u\,(D\eta)_{rh}\,\nabla^2_{hs} u + \\
&\qquad\qquad\qquad + \nabla^2_{rk} u\,(D\eta)_{ks}\,\nabla^2_{sr} u + \nabla^2_{sr} u\,\nabla^2_{rk} u\,(D\eta)_{ks} + \\
&\qquad\qquad\qquad + \nabla_l u\,D^2_{sr}\eta_l\,\nabla^2_{sr} u \;+\; \nabla^2_{sr} u\,\nabla_l u\,D^2_{sr}\eta_l\,\Big) + \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \;o(\varepsilon)\;= \\
&= |\nabla^2 u|^2 - 2\varepsilon\big(\nabla^2 u D\eta + D\eta\nabla^2 u + \nabla u D^2\eta\big) : \nabla^2 u + o(\varepsilon) \;= \\
&= |\nabla^2 u|^2 - 2\varepsilon\left(\left(D\eta + (D\eta)^t\right)\nabla^2 u + \nabla u D^2\eta\right) : \nabla^2 u + o(\varepsilon).
\end{aligned}
$$

By taking into account that

$$
\det(I+\varepsilon D\eta) = 1 + \varepsilon\,\mathrm{div}\,\eta + o(\varepsilon)\;,
$$

and by using the change of variables $y=\tau_\varepsilon(x)$, for small ε we get

$$
\begin{aligned}
&0 \le \mathcal{F}(u_\varepsilon) - \mathcal{F}(u) \qquad\qquad (5.2)\\
&= \int_\Omega |\nabla^2 u_\varepsilon(y)|^2\,dy - \int_\Omega |\nabla^2 u(x)|^2\,dx \\
&\quad + \mu\int_\Omega |u_\varepsilon(y)-g(y)|^q\,dy - \mu\int_\Omega |u(x)-g(x)|^q\,dx \\
&\quad + \alpha\mathcal{H}^{n-1}(S_{u_\varepsilon}) + \beta\mathcal{H}^{n-1}(S_{\nabla u_\varepsilon}\setminus S_{u_\varepsilon}) - \alpha\mathcal{H}^{n-1}(S_u) - \beta\mathcal{H}^{n-1}(S_{\nabla u}\setminus S_u) \\
&= \varepsilon\int_\Omega \left(|\nabla^2 u|^2\;\mathrm{div}\,\eta - 2\left(D\eta\,\nabla^2 u \;+\; (D\eta)^t\,\nabla^2 u \;+\; \nabla u\,D^2\eta\right) : \nabla^2 u\right)\,dx \\
&\quad + \mu\int_\Omega |u_\varepsilon(y)-g(y)|^q\,dy - \mu\int_\Omega |u(x)-g(x)|^q\,dx \\
&\quad + \alpha\mathcal{H}^{n-1}(S_{u_\varepsilon}) + \beta\mathcal{H}^{n-1}(S_{\nabla u_\varepsilon}\setminus S_{u_\varepsilon}) - \alpha\mathcal{H}^{n-1}(S_u) - \beta\mathcal{H}^{n-1}(S_{\nabla u}\setminus S_u) + o(\varepsilon).
\end{aligned}
$$

On the other hand, as in [S], pg.80, we have

$$\mathcal{H}^{n-1}(S_{u_\varepsilon}) - \mathcal{H}^{n-1}(S_u) = \varepsilon \int_{S_u} \sum_{j=1}^{n} \delta_j \eta^j \, d\mathcal{H}^{n-1} + o(\varepsilon). \tag{5.3}$$

and

$$\mathcal{H}^{n-1}(S_{\nabla u_\varepsilon} \setminus S_{u_\varepsilon}) - \mathcal{H}^{n-1}(S_{\nabla u} \setminus S_u) = \varepsilon \int_{S_{\nabla u} \setminus S_u} \sum_{j=1}^{n} \delta_j \eta^j \, d\mathcal{H}^{n-1} + o(\varepsilon). \tag{5.4}$$

By taking into account $g(\tau_\varepsilon(x)) = g(x) + \varepsilon\, Dg(x) \cdot \eta(x) + o(\varepsilon)$, the variation due to the contribution of the forcing term is given by μ times the following value

$$\begin{aligned}
&\int_\Omega |u_\varepsilon(y) - g(y)|^q \, dy - \int_\Omega |u(x) - g(x)|^q \, dx \\
&= \int_\Omega |u(x) - g(\tau_\varepsilon(x))|^q | \det(I + \varepsilon D\eta) \, dx - \int_\Omega |u(x) - g(x)|^q \, dx \\
&= \int_\Omega |u(x) - g(\tau_\varepsilon(x))|^q \ (1 + \varepsilon \operatorname{div} \eta + o(\varepsilon)) \ dx - \int_\Omega |u(x) - g(x)|^q \, dx \\
&= \varepsilon \int_\Omega \Big(q |u(x) - g(\tau_\varepsilon(x))|^{q-2} \, (\, u(x) - g(\tau_\varepsilon(x)) \,)(-Dg(x)) \cdot \eta(x) \\
&\qquad\qquad + \ |u(x) - g(\tau_\varepsilon(x))|^q \operatorname{div} \eta \Big) \, dx + o(\varepsilon) \\
&\overset{q \geq 1}{=} \varepsilon \int_\Omega \Big(q |u(x) - g(x)|^{q-2} \, (\, u(x) - g(x) \,)(-Dg(x)) \cdot \eta(x) \\
&\qquad\qquad + \ |u(x) - g(x)|^q \operatorname{div} \eta \Big) \, dx \ + \ O(\varepsilon^q) \ + \ O(\varepsilon^{q+1}) \ + \ o(\varepsilon)\,.
\end{aligned}$$

And the thesis follows by (5.2)-(5.4) and $q > 1$. ■

In the next Theorems we shall require some regularity conditions on $S_u \cup S_{\nabla u}$. These will enable us to speak of the normal derivatives of u and of the traces of $|\nabla^2 u|$ on both sides of K_u, which we shall denote by $\left(\frac{\partial u}{\partial \nu}\right)^{\pm}$, $|\nabla^2 u^{\pm}|$.
In order to obtain additional information on local minimizers, it is useful to perform a careful integration by parts of the volume integral of Euler equation (5.1) independent of the forcing term.

Theorem 5.2. (**Integration by parts**) *Assume $A \subset \Omega$ is Lipschitz and piecewise C^3 open set, $N = (N^1, \ldots, N^n)$ its outer normal, $u \in W^{4,r}(A)$ with $r > 1$, then we have, for every $\eta \in C^2(\overline{A}, \mathbf{R}^n)$,*

$$\int_A \Big(|\nabla^2 u|^2 \ \mathrm{div}\, \eta - 2\big(D\eta \nabla^2 u + (D\eta)^t \nabla^2 u + \nabla u D^2 \eta\big) : \nabla^2 u \Big)\, dx = -2 \int_A \Delta^2 u \ \eta \cdot \nabla u dx +$$

$$+ \int_{\partial A} \big(|\nabla^2 u|^2 N - 2 \nabla^2 u \nabla^2 u N + 2 \nabla u \frac{\partial}{\partial N} \Delta u \big) \cdot \eta d\mathcal{H}^{\,n-1} - 2 \int_{\partial A} \nabla u \cdot D\eta \nabla^2 u N d\mathcal{H}^{\,n-1}$$

where, by assuming the summation convention over repeated indexes and denoting distributional partial derivatives by subscripts and components of vectors by superscripts, we have set $\eta_k^h = D_k \eta^h$ *and*

$$\begin{aligned} |\nabla^2 u|^2 \ \mathrm{div}\, \eta &= \big(u_{ik}\big)^2 \eta_h^h \\ \big(D\eta \, \nabla^2 u\big) : \nabla^2 u &= u_{ih} \, \eta_k^h \, u_{ik} \\ \Big(\big(D\eta\big)^t \, \nabla^2 u\Big) : \nabla^2 u &= u_{ik} \, \eta_i^h \, u_{hk} \\ \big(\nabla u \, D^2 \eta\big) : \nabla^2 u &= u_h \, \eta_{ik}^h \, u_{ik} \end{aligned}$$

$$\int_{\partial A} \big(|\nabla^2 u|^2 N - 2 \nabla^2 u \nabla^2 u N + 2 \nabla u \frac{\partial}{\partial N} \Delta u \big) \cdot \eta d\mathcal{H}^{\,n-1} - 2 \int_{\partial A} \nabla u \cdot D\eta \nabla^2 u N d\mathcal{H}^{\,n-1} =$$

$$\stackrel{\mathrm{def}}{=} \int_{\partial A} \big(u_{ik}\big)^2 N^h - 2 u_{ik} u_{ih} N^k + 2 u_h u_{kki} N^i \big) \eta^h d\mathcal{H}^{\,n-1} - \int_{\partial A} 2 u_h u_{ik} N^k \eta_i^h d\mathcal{H}^{\,n-1}.$$

Proof - We evaluate separately the various terms in the left-hand side:

$$\int_A (u_{ik})^2 \eta_h^h dx = \int_{\partial A} (u_{ik})^2 \eta^h N^h d\mathcal{H}^{\,n-1} - \int_A 2 u_{ik} u_{ikh} \eta^h dx \tag{5.5}$$

$$\begin{aligned} \int_A u_{ih} \eta_k^h u_{ik} dx &= \int_{\partial A} u_{ik} u_{ih} \eta^h N^k d\mathcal{H}^{\,n-1} - \int_A \left(u_{ih} u_{ik}\right)_k \eta^h dx \\ &= \int_{\partial A} u_{ih} \eta^h u_{ik} N^k d\mathcal{H}^{\,n-1} - \int_A \left(u_{ihk} u_{ik} \eta^h + u_{ih} u_{ikk} \eta^h \right) dx \end{aligned} \tag{5.6}$$

$$\begin{aligned} \int_A u_{ik} \eta_i^h u_{hk} dx &= \int_{\partial A} u_{ik} \eta^h u_{hk} N^i d\mathcal{H}^{\,n-1} - \int_A \left(u_{ik} u_{hk}\right)_i \eta^h dx \\ &= \int_{\partial A} u_{kh} \eta^h u_{ki} N^i d\mathcal{H}^{\,n-1} - \int_A \left(u_{iik} u_{hk} \eta^h + u_{ihk} u_{ik} \eta^h \right) dx \end{aligned} \tag{5.7}$$

$$
\begin{aligned}
\int_A u_h \eta^h_{ik} u_{ik} dx &= \int_{\partial A} u_h \eta^h_i u_{ik} N^k d\mathcal{H}^{n-1} - \int_A \left(u_h u_{ik}\right)_k \eta^h_i dx \\
&= \int_{\partial A} u_h \eta^h_i u_{ik} N^k d\mathcal{H}^{n-1} - \int_A \left(u_{hk} u_{ik} \eta^h_i + u_h u_{kki} \eta^h_i\right) dx \\
&= \int_{\partial A} \left(u_h \eta^h_i u_{ik} N^k - \left(u_{hk} u_{ik} + u_h u_{kki}\right) \eta^h N^i\right) d\mathcal{H}^{n-1} \\
&+ \int_A \left(u_{hk} u_{ik} + u_h u_{kki}\right)_i \eta^h dx.
\end{aligned}
\tag{5.8}
$$

By subtracting twice (5.6),(5.7),(5.8) from (5.5) we get the thesis. ∎

It is possible to derive additional necessary conditions on a local minimizer, provided that the variation η in (5.1) is suitably chosen.

Now we perform a qualitative analysis of the singular set by assuming enough regularity to perform integration by parts of Theorem 5.2. We will use compactly supported vector fields that are normal to S_u or $S_{\nabla u}$ as test function in (5.1).

Theorem 5.3. (**Curvature of S_u and squared hessian jump**) *Let u be a local minimizer of $\mathcal{F}$ in Ω. Assume (1.3), $q > 1$, $g \in C^1(\Omega)$ and $B \subset\subset U \subset \Omega$ two open balls, such that $S_u \cap U$ is the graph of a C^4 function and B^+ (resp. B^-) the open connected epigraph (resp. subgraph) of such function in B. Assume $\overline{S_{\nabla u} \setminus S_u} \cap U = \emptyset$, $(\bar{S}_u \setminus S_u) \cap U = \emptyset$, and $u \in W^{4,r}(B^+) \cap W^{4,r}(B^-)$, $r > 1$, $r \geq (2n)/(n+2)$. Then*

$$
\left[\!\left[|\nabla^2 u|^2 + \mu |u - g|^q \right]\!\right] \;=\; (n-1)\,\alpha\,\mathcal{K}(S_u) \qquad \text{on } S_u \cap B\ ,
$$

where we denote by $[\![w]\!]$ the jump of a function w on S_u, say the trace of w in B^+ minus the trace of w in B^- and $\mathcal{K}$ is evaluated (see Definition 2.3) with the orientation on S_u induced by the normal pointing toward B^+. In particular

$$
\text{if} \qquad \left[\!\left[|\nabla^2 u|^2 \right]\!\right] \;=\; \mathcal{K}(S_u) \;=\; 0\,, \qquad \text{then} \qquad \frac{1}{2}\left(u^+ + u^-\right) \;=\; g \quad \text{on } S_u \cap B\,.
$$

Proof. The plan is to exploit equation (5.1) by a suitable choice of test function η. The assumptions entail the existence of a vector field $\nu \in C^1(B, \mathbf{R}^n)$ such that ν on $S_u \cap B$ is the unit normal to S_u pointing toward U^+. For instance we may set $\nu(x) = D\,(dist(x, U^-) - dist(x, U^+))$ for every x in a neighborhood of $S_u \cap U$ (see [Giu]), so that ν points toward B^+.
We choose $\eta = \zeta\nu$, with $\zeta \in C^\infty_0(B)$, hence $\eta \in C^1_0(B, \mathbf{R}^n)$.

Referring to Definition 2.2 of tangential derivatives along S_u we set $\delta = (\delta_1, \ldots, \delta_n)$. Then $(\delta v)\cdot \nu = 0\ \forall v$ and $(\delta\eta) = \zeta\,\delta\nu$ on $S_u \cap B$. Hence

$$\mathrm{div}^{\tau}_{\mathrm{S_u}}(\zeta\nu) \;=\; \zeta\,\mathrm{div}^{\tau}_{\mathrm{S_u}}\,\nu \;=\; (n-1)\,\zeta\,\mathcal{K}(S_u) \quad \text{on } S_u\cap B,$$

(5.9)
$$\alpha\int_{S_u\cap B}\mathrm{div}^{\tau}_{\mathrm{S_u}}\,\eta\,d\mathcal{H}^{n-1} \;=\; (n-1)\,\alpha\int_{S_u\cap B}\zeta\mathcal{K}\,d\mathcal{H}^{n-1}.$$

We integrate by parts in B^+ and B^- the volume integrals due to the forcing term g in the Euler equation (5.1):

(5.10)
$$\begin{aligned}\mu\int_{B^\pm}\Big(|u-g|^q\,\mathrm{div}\,\eta \,-\, q|u-g|^{q-2}(u-g)\,Dg\cdot\eta\Big)\,dx \;=\\ -\,\mu\,q\int_{B^\pm}|u-g|^{q-2}(u-g)\,Du\cdot\eta\,dx \;\mp\; \mu\int_{\partial B^\pm}|u-g|^q\zeta\,d\mathcal{H}^{n-1}\end{aligned}$$

By applying the integration by parts of Theorem 5.2 with the choices $A = B^+$ and $A = B^-$, we have $N_{B^-} = \nu = -N_{B^+}$ and (with summation convention from 1 to n over repeated indexes)

(5.11)
$$\int_{B^\pm}\Big(|\nabla^2 u|^2\ \mathrm{div}\,\eta - 2\big(D\eta\nabla^2 u + (D\eta)^t\nabla^2 u + \nabla u D^2\eta\big) : \nabla^2 u\Big)\,dx \;=$$

$$= -2\int_{B^\pm}\eta^h u_h\Delta^2 u\,dx \;\mp$$

$$\left(\int_{\partial B^\pm}\big(u^2_{ik}\nu^h - 2u_{ik}u_{ih}\nu^k + 2u_h u_{kki}\nu^i\big)\zeta\nu^h d\mathcal{H}^{n-1} \;-\int_{\partial B^\pm}2u_h u_{ik}\nu^k\eta^h_i d\mathcal{H}^{n-1}\right)\,.$$

By substituting (5.9), (5.10) and (5.11) in (5.1) and taking into account Theorem 3.4(i), we get

(5.12)
$$\begin{aligned}-\int_{S_u\cap B}\Big[\!\!\Big[-2u_{ik}u_{ih}\nu^k + 2u_h u_{kki}\nu^i\Big]\!\!\Big]\zeta\nu^h d\mathcal{H}^{n-1} + \int_{S_u\cap B}\Big[\!\!\Big[2u_h u_{ik}\Big]\!\!\Big]\,\nu^k\eta^h_i d\mathcal{H}^{n-1}\\ -\int_{S_u\cap B}\Big[\!\!\Big[|\nabla^2 u|^2 + \mu|u-g|^q\Big]\!\!\Big]\,\zeta\,d\mathcal{H}^{n-1} + (n-1)\,\alpha\int_{S_u\cap B}\zeta\mathcal{K}\,d\mathcal{H}^{n-1} = 0.\end{aligned}$$

We claim that the first line in (5.12) vanishes. Then

$$\int_{S_u\cap B}\Big(\Big[\!\!\Big[|\nabla^2 u|^2 + \mu|u-g|^q\Big]\!\!\Big] - (n-1)\alpha\,\mathcal{K}(S_u)\Big)\,\zeta d\mathcal{H}^{n-1} = 0,$$

and by the arbitrariness of ζ we get the thesis of the Theorem.
Eventually we prove the claim.
We choose a smooth system $\{t^p = t^p(x)\ ;\ p = 1, \ldots, n-1, x \in S_u \cap B\ \}$ of local tangential coordinates on $S_u \cap B$ and we perform an integration by parts (with respect to tangential coordinates t^p) over $S_u \cap B$ in the term of (5.12) containing η_i^h. Since

$$\frac{\partial \eta^h}{\partial x_i} = \sum_{p=1}^{n-1} \frac{\partial \eta^h}{\partial t^p} \frac{\partial t^p}{\partial x_i} + \frac{\partial \eta^h}{\partial \nu} \nu^i$$

then

$$2 \int_{S_u \cap B} \left[\!\left[u_h u_{ik} \nu^k \right]\!\right] \eta_i^h \, d\mathcal{H}^{n-1} = \tag{5.13}$$

$$2 \int_{S_u \cap B} \left[\!\left[u_h u_{ik} \nu^k \right]\!\right] \frac{\partial \eta^h}{\partial \nu} \nu^i \, d\mathcal{H}^{n-1} - 2 \int_{S_u \cap B} \left[\!\left[\sum_{p=1}^{n-1} \frac{\partial}{\partial t^p} \left(\nu^k u_h u_{ik} \frac{\partial t^p}{\partial x_i} \right) \right]\!\right] \eta^h \, d\mathcal{H}^{n-1}.$$

Hence

$$-\frac{1}{2} (\text{ first line of (5.12) }) =$$

$$= \int_{S_u \cap B} \left(\left[\!\left[-\, u_{ik}\, u_{ih}\, \nu^k + u_h\, u_{kki}\, \nu^i \right]\!\right] \zeta\, \nu^h - \left[\!\left[u_h\, u_{ik} \right]\!\right] \nu^k\, \eta_i^h \right) d\mathcal{H}^{n-1} =$$

$$= \int_{S_u \cap B} \left(- \left[\!\left[u_{ik} u_{ih} \nu^k \right]\!\right] \eta^h + \left[\!\left[\frac{\partial u}{\partial \nu} \frac{\partial}{\partial \nu} \Delta u \right]\!\right] \zeta + \right.$$

$$\left. - \left[\!\left[u_h u_{ik} \nu^k \right]\!\right] \frac{\partial \eta^h}{\partial \nu} \nu^i + \left[\!\left[\sum_{p=1}^{n-1} \frac{\partial}{\partial t^p} \left(\nu^k u_h u_{ik} \frac{\partial t^p}{\partial x_i} \right) \right]\!\right] \eta^h \right) d\mathcal{H}^{n-1} \overset{(4.3)}{=}$$

$$= \int_{S_u \cap B} \left(- \left[\!\left[u_{ik} u_{ih} \nu^k \right]\!\right] \eta^h + \left[\!\left[\frac{\partial u}{\partial \nu} \frac{\partial}{\partial \nu} \Delta u \right]\!\right] \zeta + \right.$$

$$\left. + \left[\!\left[\sum_{p=1}^{n-1} \nu^k u_{ik} \frac{\partial t^p}{\partial x_i} \frac{\partial u_h}{\partial t^p} \right]\!\right] \eta^h + \left[\!\left[\sum_{p=1}^{n-1} \frac{\partial}{\partial t^p} \left(\nu^k u_{ik} \frac{\partial t^p}{\partial x_i} \right) u_h \right]\!\right] \eta^h \, d\mathcal{H}^{n-1} \right) \overset{(4.3)}{=}$$

$$= \int_{S_u \cap B} \mp \left[\!\left[u_{ik} u_{hi} \right]\!\right] \nu^k \nu^h \zeta \, d\mathcal{H}^{n-1} + \int_{S_u \cap B} \left[\!\left[\frac{\partial u}{\partial \nu} \left(\frac{\partial}{\partial \nu} \Delta u - S(u) \right) \right]\!\right] \zeta d\mathcal{H}^{n-1} \overset{(4.4)}{=} 0\,.$$

If $\left[\!\left[|\nabla^2 u|^2 \right]\!\right] = \mathcal{K}(S_u) = 0$, then $|u^+ - g| = |u^- - g|$ and $u^+ \neq u^-$ give $u^+ + u^- = 2g$.■

Theorem 5.4. (**Curvature of $S_{\nabla u}$ and squared hessian jump**) *Let u be a local minimizer of $\mathcal{F}$ in Ω. Assume (1.3), $\alpha=\beta$, $q>1$, $g\in C^1(\Omega)$ and let $B\subset\subset U\subset\Omega$ two open balls such that $S_{\nabla u}\cap U$ be the graph of a C^4 function and B^+ (resp. B^-) be the open connected epigraph (resp. subgraph) of such function in B. Assume $\overline{S_u}\cap U=\emptyset$ and $u\in W^{4,r}(B^+)\cap W^{4,r}(B^-)$, $r>1$, $r\geq (2n)/(n+2)$. Then*

$$\left[\!\!\left[|\nabla^2 u|^2 \right]\!\!\right] \;=\; (n-1)\,\beta\,\mathcal{K}(S_{\nabla u}) \qquad \text{on } \; S_{\nabla u}\cap B\ ,$$

where we denote by $[\![w]\!]$ the jump of a function w on $S_{\nabla u}$, say the trace of w in B^+ minus the trace of w in B^- and $\mathcal{K}$ is evaluated (see Definition 2.3) with the orientation on $S_{\nabla u}$ induced by the normal pointing toward B^+.

