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Existence and stability of entire solutions to a semilinear fourth order elliptic problem

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Abstract

For a semilinear biharmonic equation with exponential nonlinearity, we study the existence and the asymptotic behavior of entire solutions. Furthermore, their stability and stability outside a compact set of $\mathbb{R}^n$ ($n \geq 2$) is discussed in any space dimension n.

Mathematics Subject Classification: 35J60, 35B08, 35B35

Keywords: biharmonic equations, radial solutions, stability

1 Introduction

The second order elliptic equation

$$-\Delta u = e^u \quad \text{in} \quad \mathbb{R}^n, \qquad n \geq 1, \tag{1}$$

describes problems of thermal self-ignition [21], diffusion phenomena induced by nonlinear sources [25] or a ball of isothermal gas in gravitational equilibrium as proposed by lord Kelvin [11]. The properties of radial solutions to (1) in the ball are related to the stability of the solutions in $\mathbb{R}^n$, see [7, 24, 32]. The stability of $C^2(\mathbb{R}^n)$ solutions to (1) is studied in [13, 18].

The purpose of the present paper is to give a contribution to a problem formulated by P.L. Lions [23, Section 4.2 (c)], namely: *Is it possible to obtain a description of the solution set for higher order semilinear equations associated to exponential nonlinearities?*

We consider entire solutions to the semilinear biharmonic equation

$$\Delta^2 u = e^u \quad \text{in} \quad \mathbb{R}^n, \qquad n \geq 1, \tag{2}$$

i.e. solutions u which exist for all $x \in \mathbb{R}^n$. As we shall see, the existence and the stability of these solutions strongly depend on the space dimension n. This is well-established in the ball where radial solutions are widely studied see [6, 15, 16]. The existence and the asymptotic behavior of solutions to the fourth order problem (2) have been partially studied in the so-called "conformal dimension" $n = 4$ (see [12, 27, 34]) and in "supercritical dimensions" $n \geq 5$ (see [5]). More recently, first characterizations to the stability properties of these solutions were determined in [33]. In the present paper, we first prove nonexistence of entire solutions to (2) in the one-dimensional case $n = 1$; this is in striking contrast with the second order equation, see Theorem 1 and the subsequent comment. Next, we turn to the "subcritical dimensions" $n = 2, 3$. When $n = 2$, we show that (2) admits no radial entire solution. On the other hand, if $n = 3$ there

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exist infinitely many radial entire solutions to (2) which are stable outside compact sets of $\mathbb{R}^n$; this result complements [33] where it is shown that no solutions to (2) are fully stable if $2 \leq n \leq 4$. In the conformal dimension $n = 4$ the existence and behavior of solutions to (2) was studied in [27, 34]; we classify these solutions according to their stability outside compact sets of $\mathbb{R}^n$, complementing again the results in [33]. In the supercritical dimensions $n \geq 5$ we take advantage of the analysis performed in [5] and we prove different behaviors in "low dimensions" $5 \leq n \leq 12$ and in "high dimensions" $n \geq 13$. In the first case we show that there exist both unstable solutions and solutions which are stable outside compact sets. In the second case we prove that any radially symmetric solution to (2) is fully stable.

This paper is organized as follows. In Section 2 we establish existence and nonexistence results for solutions to (2) and we study their asymptotic behavior as $|x| \to \infty$. In Section 3 we study the stability of radial solutions to (2). To this end, we need some Hardy-Rellich inequalities which are stated in Section 4. The remaining part of the paper is devoted to the proofs.

2 Existence and behavior of entire solutions

In the 1-dimensional case we have nonexistence of solutions.

Theorem 1. *There exists no global solution $u \in C^4(\mathbb{R})$ to the equation*

$$u''''(r) = e^{u(r)} \qquad r \in \mathbb{R}. \tag{3}$$

This result is in striking contrast with the corresponding second order ode

$$-u''(r) = e^{u(r)} \qquad r \in \mathbb{R}, \tag{4}$$

for which any local solution is global. To see this, it suffices to notice that any local solution to (4) is concave so that a blow up in finite time can occur only if $u \to -\infty$; but in such case $u'' \to 0$, contradiction. For instance, for any $c > 0$ the function

$$u(r) = 2r + \log(8c) - 2\log(1 + ce^{2r})$$

solves (4).

In the multidimensional case $n \geq 2$, any radial solution $u = u(|x|) \in C^4(\mathbb{R}^n)$ to (2) is even with respect to the $r = |x|$-variable and satisfies $u'(0) = u'''(0) = 0$. Then, for all $\alpha, \beta \in \mathbb{R}$ we are lead to consider the solutions $u_{\alpha,\beta}$ to the initial value problem

$$\begin{cases} \Delta^2 u(r) = e^{u(r)} \qquad \text{for } r \in [0, R(\alpha,\beta)) \\ u(0) = \alpha\,, \quad \Delta u(0) = \beta\,, \quad u'(0) = (\Delta u)'(0) = 0\,, \end{cases} \tag{5}$$

where $[0, R(\alpha,\beta))$ is the maximal interval of existence. If $R(\alpha,\beta) = +\infty$ then $u_{\alpha,\beta}$ is a global solution to (5) and, in turn, a radial entire solution to (2). Note that the solutions to (5) with different initial values α and γ are linked by the following rescaling

$$u_{\alpha, e^{\frac{\alpha-\gamma}{2}}\beta}(r) = u_{\gamma,\beta}(e^{\frac{\alpha-\gamma}{4}} r) + \alpha - \gamma \qquad \forall \alpha,\beta,\gamma \in \mathbb{R}\,. \tag{6}$$

The following statement is essentially [5, Theorem 2], where it was proved in the supercritical case $n \geq 5$:

Theorem 2. *For $n \geq 2$, local solutions to* (5) *satisfy*

$$u_{\alpha,\beta}(r) \geq \alpha + \frac{\beta}{2n} r^2 \quad \textit{for all } r \in [0, R(\alpha,\beta))\,. \tag{7}$$

Furthermore, for any $\alpha \in \mathbb{R}$ there exists $\beta_0 = \beta_0(\alpha) \in [-\infty, \min\{0, -\alpha\})$ such that

(i) *if* $\beta \geq 0$, *then* $R(\alpha,\beta) < +\infty$ *and* $u'_{\alpha,\beta}(r) > 0$ *on* $(0, R(\alpha,\beta))$;

(ii) *if* $\beta_0 < \beta < 0$, *then* $R(\alpha,\beta) < +\infty$ *and there exists a unique* $R_0 \in (0, R(\alpha,\beta))$ *such that* $u'_{\alpha,\beta}(R_0) = 0$, $u'_{\alpha,\beta}(r) < 0$ *on* $(0, R_0)$ *and* $u'_{\alpha,\beta}(r) > 0$ *on* $(R_0, R(\alpha,\beta))$;

(iii) *if* $\beta \leq \beta_0$, *then* $R(\alpha,\beta) = +\infty$ *and* $u'_{\alpha,\beta}(r) < 0$ *on* $(0,+\infty)$. *Furthermore, if* $\beta < \beta_0$ *there holds*

$$u_{\alpha,\beta}(r) \leq \alpha - \frac{\beta_0 - \beta}{2n} r^2 \quad \text{for all } r \in [0,+\infty)\,. \tag{8}$$

Theorem 2 states that local solutions $u_{\alpha,\beta}$ to (5) are defined globally only if $\beta \leq \beta_0(\alpha)$. If $\beta_0 = -\infty$, then no global solution exists and case (iii) never occurs. This happens if $n = 2$.