Proof. The plan consists in exploiting equation (5.1) again by a suitable choice of test function η.
The assumptions entail the existence of a vector field $\nu\in C^1(B,\mathbf{R}^n)$ such that ν on $S_{\nabla u}\cap B$ is the unit normal to $S_{\nabla u}$ pointing toward U^+. For instance we may set $\nu(x)=D\,(dist(x,U^-)-dist(x,U^+))$ for every x in a neighborhood of $S_{\nabla u}\cap U$ (see [Giu]), so that ν points toward B^+.
We choose $\eta=\zeta\nu$, with $\zeta\in C_0^\infty(B)$ as like as in the proof of Theorem 5.3, hence $\eta\in C_0^1(B,\mathbf{R}^n)$. The properties of ν entail

$$\beta\int_{S_{\nabla u}\cap B}\operatorname{div}^{\tau}_{S_{\nabla u}}\eta\,d\mathcal{H}^{n-1} \;=\; (n-1)\,\beta\int_{S_{\nabla u}\cap B}\zeta\mathcal{K}\,d\mathcal{H}^{n-1}. \tag{5.14}$$

We integrate by parts in B^+ and B^- the volume integrals due to the forcing term g in the Euler equation (5.1). We recover again (5.10) with the present choice of $B^\pm$.
By applying the integration by parts of Theorem 5.2 with the choices $A=B^+$ and $A=B^-$, we recover again (5.11).
By substituting (5.10), (5.11) and (5.14) in (5.1) and taking into account Theorem 3.4(i), as like as in the proof of Theorem 5.3, but using (4.7) and (4.9) of Theorem 4.4 instead of (4.2) and (4.4) of Theorem 4.3 and exploiting the identity $(\,\partial\Delta u/\partial\nu - S(u)\,)^\pm \;=\; 0$ (valid on $S_{\nabla u}\setminus S_u$ since $\alpha=\beta$) in the last line of (5.13) we deduce

$$\left[\!\!\left[|\nabla^2 u|^2+\mu|u-g|^q \right]\!\!\right] \;=\; (n-1)\beta\,\mathcal{K}(S_{\nabla u}) \qquad \text{on } S_{\nabla u}\cap B\,.$$

Since both u and g are continuous on $S_{\nabla u}\setminus S_u$, the last identity entails the thesis. ∎

Now we perform a qualitative analysis of the "relative boundary" of the singular set (crack-tip and crease-tip), by assuming it is a manifold as smooth as required by the computation of boundary operators. The strategy is a new choice of the test functions in Euler equation (5.1): a vector field η tangential to S_u (or $S_{\nabla u}$).

Theorem 5.5. *(* **Crack-tip** *)*
Assume (1.3), $q>1$, $g \in C^1(\Omega)$. Let u be a local minimizer of $\mathcal{F}$ in Ω, and $U \subset \Omega$ an open ball, such that $(S_{\nabla u} \setminus S_u) \cap U = \emptyset$ and $S_u \cap U$ is an $(n-1)$ dimensional oriented C^4 manifold without boundary, the orientation is given by a normal vector field $\nu \in C^3(U)$. Assume $\Xi := \overline{(S_u} \setminus S_u) \cap U$ is a non-empty connected oriented $(n-2)$ dimensional C^4 manifold (a point if $n=2$) and $u \in W^{4,r}(U \setminus \overline{S_u})$ with $r>1$, $r \geq (2n)/(n+2)$.
Choose $x_0 \in \Xi$, an open ball $B = B(x_0) \subset\subset U$, smooth tangential coordinates t^p on $S_u \cap B$ and label the associated unit vectors by $\mathbf{t}^p$, $p = 1, \dots, n-1$.
Then, assuming the summation convention from 1 to n over repeated indices (different from p), we get

$$\lim_{\varepsilon} \left(\int_{\partial \Xi_\varepsilon} \left\{ \left(|\nabla^2 u|^2 + \mu |u-g|^q \right) \eta \cdot \mathbf{n}_\varepsilon - 2T^\varepsilon(u) \frac{\partial \eta}{\partial \mathbf{n}_\varepsilon} \cdot \nabla u \right.\right.$$
$$\left.\left. -2\left(S^\varepsilon(u) - \frac{\partial}{\partial \mathbf{n}_\varepsilon} \Delta u \right) \eta \cdot \nabla u \right\} d\mathcal{H}^{n-1} \right) = \int_\Xi \left(\alpha\zeta - 2[\![u_h\, u_{ik}]\!] \nu^k \eta^h \sum_{p=1}^{n-1} \mathbf{t}^p \cdot \mathbf{n} \frac{\partial t^p}{\partial x_i} \right) d\mathcal{H}^{n-2}$$

for every $\eta \in C_0^3(B, \mathbf{R}^n)$ s.t. $\eta = \zeta\tau$, $\zeta \in C_0^\infty(B)$, $\tau \in C^3(B, S^{n-1})$, such that $\eta \cdot \nu \equiv 0$ on S_u and $\tau \cdot \mathbf{n} \equiv 1$ on Ξ, where:
$\mathbf{n}$ is a vector field on Ξ, normal to Ξ, tangent to S_u and pointing toward S_u;
$\Xi_\varepsilon = \{x \in B : dist(x, \Xi) < \varepsilon\}$, with ε s.t. $0 < \varepsilon <$ radius of B (hence $\Xi_\varepsilon \subset\subset U$);
the natural boundary operators T^ε and S^ε are defined as like as in Lemma 4.1, but using $\mathbf{n}_\varepsilon$ instead of N (which is not defined in $\partial\Xi_\varepsilon$): $\mathbf{n}_\varepsilon$ is the unit vector field on $\partial(B \setminus \Xi_\varepsilon)$ pointing outward from $B \setminus \Xi_\varepsilon$.
Explicitly, when S_u is contained in an hyperplane and the natural choices for tangential coordinates t^p and associated unit vectors $\mathbf{t}^p$ are made, then $\partial t^p / \partial x_i \equiv 0$, for $p = 1, \dots, n-1$, $i = 1, \dots, n$, hence:

$$\lim_{\varepsilon} \left(\int_{\partial \Xi_\varepsilon} \left\{ \left(|\nabla^2 u|^2 + \mu |u-g|^q \right) \eta \cdot \mathbf{n}_\varepsilon - 2T^\varepsilon(u) \frac{\partial \eta}{\partial \mathbf{n}_\varepsilon} \cdot \nabla u \right.\right.$$
$$\left.\left. -2\left(S^\varepsilon(u) - \frac{\partial}{\partial \mathbf{n}_\varepsilon} \Delta u \right) \eta \cdot \nabla u \right\} d\mathcal{H}^{n-1} \right) = \alpha \int_\Xi \zeta \, d\mathcal{H}^{n-2}, \tag{5.15}$$

if $n=2$ and S_u is flat, then $\Xi = \{x_0\}$ and for any η as above

$$\lim_{\varepsilon} \left(\int_{\partial B_\varepsilon(x_0)} \left\{ \left(|\nabla^2 u|^2 + \mu |u-g|^q \right) \eta \cdot \mathbf{n}_\varepsilon - 2T^\varepsilon(u) \frac{\partial \eta}{\partial \mathbf{n}_\varepsilon} \cdot \nabla u \right.\right.$$
$$\left.\left. -2\left(S^\varepsilon(u) - \frac{\partial}{\partial \mathbf{n}_\varepsilon} \Delta u \right) \eta \cdot \nabla u \right\} d\mathcal{H}^1 \right) = \alpha\, \zeta(x_0). \tag{5.16}$$

Proof - Fix ε as required by the assumptions and δ s.t. $0<\delta<\varepsilon$.
Set $S_u^\delta := \{x \in B \ : \ dist(x,S_u)<\delta\ \}$, $Q_\varepsilon := B\setminus \Xi_\varepsilon$, $Q_{\varepsilon,\delta} := B\setminus(\Xi_\varepsilon \cup S_u^\delta)$, $\partial\Xi_\varepsilon$ shortly denotes $\partial\Xi_\varepsilon\cap B$ in the proof, and $\mathbf{n}_{\varepsilon\,\delta}$ is the outward normal to $Q_{\varepsilon,\delta}$.

Figures 1, 2 and 3 illustrate the relevant sets in a neighborhood of Ξ.

In the Euler equation of Theorem 5.1 we choose a test function $\eta=\zeta\tau$ as required in the statement: then equation (5.1) holds true with $Q_{\varepsilon,\delta}$ in place of Ω.
We integrate by parts on S_u , by exploiting the assumption η normal to Ξ ,

$$\alpha\int_{S_u\cap B}\mathrm{div}^\tau_{\mathrm{S_u}}\,\eta\,d\mathcal{H}^{\,n-1} \ = \ -\,\alpha\int_\Xi \zeta\,d\mathcal{H}^{n-2}\,. \tag{5.17}$$

By absolute continuity of the integral and (5.17) we get

$$\lim_\varepsilon\lim_\delta\left(\int_{Q_{\varepsilon,\delta}}\Big(|\nabla^2u|^2\,\mathrm{div}\,\eta-2\big(D\eta\nabla^2u+(D\eta)^t\nabla^2u+\nabla uD^2\eta\big):\nabla^2u\Big)dx\right.$$
$$\left.+\ \mu\int_{Q_{\varepsilon,\delta}}\Big(|u-g|^q\,\mathrm{div}\,\eta-q|u-g|^{q-2}(u-g)\,Dg\cdot\eta\Big)\,dx\right)-\ \alpha\int_\Xi\zeta\,d\mathcal{H}^{n-2}\ =\ 0\,, \tag{5.18}$$

where $\big(D\eta\nabla^2u+(D\eta)^t\nabla^2u+\nabla uD^2\eta\big)_{ij}=\sum_k\big(D_k\eta_i\nabla^2_{kj}u+D_i\eta_k\nabla^2_{kj}u+\nabla_kuD^2_{ij}\eta_k\big)$.
The integration by parts of Theorem 5.2 holds with the choice $A=Q_{\varepsilon,\delta}$, which is a piecewise C^3 open subset, hence we get (the summation from 1 to n over repeated indexes is understood, the distributional partial derivatives are denoted by subscripts and components of vectors by superscripts)

$$\begin{aligned}\int_{Q_{\varepsilon,\delta}}&\Big(|\nabla^2u|^2\ \mathrm{div}\,\eta-2\big(D\eta\nabla^2u+(D\eta)^t\nabla^2u+\nabla uD^2\eta\big):\nabla^2u\Big)\,dx\ =\\ &=-\,2\int_{Q_{\varepsilon,\delta}}\eta^hu_h\Delta^2u\,dx\ +\\ &\quad+\int_{\partial Q_{\varepsilon,\delta}}\big((u_{ik})^2\mathbf{n}^h_{\varepsilon\,\delta}-2u_{ik}u_{ih}\mathbf{n}^k_{\varepsilon\,\delta}+2u_hu_{kki}\mathbf{n}^i_{\varepsilon\,\delta}\big)\eta^hd\mathcal{H}^{\,n-1}\\ &\quad-\int_{\partial Q_{\varepsilon,\delta}}2\,u_h\,u_{ik}\,\mathbf{n}^k_{\varepsilon\,\delta}\,\eta^h_i\,d\mathcal{H}^{\,n-1}\,.\end{aligned} \tag{5.19}$$

We pass to the limit as $\delta\to0^+$ in (5.19), hence by (5.18) we get

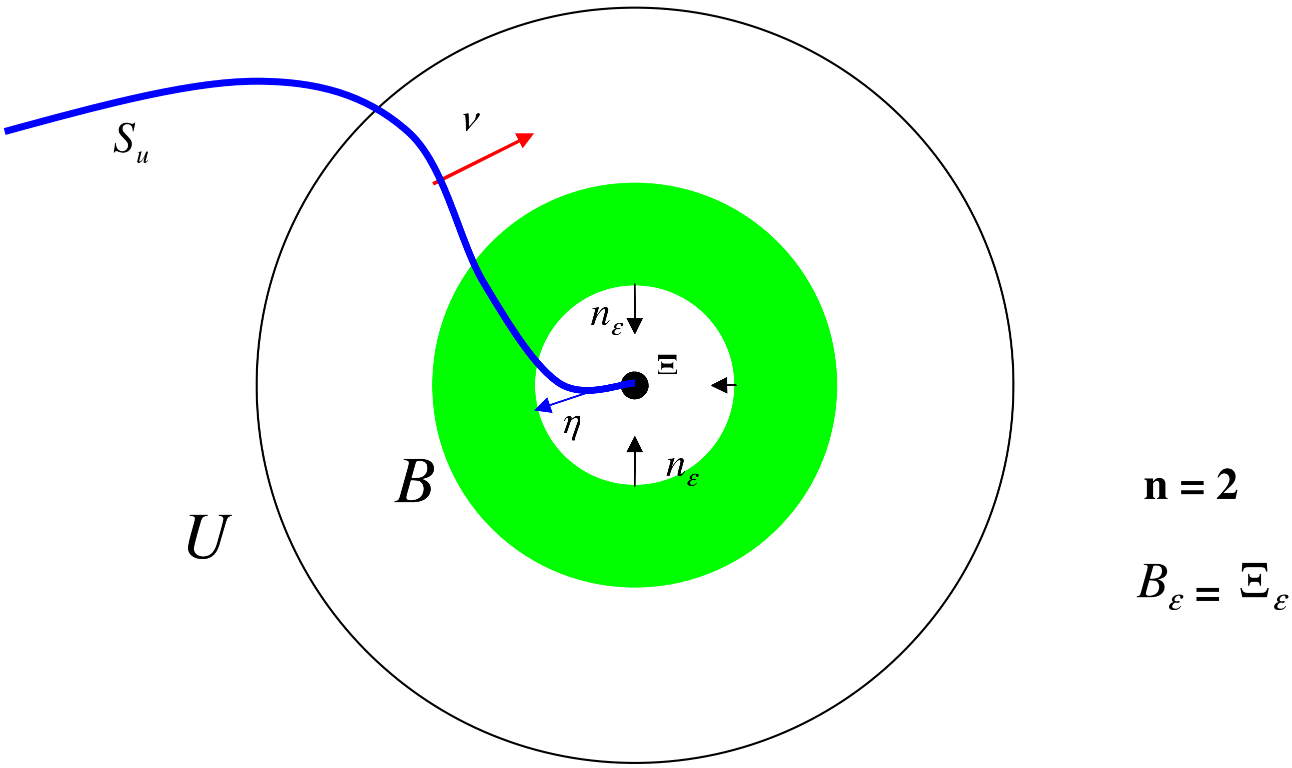

Fig.1 - A neighborhood of Ξ for $n = 2$.

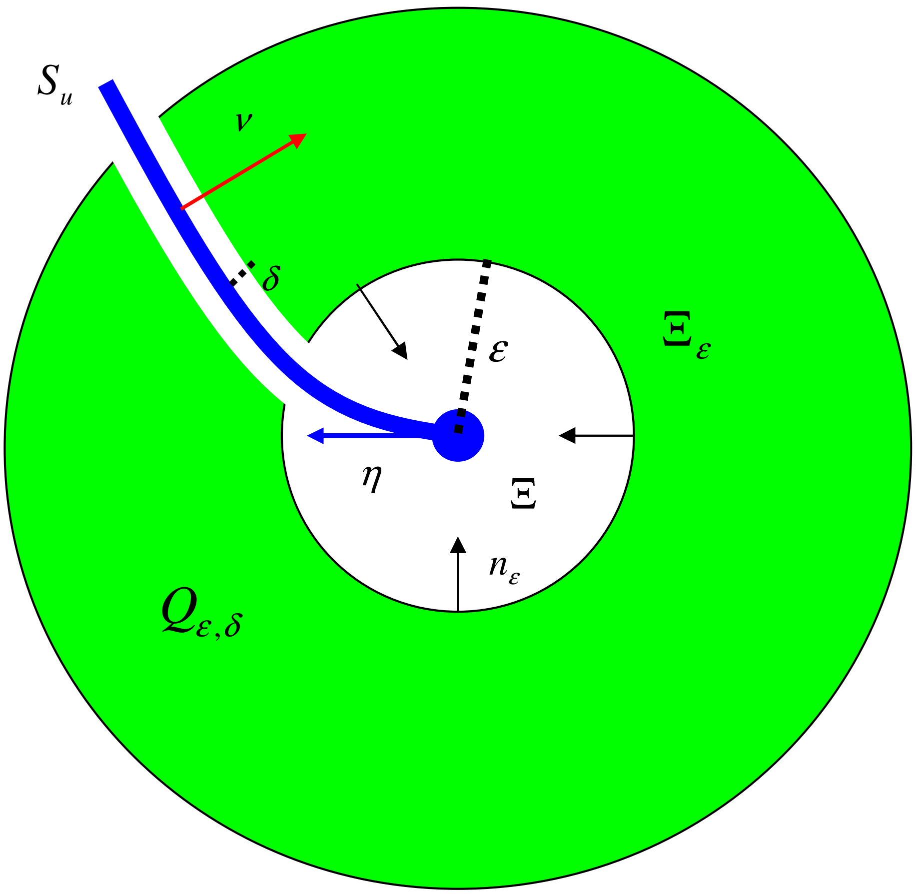

Fig.2 - The set $Q_{\varepsilon,\delta}$.

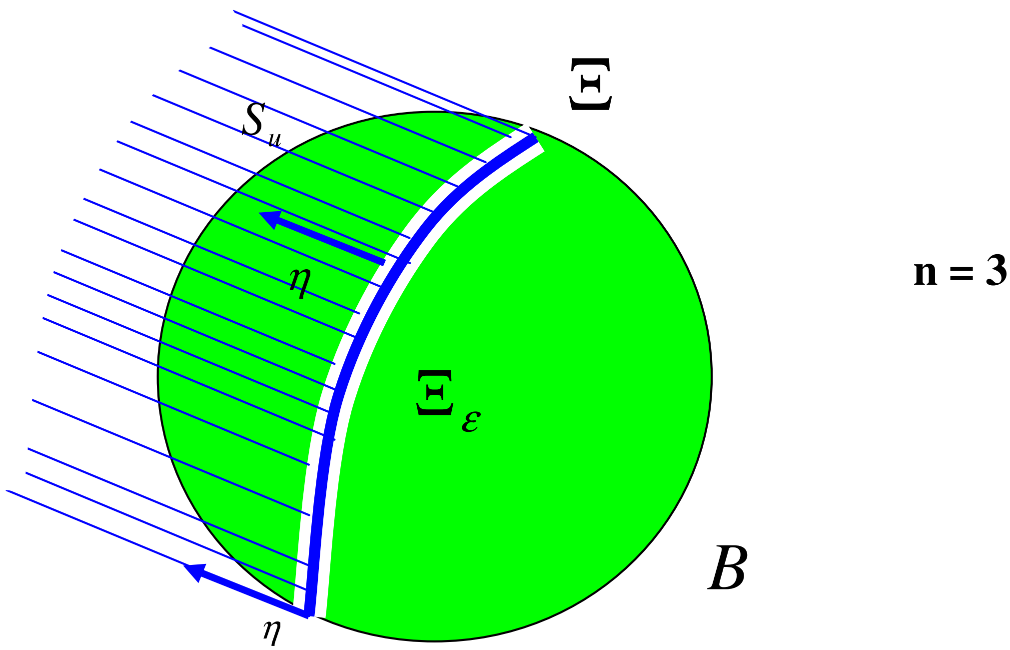

Fig.3 - A neighborhood of Ξ **for** $n = 3$.

$$
(5.20) \qquad \lim_{\varepsilon} \Big\{ -2\int_{Q_\varepsilon} \eta^h u_h \Delta^2 u \, dx \ +
$$

$$
-\int_{S_u\cap Q_\varepsilon} [\![(u_{ik})^2\nu^h - 2u_{ik}u_{ih}\nu^k + 2u_h u_{kki}\nu^i]\!] \eta^h d\mathcal{H}^{\,n-1} \ + \int_{S_u\cap Q_\varepsilon} [\![2u_h u_{ik}]\!] \nu^k \eta^h_i d\mathcal{H}^{\,n-1} +
$$

$$
\int_{\partial\Xi_\varepsilon} \big((u_{ik})^2 \mathbf{n}^h_\varepsilon - 2u_{ik}u_{ih}\mathbf{n}^k_\varepsilon + 2u_h u_{kki}\mathbf{n}^i_\varepsilon \big) \eta^h d\mathcal{H}^{\,n-1} \ - \int_{\partial\Xi_\varepsilon} 2u_h u_{ik} \mathbf{n}^k_\varepsilon \eta^h_i d\mathcal{H}^{\,n-1} \ +
$$

$$
+\mu \int_{Q_\varepsilon} \Big(|u-g|^q \operatorname{div}\eta - q|u-g|^{q-2}(u-g)\, Dg\cdot\eta \Big)\, dx \Big\} - \alpha \int_{\Xi} \zeta \, d\mathcal{H}^{n-2} \ = \ 0 \ .
$$

All along the proof we set $[\![v]\!] = v^+ - v^-$ where v^+ is the trace of v on S_u from the side where ν points.
By arguing as like as in (5.13), taking into account that ζ vanishes on $S_u \cap \partial B$ but does not vanish on $S_u \cap \partial\Xi_\varepsilon$, we integrate by parts with respect to tangential coordinates t to find a first equality

$$
(5.21) \qquad
\begin{aligned}
& 2\int_{S_u\cap Q_\varepsilon} [\![u_h u_{ik}]\!] \nu^k \eta^h_i \, d\mathcal{H}^{\,n-1} \ = \\
& \quad = \ 2\int_{S_u\cap Q_\varepsilon} [\![u_h u_{ik}]\!] \frac{\partial \eta^h}{\partial \nu} \nu^k \nu^i \, d\mathcal{H}^{\,n-1} \\
& \qquad - \ 2\int_{S_u\cap Q_\varepsilon} \left[\!\!\left[\sum_{p=1}^{n-1} \frac{\partial}{\partial t^p} \left(\nu^k u_h u_{ik} \frac{\partial t^p}{\partial x_i} \right) \right]\!\!\right] \eta^h \, d\mathcal{H}^{\,n-1} + \\
& \qquad\qquad + 2\int_{S_u\cap \partial\Xi_\varepsilon} [\![u_h \, u_{ik}]\!] \, \nu^k \, \eta^h \sum_{p=1}^{n-1} \mathbf{t}^p \cdot \mathbf{n}_\varepsilon \, \frac{\partial t^p}{\partial x_i} \, d\mathcal{H}^{n-2}(t) = \\
& \quad = \ 2\int_{S_u\cap Q_\varepsilon} [\![u_h u_{ik}]\!] \frac{\partial \eta^h}{\partial \nu} \nu^k \nu^i \, d\mathcal{H}^{\,n-1} \\
& \qquad - \qquad 2\int_{S_u\cap Q_\varepsilon} \left[\!\!\left[\sum_{p=1}^{n-1} \left(\nu^k u_{ik} \frac{\partial u_h}{\partial t^p} \frac{\partial t^p}{\partial x_i} \right) \right]\!\!\right] \eta^h \, d\mathcal{H}^{\,n-1} \\
& \qquad + \ 2\int_{S_u\cap Q_\varepsilon} [\![u_h S(u)]\!] \, \eta^h \, d\mathcal{H}^{\,n-1} + \\
& \qquad\qquad + 2\int_{S_u\cap \partial\Xi_\varepsilon} [\![u_h \, u_{ik}]\!] \, \nu^k \, \eta^h \sum_{p=1}^{n-1} \mathbf{t}^p \cdot \mathbf{n}_\varepsilon \, \frac{\partial t^p}{\partial x_i} \, d\mathcal{H}^{n-2}(t)
\end{aligned}
$$

and a second equality (without $n-2$ dimensional contribution)

$$
\begin{aligned}
2\int_{\partial\Xi_\varepsilon} u_h u_{ik}\mathbf{n}_\varepsilon^k \eta_i^h \, d\mathcal{H}^{n-1} \ &= \\
= \ 2\int_{\partial\Xi_\varepsilon} u_h u_{ik} \frac{\partial \eta^h}{\partial \mathbf{n}_\varepsilon} \mathbf{n}_\varepsilon^k \mathbf{n}_\varepsilon^i \, d\mathcal{H}^{n-1} & \\
- \ 2\int_{\partial\Xi_\varepsilon} \sum_{p=1}^{n-1} \frac{\partial}{\partial t^p}\left(\mathbf{n}_\varepsilon^k u_h u_{ik} \frac{\partial t^p}{\partial x_i}\right)\eta^h \, d\mathcal{H}^{n-1} &= \\
= \ 2\int_{\partial\Xi_\varepsilon} u_h u_{ik} \frac{\partial \eta^h}{\partial \mathbf{n}_\varepsilon} \mathbf{n}_\varepsilon^k \mathbf{n}_\varepsilon^i \, d\mathcal{H}^{n-1} & \\
- \ 2\int_{\partial\Xi_\varepsilon} \sum_{p=1}^{n-1} \left(\mathbf{n}_\varepsilon^k u_{ik} \frac{\partial u_h}{\partial t^p} \frac{\partial t^p}{\partial x_i}\right)\eta^h \, d\mathcal{H}^{n-1} & \\
+ 2\int_{\partial\Xi_\varepsilon} u_h \eta^h S^\varepsilon(u) \, d\mathcal{H}^{n-1}. &
\end{aligned}
\tag{5.22}
$$

By taking into account Theorem 3.4(i) and that $\eta = \zeta\tau$, and $\mathbf{n}_\varepsilon$ is the outward normal from Q_ε, we integrate by parts the volume integral depending on the forcing term g

$$
\begin{aligned}
-2\int_{Q_\varepsilon} \eta^h u_h \, \Delta^2 u \, dx \ &= \\
= \ \mu q \int_{Q_\varepsilon} |u-g|^{q-2}(u-g)\, Du\cdot\eta \, dx \ &= \\
= \ \mu q \int_{Q_\varepsilon} |u-g|^{q-2}(u-g)\, Dg\cdot\eta \, dx \ + & \\
+ \ \mu q \int_{Q_\varepsilon} |u-g|^{q-2}(u-g)\, (Du-Dg)\cdot\eta \, dx \ &= \\
= \ \mu q \int_{Q_\varepsilon} |u-g|^{q-2}(u-g)\, Dg\cdot\tau\zeta \, dx + \mu\int_{Q_\varepsilon} \eta^h \frac{\partial}{\partial x_h}|u-g|^q \, dx &= \\
= \ \mu\, q \int_{Q_\varepsilon} |u-g|^{q-2}(u-g)\, Dg\cdot\tau\zeta \, dx \ - \ \mu\int_{Q_\varepsilon} |u-g|^q \operatorname{div}\eta \, dx & \\
+ \ \mu \int_{\partial\Xi_\varepsilon} |u-g|^q \, \mathbf{n}_\varepsilon\cdot\tau\zeta \, d\mathcal{H}^{n-1} &
\end{aligned}
\tag{5.23}
$$

By substituting (5.21),(5.22) and (5.23) in (5.20) we have cancellation of all the volume integrals and we are left with the following:

$$\lim_{\varepsilon} \Bigg\{ \int_{S_u\cap Q_\varepsilon} [\![2u_{ik}u_{ih}\nu^k - 2u_h u_{kki}\nu^i]\!] \eta^h d\mathcal{H}^{n-1} + 2\int_{S_u\cap Q_\varepsilon} [\![u_h u_{ik}]\!] \frac{\partial \eta^h}{\partial \nu} \nu^k \nu^i \, d\mathcal{H}^{n-1}$$

$$- 2\int_{S_u\cap Q_\varepsilon} [\![\nu^k u_{ik} u_{ih}]\!] \eta^h \, d\mathcal{H}^{n-1} + 2\int_{S_u\cap Q_\varepsilon} [\![u_h S(u)]\!] \, \eta^h \, d\mathcal{H}^{n-1}$$

$$+ \int_{\partial \Xi_\varepsilon} \left(- 2u_{ik}u_{ih}\mathbf{n}^k_\varepsilon + 2u_h u_{kki}\mathbf{n}^i_\varepsilon \right) \eta^h d\mathcal{H}^{n-1} - 2\int_{\partial \Xi_\varepsilon} u_h u_{ik} \frac{\partial \eta^h}{\partial \mathbf{n}_\varepsilon} \mathbf{n}^k_\varepsilon \mathbf{n}^i_\varepsilon \, d\mathcal{H}^{n-1}$$

$$+ 2\int_{\partial \Xi_\varepsilon} u_{ik}u_{ih}\, \mathbf{n}^k_\varepsilon \eta^h \, d\mathcal{H}^{n-1} - 2\int_{\partial \Xi_\varepsilon} u_h \eta^h \, S^\varepsilon(u) \, d\mathcal{H}^{n-1}$$

$$- \int_{S_u\cap Q_\varepsilon} [\![|\nabla^2 u|^2]\!] \, \eta\cdot\nu \, d\mathcal{H}^{n-1} + \int_{\partial \Xi_\varepsilon} \left(|\nabla^2 u|^2 + \mu |u-g|^q \right) \eta \cdot \mathbf{n}_\varepsilon \, d\mathcal{H}^{n-1} \Bigg\}$$

$$- \int_{\Xi} \left(\alpha\zeta - 2\, [\![u_h\, u_{ik}]\!] \nu^k \eta^h \sum_{p=1}^{n-1} \mathbf{t}^p \cdot \mathbf{n} \frac{\partial t^p}{\partial x_i} \right) d\mathcal{H}^{n-2} = 0 \ .$$

By the definitions of $S, T, S^\varepsilon, T^\varepsilon$, by using $\eta\cdot\nu \equiv 0$ on $S_u \cap B$ and Theorem 4.3 on S_u we get the thesis. We emphasize that Theorems 4.3 and 4.4 do not hold for operators T^ε and S^ε on $\partial\Xi_\varepsilon$ (which is not a subset of S_u).
Notice that in above formula and in the thesis of the theorem, whenever an integrand f depends on two-sided values, by $\int_\Xi f\, d\mathcal{H}^{n-2}$ we mean $\lim_\varepsilon \int_{S_u\cap\partial\Xi_\varepsilon} f\, d\mathcal{H}^{n-2}$. ■

Theorem 5.6. *(* **Crease-tip** *)*
Assume (1.3), $q>1$, $g\in C^1(\Omega)$. Let u be a local minimizer of $\mathcal{F}$ in Ω, and $U\subset\Omega$ an open ball, s.t. $S_u\cap U=\emptyset$ and $S_{\nabla u}\cap U$ is an $(n-1)$ dimensional oriented C^4 manifold without boundary, the orientation is given by a normal vector field $\nu\in C^3(U)$. Assume $\Lambda := \overline{(S_{\nabla u}} \setminus S_{\nabla u})\cap U$ is a non-empty connected oriented $(n-2)$ dimensional C^4 manifold (say a point if $n=2$), and $u\in W^{4,r}(U\setminus\overline{S_{\nabla u}})$, $r>1$, $r\geq (2n)/(n+2)$. Choose $x_0\in\Lambda$, an open ball $B=B(x_0)\subset\subset U$, smooth tangential coordinates t^p on $S_{\nabla u}\cap B$ and label associated unit vectors by $\mathbf{t}^p$, $p=1,\cdots,n-1$.
Then, assuming the summation convention from 1 to n over repeated indices (different from p), we get

$$\lim_{\varepsilon} \left(\int_{\partial\Lambda_\varepsilon} \Bigg\{ \left(|\nabla^2 u|^2 + \mu|u-g|^q \right) \eta\cdot\mathbf{n}_\varepsilon - 2T^\varepsilon(u) \frac{\partial \eta}{\partial \mathbf{n}_\varepsilon} \cdot \nabla u \right.$$

$$\left. -2 \left(S^\varepsilon(u) - \frac{\partial}{\partial \mathbf{n}_\varepsilon} \Delta u \right) \eta \cdot \nabla u \Bigg\} \, d\mathcal{H}^{n-1} \right) =$$

$$= \int_\Lambda \left(\beta\zeta - 2 [\![u_h\, u_{ik}]\!] \nu^k \eta^h \sum_{p=1}^{n-1} \mathbf{t}^p \cdot \mathbf{n} \frac{\partial t^p}{\partial x_i} \right) d\mathcal{H}^{n-2} \ ,$$

for every $\eta \in C^3_0(B,\mathbf{R}^n)$ *s.t.* $\eta = \zeta\tau$, $\zeta \in C^\infty_0(B)$, $\tau \in C^3(B, S^{n-1})$, *such that* $\eta\cdot\nu \equiv 0$ *on* $S_{\nabla u}$ *and* $\tau\cdot\mathbf{n} \equiv 1$ *on* Λ, *where:*
$\mathbf{n}$ *is a vector field on* Λ*, normal to* Λ*, tangent to* $S_{\nabla u}$ *and pointing toward* $S_{\nabla u}$*;*
$\Lambda_\varepsilon = \{x \in B : dist(x,\Lambda) < \varepsilon\}$, *with* ε *s.t.* $0 < \varepsilon <$ *radius of* B *(hence* $\Lambda_\varepsilon \subset\subset U$ *);*
the natural boundary operators T^ε *and* S^ε *are defined as like as in Lemma 4.1, but using* $\mathbf{n}_\varepsilon$ *instead of* N *(which is not defined in* $\partial\Lambda_\varepsilon$*):* $\mathbf{n}_\varepsilon$ *is the unit vector field on* $\partial(B\setminus\Lambda_\varepsilon)$ *pointing outward from* $B\setminus\Lambda_\varepsilon$*.*

Explicitly, when $S_{\nabla u}$ *is contained in an hyperplane and the natural choices for tangential coordinates* t^p *and associated unit vectors* $\mathbf{t}^p$ *are made, then* $\partial t^p/\partial x_i \equiv 0$, *for* $p = 1,\dots,n-1$, $i = 1,\dots,n$, *hence:*

$$\lim_\varepsilon \left(\int_{\partial\Lambda_\varepsilon} \left\{ \left(|\nabla^2 u|^2 + \mu|u-g|^q\right)\eta\cdot\mathbf{n}_\varepsilon - 2T^\varepsilon(u)\frac{\partial\eta}{\partial\mathbf{n}_\varepsilon}\cdot\nabla u \right.\right.$$
$$\left.\left. -2\left(S^\varepsilon(u) - \frac{\partial}{\partial\mathbf{n}_\varepsilon}\Delta u\right)\eta\cdot\nabla u \right\} d\mathcal{H}^{n-1} \right) \;=\; \beta\int_\Lambda \zeta\, d\mathcal{H}^{n-2}\,, \tag{5.24}$$

if $n = 2$ *and* $S_{\nabla u}$ *is flat, then* $\Lambda = \{x_0\}$ *and for any* η *as above*

$$\lim_\varepsilon \left(\int_{\partial B_\varepsilon(x_0)} \left\{ \left(|\nabla^2 u|^2 + \mu|u-g|^q\right)\eta\cdot\mathbf{n}_\varepsilon - 2T^\varepsilon(u)\frac{\partial\eta}{\partial\mathbf{n}_\varepsilon}\cdot\nabla u \right.\right.$$
$$\left.\left. -2\left(S^\varepsilon(u) - \frac{\partial}{\partial\mathbf{n}_\varepsilon}\Delta u\right)\eta\cdot\nabla u \right\} d\mathcal{H}^{1} \right) \;=\; \beta\,\zeta(x_0)\,. \tag{5.25}$$

Proof - Fix ε as required by the assumptions and δ s.t. $0 < \delta < \varepsilon$.
Set $S^\delta_{\nabla u} := \{x \in B \,:\, dist(x, S_{\nabla u}) < \delta\}$, $M_\varepsilon := B\setminus\Lambda_\varepsilon$, $M_{\varepsilon,\delta} := B\setminus(\Lambda_\varepsilon \cup S^\delta_{\nabla u})$, $\partial\Lambda_\varepsilon$ shortly denotes $\partial\Lambda_\varepsilon \cap B$ in the proof, and $\mathbf{n}_{\varepsilon\,\delta}$ is the outward normal to $M_{\varepsilon,\delta}$.
In the Euler equation of Theorem 5.1 we choose a test function $\eta = \zeta\tau$ as required in the statement: then equation (5.1) holds with B in place of Ω.
We integrate by parts on $S_{\nabla u}$, by exploiting the assumption η normal to Λ ,

$$\beta\int_{S_{\nabla u}\cap B} \mathrm{div}^\tau_{S_{\nabla u}}\eta\, d\mathcal{H}^{n-1} \;=\; -\beta\int_\Lambda \zeta\, d\mathcal{H}^{n-2}\,. \tag{5.26}$$

By absolute continuity of the integral and (5.26) we get

$$\lim_\varepsilon\lim_\delta\left(\int_{M_{\varepsilon,\delta}} \left(|\nabla^2 u|^2\,\mathrm{div}\,\eta - 2\big(D\eta\nabla^2 u + (D\eta)^t\nabla^2 u + \nabla u D^2\eta\big) : \nabla^2 u\right)dx \right. \tag{5.27}$$

$$+\ \mu \int_{M_{\varepsilon,\delta}} \Big(|u-g|^q \operatorname{div}\eta - q|u-g|^{q-2}(u-g)\, Dg\cdot\eta \Big)\, dx \Bigg) - \beta \int_{\Lambda} \zeta\, d\mathcal{H}^{n-2} \ = \ 0\ ,$$

where $\big(D\eta\nabla^2 u+(D\eta)^t\nabla^2 u+\nabla u D^2\eta\big)_{ij} = \sum_k \Big(D_k\eta_i \nabla^2_{kj} u + D_i\eta_k \nabla^2_{kj} u + \nabla_k u D^2_{ij}\eta_k \Big)$. The integration by parts of Theorem 5.2 holds with the choice $A = M_{\varepsilon,\delta}$, which is a piecewise C^3 open subset, hence we get (the summation from 1 to n over repeated indexes is understood, the distributional partial derivatives are denoted by subscripts and components of vectors by superscripts)

$$\begin{aligned}
&\int_{M_{\varepsilon,\delta}} \Big(|\nabla^2 u|^2 \ \operatorname{div}\eta - 2\big(D\eta\nabla^2 u + (D\eta)^t\nabla^2 u + \nabla u D^2\eta\big) : \nabla^2 u \Big)\, dx \ = \\
&\qquad = - 2\int_{M_{\varepsilon,\delta}} \eta^h u_h \Delta^2 u\, dx \ + \\
&\qquad\quad + \int_{\partial M_{\varepsilon,\delta}} \big((u_{ik})^2 \mathbf{n}^h_{\varepsilon\,\delta} - 2u_{ik}u_{ih}\mathbf{n}^k_{\varepsilon\,\delta} + 2u_h u_{kki}\mathbf{n}^i_{\varepsilon\,\delta} \big)\eta^h d\mathcal{H}^{\,n-1} \\
&\qquad\quad - \int_{\partial M_{\varepsilon,\delta}} 2u_h u_{ik}\mathbf{n}^k_{\varepsilon\,\delta}\eta^h_i d\mathcal{H}^{\,n-1}\ .
\end{aligned} \tag{5.28}$$

We pass to the limit as $\delta \to 0^+$ in (5.28), hence by (5.27) we get

$$\begin{aligned}
&\lim_{\varepsilon}\ \Big\{ - 2\int_{M_\varepsilon} \eta^h u_h \Delta^2 u\, dx \ + \qquad\qquad (5.29)\\
&- \int_{S_{\nabla u}\cap M_\varepsilon} [\![(u_{ik})^2\nu^h - 2u_{ik}u_{ih}\nu^k + 2u_h u_{kki}\nu^i]\!] \eta^h d\mathcal{H}^{\,n-1} + \int_{S_{\nabla u}\cap M_\varepsilon} [\![2u_h u_{ik}]\!] \nu^k \eta^h_i d\mathcal{H}^{\,n-1} + \\
&\int_{\partial\Lambda_\varepsilon} \big((u_{ik})^2\mathbf{n}^h_\varepsilon - 2u_{ik}u_{ih}\mathbf{n}^k_\varepsilon + 2u_h u_{kki}\mathbf{n}^i_\varepsilon \big)\eta^h d\mathcal{H}^{\,n-1} \ - \int_{\partial\Lambda_\varepsilon} 2u_h u_{ik}\mathbf{n}^k_\varepsilon \eta^h_i d\mathcal{H}^{\,n-1} \ + \\
&\quad + \mu \int_{M_\varepsilon} \Big(|u-g|^q \operatorname{div}\eta - q|u-g|^{q-2}(u-g)\, Dg\cdot\eta \Big)\, dx \Big\} - \beta \int_{\Lambda} \zeta\, d\mathcal{H}^{n-2} \ = \ 0\ .
\end{aligned}$$

All along the proof we set $[\![v]\!] = v^+ - v^-$ where v^+ is the trace of v on $S_{\nabla u}$ from the side where ν points.
By arguing as like as in (5.13), taking into account that ζ vanishes on $S_{\nabla u}\cap\partial B$ but does not vanish on $S_{\nabla u}\cap\partial\Lambda_\varepsilon$, we integrate by parts with respect to tangential

coordinates t to find a first equality

$$
\begin{aligned}
2\int_{S_{\nabla u}\cap M_\varepsilon} [\![u_h u_{ik}]\!]\nu^k\eta^h_i\,d\mathcal{H}^{n-1} \;=\;& \\
=\; 2\int_{S_{\nabla u}\cap M_\varepsilon} [\![u_h u_{ik}]\!]\frac{\partial\eta^h}{\partial\nu}\nu^k\nu^i\,d\mathcal{H}^{n-1}& \\
-\; 2\int_{S_{\nabla u}\cap M_\varepsilon} \left[\!\!\left[\sum_{p=1}^{n-1}\frac{\partial}{\partial t^p}\left(\nu^k u_h u_{ik}\frac{\partial t^p}{\partial x_i}\right)\right]\!\!\right]\eta^h\,d\mathcal{H}^{n-1} +& \\
+\,2\int_{S_{\nabla u}\cap\partial\Lambda_\varepsilon} [\![u_h\, u_{ik}]\!]\,\nu^k\,\eta^h\sum_{p=1}^{n-1}\mathbf{t}^p\cdot\mathbf{n}_\varepsilon\,\frac{\partial t^p}{\partial x_i}\,d\mathcal{H}^{n-2}(t) =& \\
=\; 2\int_{S_{\nabla u}\cap M_\varepsilon} [\![u_h u_{ik}]\!]\frac{\partial\eta^h}{\partial\nu}\nu^k\nu^i\,d\mathcal{H}^{n-1}& \\
-\; 2\int_{S_{\nabla u}\cap M_\varepsilon} \left[\!\!\left[\sum_{p=1}^{n-1}\left(\nu^k u_{ik}\frac{\partial u_h}{\partial t^p}\frac{\partial t^p}{\partial x_i}\right)\right]\!\!\right]\eta^h\,d\mathcal{H}^{n-1}& \\
+\; 2\int_{S_{\nabla u}\cap M_\varepsilon} [\![u_h S(u)]\!]\,\eta^h\,d\mathcal{H}^{n-1} +& \\
+\,2\int_{S_{\nabla u}\cap\partial\Lambda_\varepsilon} [\![u_h\, u_{ik}]\!]\,\nu^k\,\eta^h\sum_{p=1}^{n-1}\mathbf{t}^p\cdot\mathbf{n}_\varepsilon\,\frac{\partial t^p}{\partial x_i}\,d\mathcal{H}^{n-2}(t)&
\end{aligned}
\tag{5.30}
$$

and a second inequality (without $n-2$ dimensional contribution)

$$
\begin{aligned}
2\int_{\partial\Lambda_\varepsilon} u_h u_{ik}\mathbf{n}^k_\varepsilon\eta^h_i\,d\mathcal{H}^{n-1} \;=\;& \\
=\; 2\int_{\partial\Lambda_\varepsilon} u_h u_{ik}\frac{\partial\eta^h}{\partial\mathbf{n}_\varepsilon}\mathbf{n}^k_\varepsilon\mathbf{n}^i_\varepsilon\,d\mathcal{H}^{n-1}& \\
-\; 2\int_{\partial\Lambda_\varepsilon}\sum_{p=1}^{n-1}\frac{\partial}{\partial t^p}\left(\mathbf{n}^k_\varepsilon u_h u_{ik}\frac{\partial t^p}{\partial x_i}\right)\eta^h\,d\mathcal{H}^{n-1}& \\
=\; 2\int_{\partial\Lambda_\varepsilon} u_h u_{ik}\frac{\partial\eta^h}{\partial\mathbf{n}_\varepsilon}\mathbf{n}^k_\varepsilon\mathbf{n}^i_\varepsilon\,d\mathcal{H}^{n-1}& \\
-\; 2\int_{\partial\Lambda_\varepsilon}\sum_{p=1}^{n-1}\left(\mathbf{n}^k_\varepsilon u_{ik}\frac{\partial u_h}{\partial t^p}\frac{\partial t^p}{\partial x_i}\right)\eta^h\,d\mathcal{H}^{n-1}& \\
+\,2\int_{\partial\Lambda_\varepsilon} u_h\eta^h S^\varepsilon(u)\,d\mathcal{H}^{n-1}\;.&
\end{aligned}
\tag{5.31}
$$

By taking into account Theorem 3.4(i) and that $\eta = \zeta\tau$, and $\mathbf{n}_\varepsilon$ is the outward normal from M_ε, we integrate by parts the volume integral depending on the forcing term g

$$
\begin{aligned}
-2 \int_{M_\varepsilon} & \eta^h \; u_h \; \Delta^2 u \; dx = \\
&= \mu q \int_{Q_\varepsilon} |u-g|^{q-2}(u-g)\, Du \cdot \eta \; dx \; = \\
&= \mu q \int_{Q_\varepsilon} |u-g|^{q-2}(u-g)\, Dg \cdot \eta \; dx \; + \\
&\qquad\qquad + \mu q \int_{Q_\varepsilon} |u-g|^{q-2}(u-g)\, (Du - Dg) \cdot \eta \; dx \; = \\
&= \mu q \int_{M_\varepsilon} |u-g|^{q-2}(u-g)\, Dg \cdot \tau\zeta \; dx + \mu \int_{M_\varepsilon} \eta^h \frac{\partial}{\partial x_h} |u-g|^q \; dx = \\
&= \mu\, q \int_{M_\varepsilon} |u-g|^{q-2}(u-g)\, Dg \cdot \tau\zeta \; dx \; - \; \mu \int_{M_\varepsilon} |u-g|^q \operatorname{div} \eta \; dx \\
&\quad + \; \mu \int_{\partial\Lambda_\varepsilon} |u-g|^q \, \mathbf{n}_\varepsilon \cdot \tau\zeta \; d\mathcal{H}^{\,n-1}
\end{aligned}
\tag{5.32}
$$

By substituting (5.30),(5.31) and (5.32) in (5.29) we have cancellation of all the volume integrals and we are left with the following:

$$
\begin{aligned}
\lim_{\varepsilon} \Bigg\{ & \int_{S_{\nabla u} \cap M_\varepsilon} [\![2u_{ik}u_{ih}\nu^k - 2u_h u_{kki}\nu^i]\!] \eta^h d\mathcal{H}^{\,n-1} + 2 \int_{S_{\nabla u} \cap M_\varepsilon} [\![u_h u_{ik}]\!] \frac{\partial \eta^h}{\partial \nu} \nu^k \nu^i \, d\mathcal{H}^{\,n-1} \\
& \quad - \; 2 \int_{S_{\nabla u} \cap M_\varepsilon} [\![\nu^k u_{ik} u_{ih}]\!] \eta^h \, d\mathcal{H}^{\,n-1} \; + \; 2 \int_{S_{\nabla u} \cap M_\varepsilon} [\![u_h S(u)]\!] \, \eta^h \, d\mathcal{H}^{\,n-1} \\
& + \int_{\partial\Lambda_\varepsilon} \big(-2u_{ik}u_{ih}\mathbf{n}_\varepsilon^k + 2u_h u_{kki} \mathbf{n}_\varepsilon^i \big) \eta^h d\mathcal{H}^{\,n-1} \; - 2 \int_{\partial\Lambda_\varepsilon} u_h u_{ik} \, \frac{\partial \eta^h}{\partial \mathbf{n}_\varepsilon} \mathbf{n}_\varepsilon^k \mathbf{n}_\varepsilon^i \, d\mathcal{H}^{\,n-1} \\
& \quad + \; 2 \int_{\partial\Lambda_\varepsilon} u_{ik} u_{ih} \, \mathbf{n}_\varepsilon^k \eta^h \, d\mathcal{H}^{\,n-1} \; - 2 \int_{\partial\Lambda_\varepsilon} u_h \eta^h \, S^\varepsilon(u) \, d\mathcal{H}^{\,n-1} \\
& - \int_{S_{\nabla u} \cap M_\varepsilon} [\![|\nabla^2 u|^2]\!] \, \eta \cdot \nu \; d\mathcal{H}^{\,n-1} \; + \; \int_{\partial\Lambda_\varepsilon} \big(|\nabla^2 u|^2 + \mu |u-g|^q \big) \, \eta \cdot \mathbf{n}_\varepsilon \, d\mathcal{H}^{\,n-1} \Bigg\} \\
& \quad - \int_{\Lambda} \left(\beta\,\zeta \; - \; 2 \, [\![u_h \, u_{ik}]\!] \nu^k \eta^h \sum_{p=1}^{n-1} \mathbf{t}^p \cdot \mathbf{n} \frac{\partial t^p}{\partial x_i} \right) d\mathcal{H}^{n-2} \; = \; 0 \; .
\end{aligned}
$$

By the definitions of $S, T, S^\varepsilon, T^\varepsilon$, by using $\eta \cdot \nu \equiv 0$ on $S_{\nabla u} \cap B$ and Theorems 4.4 on $S_{\nabla u}$ we get the thesis. We emphasize that Theorems 4.3, 4.4 do not hold for operators T^ε and S^ε on $\partial\Lambda_\varepsilon$ (which is not a subset of $S_{\nabla u}$).
Notice that in above formula and in the thesis of the theorem, whenever an integrand f depends on two-sided values, by $\int_\Lambda f \, d\mathcal{H}^{n-2}$ we mean $\lim_\varepsilon \int_{S_{\nabla u} \cap \partial\Lambda_\varepsilon} f \, d\mathcal{H}^{n-2}$. ∎

Remark 5.7. All the statements proved in the sections 3, 4, 5 for local minimizers of the functional $\mathcal{F}$ hold true also for local minimizers of $\mathcal{E}$, provided all the terms containing $(u-g)$ are dropped.
All the statements proved in the sections 3, 4, 5 for local minimizers of the functionals $\mathcal{F}$ and $\mathcal{E}$ hold true also for local essential minimizing triplets respectively of F, E.
We define a *local essential minimizing triplets* of F as a triplet (K_0, K_1, u) such that

$$K_0 \;=\; \overline{T_0 \cap K} \setminus (S_{\nabla v} \setminus S_v)\,, \qquad K_1 \;=\; \overline{T_1 \cap K} \setminus S_v, \qquad u \;=\; \tilde{v}$$

where K is the smallest closed subset of $T_0 \cup T_1$ such that $\tilde{v} \in C^2(\Omega \setminus K)$, and (T_0, T_1, v) satisfies $T_0 \cup T_1$ closed, $v \in C^2(\Omega \setminus (T_0 \cup T_1))$, v is approximately continuous in $\Omega \setminus T_0$,

$$F(T_0, T_1, v, A) < +\infty \quad \forall A \text{ Borel set },\, A \subset\subset \Omega,$$

$$F(T_0, T_1, v, A) \leq F(H_0, H_1, w, A) \quad \forall A \text{ Borel set },\, A \subset\subset \Omega$$

where (H_0, H_1, w) is any triplet such that $H_0 \cup H_1$ closed, $w \in C^2(\Omega \setminus (H_0 \cup H_1))$, w is approximately continuous in $\Omega \setminus H_0$ and

$$(H_0 \Delta T_0) \cup (H_1 \Delta T_1) \cup \{w \neq v\} \;\subset\subset\; A.$$

The local essential minimizing triplets of E are defined by analogous procedures.