Theorem 3. *If* $n = 2$, *then problem* (5) *admits no global solutions. Hence,* (2) *admits no radial entire solutions.*

If $\beta_0 > -\infty$, Theorem 2 states the existence of a separatrix u_{α,β_0}, namely a global solution which "separates" finite time blow-up solutions from globally defined solutions. According to (7) and (8), all global solutions except the separatrix decay quadratically to $-\infty$, regardless of the space dimension n. On the contrary, the behavior of the separatrix strongly depends on the dimension.

Theorem 4. *For every* $\alpha \in \mathbb{R}$ *let* $\beta_0 = \beta_0(\alpha)$ *be as defined in Theorem 2;*

(i) *if* $n = 3$, *then* $\beta_0 \geq -63e^{\alpha/2}/\sqrt{8}$, $u_{\alpha,\beta}$ *is concave for any* $\beta \leq \beta_0$ *and there exist* $C, R > 0$ *such that*

$$u_{\alpha,\beta_0}(r) \leq -Cr \quad \text{for all } r \geq R\,; \tag{9}$$

(ii) *if* $n = 4$, *then* $\beta_0 = -4e^{\alpha/2}/\sqrt{6}$ *and*

$$u_{\alpha,\beta_0}(r) = \alpha - 4\log\left(1 + \frac{e^{\alpha/2}}{8\sqrt{6}} r^2\right)\,; \tag{10}$$

(iii) *if* $n \geq 5$, *then* $\beta_0 \geq -4ne^{\alpha/2}$ *and*

$$\lim_{r\to+\infty} \left(u_{\alpha,\beta_0}(r) + 4\log r\right) = \log 8(n-2)(n-4)\,. \tag{11}$$

Statement (iii) was proved in [5] whereas statement (ii) is a consequence of [27, Theorem 1.1], see also Proposition 1 below. Statement (i) is new and shows that in the subcritical dimension $n = 3$ the decay to $-\infty$ of the separatrix u_{α,β_0} is much faster than logarithmic. Clearly, this phenomenon is not visible for the second order equation (1) since there are no subcritical dimensions in this case. We also point out that (9) may not be sharp.

Problem 1. Determine the exact asymptotic behavior of the separatrix u_{α,β_0} in the subcritical dimension $n = 3$.

In the conformal dimension $n = 4$, Lin [27] classified the (possibly nonradial) solutions to (2) such that $e^u \in L^1(\mathbb{R}^4)$. More precisely, he proved

Proposition 1. *Let* u *be a solution to* (2) *such that* $e^u \in L^1(\mathbb{R}^4)$ *and let* $\gamma := \frac{1}{32\pi^2}\int_{\mathbb{R}^4} e^u\,dx$. *The following statements hold true.*

(i) *We have $\gamma \leq 2$ and, after an orthogonal transformation, u can be represented by*

$$u(x) = -\sum_{j=1}^{4} a_j (x_j - x_j^0)^2 - 4\gamma \log |x| + c_0 + o(1) \quad \text{as } |x| \to \infty, \tag{12}$$

for some $a_j \geq 0$, $c_0 \in \mathbb{R}$ and $x^0 \in \mathbb{R}^4$. If $a_1 = a_2 = a_3 = a_4$, then u is radially symmetric with respect to x^0.

(ii) *If $u(x) = o(|x|^2)$ as $x \to \infty$ then $\gamma = 2$, $a_j = 0$ for all j, and*

$$u(x) = 4 \log \frac{2\sqrt[4]{24}\lambda}{(1 + \lambda^2 |x - x^0|^2)},$$

for some $\lambda > 0$ and $x^0 \in \mathbb{R}^4$.

In [34], given $x^0 \in \mathbb{R}^4$, $\gamma \in (0, 2)$ and $a_j > 0$, the existence of solutions satisfying (12) was proved. Here, by Theorem 4 we deduce

Corollary 1. *Let $n = 3, 4$. Then any radial entire solution u to (2) satisfies $e^u \in L^1(\mathbb{R}^n)$. Moreover, if $n = 4$ then Proposition 1 applies and*

$$u(r) = -ar^2 - 4\gamma \log r + O(1) \quad \text{as } r \to +\infty$$

with $a = a_1 = a_2 = a_3 = a_4 \geq 0$ and $\gamma \in (0, 2]$ as defined in Proposition 1.

We conclude this section with a nonexistence result for (possibly nonradial) solutions bounded from below. Any solution to (2) such that $u \geq m$ for some $m \in \mathbb{R}$, satisfies the inequality

$$\Delta^2 u \geq K(m)|u|^q \quad \text{in } \mathbb{R}^n$$

with $q = 2$ if $1 \leq n \leq 8$, and $q = \frac{n}{n-4}$ if $n > 8$, for a suitable $K(m) > 0$. Then, from [30, Theorem 4.1], we infer

Proposition 2. *For any $n \geq 2$, problem (2) admits no entire solution bounded from below.*

3 Stability of the solutions

We start by explaining what we mean by stability.

Definition 1. *A solution $u \in C^4(\mathbb{R}^n)$ to (2) is stable if*

$$\int_{\mathbb{R}^n} |\Delta\varphi|^2 \, dx - \int_{\mathbb{R}^n} e^u \varphi^2 \, dx \geq 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n). \tag{13}$$

A solution $u \in C^4(\mathbb{R}^n)$ to (2) is stable outside the compact set K if

$$\int_{\mathbb{R}^n \setminus K} |\Delta\varphi|^2 \, dx - \int_{\mathbb{R}^n \setminus K} e^u \varphi^2 \, dx \geq 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n \setminus K). \tag{14}$$

By [33, Theorem 6] we know

Proposition 3. *Let $2 \leq n \leq 4$, then equation (2) admits no stable solutions.*

However, we can prove that in subcritical dimensions the solutions found in Theorem 2 for $\beta \leq \beta_0$ are stable outside suitable compact sets:

Theorem 5. *Let $n = 3$ or $n = 4$ and let u be a radial entire solution to* (2). *Then u is stable outside a compact set.*

When $n = 2$ (the conformal dimension for the second order equation), all the $C^2(\mathbb{R}^2)$ solutions to (1) stable outside a compact set, have been completely characterized in [18, Theorem 3]. Similarly, by Definition 1, we have that any solution $u \in C^4(\mathbb{R}^n)$ to (2), stable outside a compact set, satisfies $e^u \in L^1(\mathbb{R}^n)$. On the other hand, in the conformal dimension $n = 4$, by Proposition 1, any solution u to (2) such that $e^u \in L^1(\mathbb{R}^4)$, can be represented as in (12). If $a_1 = a_2 = a_3 = a_4$ holds in (12), namely u is radially symmetric, by Theorem 5 we know that u is stable outside a compact set. On the other hand, from [34], we know that there exist solutions to (2) in the form (12) with $a_j > 0$ for any $1 \leq j \leq 4$, not necessarily radially symmetric. Arguing as in the proof of Theorem 5, these are stable outside a compact set.