6. Local minimizers of $\mathcal{E}$ in $\mathbf{R}^n$ ($n \geq 2$): Caccioppoli inequality and Liouville property.

So far we have found many necessary conditions for local minimizers of the functional $\mathcal{F}$. Now we focus the main part $\mathcal{E}$ of the functional $\mathcal{F}$ in $\mathbf{R}^n$, which is a natural procedure in the study of regularity properties of $\mathcal{F}$.
In this section we recall some properties of a local minimizer of the main part $\mathcal{E}$ of the Blake & Zisserman functional (proved in [CLT8]): an energy estimate, a Caccioppoli type inequality and the following facts: local minimizers in $\mathbf{R}^n$, with finite energy and bounded singular sets, are affine, a 1–dimensional step or an infinite dihedral are not local minimizers of $\mathcal{E}$ in $\mathbf{R}^n$. Eventually we prove a Liouville type property for bi-harmonic functions which are local minimizers in $\mathbf{R}^n$: in the case $n = 2$ it was showed in [CLT8] (Th.3.5) with a different proof.

Theorem 6.1. *([CLT8], Th.3.1.) Let $\Omega \subset \mathbf{R}^n$ be an open set and let u be a local minimizer of $\mathcal{E}$ in Ω. Then (denoting by ω_n the volume of the unit ball in $\mathbf{R}^n$, and hence $n\omega_n$ is the area of its boundary) for every ball $B_R \subset \Omega$ we have*

$$\mathcal{E}_{B_R}(u) \leq \alpha n \omega_n R^{n-1}.$$

Theorem 6.2. **(Caccioppoli inequality** *[CLT8], Th.3.2.***)** *Assume $\Omega \subset \mathbf{R}^n$ and u is a local minimizer of $\mathcal{E}$ in Ω. Then, for every $a \in \mathbf{R}$, $\mathbf{b} \in \mathbf{R}^n$ and for every $\rho > 0$ such that $B_{2\rho} \subset \Omega$, we have*

$$\int_{B_\rho} |\nabla^2 u|^2 \, dx \le \frac{c}{\rho^2} \int_{B_{2\rho} \setminus B_\rho} |\nabla u - \mathbf{b}|^2 \, dx + \frac{c}{\rho^4} \int_{B_{2\rho} \setminus B_\rho} |u - a - \mathbf{b} \cdot x|^2 \, dx \ ,$$

where c is a constant independent of u and ρ.

Theorem 6.3. *([CLT8], Th.3.3.) Let u be a local minimizer of $\mathcal{E}$ in $\mathbf{R}^n$ such that S_u and $S_{\nabla u}$ are bounded and*

$$\int_{\mathbf{R}^n} |\nabla^2 u|^2 \, dx < +\infty.$$

Then u is affine.

Theorem 6.4. **(Liouville property)** *Bi-harmonic functions in $\mathbf{R}^n$ are local minimizers for $\mathcal{E}$ in $\mathbf{R}^n$ if and only if they are affine.*
Proof - If u is affine then $\mathcal{E}(u) = 0$, hence u is a local minimizer of $\mathcal{E}$ in $\mathbf{R}^n$.
For a bi-harmonic function the following inequality holds

$$\int_{B_\rho(\mathbf{0})} |D^2 u|^2 dx \le c_n \left(\frac{\rho}{R}\right)^n \int_{B_R(\mathbf{0})} |D^2 u|^2 dx \qquad \forall \rho < R$$

where c_n is an absolute constant depending only on the dimension n (see [Gia], Chapt.3,Sect.2). Let $R > \rho > 0$. By Th.6.1 we have

$$\int_{B_\rho(\mathbf{0})} |D^2 u|^2 dx \le c_n \left(\frac{\rho}{R}\right)^n \int_{B_R(\mathbf{0})} |D^2 u|^2 dx \le \alpha n \omega_n c_n \frac{\rho^n}{R}. \tag{6.1}$$

By the arbitrariness of R we have

$$\int_{B_\rho(\mathbf{0})} |D^2 u|^2 dx = 0.$$

Since ρ is arbitrary we get the thesis. ∎

Theorem 6.5. *([CLT8], Th.3.6.) Set $x = (x_1, x')$ for $x \in \mathbf{R}^n$. Then, for any $c \neq 0$, the function $u(x) = c \,\mathrm{sign}(x_1)$ is not a local minimizer for $\mathcal{E}$ in $\mathbf{R}^n$.*

Theorem 6.6. *([CLT8], Th.3.7.) Set $x = (x_1, x')$ for $x \in \mathbf{R}^n$. Then, for any $c \neq 0$, the function $d(x) = c|x_1|$ is not a local minimizer for $\mathcal{E}$ in $\mathbf{R}^n$.*

Remark 6.7. All the statements proved in this section 6 for local minimizers of the functionals $\mathcal{E}$ hold true also for local essential minimizing triplets of E (we refer to (1.2), (1.5), Definitions 2.9, 2.11 and Remark 5.7).

7. Asymptotic expansions of bi-harmonic functions in a disk with a cut and non trivial local minimizers of $\mathcal{E}$ in $\mathbf{R}^2$.

In this section we look for a description of all functions v which are defined almost everywhere in B_ρ (where $0 < \varrho \leq +\infty$ and, for simplicity, $n = 2$), are bi-harmonic in $B_\rho \setminus \Gamma$ (where $\Gamma = K_v$ is the closed negative real axis) and fulfill all properties of local minimizers of $\mathcal{E}$ in B_ρ proved in the previous sections. These properties are so many that, at a first glance, this set must be very small (if not empty!).
Nevertheless at the end of the analysis we will be able to exhibit functions fulfilling all of them and, in addition, the equipartition between absolutely continuous part and lineic part of $\mathcal{E}$ energy.
From now on we assume

$$n = 2 \quad \text{and} \quad \Gamma \text{ is the half line } (-\infty, 0]_x \times \{0\}_y, \tag{7.1}$$

and we denote respectively by $D_{\mathbf{x}}$, $\mathrm{div}_{\mathbf{x}}$, $\Delta_{\mathbf{x}}$ the distributional gradient, divergence and Laplace operator with respect to the cartesian coordinate $\mathbf{x} = (x, y)$.
Since it is not possible to perform complete separation of variables in the bi-harmonic operator $\Delta_{\mathbf{x}}^2$ the achievement of the formal expansion requires some care. For completeness we write the details to avoid repetition of mistakes present in the literature.
We look for a suitable expansion strongly convergent in $L^2(B_\varrho)$ of functions $v \in L^2(B_\varrho)$ satisfying, in case $\Gamma = \overline{S_v}$:

$$\begin{cases} {\Delta_{\mathbf{x}}}^2 v = 0 & \text{on } B_\varrho \setminus \Gamma, \\[2mm] \dfrac{\partial^2 v}{\partial N^2} \; = \; 0 & \text{from both sides of } \Gamma, \\[2mm] \dfrac{\partial^3 v}{\partial N^3} \; + \; 2 \, \dfrac{\partial}{\partial N} \left(\dfrac{\partial^2 v}{\partial \tau^2} \right) \; = \; 0 & \text{from both sides of } \Gamma. \end{cases} \tag{7.2}$$

If in addition $v \in H^2(B_\varrho \setminus \Gamma)$ we will show an expansion strongly convergent in $H^2(B_\varrho \setminus \Gamma)$ and built with elementary functions v_k, z_k (see Definitions 7.3, 7.5, 7.6, (7.23) and Lemma 7.7), such that $S_{v_k} = S_{z_k} = \Gamma \setminus \{(0,0)\}$.

Following some classical ideas by E.Almansi, we rewrite (in modern language for reader convenience) a statement from his paper ([Al]) about the decomposition of poly-harmonic functions, then we weaken his topological assumptions on the domain. This provides us some heuristic about the correct expansion of v in $B_\varrho \setminus \Gamma$.

Theorem 7.1. ([Al]) - *Let $n \geq 1$, $\Omega \subset \mathbf{R}^n$ open set, Ω star-shaped with respect to the origin, $u \in C^4(\Omega)$ then*

$$\Delta_{\mathbf{x}}{}^2 u = 0 \quad \textit{in } \Omega \qquad\qquad \textit{iff}$$

$$\exists\, \varphi, \psi : \qquad u(\mathbf{x}) \;=\; \psi(\mathbf{x}) + \|\mathbf{x}\|^2 \varphi(\mathbf{x}), \qquad \Delta_{\mathbf{x}}\, \varphi(\mathbf{x}) \;=\; \Delta_{\mathbf{x}}\, \psi(\mathbf{x}) \;\equiv\; 0 \qquad \mathbf{x} \in \Omega\,.$$

For our purposes we adapt the above statement to 2-dimensional domains with a cut.

Theorem 7.2. - *Let $n = 2$, $0 < R \leq +\infty$, $u \in C^4(B_R \setminus \Gamma)$. Then*

$$\Delta_{\mathbf{x}}{}^2 u = 0 \quad \textit{in } B_R \setminus \Gamma \qquad\qquad \textit{iff}$$

$$(7.3)\;\; \exists\, \varphi, \psi : \quad u(\mathbf{x}) \;=\; \psi(\mathbf{x}) + \|\mathbf{x}\|^2\, \varphi(\mathbf{x})\,,\; \Delta_{\mathbf{x}}\, \varphi(\mathbf{x}) \;=\; \Delta_{\mathbf{x}}\, \psi(\mathbf{x}) \;\equiv\; 0\,,\; \mathbf{x} \in B_R \setminus \Gamma\,.$$

Moreover for any u s.t. $\Delta_{\mathbf{x}}{}^2 u = 0$ in $B_R \setminus \Gamma$ and $\Delta_{\mathbf{x}}\, u$ in $C^0\big((B_R \setminus \Gamma) \cup \{0\}\big)$ the decomposition (7.3) is unique.

As it will be clear from the proof, in the last statement the assumption $\Delta_{\mathbf{x}}\, u$ in $C^0\big((B_R \setminus \Gamma) \cup \{0\}\big)$ can be dropped whenever $(\partial/\partial\varrho) \int_0^\varrho \Delta_{\mathbf{x}} u(r,\vartheta) dr = \Delta_{\mathbf{x}} u(\varrho,\vartheta)$, and this is exactly what happens (by computations (10.20) in the Appendix) if u is one of the candidates W and Φ defined in the next section ((8.2),(8.3)):

$$\Delta_{\mathbf{x}}\, W(\varrho,\vartheta) \;=\; 2\,\sqrt{\alpha}\,\big(\,\sqrt{21}\,\sin(\vartheta/2) \;+\; \cos(\vartheta/2)\,\big)\,/\,\sqrt{193\,\pi\,\varrho}\,.$$

$$\Delta_{\mathbf{x}}\; \Phi(\varrho,\vartheta) \;=\; 2\,\sqrt{\alpha}\,\big(\,\sqrt{21}\;\sin(\vartheta/2) \;-\; \cos(\vartheta/2)\,\big)\,/\,\sqrt{193\,\pi\,\varrho}\,.$$

Proof - - We use the identities

$$(7.4) \qquad \Delta_{\mathbf{x}}(pq) \;=\; p\,\Delta_{\mathbf{x}} q \;+\; q\,\Delta_{\mathbf{x}} p \;+\; 2\, D_{\mathbf{x}} p \cdot D_{\mathbf{x}} q \qquad\qquad \forall p, q \in C^2(B_R \setminus \Gamma)\,,$$

$$(7.5)\;\; \Delta_{\mathbf{x}}(\mathbf{p} \cdot \mathbf{q}) \;=\; \mathbf{p} \cdot \Delta_{\mathbf{x}} \mathbf{q} \;+\; \mathbf{q} \cdot \Delta_{\mathbf{x}} \mathbf{p} \;+\; 2\,(\mathrm{div}_{\mathbf{x}}\, \mathbf{p})\,(\mathrm{div}_{\mathbf{x}}\, \mathbf{q}) \quad \forall \mathbf{p}, \mathbf{q} \in C^2(B_R \setminus \Gamma)\;.$$

(**if part**) - If $\exists\, \varphi, \psi \,:\; u(\mathbf{x}) \;=\; \psi(\mathbf{x}) + \|\mathbf{x}\|^2 \varphi(\mathbf{x})\,,\; \Delta_{\mathbf{x}}\, \varphi \;=\; \Delta_{\mathbf{x}}\, \psi \;\equiv\; 0\,,\; \mathbf{x} \in B_R \setminus \Gamma$, then by (7.4),(7.5), with $p = \|\mathbf{x}\|^2$, $q = \varphi$, $\varrho = \|\mathbf{x}\|$, we get $D_{\mathbf{x}}\, p = 2\mathbf{x}$, $\Delta_{\mathbf{x}} p = 2n$, and by $\Delta_{\mathbf{x}} \varphi = 0$

$$\Delta_{\mathbf{x}} \left(\|\mathbf{x}\|^2 \varphi \right) \;=\; \|\mathbf{x}\|^2 \Delta_{\mathbf{x}} \varphi + 2n\varphi + 4\mathbf{x} \cdot D_{\mathbf{x}} \varphi \;=\; 4\,\varphi \,+\, 4\,\varrho\, \frac{\partial \varphi}{\partial \varrho} \tag{7.6}$$

$$\Delta_{\mathbf{x}}{}^2 \left(\|\mathbf{x}\|^2 \varphi \right) \;=\; 2n\Delta_{\mathbf{x}} \varphi + 4\mathbf{x} \cdot (D_{\mathbf{x}} \Delta_{\mathbf{x}} \varphi) + 4\, D_{\mathbf{x}} \varphi \cdot \Delta_{\mathbf{x}} \mathbf{x} \,+\, 16\, \Delta_{\mathbf{x}} \varphi \;=\; 0\;. \tag{7.7}$$

(**only if part**) - If $\Delta_{\mathbf{x}}{}^2 u \;=\; 0$ then there are φ, ψ satisfying (7.3). In fact the harmonic function $\psi = u - \|\mathbf{x}\|^2 \varphi$ will match the thesis thank to (7.6), once φ is chosen (provided such φ exists) as the solution in $B_R \setminus \Gamma$ of

$$\begin{cases} (i) & \Delta_{\mathbf{x}}\, \varphi \;=\; 0 \\ (ii) & \varrho\, \dfrac{\partial \varphi}{\partial \varrho} \,+\, \varphi \;=\; \sigma \end{cases}$$

where we set $\sigma \;=\; \dfrac{1}{4}\, \Delta_{\mathbf{x}}\, u$.
The thesis follows by showing that both (i),(ii) are satisfied by

$$\varphi(\varrho, \vartheta) \;=\; \varrho^{-1} \int_0^{\varrho} \sigma(r, \theta)\, dr \qquad 0 \le \varrho < R,\; |\vartheta| \le \pi \tag{7.8}$$

where ϱ and θ denote the polar coordinates in $\mathbf{R}^2 \setminus \Gamma$ and such φ has a continuous extension up to the origin.

(ii) By (7.8) and the continuity of σ up to the origin, we get

$$\frac{\partial \varphi}{\partial \varrho} \;=\; \varrho^{-1}\, \sigma(\varrho, \theta) \;-\; \varrho^{-2} \int_0^{\varrho} \sigma(r, \theta)\, dr$$

hence

$$\varrho\, \frac{\partial \varphi}{\partial \varrho} + \varphi \;=\; \sigma(\rho, \theta) \;\mp\; \varrho^{-1} \int_0^{\varrho} \sigma(r, \theta)\, dr \;=\; \sigma\,.$$

(i) In order to verify (i), by recalling that σ is harmonic, let $\mathbf{x} = (x, y)$, and $S(x{+}iy) = \sigma(x, y) + i\eta(x, y)$ be an holomorphic function (defined in the simply connected open set $B_R \setminus \Gamma$) whose real part is σ. Then, by $z = \varrho e^{i\theta}$, $w = re^{i\theta}$, $dw = e^{i\theta} dr$, the following function is holomorphic in $B_R \setminus \Gamma$

$$\Phi(z) \;\equiv\; \frac{1}{z} \int_0^z S(w)\, dw \;=\; \frac{1}{\varrho} \int_0^{\varrho} S(re^{i\theta})\, dr \qquad \forall z \in \mathbf{C} \setminus \Gamma :\; |z| < R\,.$$

The above integration is performed on the segment joining 0 and z.
Then $Re\,\Phi \;=\; \frac{1}{\varrho}\int_0^{\varrho}\sigma(r,\theta)\,dr$ is harmonic and coincides with φ.

About uniqueness, if $\sigma\equiv 0$ then the linear problem (i)(ii) has only the trivial solution: in fact (ii) entails $\varphi\in C^1$ up to the origin hence, by (ii), $\varphi(0,\theta)=0\ \forall\vartheta$ and (ii) has only one solution. ■

Now we seek an asymptotic expansion compatible with Almansi decomposition for functions v which are defined almost everywhere in B_ρ, are bi-harmonic in $B_\rho\setminus\Gamma$ and fulfill all properties of local minimizers of $\mathcal{E}$ in B_ρ: the idea is to write an explicit formal expansion in terms of eigenfunctions of the operator $\Delta^2_{\mathbf{x}}$ with conditions (4.5), (4.6) on Γ (see (7.2)) and Dirichlet boundary conditions on ∂B_ρ.

If we look for bi-harmonic functions p homogeneous in the radial coordinate, say

$$v=v(r,\vartheta)=r^p\psi(\vartheta)\ ,\qquad p\in\mathbf{R}\,,$$

we get the following indicial equation for the function ψ:

$$\psi^{(IV)}+2(p^2-2p+2)\psi''+p^2(p-2)^2\psi=0. \tag{7.9}$$

If $p\neq 0$, and $p\neq 2$ then the characteristic roots are $\pm ip$, $\pm i(p-2)$, hence the related solutions are $r^p\cos(p\vartheta)$, $r^p\sin(p\vartheta)$, $r^p\cos\big((p-2)\vartheta\big)$, $r^p\sin\big((p-2)\vartheta\big)$.
If $p=0$ then the characteristic roots are 0 with multiplicity 2 and $\pm 2i$, hence the related solutions are 1 , ϑ , $\cos(2\vartheta)$, $\sin(2\vartheta)$.
If $p=2$ then the characteristic roots are 0 with multiplicity 2 and $\pm 2i$, hence the related solutions are r^2 , $r^2\vartheta$, $r^2\cos(2\vartheta)$, $r^2\sin(2\vartheta)$.
The functions

$$\vartheta\,,\ \cos(2\vartheta)\,,\ \sin(2\vartheta)$$

do not have distributional hessian in $L^2(B_\varrho\setminus\Gamma)$.
The conditions (4.5),(4.6) on both sides of Γ read

$$\begin{array}{r} \text{indicial equation in case}\\ \Gamma=\overline{S_v}\end{array}\qquad \left\{\begin{array}{rcl}\psi''(\pm\pi)+p\,\psi(\pm\pi) &=& 0\\ \psi'''(\pm\pi)+(2p^2-3p+2)\,\psi'(\pm\pi) &=& 0\end{array}\right\}$$

By imposing indicial equation on

$$z(r,\vartheta)\;=\;r^2\,(\,d_1\,+\,d_2\vartheta\,+d_3\cos(2\vartheta)\,+\,d_4\sin(2\vartheta)\,)\qquad p\;=\;2$$

we get $d_2=0,\ d_1=d_3$ say $\ z\;=\;r^2\,(\,d_1\,+\,d_1\cos(2\vartheta)\,+\,d_4\sin(2\vartheta)\,)$ and, by setting

$$v_p(r,\vartheta)\;=\;r^p\,(\,c_1\cos(p\vartheta)+c_2\sin(p\vartheta)+c_3\cos((p-2)\vartheta)+c_4\sin((p-2)\vartheta)\,)\quad p\;\neq\;0,2$$

we notice (see (4.1)) that $a_{B_\varrho}(v_p, v_p) < +\infty$ iff $p > 1$.
Now we take into account only the $p > 1$ and impose the indicial equation. We obtain the system $M\mathbf{c} = 0$, where $\mathbf{c} = (c_1, c_2, c_3, c_4)^t$ and M is the tridiagonal matrix

$$\begin{bmatrix} p(p-1)\cos(p\pi) & 0 & (p-1)(p-4)\cos(p\pi) & 0 \\ 0 & p(p-1)\sin(p\pi) & 0 & (p-1)(p-4)\sin(p\pi) \\ p(p-1)(p-2)\sin(p\pi) & 0 & (p-1)(p^2-4)\sin(p\pi) & 0 \\ 0 & p(p-1)(p-2)\cos(p\pi) & 0 & (p-1)(p^2-4)\cos(p\pi) \end{bmatrix}$$

whose determinant is $9p^2(p-1)^4(p-2)^2(\sin(2p\pi))^2$. Since we look for non trivial solutions $\mathbf{c}$, we solve $\det M = 0$ taking into account $p \neq 0$, $p \neq 2$, $p > 1$. We find $p = 1 + \frac{h}{2}$ with $h = 1, 2, \ldots$, hence $p \geq \frac{3}{2}$.