Problem 2. Study the stability outside compact sets of all the functions represented in (12). This appears challenging when some (but not all) of the a_j vanish and for small γ.

If $n \geq 5$, due to the stability behavior of the separatrix, a further "critical" dimension arises. Namely, we prove

Theorem 6. *Let $n \geq 5$, $\beta_0 = \beta_0(\alpha)$ be as defined in Theorem 2 and $u_{\alpha,\beta}$ be a solution to* (5). *The following statements hold*

(i) *if $5 \leq n \leq 12$, then $u_{\alpha,\beta}$ is stable outside a compact set for every $\beta < \beta_0$ while u_{α,β_0} is unstable outside every compact set and, in particular, it is unstable;*

(ii) *if $n \geq 13$, then $u_{\alpha,\beta}$ is stable for every $\beta \leq \beta_0$.*

In Lemma 9 below one can find estimates of the "size" of compact sets outside of which one has stability of $u_{\alpha,\beta}$. Statement (i) of Theorem 6 is surprising if compared with [13, Theorem 1], in the second order case, where the authors show that (1) admits no $C^2(\mathbb{R}^n)$ solutions stable outside a compact set if $3 \leq n \leq 9$.

The dimension $n = 13$ is somehow "critical" also for Dirichlet or Navier boundary value problems associated to (2) in the ball, see [6, 9, 15], although this fact is strongly related to the boundary conditions considered, see [8, Theorem 7].

In Table 1 we summarize the stability results obtained in this section.

	$n = 1, 2$	$n = 3, 4$	$5 \leq n \leq 12$	$n \geq 13$
$u_{\alpha,\beta}$ stable $\forall \beta < \beta_0$	$\nexists$	NO	?	YES
u_{α,β_0} stable	$\nexists$	NO	NO	YES
$u_{\alpha,\beta}$ stable outside a compact $\forall \beta < \beta_0$	$\nexists$	YES	YES	YES
u_{α,β_0} stable outside a compact	$\nexists$	YES	NO	YES

Table 1: The stability of global solutions $u_{\alpha,\beta}$ to (5) as β and n vary.

Problem 3. For $5 \leq n \leq 12$, study the stability of $u_{\alpha,\beta}$ when $\beta < \beta_0$.

4 Some Hardy-Rellich inequalities in exterior domains

For $n \geq 5$, a useful tool to check conditions (13) and (14) is the so-called *Hardy-Rellich inequality* [31] (see also [4, 14, 29]):

$$\int_{\mathbb{R}^n} |\Delta u|^2 \, dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{u^2}{|x|^4} \, dx \qquad \forall \, u \in C_c^\infty(\mathbb{R}^n) \, , \tag{15}$$

where the constant $\frac{n^2(n-4)^2}{16}$ is optimal, in the sense that it is the largest possible. Inequality (15) is the second order version of the celebrated first order Hardy inequality which holds for $n \geq 3$:

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \, dx \qquad \forall \, u \in C_c^\infty(\mathbb{R}^n) \, . \tag{16}$$

We first show that the constant in (15) is also optimal in exterior domains:

Proposition 4. *Let B_R be the ball in $\mathbb{R}^n$ of radius $R > 0$ ($n \geq 5$) centered at the origin. The following inequality holds*

$$\int_{\mathbb{R}^n \setminus B_R} |\Delta u|^2 \, dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n \setminus B_R} \frac{u^2}{|x|^4} \, dx \qquad \forall \, u \in C_c^\infty(\mathbb{R}^n \setminus \overline{B}_R) \, , \tag{17}$$

and the constant $\dfrac{n^2(n-4)^2}{16}$ *is optimal.*

Proof. Let $\eta \in C^\infty(\mathbb{R}_+)$ be such that $\eta(t) = 0$ for $0 \leq t \leq 1$ and $\eta(t) = 1$ for $t \geq 2$. Then, for a given $u \in C_c^\infty(\mathbb{R}^n)$ and a given integer $k \geq 1$, we set $u_k(x) := \eta(k|x|)u(x) \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$. Since $n \geq 5$, one has that $u_k \to u$, as $k \to +\infty$, with respect to the norm $\|u\| := \left(\int_{\mathbb{R}^n} |\Delta u|^2 \, dx \right)^{1/2}$. Hence, $C_c^\infty(\mathbb{R}^n \setminus \{0\})$ is dense in $C_c^\infty(\mathbb{R}^n)$ with respect to this norm and this fact shows that the constant in (15) is optimal also for test functions in $C_c^\infty(\mathbb{R}^n \setminus \{0\})$.

Thanks to the invariance of both sides of inequality (17) with respect to the rescaling $u_\lambda(x) = \lambda^{\frac{n-4}{2}} u(\lambda x)$, the result follows immediately. $\square$

In dimensions $n \leq 4$, (15) is no longer true and one should also expect a different statement for Proposition 4. See [2] for Hardy-Rellich inequalities on bounded domains in the conformal dimension $n = 4$. Here we prove:

Proposition 5. *Let B_R be the ball of radius $R > 0$, centered at the origin in $\mathbb{R}^n$ ($2 \leq n \leq 4$). The following Hardy-type inequalities hold true*

$$4 \int_{\mathbb{R}^2 \setminus B_R} |\Delta u|^2 \, dx \geq \int_{\mathbb{R}^2 \setminus B_R} \frac{u^2}{|x|^4 \log^2 |x/R|} \, dx \qquad \forall u \in C_c^\infty(\mathbb{R}^2 \setminus \overline{B}_R) \, , \tag{18}$$

$$16 \int_{\mathbb{R}^3 \setminus B_R} |\Delta u|^2 \, dx \geq \int_{\mathbb{R}^3 \setminus B_R} \frac{u^2}{|x|^4} \, dx \qquad \forall u \in C_c^\infty(\mathbb{R}^3 \setminus \overline{B}_R) \, , \tag{19}$$

$$4 \int_{\mathbb{R}^4 \setminus B_R} |\Delta u|^2 \, dx \geq \int_{\mathbb{R}^4 \setminus B_R} \frac{u^2}{|x|^4 \log^2 |x/R|} \, dx \qquad \forall u \in C_c^\infty(\mathbb{R}^4 \setminus \overline{B}_R) \, . \tag{20}$$

Proof. Thanks to scaling it suffices to prove the inequalities for $R = 1$. Denote by B the unit ball centered at the origin. Let $u \in C_c^\infty(\mathbb{R}^n \setminus \overline{B})$ and let $\alpha \geq 0$ to be fixed later. By the divergence Theorem we have

$$\int_{\mathbb{R}^n \setminus B} \operatorname{div} \left(\frac{x u^2}{|x|^4 \log^\alpha |x|} \right) dx = 0.$$