At this point we have some heuristic about the expansion of bi-harmonic functions in $B_\varrho \setminus \Gamma$, therefore we are looking for:

$$\begin{aligned} v \;=\; & A + B\, r \cos\theta + C\, r \sin\theta + D\, r^2 + E \ln r + F\, r^2 \ln r + \\ & + r^{3/2}\left(c1_0 \cos\left(\frac{3}{2}\theta\right) + c2_0 \sin\left(\frac{3}{2}\theta\right) + c3_0 \cos\left(\frac{\theta}{2}\right) + c4_0 \sin\left(\frac{\theta}{2}\right)\right) + \\ & + \sum_{h=0}^{+\infty} r^{2+\frac{h}{2}}\left(C1_h \cos\left((2+h/2)\,\vartheta\right) + C2_h \sin\left((2+h/2)\,\vartheta\right) + \right. \\ & \left. + C3_h \cos\left((h/2)\vartheta\right) + C4_h \sin\left((h/2)\vartheta\right)\right). \end{aligned} \tag{7.10}$$

Actually the expansion (7.10) is redundant: not all the terms are compatible with Euler equations; moreover some terms do not belong to $H^2(B_\varrho \setminus \Gamma)$, and some pairs may be not orthogonal in $H^2(B_\varrho \setminus \Gamma)$.
We look for an essential (in $\{z \in L^2(B_\varrho) : (\Delta_{\mathbf{x}})^2 z = 0 \text{ in } B_\varrho \setminus \Gamma\}$) expansion with respect to a suitably chosen orthogonal (in $H^2(B_\varrho \setminus \Gamma)$) set. Now we proceed aiming to precise statements about such expansion.

On one hand an expansion of the following kind (only integer powers of r)

$$\begin{aligned} v \;=\; & A + B\, r \cos\theta + C\, r \sin\theta + D\, r^2 + F\, r^2 \ln r + \\ & + \sum_{h=2}^{+\infty} r^h \left(d1_h\, r^2 \cos(h\vartheta) + d2_h\, r^2 \sin(h\vartheta) + \right. \\ & \left. + d3_h \cos(h\vartheta) + d4_h \sin(h\vartheta)\right) \end{aligned} \tag{7.11}$$

describes any bi-harmonic function in $H^2(B_\varrho)$ with strong convergence in $H^2(B_\varrho)$.

On the other hand we are going to show (Lemma 7.7) that an expansion of the following type (only semi-integer powers of r)

$$\begin{aligned} v \;=\; & r^{3/2}\left(c1_0\,\cos\big(\frac{3}{2}\theta\big) \,+\, c2_0\,\sin\big(\frac{3}{2}\theta\big) \,+\, c3_0\,\cos\big(\frac{\theta}{2}\big) \,+\, c4_0\,\sin\big(\frac{\theta}{2}\big)\right) + \\ & \sum_{k=1}^{+\infty} r^{k+3/2}\big(c1_k\,\cos\left((k+3/2)\,\vartheta\right) + c2_k\,\sin\left((k+3/2)\,\vartheta\right) + \\ & \qquad + c3_k\,\cos\left((k-1/2)\vartheta\right) + c4_k\,\sin\left((k-1/2)\vartheta\right)\big) \end{aligned} \tag{7.12}$$

describes (with strong $H^2(B_\varrho\backslash\Gamma)$ convergence) any bi-harmonic function in $H^2(B_\varrho\backslash\Gamma)$ which is also in the $H^2(B_\varrho\setminus\Gamma)$ orthogonal complement of $H^2(B_\varrho)$ (closed subspace of $H^2(B_\varrho\setminus\Gamma)$).
The coefficients $c1_k\,, c2_k\,, c3_k\,, c4_k$ in (7.12) are given explicitly in Lemma 7.7 via formulas (7.19') and (7.22).
Moreover, since expansion (7.12) strongly converges in $H^2(B_\varrho\setminus\Gamma)$, then (7.12) converges uniformly to the two-sided values of v, up to both sides of the cut Γ.
As it will be clarified in Lemma 7.4, for any function v in $H^2(B_\varrho\backslash\Gamma)$ and bi-harmonic in $B_\varrho\setminus\Gamma$, an expansion of type (7.12) can be found such that (if added with a suitable logarithmic term) it converges in $L^2(B_\varrho)$ to v (the expansion refers to a non L^2 orthogonal basis).

Definition 7.3. We introduce two relevant subspaces

$$A^1_\varrho := \{v \in L^2(B_\varrho) \text{ s.t. } \Delta_{\mathbf{x}} v = 0 \text{ in } B_\varrho\setminus\Gamma\,\}\,,$$

$$A^2_\varrho := \{z \in L^2(B_\varrho) \text{ s.t. } \left(\Delta_{\mathbf{x}}\right)^2 z = 0 \text{ in } B_\varrho\setminus\Gamma\,\}\,,$$

and we label two complex sequences (and the real counterpart) which are relevant in the expansion we are looking for (here $r>0,\ |\vartheta|<\pi$):

$$\begin{aligned} v_k(r,\theta) &:= r^{|k|-1/2}e^{i(k-1/2)\vartheta} && k\in\mathbf{Z} \\ z_k(r,\theta) &:= r^{|k|+3/2}e^{i(k-1/2)\vartheta} && k\in\mathbf{Z} \\ f1_k(r,\vartheta) &:= r^{(k+3/2)}\cos((k+3/2)\vartheta) && k=0,1,\dots \\ f2_k(r,\vartheta) &:= r^{(k+3/2)}\sin((k+3/2)\vartheta) && k=0,1,\dots \\ f3_k(r,\vartheta) &:= r^{(k+3/2)}\cos((k-1/2)\vartheta) && k=0,1,\dots \\ f4_k(r,\vartheta) &:= r^{(k+3/2)}\sin((k-1/2)\vartheta) && k=0,1,\dots \end{aligned}$$

We emphasize that the logarithmic-type terms listed below must be taken into account only in the L^2 framework. In fact the functions

$$
\begin{aligned}
v_0^*(r,\vartheta) &:= (1/2 - \ln\varrho + \ln r)\ r^{-1/2}\, e^{-i\vartheta/2} \\
z_0^*(r,\vartheta) &:= (1/2 - \ln\varrho + \ln r)\ r^{3/2}\, e^{-i\vartheta/2} \\
f1_0^*(r,\vartheta) &:= (1/2 - \ln\varrho + \ln r)\ r^{3/2}\ \cos(3\vartheta/2) \\
f2_0^*(r,\vartheta) &:= (1/2 - \ln\varrho + \ln r)\ r^{3/2}\ \sin(3\vartheta/2) \\
f3_0^*(r,\vartheta) &:= (1/2 - \ln\varrho + \ln r)\ r^{3/2}\ \cos(\vartheta/2) \\
f4_0^*(r,\vartheta) &:= (1/2 - \ln\varrho + \ln r)\ r^{3/2}\ \sin(\vartheta/2)
\end{aligned}
$$

are neither harmonic nor bi-harmonic in $B_\varrho\setminus\Gamma$, while

$$
v_0^{**}(r) := \ln\left(\frac{\sqrt{e}}{\varrho}\, r\right) = \left(\frac{1}{2} - \ln\varrho + \ln r\right) \quad \text{is harmonic in } B_\varrho\setminus\Gamma,
$$

$$
r^2\, v_0^{**}(r) := r^2\, \ln\left(\frac{\sqrt{e}}{\varrho}\, r\right) = r^2\, \left(\frac{1}{2} - \ln\varrho + \ln r\right) \quad \text{is bi-harmonic in } B_\varrho\setminus\Gamma\,. \ \blacksquare
$$

Lemma 7.4. *Referring to definition 7.3, for any $\varrho \in (0,1]$, the system*

$$
\left\{\ v_k(r,\theta),\ k \in \mathbf{Z}\ \right\} \tag{7.13}
$$

is orthogonal in $L^2(B_\varrho)$. Moreover system (7.13) together with v_0^{**} is $L^2(B_\varrho)$ complete in $A^1_\varrho := \{v \in L^2(B_\varrho)$ s.t. $\Delta_{\mathbf{x}} v = 0$ in $B_\varrho\setminus\Gamma\ \}$.
The system

$$
\left\{\ z_k(r,\theta),\ k \in \mathbf{Z}\ \right\} \tag{7.14}
$$

is orthogonal in $L^2(B_\varrho)$. Moreover system (7.14) together with $r^2 v_0^{**}$ is an $L^2(B_\varrho)$ complete system in $A^2_\varrho \setminus A^1_\varrho = \{z \in L^2(B_\varrho)$ s.t. $(\Delta_{\mathbf{x}})^2 z = 0 \neq \Delta_{\mathbf{x}} z$ in $B_\varrho\setminus\Gamma\,\}$.
The system

$$
\left\{\ \{f3\}_k\,,\ \ \{f4\}_k\ \right\}_{k=0,1,2,\dots} \tag{7.15}
$$

is orthogonal in $L^2(B_\varrho)$.
Moreover (7.15) together with $r^2 v_0^{**}$ is $L^2(B_\varrho)$ complete system in $A^2_\varrho\setminus A^1_\varrho$.

The system

$$(7.16)\qquad \Big\{\ \{f1\}_k\,,\ \ \{f2\}_k\ \Big\}_{k=0,1,2,\ldots}$$

is orthogonal with respect to the norm $L^2(B_\varrho)$.
Moreover (7.16) together with v_0^{**} is an L^2 complete system in the closed subspace

$$\left\{ v \in A^1_\varrho \ \ s.t.\ \int_{B_\varrho(0)} \overline{v}\, r^{-1/2}\, e^{-i\theta/2} = \int_{B_\varrho(0)} \overline{v}\, r^{1/2}\, e^{i\theta/2} = 0 \right\}.$$

All the above systems lead to a unique representation of coefficients.
The whole system formed by (7.13) and (7.14) together with v_0^{**} and $r^2 v_0^{**}$ is a complete (but not orthogonal) system in A^2_ϱ.
The whole system formed by (7.15) and (7.16) together with v_0^{**} and $r^2 v_0^{**}$ is a complete (but not orthogonal) system in A^2_ϱ.

Proof - We use polar coordinates (r,θ) in both balls B_1 and B_ϱ.
All the orthogonality statements easily follow by integrating with respect to θ since the variables are separated. We have only to show the completeness in each case.
The system of unit vectors

$$\left\{ \sqrt{\frac{|k|+1}{\pi}}\, e^{ik\theta}\, r^{|k|}\,, \quad k \in \mathbf{Z}; \qquad \frac{2}{\sqrt{\pi}} \left(\frac{1}{2} + \ln r \right) \right\}$$

is orthonormal in $L^2(B_1)$ and $L^2(B_1)$ complete in A^1_1.
By, dilation, the system

$$\left\{ w_k(r,\theta)\ := e^{ik\theta}\, r^{|k|}\,, \quad k \in \mathbf{Z}\,; \quad \left(\frac{1}{2} - \ln\varrho + \ln r \right) \right\}$$

is orthogonal and complete in A^1_ϱ.
We claim that the system

$$\Big\{\ v_k(r,\theta)\ =\ e^{i(k-1/2)\theta}\, r^{|k|-1/2}\,, \qquad k \in \mathbf{Z} \Big\}$$

is orthogonal and together with v_0^{**} is complete in A^1_ϱ. The orthogonality is a straightforward consequence of integration with respect to the angular coordinate; about completeness property: if $f \in A^1_\varrho$, and $(f, r^{|k|-1/2} e^{i\theta(k-1/2)})_{L^2(B_\varrho)} = 0\ \forall k$ and $(f, v_0^{**})_{L^2(B_\varrho)} = 0$ then $(r^{1/2} e^{i\theta/2} f, w_k)_{L^2(B_\varrho)} = (f, v_0^{**})_{L^2(B_\varrho)} = 0\ \forall k$, hence $f \equiv 0$ by completeness of the previous system. Then $\{\{v_k\}_{k\in\mathbf{Z}}, v_0^{**}\}$ is an L^2 complete system in A^1_ϱ, hence (7.16) together with v_0^{**} is an L^2 complete system in A^1_ϱ too.

By Almansi characterization in the form of Theorem 7.2 all the functions $z_k = r^2 v_k$ are bi-harmonic in $B_\varrho \setminus \Gamma$, hence $z_k \in A^2_\varrho$ for any k; moreover, given $f \in A^2_\varrho \setminus A^1_\varrho$, the properties $(f, z_k)_{L^2(B_\varrho)} = 0\ \forall k$ and $(f, r^2 v_0^{**})_{L^2(B_\varrho)} = 0$ entail (by completeness of (7.13) in A^1_ϱ) the identity $r^2 f \equiv 0$ in B_ϱ, say the system (7.14) is $L^2(B_\varrho)$ complete in $A^2_\varrho \setminus A^1_\varrho$, and hence (7.15) is complete too (both together with $r^2 v_0^{**}$).
Let us consider the statement of completeness in A^2_ϱ : take any $f \in A^2_\varrho = A^1_\varrho \cup (A^2_\varrho \setminus A^1_\varrho)$, notice that A^2_ϱ is a (not orthogonal) direct sum: $A^2_\varrho = \left(A^2_\varrho \setminus A^1_\varrho\right) \oplus A^1_\varrho$. Then we can choose either $f \in A^1_\varrho$ or $f \in A^2_\varrho \setminus A^1_\varrho$.
If $f \in A^1_\varrho$ and $(f, v_0^{**})_{L^2} = (f, v_k)_{L^2} = 0,\ \forall k$, then $f = 0$ by completeness in A^1_ϱ of (7.13) with v_0^{**}.
If $f \in A^2_\varrho \setminus A^1_\varrho$ and $(f, r^2 v_0^{**})_{L^2} = (f, z_k)_{L^2} = 0,\ \forall k$, then $f = 0$ by completeness in $A^2_\varrho \setminus A^1_\varrho$ of (7.14) with $r^2 v_0^{**}$.

Then the last two statement easily follow since the systems (7.15),(7.16) correspond to the real form of (7.13),(7.14): $f3_k = \mathit{Re}\, z_k,\ f4_k = \mathit{Im}\, z_k$, for the first system and $f1_k = \mathit{Re}\, v_{k+2},\ f2_k = \mathit{Im}\, v_{k+2},\ k = 0, 1, 2, \ldots$ for the second one. ■

We emphasize the algebraic direct sum

$$A^2_\varrho \ = \ A^1_\varrho \ \oplus \ \left(A^2_\varrho \setminus A^1_\varrho \cup \{0\}\right),$$

nevertheless the two terms in the (algebraic) direct sum are not orthogonal in $L^2(B_\varrho)$.
We proceed to show that A^1_ϱ and $A^2_\varrho \setminus A^1_\varrho$ are mutually orthogonal in $H^2(B_\varrho \setminus \Gamma)$.
Then any function in A^2_ϱ can be expanded in the explicit redundant form (7.10) as it will be clarified by Lemma 7.7, which refers to the right topology ($H^2(B_\varrho \setminus \Gamma)$), moreover the splitting allows us to get rid of redundancy related to functions without jump in Γ.

Definition 7.5. Define the bilinear form

$$(\varphi, \psi)_\varrho \ := \ a_{B_\varrho \setminus \Gamma}(\varphi, \psi) \ = \ \int_{B_\varrho \setminus \Gamma} D^2\varphi \ : \ \overline{D^2\psi}\ dx\, dy \ .$$

which induces a semi-norm in $H^2(B_\varrho(0) \setminus \Gamma)$:

$$|v|_{2,\varrho} = (v, v)_\varrho^{1/2} \, .$$

Set

$$\|v\|^2_{H^2(B_\varrho \setminus \Gamma)} \ = \ \|v\|^2_{L^2(B_\varrho)} + \|D^2 v\|^2_{L^2(B_\varrho)} \ = \ \|v\|^2_{L^2(B_\varrho)} + (v, v)_\varrho \tag{7.17}$$

Definition 7.6. Let V be the space of bi-harmonic functions which are orthogonal to the smooth functions in B_ϱ with respect to the scalar product in $H^2(B_\varrho \setminus \Gamma)$

associated to the semi-norm $|\cdot|_{2,\varrho}$ and orthogonal to affine functions with respect to the $L^2(B_\varrho)$ scalar product; precisely we set:

$$
\begin{aligned}
V \quad &:= \quad A^2_\varrho \,\cap\, \{\ H^2(B_\varrho)\ \}^{\perp\, H^2(B_\varrho\setminus\Gamma)} \,\cap\, \{\text{affine functions}\}^{\perp\, L^2(B_\varrho)}\,, \\
&\text{where} \quad \{H^2(B_\varrho)\}^{\perp\, H^2(B_\varrho\setminus\Gamma)} \ = \ \left\{ v\in V:\ (v,w)_\varrho = 0\ \forall w\in H^2(B_\varrho) \right\}\,.
\end{aligned}
\tag{7.18}
$$

Notice that, thank to the completeness of systems proven in Lemma 7.4, $V\subset A^2_\varrho$ is a subspace orthogonal to affine functions with respect to the L^2 norm too.

Lemma 7.7. *V is a Hilbert space when endowed with the natural norm $|\cdot|_{2,\varrho}$, which turn out to be equivalent to (7.17) in V.*
The two sets

$$\left\{\,\{\,v_k\,\}_{k\in Z,\ k\neq 0,\pm 1}\,,\ \{\,z_k\,\}_{k\in\mathbf{Z}}\,\right\}$$

and

$$\left\{\ \{\,f1_k\,,\ f2_k\,,\ f3_k\,,\ f4_k,\,\}_{k=0,1,\ldots}\ \right\}$$

are (separately) both orthogonal with respect to the scalar product $(\cdot,\cdot)_\varrho$ in V and are $H^2(B_\varrho\setminus\Gamma)$ complete in V.
Therefore we can eliminate the redundant part in (7.10). Say, for any v in V there is a unique expansion, convergent to v in $H^2(B_\varrho\setminus\Gamma)$, respect to the two sets: either (complex form) $\{v_k, z_k\}_{k\in\mathbf{Z}}\setminus\{v_0, v_1, v_{-1}\}$ or (real form) $\{f1_k,\ f2_k,\ f3_k,\ f4_k,\}_{k=0,1,\ldots}$, as follows

$$
v \ = \sum_{\substack{h\in\mathbf{Z} \\ C_h=0\ \ if\ \ |h|\le 1}} (C_h v_h + E_h z_h)
\tag{7.19}
$$

or equivalently, in real form, the complete expansion of any $v\in V$ is given by

$$
\begin{aligned}
v \quad = \quad \sum_{h=0}^{+\infty} r^{h+\frac{3}{2}} \Bigg(c1_h\cos\left(\left(h+\frac{3}{2}\right)\vartheta\right) + c2_h\sin\left(\left(h+\frac{3}{2}\right)\vartheta\right) + \\
+\ c3_h\cos\left(\left(h-\frac{1}{2}\right)\vartheta\right) + c4_h\sin\left(\left(h-\frac{1}{2}\right)\vartheta\right)\Bigg)
\end{aligned}
\tag{7.19'}
$$

where for any $v\in V$ all coefficients $C_h, E_h, c1_h, c2_h, c3_h, c4_h$ are uniquely defined by

$$
C_h \ = \ \frac{1}{(|v_h|_{2,\varrho})^2}\int_{B_\varrho} D^2 v\ :\ \overline{D^2 v_h}\, d\mathbf{x} \qquad h\in\mathbf{Z},\ |h|>1,
\tag{7.20}
$$

$$E_h \;=\; \frac{1}{(|z_h|_{2,\varrho})^2}\int_{B_\varrho} D^2v \;:\; \overline{D^2 z_h}\, d\mathbf{x} \qquad h\in\mathbf{Z}\,, \tag{7.21}$$

$$cj_h \;=\; \frac{1}{(|fj_h|_{2,\varrho})^2}\int_{B_\varrho} D^2v \;:\; D^2 fj_h\, d\mathbf{x} \qquad h=0,1,2,\dots\,,\; j=1,2,3,4. \tag{7.22}$$

Both expansions (7.19) and (7.19') are strongly convergent in $H^2(B_\varrho\setminus\Gamma)$.
We emphasize that all the coefficients are independent of the radius ϱ.

Proof - If $\varphi_h\in V$ and $\varphi_h\to\varphi$ strongly in $H^2(B_\varrho\setminus\Gamma)$ then ${\Delta_{\mathbf{x}}}^2\varphi=0$ in $B_\varrho\setminus\Gamma$; moreover $(\varphi_h,w)_{L^2(B_\varrho)}=0$ for any affine w, $\varphi_h\in V$ and $\varphi_h\to\varphi$ strongly in $H^2(B_\varrho\setminus\Gamma)$ then $(\varphi,w)_{L^2(B_\varrho)}=0$ for any affine w. Hence V is complete with respect to the norm induced by the scalar product $(\,.\,,\,.\,)_\varrho$.
Notice that $v_0^{**}\notin H^2(B_\varrho\setminus\Gamma)$, $r^2v_0^{**}\in H^2(B_\varrho)$, and v_h does not belong to $H^2(B_\varrho\setminus\Gamma)$ when $h=0,\pm1$, hence neither of them belong to V. Then, thank to Theorem 7.4 we have only to show orthogonality and completeness of the two sets.
By performing very long computations, checked also with software Mathematica 5.0 ©(see (10.32) (10.33)), we evaluate the hessian matrices and find that the bi-harmonic functions v_k, for $k\in\mathbf{Z}\setminus\{0,\pm1\}$, $|\vartheta|<\pi$, and the bi-harmonic functions z_k, for $k\in\mathbf{Z}$, $|\vartheta|<\pi$, fulfil:

$$\frac{\partial^2 v_k}{\partial x^2} \;=\; \frac{e^{i\left(k-\frac12\right)\vartheta}\; r^{|k|-\frac52}}{4}\times$$
$$\times\left(-4\,k^- + (4k^2-4k^+-4|k|+3)\cos(2\vartheta) - (4k|k|-4k-4k^++3)\,i\,\sin(2\vartheta)\right)$$

$$\frac{\partial^2 v_k}{\partial y^2} \;=\; -\,\frac{e^{i\left(k-\frac12\right)\vartheta}\; r^{|k|-\frac52}}{4}\times$$
$$\times\left(4k^- + (4k^2-4k^+-4|k|+3)\cos(2\vartheta) - (4k|k|-4k-4k^++3)\,i\,\sin(2\vartheta)\right)$$

$$\frac{\partial^2 v_k}{\partial x\partial y} \;=\; \frac{e^{i\left(k-\frac12\right)\vartheta}\; r^{|k|-\frac52}}{4}\times$$
$$\times\left((4k|k|-4k-4k^++3)\,i\,\cos(2\vartheta) + (4k^2-4|k|-4k^++3)\sin(2\vartheta)\right)$$

$$\frac{\partial^2 z_k}{\partial x^2} \;=\; \frac{e^{i\left(k-\frac12\right)\vartheta}\; r^{|k|-\frac12}}{4}\times$$
$$\times\left(4\,(1+k^++|k|) + \left(4k^2+4k^--1\right)\cos(2\vartheta) - (4k|k|-4k^--1)\,i\,\sin(2\vartheta)\right)$$

$$\frac{\partial^2 z_k}{\partial y^2} = \frac{e^{i\left(k-\frac{1}{2}\right)\vartheta}\ r^{|k|-\frac{1}{2}}}{4} \times$$
$$\times \left(4\,(1+k^+ + |k|) - (4k^2+4k^- - 1)\cos(2\vartheta) + (4k|k| - 4k^- - 1)\, i\, \sin(2\vartheta) \right)$$

$$\frac{\partial^2 z_k}{\partial x \partial y} = \frac{e^{i\left(k-\frac{1}{2}\right)\vartheta}\ r^{|k|-\frac{1}{2}}}{4} \times$$
$$\times \left((4k|k| - 4k^- - 1)\, i\, \cos(2\vartheta) + (4k^2+4k^- - 1)\sin(2\vartheta) \right)$$

moreover $f1_k,\ f2_k,\ f3_k,\ f4_k,$ for $k = 0, 1, \ldots,\ |\vartheta| < \pi$, (see (10.35)-(10.38)) fulfil

$$\frac{\partial^2 f1_k}{\partial x^2} = \frac{\left(3+8\,k+4\,k^2\right)\ r^{-\left(\frac{1}{2}\right)+k}\ \cos(\frac{\vartheta-2\,k\,\vartheta}{2})}{4}$$

$$\frac{\partial^2 f1_k}{\partial x\,\partial y} = \frac{\left(3+8\,k+4\,k^2\right)\ r^{-\left(\frac{1}{2}\right)+k}\ \sin(\frac{\vartheta-2\,k\,\vartheta}{2})}{4}$$

$$\frac{\partial^2 f1_k}{\partial y^2} = \frac{-\left(\left(3+8\,k+4\,k^2\right)\ r^{-\left(\frac{1}{2}\right)+k}\ \cos(\frac{\vartheta-2\,k\,\vartheta}{2})\right)}{4}$$

$$\frac{\partial^2 f2_k}{\partial x^2} = \frac{-\left(\left(3+8\,k+4\,k^2\right)\ r^{-\left(\frac{1}{2}\right)+k}\ \sin(\frac{\vartheta-2\,k\,\vartheta}{2})\right)}{4}$$

$$\frac{\partial^2 f2_k}{\partial x\,\partial y} = \frac{\left(3+8\,k+4\,k^2\right)\ r^{-\left(\frac{1}{2}\right)+k}\ \cos(\frac{\vartheta-2\,k\,\vartheta}{2})}{4}$$

$$\frac{\partial^2 f2_k}{\partial y^2} = \frac{\left(3+8\,k+4\,k^2\right)\ r^{-\left(\frac{1}{2}\right)+k}\ \sin(\frac{\vartheta-2\,k\,\vartheta}{2})}{4}$$

$$\frac{\partial^2 f3_k}{\partial x^2} = \frac{(1+2\,k)\ r^{-\left(\frac{1}{2}\right)+k}\ \left((-1+2\,k)\ \cos(\left(\frac{5}{2}-k\right)\ \vartheta) + 4\ \cos(\left(-\left(\frac{1}{2}\right)+k\right)\ \vartheta)\right)}{4}$$

$$\frac{\partial^2 f3_k}{\partial x\,\partial y} = \frac{(-1+2\,k)\ (1+2\,k)\ r^{-\left(\frac{1}{2}\right)+k}\ \sin(\left(\frac{5}{2}-k\right)\ \vartheta)}{4}$$

$$\frac{\partial^2 f3_k}{\partial y^2} = \frac{-\left((1+2\,k)\ r^{-\left(\frac{1}{2}\right)+k}\ \left((-1+2\,k)\ \cos(\left(\frac{5}{2}-k\right)\ \vartheta) - 4\ \cos(\left(-\left(\frac{1}{2}\right)+k\right)\ \vartheta)\right)\right)}{4}$$

$$\frac{\partial^2 f4_k}{\partial x^2} = \frac{(1+2\,k)\;r^{-\left(\frac{1}{2}\right)+k}\;\left((1-2\,k)\;\sin(\left(\frac{5}{2}-k\right)\;\vartheta)+4\;\sin(\left(-\left(\frac{1}{2}\right)+k\right)\;\vartheta)\right)}{4}$$

$$\frac{\partial^2 f4_k}{\partial x\,\partial y} = \frac{(-1+2\,k)\;(1+2\,k)\;r^{-\left(\frac{1}{2}\right)+k}\;\cos(\left(\frac{5}{2}-k\right)\;\vartheta)}{4}$$

$$\frac{\partial^2 f4_k}{\partial y^2} = \frac{(1+2\,k)\;r^{-\left(\frac{1}{2}\right)+k}\;\left((-1+2\,k)\;\sin(\left(\frac{5}{2}-k\right)\;\vartheta)+4\;\sin(\left(-\left(\frac{1}{2}\right)+k\right)\;\vartheta)\right)}{4}$$

Notice that all the second derivatives of $\{f1_k,\ f2_k,\ f3_k,\ f4_k\}$ are linear combinations of $\{f1_{k-2},\ f2_{k-2},\ f3_{k-2},\ f4_{k-2}\}$, when $k \geq 2$. Moreover, by performing integrations with Mathematica 5.0 ©(see (10.34) and (10.39)) the following list of orthogonality properties hold true.

$$\text{(7.23)}\quad \begin{cases} (v_k, v_h)_\varrho \;=\; 0 & k \neq h,\ h,k \in \mathbf{Z}\setminus\{0,\pm 1\}, \\ (z_k, z_h)_\varrho \;=\; 0 & k \neq h,\ \ k,h \in \mathbf{Z}, \\ (v_k, z_h)_\varrho \;=\; 0 & k \in \mathbf{Z}\setminus\{0,\pm 1\},\ \ h \in \mathbf{Z}, \\ & \\ (fi\,_k, fj\,_l)_\varrho \;=\; 0 \ \text{ if, either } i \neq j \text{ or } k \neq l,\ i,j = 1,2,3,4,\ \ k,l = 0,1,\ldots, & \\ (f1_k, f1_k)_\varrho \;=\; \frac{\pi}{4}\,(1+2\,k)\;(3+2\,k)^2\;\varrho^{1+2k}, & \\ (f2_k, f2_k)_\varrho \;=\; \frac{\pi}{4}\,(1+2\,k)\;(3+2\,k)^2\;\varrho^{1+2k}, & \\ (f3_k, f3_k)_\varrho \;=\; \frac{\pi}{4}\;(1+2\,k)\;\left(9-4k+4k^2\right)\;\varrho^{1+2k}, & \\ (f4_k, f4_k)_\varrho \;=\; \frac{\pi}{4}\;(1+2\,k)\;\left(9-4k+4k^2\right)\;\varrho^{1+2k}. & \end{cases}$$

Then the sets $\{v_k, z_k\}_{k\in\mathbf{Z}}$ and $\{f1_k,\ f2_k,\ f3_k,\ f4_k,\}_{k=0,1,\ldots}$ are built with independent and $(.,.)_\varrho$ orthogonal functions.

By recalling that $r^2 v_0^{**} \in H^2(B_\varrho)$ and $v_0^{**} \notin H^2(B_\varrho \setminus \Gamma)$, hence neither of them belong to V, due to Lemma 7.4, an expansion of type (7.19) exists (with possibly different coefficients not necessarily evaluated by (7.20)(7.21)) and is strongly convergent at least in L^2 for any $v \in V \subset A^2_\varrho$. Orthogonality relationship (7.23) entails pairwise orthogonality in V of terms in the expansion hence uniqueness of expansion (7.19) (if it exists); so we are left to show the existence of such expansion for any $v \in V$, or equivalently the $H^2(B_\varrho \setminus \Gamma)$ completeness in V of the joint system $\left\{\,\{\,v_k\,\}_{k\in Z,\ k\neq 0,\,\pm 1}\,,\ \{\,z_k\,\}_{k\in\mathbf{Z}}\,\right\}$.

It will be enough showing

$$\text{(7.24)}\quad \{z \in V,\ (v, v_k)_\varrho = 0\ \forall\ k \in \mathbf{Z}\setminus\{0,\pm1\},\ (v, z_k)_\varrho = 0,\ \forall\ k \in \mathbf{Z}\}\ \Rightarrow\ z \equiv 0\,.$$

For any fixed $v \in V$, by uniqueness of projections and Parseval inequality there are coefficients C_h, E_h and a function $w \in H^2(B_\varrho \setminus \Gamma)$ such that

$$w = \sum_{h \in \mathbf{Z},\, C_h = 0\, if\, |h| \le 1} \big(C_h\, v_h \; + \; E_h\, z_h \big) \text{ strongly convergent in } H^2(B_\varrho \setminus \Gamma) \text{ hence in } L^2$$

where

$$\begin{aligned} C_h \;&=\; \frac{1}{(|v_h|_{2,\varrho})^2} \int_{B_\varrho} D^2 v \;:\; \overline{D^2 v_h}\, d\mathbf{x} \qquad h \in \mathbf{Z},\; |h| > 1, \\ E_h \;&=\; \frac{1}{(|z_h|_{2,\varrho})^2} \int_{B_\varrho} D^2 v \;:\; \overline{D^2 z_h}\, d\mathbf{x} \qquad h \in \mathbf{Z}\,. \end{aligned}$$

So that (7.19) is strongly convergent in $H^2(B_\varrho \setminus \Gamma)$. Now we show that $w = v$.

Both systems $\{v_h\}_{h \in \mathbf{Z},\, |h| > 1}$, $\{z_h\}_{h \in \mathbf{Z}}$ are orthogonal in $L^2(B_\varrho)$; moreover the two systems joined together with $v_0^{**} = r^2 \ln r$ are $L^2(B_\varrho)$ complete in A^2_ϱ (by Lemma 7.4) and neither of them contains the affine functions; V does not contain neither v_0^{**} nor $r^2 v_0^{**}$. Then, by Lemma 7.4, every $v \in V$ is represented by an expansion which is uniquely defined and strongly convergent in L^2:

$$v \;=\; \sum_{\substack{h \in \mathbf{Z}, \\ c_h = 0 \;\; if \;\; |h| \le 1}} (c_h\, v_h \; + \; e_h\, z_h)\,. \tag{7.25}$$

In the above expansion $(c_h\, v_h \; + \; e_h\, z_h)$ is the unique $L^2(B_\varrho)$ projection of v on 2 dimensional spaces $V_h := \text{span}\{v_h, z_h\}$, $h \in \mathbf{Z}$, $|h| > 1$, and on 1 dimensional spaces $V_h := \text{span}\{z_h\}$ $h = 0, \pm 1$. The orthogonality $V_h \perp V_l$ holds true both with respect to both scalar products $(\cdot, \cdot)_{L^2(B_\varrho)}$ and $(\cdot, \cdot)_\varrho$. Notice that the coefficients c_h, e_h are not obtained by scalar products with v_h and z_h since v_h, z_h are not $L^2(B_\varrho)$ mutually orthogonal.

Eventually we show that the expansion (7.25) is strongly convergent also in $H^2(B_\varrho \setminus \Gamma)$, that is $c_h = C_h,\; e_h = E_h$.

We recall that $(v_h, z_l)_{L^2(B_\varrho)} = (v_h, v_l)_{L^2(B_\varrho)} = (z_h, z_l)_{L^2(B_\varrho)} = 0$, $h \ne l$, so that v is obtained as an infinite sum of terms belonging to a sequence of 2 dimensional complex subspaces V_h (each one spanned by v_h, z_h for any fixed $h \in \mathbf{Z}$, $|h| > 1$, or spanned by z_h if $h = 0, \pm 1$). These 2 dimensional spaces V_h are pairwise orthogonal in $L^2(B_\varrho)$.

If $v \in H^2(B_\varrho \setminus \Gamma)$ then every finite truncated sum from (7.25) belong to $H^2(B_\varrho \setminus \Gamma)$, by subtraction the residual series belongs to $H^2(B_\varrho \setminus \Gamma)$, too. Notice that the residual series is a priori convergent only in L^2.

We claim that all the expansion (7.25) converges also in $H^2(B_\varrho \setminus \Gamma)$: this property follows from uniform boundedness in $H^2((B_\varrho \setminus \Gamma))$ of finite truncated sums of (7.25):

$$\exists C \ s.t. \ \forall N \qquad \left| \sum_{\substack{h=-N \\ c_h=0 \ \ if \ \ |h|\le 1}}^{N} (c_h\, v_h \,+\, e_h\, z_h) \right|^2_{2,\varrho} \le C \ < \ +\infty$$

since this boundedness, together with $V_h \perp V_l$ in $H^2(B_\varrho \setminus \Gamma)$, entails

$$\exists C \ s.t. \ \forall N \qquad \sum_{\substack{h=-N \\ c_h=0 \ \ if \ \ |h|\le 1}}^{N} |c_h\, v_h \,+\, e_h\, z_h|^2_{2,\varrho} \le C \ < \ +\infty$$

hence exists $w \in H^2(B_\varrho \setminus \Gamma)$ s.t.

$$w \ = \sum_{\substack{h\in\mathbf{Z}, \\ c_h=0 \ \ if \ \ |h|\le 1}} (c_h\, v_h \,+\, e_h\, z_h) \qquad \text{with strong } H^2(B_\varrho \setminus \Gamma) \text{ convergence}$$

and this w must coincide with v for uniqueness of limit in L^2.
Otherwise, assuming by contradiction that uniform boundedness in $H^2((B_\varrho \setminus \Gamma))$ of truncated sums of (7.25) does not hold true, we would obtain

$$\sum_{\substack{h\in\mathbf{Z}, \\ c_h=0 \ \ if \ \ |h|\le 1}} |\, c_h\, v_h \,+\, e_h\, z_h\,|^2_{2,\varrho} \ = \ +\infty\,,$$

from the L^2 convergence we obtain (up to subsequences) the following convergence

$$\sum_{\substack{h\in\mathbf{Z}, \\ c_h=0 \ \ if \ \ |h|\le 1}} (c_h\, v_h \,+\, e_h\, z_h) \ = \ v \qquad \text{a.e in } B_\varrho$$

and applying Parseval inequality to this last relationship together with $V_h \perp V_l$ in $H^2(B\varrho \setminus \Gamma)$ we get a contradiction with $v \in H^2(B\varrho \setminus \Gamma)$:

$$|v|^2_{2,\varrho} \ \ge \sum_{\substack{h\in\mathbf{Z}, \\ c_h=0 \ \ if \ \ |h|\le 1}} |\, c_h\, v_h \,+\, e_h\, z_h\,|^2_{2,\varrho} \ = \ +\infty\,.$$

Then we have proved that (7.25) strongly converges in $H^2(B\varrho \setminus \Gamma)$ and hence for any fixed $k \in \mathbf{Z} \setminus \{0, \pm 1\}$ and $N \in \mathbf{N}$, $N \ge k$, we have

$$C_k \ = \ \frac{1}{(|v_k|_{2,\varrho})^2}\,(v, v_k)_\varrho \ =$$

$$= \frac{1}{(|v_k|_{2,\varrho})^2} \left(\sum_{h\in\mathbf{Z},\ h\le N} (c_h v_h + e_h z_h)\,,\ v_k \right)_{\varrho} +$$

$$+ \frac{1}{(|v_k|_{2,\varrho})^2} \left(\sum_{h\in\mathbf{Z},\ h> N} (c_h v_h + e_h z_h)\,,\ v_k \right)_{\varrho} \ = \ c_k$$

the last equality is due to the fact the first sum is a finite sum of $H^2(B_\varrho\setminus\Gamma)$ functions so we can exploit (7.23), while the second one (infinite sum, a priori convergent only in L^2) is an H^2 function belonging to the $H^2(B_\varrho\setminus\Gamma)$ orthogonal space to V_k.
In the same way one gets $E_k = e_k$ for any k.
So (7.24) is proved, hence the system $\big\{ \{ v_k \}_{k\in Z,\ k\neq 0,\pm1}\,,\ \{ z_k \}_{k\in\mathbf{Z}} \big\}$ is complete and has no redundancy.
The completeness and non redundancy of $\{f1_k, f2_k, f3_k, f4_k\}$ follows by considering real and imaginary parts of $\big\{ \{ v_k \}_{k\in Z,\ k\neq 0,\pm1}\,,\ \{ z_k \}_{k\in\mathbf{Z}} \big\}$. ■
Now we obtain information about coefficients in the asymptotic expansion (7.19) in the particular case of a local minimizers with jump set on Γ, by choosing an horizontal vector field η pointing toward the crack in the crack-tip condition of Theorem 5.5 (n=2).

Lemma 7.8. *Assume $u = r^p\psi(\vartheta)$ is a local minimizer of $\mathcal{E}$ in $\mathbf{R}^2$ such that $K_u = \overline{S_u}$ is the closed negative real axis.*
Then $p = \frac{3}{2}$ and there are constants c_1, c_2, c_3, c_4 such that $u = W_0$ in $\mathbf{R}^2\setminus\Gamma$, where W_0 is expressed in polar co-ordinates ($r>0,\ |\vartheta|<\pi$) by

$$(7.26)\quad W_0(r,\vartheta) := r^{\frac{3}{2}}\left(c_1\cos\left(\frac{3}{2}\vartheta\right) + c_2\sin\left(\frac{3}{2}\vartheta\right) + c_3\cos\left(\frac{\vartheta}{2}\right) + c_4\sin\left(\frac{\vartheta}{2}\right)\right).$$

Even without assuming homogeneity in r we can say a lot about local minimizers: if u is a local minimizer of $\mathcal{E}$ in $\mathbf{R}^2$ such that $K_u = \overline{S_u}$ is the closed negative real axis, then in the expansions (7.19) and (7.19') the terms at level $h=0$ cannot vanish altogether, that is referring to (7.26) the local minimizer has the following form

$$(7.27)\qquad u(r,\vartheta) \ = \ W_0(r,\vartheta) \ + \ o(r^{3/2}) \ = \ W_0(r,\vartheta) \ + \ O(r^{5/2})\,.$$

The coefficients c_1, c_2, c_3, c_4 cannot vanish altogether neither in (7.26) nor in (7.27).

Proof - Assume first $u = r^p\psi(\vartheta)$. By taking into account Remark 5.7, in Theorem 5.5 we can choose $n=2$, $\Xi = \{\mathbf{0}\}$, $\mathbf{n} = (-1,0)$ and $\zeta\equiv 1$ in a neighborhood of $\mathbf{0}$.
In $\partial B_\varepsilon(0)$ we have $\eta\equiv(1,0)$, $\mathbf{n}_\epsilon = -(\cos\vartheta,\sin\vartheta)$ and $\frac{\partial\eta}{\partial\mathbf{n}_\varepsilon} = 0$.
Moreover, if $s = \varepsilon\vartheta$, then for suitable $\varphi = \varphi(\vartheta)$ and $\xi = \xi(\vartheta)$

$d\mathcal{H}^1(s) = \varepsilon d\vartheta,$
$|\nabla^2 u(\varepsilon,\vartheta)|^2 = \varphi(\vartheta)\varepsilon^{2p-4},$
$|\nabla^2 u(\varepsilon,\vartheta)|^2 \,\cos\vartheta \, d\mathcal{H}^1(s) = \varphi(\vartheta)\, \cos\vartheta\, \varepsilon^{2p-3}\, d\vartheta,$
$-2\left(S^\varepsilon(u) - \frac{\partial}{\partial \mathbf{n}_\varepsilon}\Delta u\right)\eta\cdot\nabla u \;=\; \xi(\vartheta)\,\varepsilon^{2p-4},$
$-2\left(S^\varepsilon(u) - \frac{\partial}{\partial \mathbf{n}_\varepsilon}\Delta u\right)\eta\cdot\nabla u \, d\mathcal{H}^1(s) \;=\; \xi(\vartheta)\,\varepsilon^{2p-3}\, d\vartheta.$
We set

$$k_1 = k_1(0) \;:=\; -\int_{-\pi}^{\pi}\varphi(\vartheta)\cos\vartheta\, d\vartheta\ , \qquad\qquad k_2 = k_2(0) := \int_{-\pi}^{\pi}\xi(\vartheta)\, d\vartheta\ .$$

Then for ε small enough we get

$$\int_{\partial B_\varepsilon(0)} |\nabla^2 u|^2 \eta\cdot\mathbf{n}_\varepsilon\, d\mathcal{H}^1(s) \;=\; -\,\varepsilon^{2p-3}\int_{-\pi}^{\pi}\varphi(\vartheta)\,\cos\vartheta\, d\vartheta \;=\; k_1\,\varepsilon^{2p-3},$$

$$\int_{\partial B_\varepsilon(0)} T^\varepsilon(u)\frac{\partial\eta}{\partial\mathbf{n}_\varepsilon}\cdot\nabla u\;\; d\mathcal{H}^1(s) \;=\; 0,$$

$$\int_{\partial B_\varepsilon(0)} -2\left(S^\varepsilon(u) - \frac{\partial}{\partial\mathbf{n}_\varepsilon}\Delta u\right)\eta\cdot\nabla u\, d\mathcal{H}^1(s) \;=\; \varepsilon^{2p-3}\int_{-\pi}^{\pi}\xi(\vartheta)\, d\vartheta \;=\; k_2\,\varepsilon^{2p-3}.$$

By (5.16) we must have $k_1 + k_2 = \alpha \neq 0$ and $2p-3 = 0$. Hence (7.26) holds true and the coefficients c_1, c_2, c_3, c_4 cannot vanish altogether.
By substituting $W_0 = r^{3/2}\psi(\vartheta)$ in the relationship $\Delta^2 W_0 = 0$, in $\mathbf{R}^2\setminus\Gamma$, and by taking into account Lemma 7.7 and $W_0\in V$ we get (7.27).