Since

$$\operatorname{div}\left(\frac{x}{|x|^4\log^\alpha|x|}\right)=\frac{n-4}{|x|^4\log^\alpha|x|}-\frac{\alpha}{|x|^4\log^{\alpha+1}|x|},$$

we readily obtain

$$\int_{\mathbb{R}^n\setminus B}\frac{u(\nabla u\cdot x)}{|x|^4\log^\alpha|x|}\,dx=\frac{4-n}{2}\int_{\mathbb{R}^n\setminus B}\frac{u^2}{|x|^4\log^\alpha|x|}\,dx+\frac{\alpha}{2}\int_{\mathbb{R}^n\setminus B}\frac{u^2}{|x|^4\log^{\alpha+1}|x|}\,dx\,. \tag{21}$$

Assume first that $n=2$. Take $u\in C_c^\infty(\mathbb{R}^2\setminus\overline{B})$ and fix $\alpha=2$. Then by dropping the last term in (21) and using Hölder inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B}\frac{u^2}{|x|^4\log^2|x|}\,dx&\le\int_{\mathbb{R}^2\setminus B}\frac{u(\nabla u\cdot x)}{|x|^4\log^2|x|}\,dx\\&\le\int_{\mathbb{R}^2\setminus B}\frac{|\nabla u|}{|x|\log|x|}\frac{|u|}{|x|^2\log|x|}\,dx\le\left(\int_{\mathbb{R}^2\setminus B}\frac{|\nabla u|^2}{|x|^2\log^2|x|}\,dx\right)^{1/2}\left(\int_{\mathbb{R}^2\setminus B}\frac{u^2}{|x|^4\log^2|x|}\,dx\right)^{1/2}\end{aligned}$$

and hence

$$\int_{\mathbb{R}^2\setminus B}\frac{u^2}{|x|^4\log^2|x|}\,dx\le\int_{\mathbb{R}^2\setminus B}\frac{|\nabla u|^2}{|x|^2\log^2|x|}\,dx=\sum_{i=1}^2\int_{\mathbb{R}^2\setminus B}\frac{\left(\frac{\partial u}{\partial x_i}\right)^2}{|x|^2\log^2|x|}\,dx\,. \tag{22}$$

At this point we recall a Hardy-type inequality in dimension $n=2$, see [1] and [18, proof of Theorem 3]. For any $R>0$, we have

$$\int_{\mathbb{R}^2\setminus B_R}\frac{w^2}{|x|^2\log^2|x|}\,dx\le4\int_{\mathbb{R}^2\setminus B_R}|\nabla w|^2dx\qquad\forall w\in C_c^\infty(\mathbb{R}^2\setminus\overline{B}_R). \tag{23}$$

We apply (23) to the partial derivatives of u so that, by (22), we obtain

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B}\frac{u^2}{|x|^4\log^2|x|}\,dx&\le4\sum_{i=1}^2\int_{\mathbb{R}^2\setminus B}\left|\nabla\left(\frac{\partial u}{\partial x_i}\right)\right|^2dx\\&=4\sum_{i,j=1}^2\int_{\mathbb{R}^2\setminus B}\left(\frac{\partial^2u}{\partial x_i\partial x_j}\right)^2dx=4\int_{\mathbb{R}^2\setminus B}|\Delta u|^2dx\,.\end{aligned}$$

The last equality follows with two integrations by parts. This completes the proof of (18).

Assume now that $n=3$. Take $u\in C_c^\infty(\mathbb{R}^3\setminus\overline{B})$ and fix $\alpha=0$ in (21) to obtain

$$\int_{\mathbb{R}^3\setminus B}\frac{u(\nabla u\cdot x)}{|x|^4}\,dx=\frac12\int_{\mathbb{R}^3\setminus B}\frac{u^2}{|x|^4}\,dx\,.$$

Therefore, by Hölder inequality, we have

$$\frac12\int_{\mathbb{R}^3\setminus B}\frac{u^2}{|x|^4}\,dx\le\int_{\mathbb{R}^3\setminus B}\frac{|\nabla u|}{|x|}\frac{|u|}{|x|^2}\,dx\le\left(\int_{\mathbb{R}^3\setminus B}\frac{|\nabla u|^2}{|x|^2}\,dx\right)^{1/2}\left(\int_{\mathbb{R}^3\setminus B}\frac{u^2}{|x|^4}\,dx\right)^{1/2}\,. \tag{24}$$

By (24), by applying (16) in dimension $n=3$ to the partial derivatives of u, and the argument of the previous case, it follows that

$$\frac14\int_{\mathbb{R}^3\setminus B}\frac{u^2}{|x|^4}\,dx\le\int_{\mathbb{R}^3\setminus B}\frac{|\nabla u|^2}{|x|^2}\,dx\le4\int_{\mathbb{R}^3\setminus B}|\Delta u|^2dx\,.$$

This completes the proof of (19).

Finally, we consider the case where $n=4$. Let $u \in C_c^\infty(\mathbb{R}^4 \setminus \overline{B})$ and fix $\alpha = 1$ in (21) to obtain

$$\int_{\mathbb{R}^4\setminus B} \frac{u(\nabla u \cdot x)}{|x|^4 \log|x|}\, dx = \frac{1}{2}\int_{\mathbb{R}^n\setminus B} \frac{u^2}{|x|^4\log^2|x|}\, dx\, .$$

Let us estimate the left hand side by

$$\begin{aligned}\int_{\mathbb{R}^4\setminus B} \frac{u(\nabla u \cdot x)}{|x|^4 \log|x|}\, dx &\leq \int_{\mathbb{R}^4\setminus B} \frac{|u|}{|x|^2\log|x|}\frac{|\nabla u|}{|x|}\, dx \\ &\leq \left(\int_{\mathbb{R}^4\setminus B} \frac{|u|^2}{|x|^4\log^2|x|}\, dx\right)^{1/2} \left(\int_{\mathbb{R}^4\setminus B} \frac{|\nabla u|^2}{|x|^2}\, dx\right)^{1/2}\end{aligned}$$

so that

$$\int_{\mathbb{R}^n\setminus B} \frac{u^2}{|x|^4\log^2|x|}\, dx \leq 4\int_{\mathbb{R}^4\setminus B} \frac{|\nabla u|^2}{|x|^2}\, dx \leq 4\int_{\mathbb{R}^4\setminus B} |\Delta u|^2\, dx$$

where the last inequality follows from (16) in dimension $n=4$ applied to the partial derivatives of u. $\square$

Contrary to Proposition 4, we do not know if the constants in Proposition 5 are optimal. We recently learned that results in this direction were found in [10].