Assume now u is a (not necessarily homogeneous) local minimizer of $\mathcal{E}$ in $\mathbf{R}^2$ such that $K_u = \overline{S_u}$ is the closed negative real axis. Then by Lemma (7.7) there are (unique) expansions of type (7.19),(7.19') convergent to u in $H^2(B_\varrho\setminus\Gamma)$, for any $\varrho > 0$.
By contradiction, assume $c_1 = c_2 = c_3 = c_4 = 0$, then there is $\widetilde{h} > 0$:
$cj_h = 0,\ j = 1,2,3,4,\ \forall h:\ h < \widetilde{h}$, in (7.19'); set

$$\begin{aligned} u_{\widetilde{h}} \;&=\; r^{\widetilde{h}+\frac{3}{2}}\left(c1_{\widetilde{h}}\cos\left(\left(\widetilde{h}+\frac{3}{2}\right)\vartheta\right) + c2_{\widetilde{h}}\sin\left(\left(\widetilde{h}+\frac{3}{2}\right)\vartheta\right) + \right.\\ &\left. \qquad + \, c3_{\widetilde{h}}\cos\left(\left(\widetilde{h}-\frac{1}{2}\right)\vartheta\right) + c4_{\widetilde{h}}\sin\left(\left(\widetilde{h}-\frac{1}{2}\right)\vartheta\right)\right) \end{aligned}$$

Then we have the following splitting between the (($\widetilde{h}+3/2$) homogeneous) leading term and the remaining part of the expansion

$$\begin{aligned} u \;&=\; u_{\widetilde{h}} \,+\, \sum_{h=\widetilde{h}+1}^{+\infty} r^{h+\frac{3}{2}}\left(c1_h\cos\left(\left(h+\frac{3}{2}\right)\vartheta\right) + c2_h\sin\left(\left(h+\frac{3}{2}\right)\vartheta\right) + \right.\\ &\left. \qquad + \, c3_h\cos\left(\left(h-\frac{1}{2}\right)\vartheta\right) + c4_h\sin\left(\left(h-\frac{1}{2}\right)\vartheta\right)\right) \end{aligned}$$

still convergent to u in $L^2(B_\varrho \setminus \Gamma))$ and in $H^{3/2}(\partial B_\varrho)$. Hence by standard argument the expansion (and, respectively every term-wise partial derivative of the expansion) is uniformly convergent to u (respectively to the related term-wise partial derivative expansion) in ∂B_ε for any $0 < \varepsilon < \varrho$.
So we can repeat the above computations (in the previous case p homogeneous meant $h + 3/2$ homogeneous): by taking into account Remark 5.7, in Theorem 5.5 we can choose $n = 2$, $\mathbf{n} = (-1, 0)$ and $\zeta \equiv 1$ in a neighborhood of 0.
In $\partial B_\varepsilon(0)$ we have $\eta \equiv (1,0)$, $\mathbf{n}_\epsilon = -(\cos\vartheta, \sin\vartheta)$ and $\frac{\partial \eta}{\partial \mathbf{n}_\varepsilon} = 0$.
Moreover, if $s = \varepsilon\vartheta$, for suitable $\varphi_{\widetilde{h}}$, $\xi_{\widetilde{h}}$ then
$d\mathcal{H}^1(s) = \varepsilon d\vartheta$,
$|\nabla^2 u_{\widetilde{h}}(\varepsilon,\vartheta)|^2 = \varphi_{\widetilde{h}}(\vartheta)\varepsilon^{2\widetilde{h}-1}$,
$|\nabla^2 u_{\widetilde{h}}(\varepsilon,\vartheta)|^2 \,\cos\vartheta \, d\mathcal{H}^1(s) = \varphi_{\widetilde{h}}(\vartheta)\, \cos\vartheta\, d\vartheta\, \varepsilon^{2\widetilde{h}}$,
$-2\left(S^\varepsilon(u_{\widetilde{h}}) - \frac{\partial}{\partial \mathbf{n}_\varepsilon}\Delta u_{\widetilde{h}}\right)\eta\cdot\nabla u \;=\; \xi_{\widetilde{h}}(\vartheta)\,\varepsilon^{2\widetilde{h}-1}$,
$-2\left(S^\varepsilon(u_{\widetilde{h}}) - \frac{\partial}{\partial \mathbf{n}_\varepsilon}\Delta u_{\widetilde{h}}\right)\eta\cdot\nabla u_{\widetilde{h}}\, d\mathcal{H}^1(s) \;=\; \xi_{\widetilde{h}}(\vartheta)\, d\vartheta\, \varepsilon^{2\widetilde{h}}$.
We set

$$k_1(\widetilde{h}) \;:=\; -\int_{-\pi}^{\pi} \varphi_{\widetilde{h}}(\vartheta)\cos\vartheta\, d\vartheta\ , \qquad\qquad k_2(\widetilde{h}) = \int_{-\pi}^{\pi} \xi_{\widetilde{h}}(\vartheta)\, d\vartheta\ .$$

Then for ε small enough we get

$$\int_{\partial B_\varepsilon(0)} |\nabla^2 u|^2 \eta\cdot\mathbf{n}_\varepsilon\, d\mathcal{H}^1(s) = -\,\varepsilon^{2\widetilde{h}}\int_{-\pi}^{\pi}\varphi_{\widetilde{h}}(\vartheta)\,\cos\vartheta\, d\vartheta + o(\varepsilon^{2\widetilde{h}}) \;=\; k_1(\widetilde{h})\,\varepsilon^{2\widetilde{h}} + o(\varepsilon^{2\widetilde{h}}),$$

$$\int_{\partial B_\varepsilon(0)} T^\varepsilon(u)\frac{\partial \eta}{\partial \mathbf{n}_\varepsilon}\cdot\nabla u\;\; d\mathcal{H}^1(s) \;=\; 0,$$

$$\int_{\partial B_\varepsilon(0)} -2\left(S^\varepsilon(u) - \frac{\partial}{\partial \mathbf{n}_\varepsilon}\Delta u\right)\eta\cdot\nabla u\, d\mathcal{H}^1(s) \;=\; \varepsilon^{2\widetilde{h}}\int_{-\pi}^{\pi}\xi_{\widetilde{h}}(\vartheta)\, d\vartheta + o(\varepsilon^{2\widetilde{h}}) \;=$$

$$=\; k_2(\widetilde{h})\,\varepsilon^{2\widetilde{h}} + o(\varepsilon^{2\widetilde{h}}).$$

By (5.16) we get $\lim_{\varepsilon\to 0_+}\left(k_1(\widetilde{h}) + k_2(\widetilde{h})\right)\varepsilon^{2\widetilde{h}} = \alpha \neq 0$ and this contradicts $\widetilde{h} > 0$. ∎

So the leading term of expansion for any minimizer is always of type W_0 as in (7.26).
In the following the notation k_1, k_2 will always be referred to $k_1(0), k_2(0)$.
Notice that both k_1 and k_2 depend on the c_j and nothing else.

We do not know yet wether the equation $k_1 + k_2 = \alpha$ can be solved or not for any choice of the c_j. Actually this can be done only for precise choices of the coefficients c_1, c_2, c_3, and c_4.

In the subsequent statements we perform the computation of admissible coefficients c_1, c_2, c_3, and c_4 of W_0 and related values of k_1, k_2 entailing $k_1 + k_2 = \alpha$, and show that they are uniquely defined: multiples of W_0 fail to be admissible, that is $t\,W_0$ *is not a minimizer for any* $t \neq 1$.
We start with Theorem 7.9, by imposing natural boundary conditions (4.5) and (4.6) at Γ, and we deduces some links between c_1, c_2, c_3, and c_4, and, for any $h = 1, 2, \dots$, between $c1_h, c2_h, c3_h$, and $c4_h$. So that only two sequences of coefficients are enough to describe any local minimizer. The achievement of the leading coefficient of the main part of expansion (the so-called S.I.F., stress intensity factor) will follow in Theorem 7.11 by forcing crack-tip condition and equipartition of energy (Definition 7.10) together.

Theorem 7.9. *Assume there exists a local minimizer u of $\mathcal{E}$ in $\mathbf{R}^2$ such that $K_u = \overline{S_u}$ is the closed negative real axis.*
Then there are constants A, B s.t. $(A, B) \neq (0, 0)$ and
(7.28)

$$\begin{aligned} u(r,\theta) \;=\; r^{3/2}\, & \left(A \Big(\sin\big(\frac{\theta}{2}\big) \;-\; \frac{5}{3}\sin\big(\frac{3}{2}\theta\big)\Big) \;+\; B\Big(\cos\big(\frac{\theta}{2}\big) \;-\; \frac{7}{3}\cos\big(\frac{3}{2}\theta\big)\Big)\right) + \\ & + \sum_{h=1}^{+\infty} r^{h+\frac{3}{2}} \left(c1_h \cos\Big(\big(h+\frac{3}{2}\big)\vartheta\Big) + c2_h \sin\Big(\big(h+\frac{3}{2}\big)\vartheta\Big) + \right. \\ & \left. \qquad - \frac{2h+3}{2h+7}\, c1_h \cos\Big(\big(h-\frac{1}{2}\big)\vartheta\Big) - \frac{2h+3}{2h-5}\, c2_h \sin\Big(\big(h-\frac{1}{2}\big)\vartheta\Big)\right). \end{aligned}$$

when expressed in polar co-ordinates in $\mathbf{R}^2$ with $\theta \in (-\pi, \pi)$ and $r \in (0, +\infty)$.

More explicitly, referring to (7.26), the first term ($h = 0$) in the expansion (7.27) must have the following form

$$W_0 \;=\; \left(A\,\omega(\vartheta) \;+\; B\,w(\vartheta)\right) r^{3/2} \qquad\qquad B_\varrho \setminus \Gamma\,,$$

for two suitable **modes** ω, w,

$$\textit{Mode 1 (Jump):} \qquad \omega(\vartheta) \;=\; \left(\sin\left(\frac{\vartheta}{2}\right) - \frac{5}{3}\sin\left(\frac{3}{2}\vartheta\right)\right) \qquad \vartheta \in (-\pi, \pi)\,,$$

$$\textit{Mode 2 (Crease):} \qquad w(\vartheta) \;=\; \left(\cos\left(\frac{\vartheta}{2}\right) - \frac{7}{3}\cos\left(\frac{3}{2}\vartheta\right)\right) \qquad \vartheta \in (-\pi, \pi)\,,$$

and constants A, B, satisfying

(7.29)
$$35\,A^2 \;+\; 37\,B^2 \;=\; \frac{4\,\alpha}{\pi}\,. \;\blacksquare$$

Proof - Since the set $\overline{S_u}$ coincides with the negative real axis Γ, referring to Definitions 7.3, 7.5, 7.6 and Remark 5.7 the local minimizer u belongs to the vector space $V = A^2_\varrho \cap \{ H^2(B_\varrho) \}^{\perp H^2(B_\varrho \backslash \Gamma)} \cap \{\text{affine functions}\}^{\perp L^2(B_\varrho)}$.
Then by Lemma 7.7 u can be expanded in the real form (7.19'):

$$\begin{aligned} u \;&=\; \sum_{h=0}^{+\infty} r^{h+\frac{3}{2}} \left(c1_h \cos\left(\left(h+\frac{3}{2}\right)\vartheta\right) + c2_h \sin\left(\left(h+\frac{3}{2}\right)\vartheta\right) + \right. \\ &\qquad\qquad \left. + \, c3_h \cos\left(\left(h-\frac{1}{2}\right)\vartheta\right) + c4_h \sin\left(\left(h-\frac{1}{2}\right)\vartheta\right)\right) \;= \\ &= \sum_{h=0}^{+\infty} r^{h+\frac{3}{2}}\, \psi_h(\vartheta)\,. \end{aligned} \tag{7.30}$$

To establish a relationship with parameters in Lemma 7.8, when $h = 0$ for sake of simplicity, we choose

$$c_1 \,=\, c1_0\,, \qquad c_2 \,=\, c2_0\,, \qquad c_3 \,=\, c3_0\,, \qquad c_4 \,=\, -\,c4_0\,.$$

The Euler conditions in Theorem 4.3 on S_u ($u_{yy} = 0,\ \theta = \pm\pi,\ \ u_{yyy} + 2u_{xxy},\ \theta = \pm\pi$) entail an infinity ($h \in \mathbf{N}$) of mutually uncoupled 4-tuples of conditions on the coefficients $c1_h, c2_h, c3_h, c4_h$, which actually reduces to an infinity ($h \in \mathbf{N}$) of mutually uncoupled pairs of conditions on the coefficients $c1_h, c2_h, c3_h, c4_h$, due to the change of sign of the two branches of the complex square root along a cut of the associated Riemann surface and the fact that they correspond to a homogeneous relationship.
Some of the following computations are performed with the help of software Mathematica © (see the Notebook in the Appendix: (10.3)-(10.6) and (10.12)-(10.15)).
The most interesting are related to the first 4-tuple (say, for $h = 0$: $c_1 = c1_0, c_2 = c2_0, c_3 = c3_0, c_4 = -c4_0$): by setting

$$W_0 \;=\; r^{3/2}\left(c_4 \,\sin\left(\frac{\theta}{2}\right) + c_2\,\sin\left(\frac{3}{2}\theta\right)\right) + r^{3/2}\left(c_3\,\cos\left(\frac{\theta}{2}\right) + c_1\,\cos\left(\frac{3}{2}\theta\right)\right). \tag{7.31}$$

Let us solve this system

$$\begin{cases} (W_0)_{yy} = 0 & \theta = \pm\pi, \\ (W_0)_{yyy} + 2(W_0)_{xxy}, & \theta = \pm\pi\,. \end{cases}$$

We find twice (see (10.3),(10.4))

$$(5\,c_4 \,+\, 3\,c_2)/(4\sqrt{r}) = \,0\,,$$

by the first equation at $\pm\pi$, and twice (see (10.5),(10.6))

$$(7\,c_3\;+\;3\,c_1)/(8\,r^{3/2}) = \;0\,,$$

by the second one at $\pm\pi$.
Set, for $h = 0, 1, 2\ldots$,

$$\begin{aligned} W_h \;&= r^{h+3/2}\left(c4_h\;\sin\big((h-\frac{1}{2})\theta\big)\;+c2_h\;\sin\big((h+\frac{3}{2})\theta\big)\right)+\\ &+\;r^{h+3/2}\left(c3_h\;\cos\big((h-\frac{1}{2})\theta\big)\;+\;c1_h\;\cos\big((h+\frac{3}{2})\theta\big)\right).\end{aligned}$$

Let us solve this system

$$\begin{cases} (W_h)_{yy} = 0 & \theta = \pm\pi,\\ (W_h)_{yyy} + 2(W_h)_{xxy}, & \theta = \pm\pi \end{cases}$$

We find twice (Notebook (10.12),(10.14))

$$\frac{1}{4}\,r^{h-1/2}\,(1+2h)\,\big((2h-5)\,c4_h\;+\;(2h+3)\,c2_h)\big)\;\;=\;\;0\,,$$

by the first equation evaluated at $\vartheta = \pm\pi$ of Γ, and twice (Notebook (10.13),(10.15))

$$-\,\frac{1}{8}\,r^{h-3/2}\,(4h^2-1)\,\big((2h+7)\,c3_h\;+\;(2h+3))\,c1_h\big)\;\;=\;\;0\,,$$

by the second one at $\pm\pi$. So

$$\begin{cases} c3_h \;\;=\;\; -\;\dfrac{2h+3}{2h+7}\;c1_h \\[2ex] c4_h \;\;=\;\; -\;\dfrac{2h+3}{2h-5}\;c2_h\;. \end{cases}$$

By taking into account Remark 5.7, in Theorem 5.5 we can choose $n = 2$, $\mathbf{n} = (-1, 0)$ and $\zeta \equiv 1$ in a neighborhood of 0. Then for ε small enough, since in $\partial B_\varepsilon(0)$ we have $\eta \equiv (1, 0)$, $\mathbf{n}_\epsilon = -(\cos\vartheta, \sin\vartheta)$ and $\frac{\partial\eta}{\partial\mathbf{n}_\varepsilon} = 0$, by Lemma 7.8 we get

$$\int_{\partial B_\varepsilon(0)} |\nabla^2 u|^2 \eta\cdot\mathbf{n}_\varepsilon\,d\mathcal{H}^1(s)\;\;=\;\;k_1\;+\;O(\varepsilon)$$

$$\int_{\partial B_\varepsilon(0)} T^\varepsilon(u)\frac{\partial\eta}{\partial\mathbf{n}_\varepsilon}\cdot\nabla u\;\;d\mathcal{H}^1(s)\;\;=\;\;0,$$

$$\int_{\partial B_\varepsilon(0)} -2\left(S^\varepsilon(u)-\frac{\partial}{\partial \mathbf{n}_\varepsilon}\Delta u\right)\eta\cdot\nabla u\, d\mathcal{H}^1(s) \;=\; k_2 \,+\, O(\varepsilon)\,,$$

and, by the proof of Lemma 7.8,

$$\int_{\partial B_\varepsilon(0)} |\nabla^2 W_0|^2 \eta\cdot\mathbf{n}_\varepsilon\, d\mathcal{H}^1(s) \;=\; -\int_{-\pi}^{\pi}\varphi(\vartheta)\,\cos\vartheta\, d\vartheta \;=\; k_1$$

$$\int_{\partial B_\varepsilon(0)} T^\varepsilon(W_0)\frac{\partial\eta}{\partial\mathbf{n}_\varepsilon}\cdot\nabla W_0 \;\, d\mathcal{H}^1(s) \;=\; 0,$$

$$\int_{\partial B_\varepsilon(0)} -2\left(S^\varepsilon(W_0)-\frac{\partial}{\partial \mathbf{n}_\varepsilon}\Delta W_0\right)\eta\cdot\nabla W_0\, d\mathcal{H}^1(s) \;=\; \int_{-\pi}^{\pi}\;\xi(\vartheta)\,d\vartheta \;=\; k_2\,.$$

We deduce by Euler equations at $h=0$ that $c_3=-3\,c1\,/\,7\,,\; c_4=-3\,c_2/5$.
We emphasize that $c3_h = -\,(3{+}2h)\,c1_h\,/(7{+}2h),\; c4_h = +(3{+}2h)\,c2_h\,/\,(5{-}2h),\; h\ge 1$ while $c_1=c1_0,\; c_2=c2_0,\; c_3=c3_0,\; c_4=-c4_0,$ entail $c3 \,=\, -\dfrac{3}{7}\,c1,\; c4 \,=\, -\dfrac{3}{5}c2\,.$
By referring to the two modes ω and w we get (7.27) and, for suitable A, B (with $(A,B)\neq(0,0)$ by Lemma 7.8), referring to (7.26), the first term ($h=0$) in (7.27) must have the following form

$$W_0 \;=\; (A\,\omega(\vartheta)\,+\,B\,w(\vartheta))\;r^{3/2} \qquad\qquad B_\varrho\setminus\Gamma\,.$$

By (5.16) we must have $k_1+k_2=\alpha\neq 0$, this identity adds a condition only on $c1, c2, c3, c4$ ($h=0$) (and not on $c1_h, c2_h, c3_h$ and $c4_h,\; h>0$) because of the different weights of ε powers: by imposing $k_1+k_2=\alpha$ we get (by (10.10) and by taking into account $A=3a,\; B=3b$ in the Notebook)

$$\frac{35}{4}\,A^2\,+\,\frac{37}{4}\,B^2 \;=\; \frac{\alpha}{\pi}\,.\;\blacksquare \tag{7.32}$$

We introduce a definition in order to describe the situation when the localization of absolutely continuous part of $\mathcal{E}$ and of its surface (lineic if $n=2$) part coincide on each dilation of $B_1(0)$.

Definition 7.10. (**Energy equipartition for $\mathcal{E}$**) We say that an admissible function v (with $S_{\nabla v}=\emptyset$) fulfills equipartition of energy (around the origin) in $\mathbf{R}^n$ if

$$\int_{B_\varrho} |\nabla^2 v|^2\, d\mathbf{x} \;=\; \alpha\,\mathcal{H}^{\,n-1}(S_v\cap B_\varrho) \qquad\quad \forall\,\varrho>0\,. \tag{7.33}$$

Theorem 7.11. *Assume there exists a local minimizer u of $\mathcal{E}$ in $\mathbf{R}^2$ such that $K_u=\overline{S_u}$ is the closed negative real axis and u fulfills the equipartition of energy around the origin. Then*

$$W_0 \;=\; \pm\sqrt{\frac{\alpha}{193\,\pi}}\; r^{3/2}\left(\sqrt{21}\,\omega(\vartheta)\,\pm\, w(\vartheta)\right) \;+\; o(r^{3/2}) \tag{7.34}$$

where modes ω, w and the main part W_0 of u are the ones introduced by Lemmas 7.8, 7.9. Moreover the coefficients A, B, k_1, k_2 related to W_0 (and defined in the proofs of Lemmas 7.8, 7.9) take the values

$$A \;=\; \pm\sqrt{\frac{21\,\alpha}{193\,\pi}}\,,\quad B \;=\; \pm\sqrt{\frac{\alpha}{193\,\pi}}\,,\quad k_1 \;=\; -\,\frac{20}{193}\,\alpha\,,\quad k_2 \;=\; \frac{213}{193}\,\alpha\,. \tag{7.35}$$

Proof -
Referring to (10.9) in the Notebook we have

$$\int_{B_\varrho} |\nabla^2 W_0|^2\, d\mathbf{x} \;=\; \frac{\pi\,\varrho}{2}\left(17A^2 \,+\, 29B^2\right)\,,$$

then the equipartition for W_0 reads

$$\frac{\pi\,\varrho}{2}\left(17A^2 \,+\, 29B^2\right) \;=\; \alpha\varrho$$

We summarize crack-tip condition (7.29) and equipartition of energy (7.35) in the following system

$$\begin{cases} \dfrac{35}{4}\,A^2 \,+\, \dfrac{37}{4}\,B^2 \;=\; \dfrac{\alpha}{\pi} \\ \dfrac{17}{2}A^2 \,+\, \dfrac{29}{2}B^2 \;=\; \dfrac{\alpha}{\pi}\,. \end{cases} \tag{7.36}$$

Then, by referring to (10.11) in the Notebook taking into account $A = 3a,\ B = 3b$, there are only four admissible solutions:
(7.37)

$$\begin{cases} A \;=\; \sqrt{\dfrac{21\,\alpha}{193\,\pi}} \\ B \;=\; \sqrt{\dfrac{\alpha}{193\,\pi}} \end{cases} \quad \begin{cases} A \;=\; \sqrt{\dfrac{21\,\alpha}{193\,\pi}} \\ B \;=\; -\sqrt{\dfrac{\alpha}{193\,\pi}} \end{cases} \quad \begin{cases} A \;=\; -\sqrt{\dfrac{21\,\alpha}{193\,\pi}} \\ B \;=\; \sqrt{\dfrac{\alpha}{193\,\pi}} \end{cases} \quad \begin{cases} A \;=\; -\sqrt{\dfrac{21\,\alpha}{193\,\pi}} \\ B \;=\; -\sqrt{\dfrac{\alpha}{193\,\pi}} \end{cases}$$

For everyone of the above four choices of A and B we get the same values of k_1, k_2: by imposing $k_1 + k_2 = \alpha$ (due to crack-tip conditions) and equipartition of energy we can evaluate k_1, k_2 (see (10.10) in Notebook Appendix):

$$\begin{cases} k_1 \;=\; \pi\,(B^2 \,-\, A^2) \\ k_2 \;=\; \dfrac{3}{4}\,\pi\,\left(13\,A^2 \,+\, 11\,B^2\right) \end{cases} \tag{7.38}$$

$$
\left\{
\begin{array}{rcl}
k_1 & = & -\dfrac{20}{193}\,\alpha \\
k_2 & = & \dfrac{213}{193}\,\alpha \quad . \quad \blacksquare
\end{array}
\right.
\tag{7.39}
$$

Remark 7.12. By arguing as in the proof of Theorem 7.11 we can show that any local minimizer u of $\mathcal{E}$ in $\mathbf{R}^2$, fulfilling the equipartition of energy 7.10 around the origin and such that $K_u = \overline{S_u}$ is the closed negative real axis, has an expansion of type (7.28), where only one sequence of coefficients is free, for $h > 0$, say there is an explicit relationship between $c2_h$ and $c1_h$.

Remark 7.13. The **asymptotic energy equipartition for** $\mathcal{E}$ (a weaker assumption than energy equipartition (7.33)) :

$$
\lim_{\varepsilon \to 0} \int_{B_\varrho} |\nabla^2 v|^2 \, d\mathbf{x} \;/\; \left(\alpha\, \mathcal{H}^{n-1}(S_v \cap B_\varrho)\right) \;=\; 1
$$

would entail the exact computation of all coefficients of W_0 too, but would not provide any relationship between sequences of coefficients cj_h, $j = 1, 2$, $h > 0$. In that case previous remark fails and asymptotic energy equipartition allow two independent sequence of coefficients cj_h.

8. A candidate for minimality of $\mathcal{E}$ in $\mathbf{R}^2$.

On one hand Theorems of Sections 3, 4 and 5 prove the Euler conditions announced in [CLT7], [CLT8], and make explicit a lot of new and strong geometric constraint at crack-tip and crease-tip on local minimizers of $\mathcal{F}$ and $\mathcal{E}$.
On the other hand, by results and computations in section 7 we know the existence and structure of an expansion for any (admissible) local minimizer of $\mathcal{E}$ in $\mathbf{R}^2$ with $K_u = \overline{S_u} =$ negative real axis, with respect to an $H^2(B_\varrho \setminus \Gamma)$ orthogonal set complete in V (see Definition 7.6 and Theorem 7.7 and (7.28)), and the unique admissible choice (see (7.34)) for the coefficients of the leading term when equipartition of energy is fulfilled. In the leading term of the expansion the following two angular modes times the natural homogeneous term $r^{3/2}$ are mixed (by introducing polar co-ordinates r, ϑ in $\mathbf{R}^2$, $\theta \in (-\pi, \pi)$ and $r \in (0, +\infty)$) :

jump on Γ: $\qquad r^{3/2}\,\omega(\vartheta) \;=\; r^{3/2}\left(\sin\left(\dfrac{\vartheta}{2}\right) - \dfrac{5}{3}\sin\left(\dfrac{3}{2}\vartheta\right)\right)$,

continuous with crease on Γ: $\qquad r^{3/2}\,w(\vartheta) \;=\; r^{3/2}\left(\cos\left(\dfrac{\vartheta}{2}\right) - \dfrac{7}{3}\cos\left(\dfrac{3}{2}\vartheta\right)\right)$.

The leading term is obtained by a weighted sum of the modes times the Stress Intensity Factor $\sqrt{\dfrac{\alpha}{193\,\pi}}$ (S.I.F. evaluated in Theorem 7.11) in such a way to produce functions with jump on Γ = negative real axis and empty crease set:

$$\begin{aligned} W(r,\vartheta) &= \sqrt{\frac{\alpha}{193\,\pi}}\; r^{3/2}\left(\sqrt{21}\,\omega(\vartheta) + w(\vartheta)\right). \\ \Phi(r,\vartheta) &= \sqrt{\frac{\alpha}{193\,\pi}}\; r^{3/2}\left(\sqrt{21}\,\omega(\vartheta) - w(\vartheta)\right). \end{aligned}$$

We emphasize that (by Theorem 7.11) S.I.F. is uniquely defined (up to sign change) as soon as crack-tip extremal condition (of Theorem 5.5) and equipartition of energy (Definition 7.10) are fulfilled.
Then the candidate minimizer is expressed by the sum or difference of two modes, as follows:

$$\pm\sqrt{\frac{\alpha}{193\,\pi}}\, r^{3/2}\left(\sqrt{21}\omega(\vartheta) \pm w(\vartheta)\right), \tag{8.1}$$

more explicitly, up to the $\pm$ sign in front:
(8.2)

$$W(r,\theta) = \sqrt{\frac{\alpha}{193\,\pi}}\; r^{3/2}\left(\sqrt{21}\left(\sin\frac{\theta}{2} - \frac{5}{3}\sin\left(\frac{3}{2}\theta\right)\right) + \left(\cos\frac{\theta}{2} - \frac{7}{3}\cos\left(\frac{3}{2}\theta\right)\right)\right)$$

(8.3)

$$\Phi(r,\theta) = \sqrt{\frac{\alpha}{193\,\pi}}\; r^{3/2}\left(\sqrt{21}\left(\sin\frac{\theta}{2} - \frac{5}{3}\sin\left(\frac{3}{2}\theta\right)\right) - \left(\cos\frac{\theta}{2} - \frac{7}{3}\cos\left(\frac{3}{2}\theta\right)\right)\right).$$

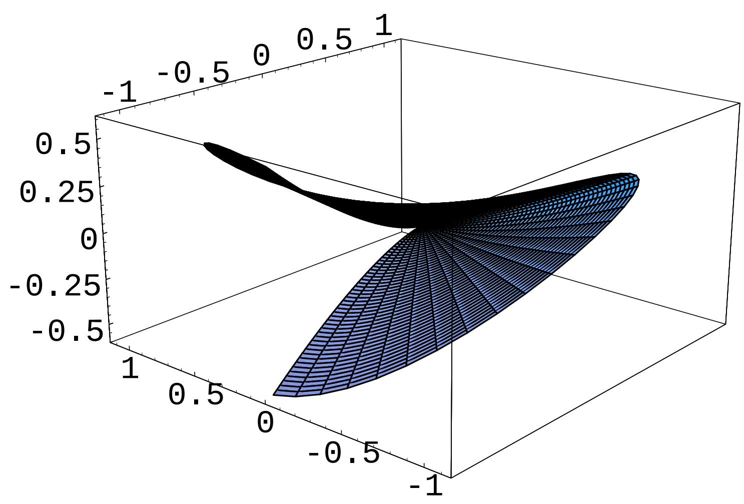

Fig.4 - Graph of W $(\alpha = 1)$

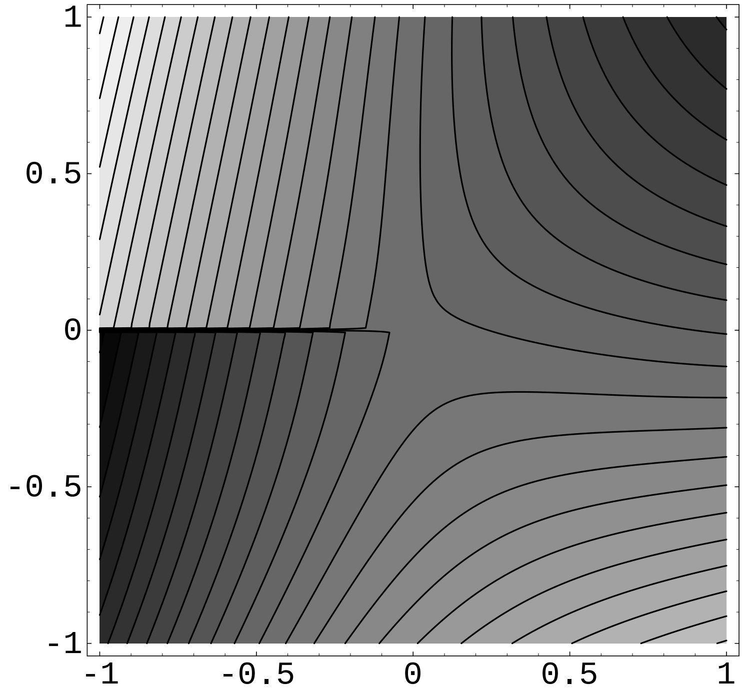

Fig.5 - Level lines of W
(picture gray levels correspond to candidate gray levels).

Candidate W, expressed by cartesian co-ordinates in $\mathbf{R}^2$ with $\mathbf{x} = (x, y)$, reads:

$$
\begin{aligned}
W(\mathbf{x}) \;=\; & \sqrt{\frac{\alpha}{386\,\pi}}\;\times \\
& \times\Bigg(\sqrt{21}\;\Big(\,\mathrm{sign}(y)\,\sqrt{x^2+y^2}\,\sqrt{\sqrt{x^2+y^2}-x}\;+ \\
& \qquad -\frac{5}{3}\,y\,\sqrt{\sqrt{x^2+y^2}+x}\;-\;\frac{5}{3}\,x\;\mathrm{sign}(y)\,\sqrt{\sqrt{x^2+y^2}-x}\Big)+ \\
& \quad +\;\sqrt{x^2+y^2}\;\sqrt{\sqrt{x^2+y^2}+x}\,+ \\
& \qquad -\frac{7}{3}\Big(x\,\sqrt{\sqrt{x^2+y^2}+x}\;-\;|y|\,\sqrt{\sqrt{x^2+y^2}-x}\Big)\Bigg)\;.
\end{aligned}
\tag{8.4}
$$

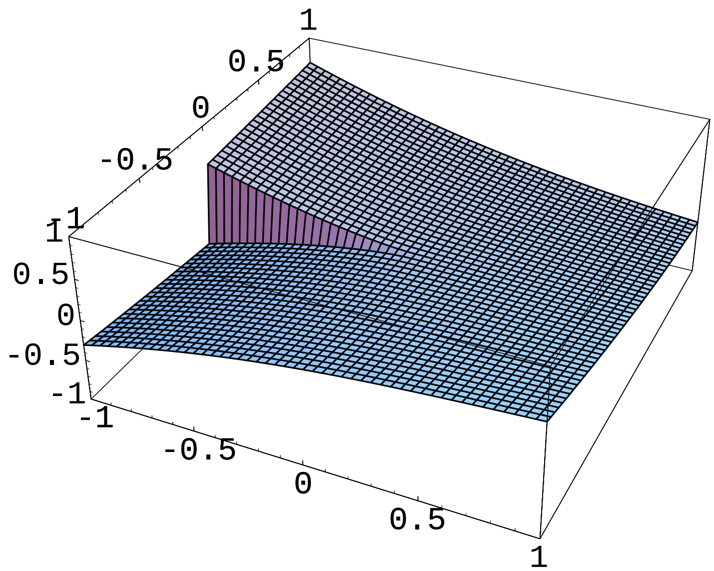

Fig.6 - Graph of W $(\alpha = 1)$

Analogously, Φ expressed by cartesian co-ordinates in $\mathbf{R}^2$ with $\mathbf{x} = (x, y)$, reads:

$$
\begin{aligned}
\Phi(\mathbf{x}) \;=\; & \sqrt{\frac{\alpha}{386\,\pi}} \times \\
& \times \Bigg(\sqrt{21}\, \Big(\operatorname{sign}(y) \sqrt{x^2+y^2} \sqrt{\sqrt{x^2+y^2} - x} \, + \\
& \qquad - \frac{5}{3}\, y \sqrt{\sqrt{x^2+y^2} + x} \,-\, \frac{5}{3}\, x \operatorname{sign}(y) \sqrt{\sqrt{x^2+y^2} - x} \Big) + \\
& \quad - \sqrt{x^2+y^2} \sqrt{\sqrt{x^2+y^2} + x} \, + \\
& \qquad + \frac{7}{3} \Big(x \sqrt{\sqrt{x^2+y^2} + x} \,-\, |y| \sqrt{\sqrt{x^2+y^2} - x} \Big) \Bigg) \; .
\end{aligned}
\tag{8.5}
$$

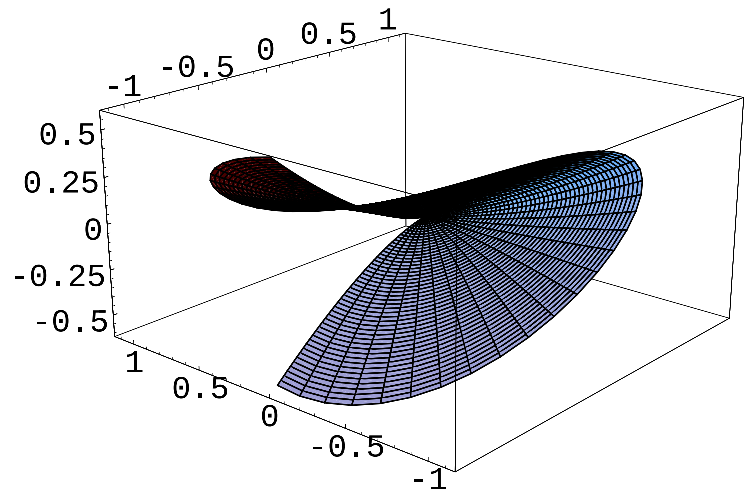

Fig.7 - Graph of Φ $(\alpha = 1)$

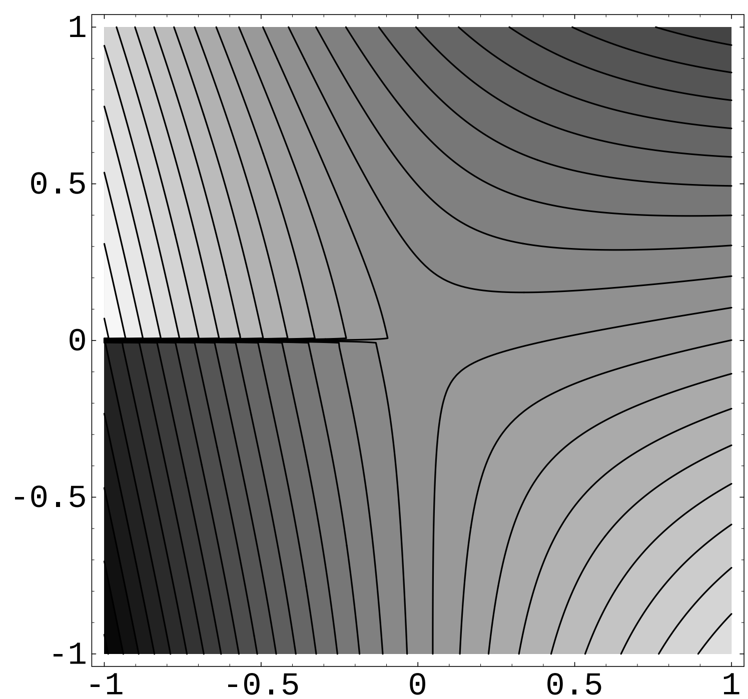

Fig.8 - Level lines of Φ
(picture gray levels correspond to candidate gray levels).

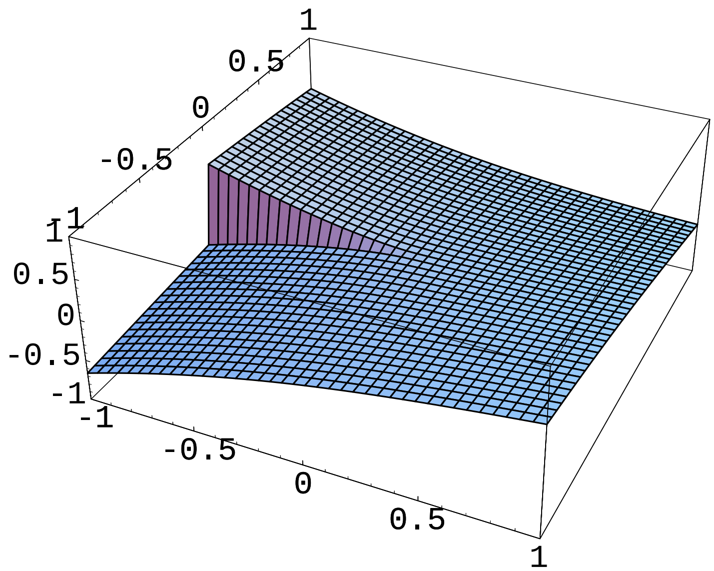

Fig.9 - Graph of Φ $(\alpha = 1)$

Notice that the candidates W, Φ are invariant (see Remark 3.2) with respect to the self-similarities $W(\cdot) \to \varrho^{-3/2}\, W(\varrho\, \cdot)\,,\ \Phi(\cdot) \to \varrho^{-3/2}\, \Phi(\varrho\, \cdot)$:

$$W(\mathbf{x}) \;=\; \varrho^{-3/2}\, W(\varrho\, \mathbf{x})\,,$$

$$\Phi(\mathbf{x}) \;=\; \varrho^{-3/2}\, \Phi(\varrho\, \mathbf{x})\,.$$

Functions (8.1) exhibit the only homogeneity in ρ (3/2 due to Lemma 7.8) compatible with minimality and fulfil the following requirements: being bi-harmonic outside the singular set, scaling invariance of the energy, all the necessary conditions on the jump set, local finiteness of energy and the proper decay rate of energy around the origin (tip of the crack) .
The following list of properties shows that W fulfills the necessary conditions of Theorems 3.4, 4.3, 5.3, Euler condition at crack-tip (Theorem 5.5) and in addition fulfils a variational principle of equi-partition of bulk and surface energy:

$$
(8.6) \qquad \begin{cases}
S_W \;=\; \text{negative real axis}, \quad S_{\nabla W} \;=\; \emptyset, \\
\Delta^2 W \;=\; 0 \quad \text{on } \mathbf{R}^2 \setminus \overline{S_W}, \\
W_{yy} \;=\; 0\,, \quad W_{yyy} + 2W_{xxy} \;=\; 0\,, \qquad \text{on both sides of } S_W, \\
W(r, \pm\pi) \;=\; \pm \dfrac{8}{3}\sqrt{\dfrac{21\,\alpha\, r^3}{193\,\pi}} \qquad\qquad \text{on } S_W, \\
\left|\, \nabla^2 W(r, \pm\pi) \,\right|^2 \;=\; \dfrac{420\,\alpha}{193\,\pi\, r} \qquad\qquad \text{on } S_W \\
[\![\, \left|\, \nabla^2 W \,\right|^2 \,]\!] \;=\; 0 \qquad\qquad\qquad\qquad \text{on } S_W\,, \\
\displaystyle\int_{B_\rho(\mathbf{0})} |\nabla^2 W|^2 dxdy \;=\; \alpha\,\varrho \;=\; \alpha\,\mathcal{H}^1\left(S_W \cap B_\varrho(\mathbf{0})\right) \qquad \forall \varrho > 0\,.
\end{cases}
$$

Properties identical to (8.6) hold true for Φ.
Moreover, due to orthogonality relationship (7.23) we have

$$
(8.7) \qquad \mathcal{E}(W, \Omega) \;=\; \mathcal{E}(\Phi, \Omega) \quad \text{for any bounded open set } \Omega.
$$

Here the analysis of Section 7 provides strong motivation for the conjecture announced in [CLT7], [CLT8] and leads to a refinement of its statement, since (8.1) must be the leading term in the asymptotic expansion of any local minimizer (see (7.34)).
As far as we know neither the calibration techniques of [ABD] nor the method used in [BD] (both successfully applied to Mumford & Shah functional to test non trivial minimizers) seem to apply to the present context of second order functionals.
Nevertheless analogously to the case of $Im\sqrt{z}$, (in the first order case [CLPP]), we announced explicitly W as an admissible non trivial candidate (with non empty jump set) to be a local minimizer for the main part $\mathcal{E}$ of Blake & Zisserman functional $\mathcal{F}$ in $\mathbf{R}^2$ (see [CLT7]).
Here we refine the conjecture as follows.

Conjecture *([CLT7]) - Functions (8.1) are local minimizers of $\mathcal{E}$ in $\mathbf{R}^2$, and there are no other nontrivial local minimizers besides W, up to (possibly independent in each mode ω and w) sign change, rigid motions of $\mathbf{R}^2$ co-ordinates and/or addition of affine functions.*

9. References

[Ad] R. ADAMS: *Sobolev Spaces,* Acadamic Press, N.Y., 1975.

[Al] ALMANSI *Sull'integrazione dell'equazione differenziale* $\Delta^{2n} = 0$ Ann.Mat. Pura Appl., **III** (1899) 1-51

[ABD] G.ALBERTI, G.BOUCHITTÉ & G.DAL MASO, *The calibration method for the Mumford-Shah functional and free-discontinuity problems,* Calc. Var. Partial Differential Equations, **16** (2003), no. 3, 299–333.

[AFM] L.AMBROSIO, L.FAINA & R.MARCH: *Variational approximation of a second order free discontinuity problem in computer vision,* SIAM J. Math. Anal., **32** (2001), 1171–1197.

[AFP] L.AMBROSIO, N.FUSCO & D.PALLARA, *Functions of bounded variations and free discontinuity problems,* Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[BoT] T.BOCCELLARI & F.TOMARELLI: *Generic uniqueness of minimizer for 1D Blake & Zisserman functional,* to appear.

[BZ] A.BLAKE & A.ZISSERMAN: *Visual Reconstruction,* The MIT Press, Cambridge, 1987.

[BD] A.BONNET & G.DAVID, *Crack-tip is a global Mumford-Shah minimizer,* Astérisque, **274** (2001).

[CFS] M.CARRIERO, A.FARINA & I.SGURA: *Image segmentation in the framework of free discontinuity problems,* in: Calculus of Variations: Topics from the Mathematical Heritage of E. De Giorgi, Quad. Mat., 14, Dept. Math., Seconda Univ. Napoli, Caserta, (2004), pp. 85–133.

[CLPP] M.CARRIERO, A.LEACI, D.PALLARA & E.PASCALI: *Euler Conditions for a Minimum Problem with Free Discontinuity Surfaces.* Preprint Dip. di Matematica n.8, Lecce (1988).

[CLT1] M.CARRIERO, A.LEACI & F.TOMARELLI: *Special Bounded Hessian and elastic-plastic plate,* Rend. Accad. Naz. Sci. XL Mem. Mat. (5) 16 (1992), 223–258.

[CLT2] M.CARRIERO, A.LEACI & F.TOMARELLI: *Free gradient discontinuities,* in "Calculus of Variations, Homogeneization and Continuum Mechanics", (Marseille 1993), 131-147, Ser.Adv.Math Appl.Sci., 18, World Sci. Publishing, River Edge, NJ, (1994), .

[CLT3] M.CARRIERO, A.LEACI & F.TOMARELLI: *A second order model in image segmentation: Blake & Zisserman functional,* in "Variational Methods for Discontinuous Structures" (Como, 1994), Progr. Nonlinear Differential Equations Appl. **25,** Birkhäuser, Basel, (1996) 57-72.

[CLT4] M.CARRIERO, A.LEACI & F.TOMARELLI: *Strong minimizers of Blake & Zisserman functional,* Ann. Scuola Norm. Sup. Pisa Cl.Sci. (4), **25** (1997), n.1-2, 257-285.

[CLT5] M.CARRIERO, A.LEACI & F.TOMARELLI, *Density estimates and further properties of Blake & Zisserman functional,* in "From Convexity to Nonconvexity", P.D.Panagiotopoulos, R.Gilbert & Pardalos Eds., Nonconvex Optim.

Appl., **55** Kluwer Acad. Publ., Dordrecht (2001), 381–392.

[CLT6] M.Carriero, A.Leaci & F.Tomarelli, *Second order functionals for image segmentation,* in: Advanced Mathematical Methods in Measurement and Instrumentation (Como 1998), Esculapio, (2000) 169–179.

[CLT7] M.Carriero, A.Leaci & F.Tomarelli: *Necessary conditions for extremals of Blake & Zisserman functional,* C. R. Math. Acad. Sci. Paris, **334** (2002) n.4, 343–348.

[CLT8] M.Carriero, A.Leaci & F.Tomarelli: *Local minimizers for a free gradient discontinuity problem in image segmentation,* in Variational Methods for Discontinuous Structures, Progr. Nonlinear Differential Equations Appl., **51,** Birkhäuser, Basel, (2002) 67-80.

[CLT9] M.Carriero, A.Leaci & F.Tomarelli, *Calculus of Variations and image segmentation,* J. of Physiology, Paris, **vol. 97,** 2-3, (2003), pp. 343-353.

[CLT10] M.Carriero, A.Leaci & F.Tomarelli, *Second Order Variational Problems with Free Discontinuity and Free Gradient Discontinuity,* in: Calculus of Variations: Topics from the Mathematical Heritage of E. De Giorgi, Quad. Mat., 14, Dept. Math., Seconda Univ. Napoli, Caserta, (2004), 135–186.

[DG] E. De Giorgi: *Free discontinuity problems in calculus of variations,* Frontiers in Pure & Appl.Math., R.Dautray Ed., North–Holland, Amsterdam, 1991, 55–61.

[DA] E. De Giorgi & L.Ambrosio: *Un nuovo tipo di funzionale del Calcolo delle Variazioni,* Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur., **82** (1988), 199–210.

[DCL] E. De Giorgi, M.Carriero & A.Leaci: *Existence theorem for a minimum problem with free discontinuity set,* Arch. Rational Mech. Anal., **108** (1989), 195–218.

[F] H.Federer: *Geometric Measure Theory,* Springer, Berlin, 1969.

[FLP] I.Fonseca, G.Leoni & R.Paroni: *On hessian matrices in the space BH,* Commun.Contemp.Math., **7** (2005), 401–420.

[Gia] M.Giaquinta: *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems,* Ann. Math. Stud., Princeton U. P., 1983.

[Giu] E.Giusti: *Minimal Surfaces and Functions of Bounded Variations,* Birkhäuser Monographs in Mathematics, Boston (1984).

[Li] J.L.Lions, *Problèmes aux limites en théorie des distributions* Acta Math. **94** (1955) 13-153.

[Ma] R.March, *Visual reconstruction with discontinuities using variational methods,* Image and Vision Computing **10** (1992) 30-38.

[MoSo] J.M.Morel & S.Solimini, *Variational Models in Image Segmentation,* PNLDE, **N.14,** Birkhäuser, Basel, (1995).

[MSh] D.Mumford & J.Shah: *Optimal approximation by piecewise smooth functions and associated variational problems,* Comm. Pure Appl. Math., **XLII**

(1989), 577–685.

[P] D.Pallara: *Some new results on functions of bounded variation,* Rend. Accad. Naz. Sci. dei XL, Mem Mat., **XIV** (1990), 295-321.

[S] L. Simon, *Lectures on Geometric Measure Theory,* Proc. of the Center for Mathematical Analysis, **n.3,** Australian National University (1983), Canberra.