5 Proof of Theorem 1

For contradiction, assume that there exists a global solution u to (3). Then by (3) we infer that the map $r \mapsto u''(r)$ is strictly convex so that either $\lim_{r\to-\infty} u''(r) = +\infty$ or $\lim_{r\to+\infty} u''(r) = +\infty$ (or both!). By possibly performing the change of variable $r \mapsto -r$ we may assume that the latter occurs. By this and by using (3), we also have

$$\lim_{r\to+\infty} u(r) = +\infty \qquad \text{and} \qquad \lim_{r\to+\infty} u'''(r) = +\infty. \tag{25}$$

Then there exists $R \in \mathbb{R}$ such that

$$u''''(r) = e^{u(r)} \geq u(r)^2\, , \qquad u'''(r) \geq 0 \qquad \forall r \geq R. \tag{26}$$

Since the problem is autonomous, we may assume that $R=0$. We now apply the test function method developed by Mitidieri-Pohožaev [30]. More precisely, fix $\rho > 0$ and a nonnegative function $\phi \in C_c^4([0,\infty))$ such that

$$\phi(r) = \begin{cases} 1 & \text{for } r \in [0,\rho] \\ 0 & \text{for } r \geq 2\rho\, . \end{cases}$$

In particular, these properties imply that

$$\phi(0) = 1, \qquad \phi'(0) = \phi''(0) = \phi'''(0) = \phi(2\rho) = \phi'(2\rho) = \phi''(2\rho) = \phi'''(2\rho) = 0.$$

Hence, multiplying inequality (26) by $\phi(r)$, integrating four times by parts, and recalling (26) yields

$$\int_\rho^{2\rho} \phi''''(r)u(r)dr = \int_0^{2\rho} \phi''''(r)u(r)dr \geq \int_0^{2\rho} u(r)^2\phi(r)dr + u'''(0) \geq \int_0^{2\rho} u(r)^2\phi(r)dr. \tag{27}$$

For further estimates, we make use of Young's inequality in the following form:

$$u\phi'''' = u\phi^{1/2}\frac{\phi''''}{\phi^{1/2}} \leq \frac{1}{2}\left(u^2\phi + \frac{|\phi''''|^2}{\phi}\right).$$

Then (27) becomes

$$\int_{\rho}^{2\rho} \frac{\phi''''(r)^2}{\phi(r)} dr \geq \int_0^{\rho} u(r)^2 dr. \tag{28}$$

We now choose $\phi(r) = \phi_\rho(r) = \phi_0(\frac{r}{\rho})$, where $\phi_0 \in C_c^4([0,\infty))$, $\phi_0 \geq 0$ and

$$\phi_0(\tau) = \begin{cases} 1 & \text{for } \tau \in [0,1] \\ 0 & \text{for } \tau \geq 2\,. \end{cases}$$

As noticed in [30], there exists a function ϕ_0 in such class satisfying moreover

$$\int_1^2 \frac{\phi_0''''(\tau)^2}{\phi_0(\tau)} d\tau =: A < \infty.$$

Then, thanks to a change of variables in the integrals, (28) yields

$$A\rho^{-7} = \rho^{-7} \int_1^2 \frac{\phi_0''''(\tau)^2}{\phi_0(\tau)} d\tau = \rho^{-8} \int_{\rho}^{2\rho} \frac{\phi_0''''(\frac{r}{\rho})^2}{\phi_0(\frac{r}{\rho})} dr = \int_{\rho}^{2\rho} \frac{\phi''''(r)^2}{\phi(r)} dr \geq \int_0^{\rho} u(r)^2 dr \qquad \forall \rho > 0.$$

Letting $\rho \to \infty$, the previous inequality contradicts (25). $\square$

6 Proof of Theorem 2

We first recall that (2) written in the radial variable becomes

$$u''''(r) + \frac{2(n-1)}{r} u'''(r) + \frac{(n-1)(n-3)}{r^2} u''(r) - \frac{(n-1)(n-3)}{r^3} u'(r) = e^{u(r)}. \tag{29}$$

Then we state some preliminary lemmas. In the sequel a crucial tool will be the following comparison principle by McKenna and Reichel, see [28, Lemma 3.2] and [5, Lemma 2].

Lemma 1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and monotonically increasing. Let $u, v \in C^4([0,R))$ be such that*

$$\begin{cases} \forall r \in [0,R) : \Delta^2 u(r) - f(u(r)) \geq \Delta^2 v(r) - f(v(r))\,, \\ u(0) \geq v(0)\,, \quad u'(0) = v'(0) = 0\,, \quad \Delta u(0) \geq \Delta v(0)\,, \quad (\Delta u)'(0) = (\Delta v)'(0) = 0\,. \end{cases}$$

Then, for all $r \in [0,R)$ we have

$$u(r) \geq v(r)\,, \quad u'(r) \geq v'(r)\,, \quad \Delta u(r) \geq \Delta v(r)\,, \quad (\Delta u)'(r) \geq (\Delta v)'(r)\,.$$

Moreover, the initial point 0 can be replaced by any initial point $\rho > 0$ if all the four initial data are weakly ordered and a strict inequality in one of the initial data at $\rho \geq 0$ or in the differential inequality in (ρ, R) implies a strict ordering of $u, u', \Delta u, (\Delta u)'$ and $v, v', \Delta v, (\Delta v)'$ on (ρ, R).

Next we show

Lemma 2. *Let $n \geq 1$ and $u_{\alpha,\beta}(r)$ be a (local) solution to (5) defined on its maximal interval of existence $(0, R(\alpha,\beta))$.*

(i) If there exists $R_0 > 0$ such that $u'_{\alpha,\beta}(R_0) \geq 0$, then $R(\alpha,\beta) < +\infty$, $u'_{\alpha,\beta}(r) > 0$ for every $r \in (R_0, R(\alpha,\beta))$, and $\lim_{r \nearrow R(\alpha,\beta)} u_{\alpha,\beta}(r) = +\infty$.

(ii) If $\beta \geq 0$, then $R(\alpha,\beta) < +\infty$, $u'_{\alpha,\beta}(r) > 0$ for every $r \in (0, R(\alpha,\beta))$, and $\lim_{r \nearrow R(\alpha,\beta)} u_{\alpha,\beta}(r) = +\infty$.

Proof. For shortness, we write $u = u_{\alpha,\beta}$. Assume first that $u'(R_0) > 0$ and, for contradiction, assume that there exists $R > R_0$ such that $u'(R) = 0$. We may choose R minimal such that $u'(r) > 0$ in (R_0, R) and, of course, $u(R) > u(R_0)$. Then, putting $v(r) := u(r) - u(R)$, we see that v solves the boundary value problem

$$\begin{cases} \Delta^2 v(r) = e^{u(r)} > 0 & \text{for } r \in [0, R) \\ v(R) = v'(R) = 0\,. \end{cases}$$

Then by Boggio's maximum principle in the ball (see e.g. [20, Lemma 2.27]), we have that $v(r) > 0$ for every $r \in [0, R)$, contradicting $v(R_0) < 0$.

Assume now that $u'(R_0) = 0$ and put $w(r) := u(r) - u(R_0)$. Then w solves the boundary value problem

$$\begin{cases} \Delta^2 w(r) = e^{u(r)} > 0 & \text{for } r \in [0, R_0) \\ w(R_0) = w'(R_0) = 0\,. \end{cases}$$

Then, by [22, Theorem 3.2], not only we have that $w(r) > 0$ for every $r \in [0, R_0)$ but also that $w''(R_0) > 0$. Therefore, $u'(r) = w'(r) > 0$ in a right neighborhood of R_0 and we are back to the case $u'(R_0) > 0$.

Summarizing, if $u'(R_0) \geq 0$ for some $R_0 > 0$, then $u'(r) > 0$ for $r \in (R_0, R(\alpha, \beta))$. If $R(\alpha, \beta) = +\infty$, then we would have an entire solution to (2) bounded from below, against Proposition 2. Hence, $R(\alpha, \beta) < +\infty$ and

$$\lim_{r \nearrow R(\alpha,\beta)} u_{\alpha,\beta}(r) = +\infty$$

by standard theory of ordinary differential equations. This completes the proof of statement (i).

Statement (ii) is a straightforward consequence of (i). Indeed, since all the derivatives of u up to order 4 are nonnegative at $r = 0$ and the fourth derivative is strictly positive, we have $u'(r) > 0$ in a right neighborhood of 0. □

At this point, we need a monotonicity and continuous dependence result.

Lemma 3. *For any $\alpha \in \mathbb{R}$:*
(i) the map $\beta \mapsto R(\alpha, \beta) \in (0, +\infty]$ is nonincreasing;
(ii) there exists $\beta_0 \in [-\infty, 0)$ such that $R(\alpha, \beta) < \infty$ if and only if $\beta > \beta_0$.

Proof. (i) Since by Lemma 1 the solutions of (29) are ordered, the map $\beta \mapsto R(\alpha, \beta)$ is nonincreasing.

(ii) Let β_0 be the infimum of the β's such that $R(\alpha, \beta) < \infty$. If $\beta_0 = -\infty$ there is nothing to prove. Otherwise, by Lemma 1, we know that $R(\alpha, \beta) < \infty$ for all $\beta > \beta_0$ and $R(\alpha, \beta) = \infty$ for all $\beta < \beta_0$. So, we just need to study the case $\beta = \beta_0$. If $R(\alpha, \beta_0) < \infty$, then by Lemma 2 there exists $R_0 > 0$ such that $u'_{\alpha,\beta_0}(r) > 0$ for all $r \in (R_0, R(\alpha, \beta_0))$. Take a sequence $\beta_k \nearrow \beta_0$, then $R(\alpha, \beta_k) = \infty$ and by Lemma 2 $u'_{\alpha,\beta_k}(r) < 0$ for all $r > 0$ (in particular, for all $r \in (R_0, R(\alpha, \beta_0))$) and all k. This is against the continuous dependence, which can be proved as in [19, Proposition A3]. □

Finally, we determine an upper bound for the existence of global solutions.

Lemma 4. *Let $n \geq 2$. For every $\alpha \in \mathbb{R}$, if $\beta \geq \min\{0, -\alpha\}$, then the solution $u_{\alpha,\beta}$ to (5) blows up in finite time.*

Proof. For $\beta \geq 0$ the statement has already been proved in Lemma 2.

Let J_ν denote the Bessel functions, see [3]. It is known [17, (4.19)] that the function

$$\bar{y}(r) = r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r) \quad r > 0\,,$$

is a radial smooth solution to

$$\Delta^2 y = y \quad \text{on } \mathbb{R}^n\,.$$

Furthermore, since

$$J_{\frac{n}{2}-1}(r) \simeq \frac{1}{\Gamma(n/2)}\left(\frac{r}{2}\right)^{\frac{n}{2}-1} \quad (\text{as } r \to 0)\,, \quad J_{\frac{n}{2}-1}(r) \simeq \sqrt{\frac{2}{\pi r}}\cos\left(r - \frac{\pi}{4}(n-1)\right) \quad (\text{as } r \to +\infty)\,,$$

where Γ is the Gamma function, we conclude that $\bar{y}(r)$ is bounded on $[0, +\infty)$. On the other hand, one has

$$\bar{y}(0) = \frac{2^{1-\frac{n}{2}}}{\Gamma(n/2)}\,, \quad \bar{y}'(0) = (\Delta\bar{y})'(0) = 0 \quad \text{and} \quad (\Delta\bar{y})(0) = -\frac{2^{1-\frac{n}{2}}}{\Gamma(n/2)}\,.$$

Finally, for every $\alpha \in \mathbb{R}$, we define $y_\alpha(r) := \alpha 2^{\frac{n}{2}-1}\Gamma(n/2)\bar{y}(r)$, see Figure 1. Then $y_\alpha(r)$ solves

$$\begin{cases} \Delta^2 y_\alpha(r) - y_\alpha(r) = 0 \quad \forall r > 0\,, \\ y_\alpha(0) = \alpha\,, \quad y'_\alpha(0) = 0\,, \quad \Delta y_\alpha(0) = -\alpha\,, \quad (\Delta y_\alpha)'(0) = 0\,. \end{cases}$$

Since $\Delta^2 u_{\alpha,\beta}(r) - u_{\alpha,\beta}(r) \geq \Delta^2 u_{\alpha,\beta}(r) - e^{u_{\alpha,\beta}(r)} = 0$ for all $r \in [0, R(\alpha,\beta))$, an application of Lemma 1 with $f(u) = u$, yields

$$u_{\alpha,\beta}(r) \geq y_\alpha(r) \quad \forall r \in [0, R(\alpha,\beta))\,, \quad u'_{\alpha,\beta}(r) \geq y'_\alpha(r) \quad \forall r \in [0, R(\alpha,\beta))\,, \tag{30}$$

for every $\beta \geq -\alpha$. Being y_α a sign changing function at infinity, it admits an infinite number of stationary points and hence by (30), $u_{\alpha,\beta}$ admits a point with nonnegative derivative. The statement of the lemma then follows from Lemma 2. $\square$

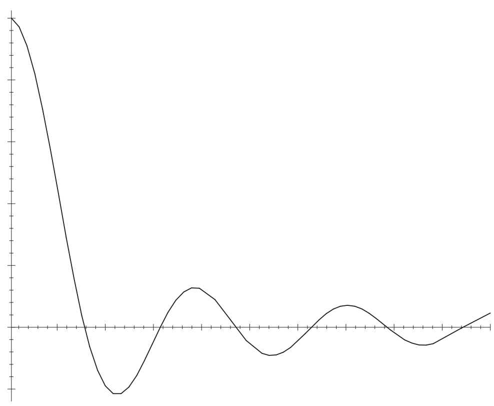

Figure 1: The plot of the subsolutions y_α for $\alpha = 1$ when $n = 3$. For other values of n the function y_α displays the same behavior.

We are now ready to complete the proof of Theorem 2.

Since the right hand side of the equation in (5) is nonnegative, Lemma 1 applied with $f = 0$, yields (7).

If $\beta \geq 0$, statement (i) follows from Lemma 2-(ii). When $\beta < 0$, by Lemma 2-(i) we see that either $u'_{\alpha,\beta}(r) < 0$ for every $r \in (0, R(\alpha,\beta))$ and $R(\alpha,\beta) = +\infty$ by (7) or $R(\alpha,\beta) < +\infty$.

Fix $\alpha \in \mathbb{R}$. If for any $\beta \in \mathbb{R}$ the solutions to (5) blow up in finite time, then $\beta_0(\alpha) = -\infty$ and we conclude. If there exists β such that $u_{\alpha,\beta}$ is global, then by Lemmas 1 and 3 there exists $\beta_0 = \beta_0(\alpha) < 0$ such that $u_{\alpha,\beta}$ is global for all $\beta \leq \beta_0$ and $u_{\alpha,\beta}$ blows up in finite time for all $\beta > \beta_0$. Moreover, by Lemma 4, we deduce $\beta_0 < \min\{0, -\alpha\}$.

For the proof of (8), see [5, Lemma 8].

7 Proof of Theorem 3

Assume that $n = 2$. Let $\alpha, \beta \in \mathbb{R}$ and let $u_{\alpha,\beta}$ be the local solution to (5). For shortness we write $u = u_{\alpha,\beta}$. When $n = 2$, equation (29) reads

$$\left(ru'''(r) + u''(r) - \frac{u'(r)}{r} \right)' = r\, e^{u(r)} > 0 \quad \forall r > 0\,.$$

Hence, the map $r \mapsto ru'''(r) + u''(r) - \frac{u'(r)}{r}$ is increasing and since it vanishes as $r \to 0^+$, we infer that

$$ru'''(r) + u''(r) - \frac{u'(r)}{r} > u'''(1) + u''(1) - u'(1) =: \gamma > 0 \quad \forall r > 1\,.$$

Multiplying by r we obtain

$$(r^2 u''(r) - ru'(r))' = r^2 u'''(r) + ru''(r) - u'(r) > \gamma r \quad \forall r > 1\,.$$

A further integration shows that there exists $\rho > 1$ such that

$$r^2 u''(r) - ru'(r) > \delta r^2 \quad \forall r > \rho\,,$$

for some $\delta > 0$. Dividing by r^3, we get

$$\left(\frac{u'(r)}{r} \right)' = \frac{u''(r)}{r} - \frac{u'(r)}{r^2} > \frac{\delta}{r} \quad \forall r > \rho$$

and, integrating over (ρ, r), we finally conclude that

$$\frac{u'(r)}{r} > C_0 \log r \quad \forall r \geq R > \rho\,,$$

for some positive C_0 and some $R > \rho$. Then Theorem 3 follows at once from Lemma 2.

8 Proofs of Theorem 4 and Corollary 1

In [5, Lemma 5] the following lower bound for the switch between global and blow-up solutions was found in supercritical dimensions.

Lemma 5. *Let $n \geq 5$ and $\alpha \in \mathbb{R}$. Then, for all $\beta \leq -4ne^{\alpha/2}$, the solution $u_{\alpha,\beta}$ to* (5) *is global and* $\lim\limits_{r\to+\infty} u_{\alpha,\beta}(r) = -\infty$.

By [27] we know that in the conformal dimension $n = 4$ there exists at least one global solution to (5). Indeed, the function (10) solves (5) for every $\alpha \in \mathbb{R}$ and for $\beta = -4e^{\alpha/2}/\sqrt{6}$. Hence, by Theorem 2 we get

Lemma 6. *Let $n = 4$ and $\alpha \in \mathbb{R}$. Then, for all $\beta \leq -4e^{\alpha/2}/\sqrt{6}$, the solution $u_{\alpha,\beta}$ to* (5) *is global and* $\lim\limits_{r\to+\infty} u_{\alpha,\beta}(r) = -\infty$.

We now show that also in the subcritical dimension $n = 3$ global solutions to (5) exist.

Lemma 7. *Let $n = 3$ and $\alpha \in \mathbb{R}$. There exists $\bar{\beta} < 0$ such that for any $\beta \leq \bar{\beta}$ the solution $u_{\alpha,\beta}$ to* (5) *is global and* $\lim\limits_{r\to+\infty} u_{\alpha,\beta}(r) = -\infty$.

Proof. If u is a solution to (5), then it solves the ordinary differential equation (29) which also reads

$$(r^4\, u'''(r))' = r^4 e^{u(r)} \qquad \forall r > 0\,. \tag{31}$$

We seek a global supersolution of (31), i.e. a function $\overline{u} \in C^4(0,+\infty)$ which satisfies

$$(r^4\, \overline{u}'''(r))' \geq r^4 e^{\overline{u}(r)} \qquad \forall r > 0\,. \tag{32}$$

We consider functions $\overline{u}$ of the form

$$\overline{u}(r) = -r^2 + \log(r+1) - b$$

for some $b > 0$. By direct computation we see that

$$(r^4\, \overline{u}'''(r))' = \frac{2r^3(r+4)}{(r+1)^4}\,, \qquad r^4 e^{\overline{u}(r)} = e^{-b} r^4 (r+1) e^{-r^2}\,.$$

Consider the function

$$\psi(r) := \frac{r(r+1)^5}{2(r+4)}\, e^{-r^2} \qquad \forall r > 0\,. \tag{33}$$

We have that $\psi(r) > 0$ for any $r > 0$ and

$$\lim_{r\to 0^+} \psi(r) = \lim_{r\to +\infty} \psi(r) = 0\,,$$

and hence the function ψ is bounded in $(0,+\infty)$. In order to ensure that (32) holds true, we choose

$$b \geq \log\big(\max_{(0,+\infty)} \psi\big). \tag{34}$$

From now on we fix b satisfying (34). Note that $\overline{u}(0) = -b$, $\overline{u}'(0) = 1$, $\overline{u}''(0) = -3$, and $\overline{u}'''(0) = 2$. Let $u = u_{-b,-9}$ be the local solution to (5) with $\alpha = -b$ and $\beta = -9$. Since $u'(0) = 0 < \overline{u}'(0)$, there exists $0 < \rho < R(-b,-9)$, such that

$$u(r) < \overline{u}(r) \qquad \forall r \in (0,\rho)\,.$$

Together with (31) and (32), this yields

$$(r^4\, \overline{u}'''(r))' > (r^4\, u'''(r))' \quad \forall r \in (0,\rho)\,.$$

By integrating twice this inequality over $(0,r)$ we deduce

$$\overline{u}''(r) > u''(r) \quad \forall r \in (0,\rho)\,,$$

where we used the fact that $u''(0) = \overline{u}''(0) = -3$. Let $(0,\overline{\rho})$ be the maximal interval where $\overline{u}''(r) > u''(r)$. We claim that $\overline{\rho} = R(-b,-9)$.

If not, by integrating twice this inequality over $(0,r)$, we deduce

$$\overline{u}'(r) > u'(r) + 1 \quad \text{and} \quad \overline{u}(r) > u(r) + r \quad \forall r \in (0,\overline{\rho})\,.$$

Then, exploiting once more (31) and (32), one concludes that $\overline{u}'''(r) > u'''(r)$ on $(0,\overline{\rho})$ and, thanks to a further integration, that $\overline{u}''(r) > u''(r)$ on $(0,\overline{\rho}]$, a contradiction.

Summarizing, we have proved that

$$u(r) < \overline{u}(r) \qquad \forall r \in (0, R(-b,-9))\,.$$

In particular, this yields $R(-b,-9) = +\infty$.

For any $\alpha \in \mathbb{R}$ define the function $w(r) = u(e^{\frac{\alpha+b}{4}} r) + b + \alpha$ which is globally defined on $[0, +\infty)$. By (6), w is a solution of (31) satisfying

$$w(0) = \alpha, \quad w'(0) = 0, \quad \Delta w(0) = -9e^{\frac{\alpha+b}{2}}, \quad (\Delta w)'(0) = 0\,.$$

Namely, w is the unique global solution to (5) with $\beta = \overline{\beta} = -9e^{\frac{\alpha+b}{2}}$. Hence, by Lemma 3, global solutions exist for all $\beta \leq \overline{\beta}$. □

Remark 1. Due to the fact that $\overline{u}'(0)$ and $\overline{u}'''(0) \neq 0$, the proof of Lemma 7 cannot be reached by simply invoking Lemma 1. Indeed, this fact makes the functions $\Delta\overline{u}$ and $(\Delta\overline{u})'$ singular at the origin. The proof of Lemma 1 is reached by successive integrations on $(0, r)$ of the second order equations arising by writing the equation in (5) as a system. Namely, putting $U := \Delta u$, one has $(r^{n-1}(u(r))')' = r^{n-1}\,U(r)$ and $(r^{n-1}\,U')' = r^{n-1}e^{u(r)}$. Hence, when U or U' are singular at $r = 0$, one cannot proceed by integrating on $(0, r)$, see [28, Lemma 3.2]. However, at least when $n = 3$, this problem can be overcome by integration of the equation as shown in the proof of Lemma 7.

When $n = 2$, $\overline{u}$ is still a supersolution to (29) for a different choice of the parameter b. But, as Theorem 3 suggests, it cannot be exploited as in the proof of Lemma 7. Recall that (29) with $n = 2$ reads

$$\left(ru'''(r) + u''(r) - \frac{u'(r)}{r}\right)' = r\,e^{u(r)} > 0 \quad \forall r > 0\,.$$

Hence, when $u'(0) \neq 0$, the antiderivative of the function on the left hand side is singular at $r = 0$.

We now estimate the decay of the separatrix in the subcritical dimension $n = 3$.

Lemma 8. *Let $n = 3$ and u be an entire radial solution to* (2)*. Then, u is concave and there exists $R > 0$ such that*

$$u(r) \leq -C\,r \quad \textit{for all } r \geq R\,,$$

for some positive constant C.

Proof. When $n = 3$, (29) reads

$$(r^4 u'''(r))' = r^4\,e^{u(r)} > 0 \quad \forall r > 0\,.$$

Then, integrating on $(0, r)$, we deduce that $u'''(r) > 0$ for all $r > 0$. Hence, $u'(r)$ is a convex function and $u''(r)$ is increasing. Being u global, by Theorem 2 we have that $u''(0) < 0$ and $u'(r) < 0$ for all $r > 0$. If there would exist $r_0 > 0$ such that $u''(r_0) = 0$, then $u''(r) > 0$ for $r > r_0$ and, being u' convex, $\lim\limits_{r\to+\infty} u'(r) = +\infty$, a contradiction. Hence, $u''(r) < 0$ for every $r > 0$ and u is concave.

On the other hand, being u' a decreasing and negative function and $u'(0) = 0$, there exists

$$\lim_{r\to+\infty} u'(r) = l \in [-\infty, 0)$$

and we conclude. □

Proof of Theorem 4. When $n = 3$, the decay rate (9) and the concavity of global solutions are proved in Lemma 8, whereas to obtain the lower bound for β_0 we proceed as follows. The function ψ defined in (33) may be estimated by

$$2\psi(r) \leq (r+1)^5 e^{-r^2} =: \varphi(r)\,.$$

In turn, the function φ attains its maximum over $(0, +\infty)$ at $r = (-1 + \sqrt{11})/2$ and

$$\varphi\left(\frac{-1+\sqrt{11}}{2}\right) = \left(\frac{1+\sqrt{11}}{2}\right)^5 e^{(\sqrt{11}-6)/2} < \frac{49}{4}\,.$$

Hence, $\psi(r) < \frac{49}{8}$ for all r and by (34) we may take $b = \log(49/8)$. Therefore, we obtain $\beta_0 \geq -63e^{\alpha/2}/\sqrt{8}$.

When $n = 4$, problem (5) admits the global solution (10), for every $\alpha \in \mathbb{R}$ and for $\beta = -4e^{\alpha/2}/\sqrt{6}$, see also Lemma 6. Statement (ii) then follows by noting that the function in (10) does not satisfy the condition (8).

For the proof of statement (iii) see [5, Theorem 2] and also Lemma 5.

Proof of Corollary 1. By Lemma 1 the solutions to (5) are ordered. Therefore, for every $\alpha \in \mathbb{R}$ and $\beta \leq \beta_0$ we have that $u_{\alpha,\beta}(r) \leq u_{\alpha,\beta_0}(r)$ for all $r > 0$. Hence, it suffices to prove that $e^{u_{\alpha,\beta_0}} \in L^1(\mathbb{R}^n)$, for $n = 3, 4$. If $n = 3$, this follows from the estimate (9). When $n = 4$, by (10) we have that $u_{\alpha,\beta_0}(r) \sim -8\log r$ as $r \to +\infty$ and the thesis follows from the integrability of r^{-8} at ∞ in $\mathbb{R}^4$.

9 Proofs of Theorems 5 and 6

We first study the stability of fast decaying solutions.

Lemma 9. *Let $n \geq 3$ and let $\alpha \in \mathbb{R}$. Assume that $\beta_0 = \beta_0(\alpha)$ and $u_{\alpha,\beta}$ are as in Theorem 2. If $\beta < \beta_0$ then $u_{\alpha,\beta}$ is stable outside suitable compact sets.*

Proof. Let $n \geq 3$, for any $r > 1$, we define

$$V_n(r) := \begin{cases} \dfrac{1}{16r^4} & \text{if } n = 3\,; \\ \dfrac{1}{4r^4\log^2 r} & \text{if } n = 4\,; \\ \dfrac{n^2(n-4)^2}{16\,r^4} & \text{if } n \geq 5\,. \end{cases}$$

By (8) we know that

$$e^{u_{\alpha,\beta}(r)} \leq e^{\alpha}e^{-\frac{\beta_0-\beta}{2n}r^2} \quad \forall r \in [0,+\infty)\,.$$

Note that the map

$$r \mapsto \frac{e^{\alpha}e^{-\frac{\beta_0-\beta}{2n}r^2}}{V_n(r)} \tag{35}$$

is well-defined, vanishes as $r \to +\infty$ and is eventually decreasing. Therefore there exists $R_\alpha > 1$ such that the map in (35) is decreasing in $(R_\alpha, +\infty)$ and satisfies

$$\frac{e^{\alpha}e^{-\frac{\beta_0-\beta}{2n}r^2}}{V_n(r)} < 1 \quad \forall r > R_\alpha\,.$$

Then by (17), (19), and (20) we have

$$\begin{aligned} \int_{\mathbb{R}^n\setminus B_{R_\alpha}} |\Delta\varphi|^2 dx - \int_{\mathbb{R}^n\setminus B_{R_\alpha}} e^{u_{\alpha,\beta}}\varphi^2\,dx &\geq \int_{\mathbb{R}^n\setminus B_{R_\alpha}} |\Delta\varphi|^2 dx - \int_{\mathbb{R}^n\setminus B_{R_\alpha}} \frac{e^{\alpha}e^{-\frac{\beta_0-\beta}{2n}|x|^2}}{V_n(|x|)} V_n(|x|)\varphi^2\,dx \\ &\geq \left(1 - \frac{e^{\alpha}e^{-\frac{\beta_0-\beta}{2n}R_\alpha^2}}{V_n(R_\alpha)}\right)\int_{\mathbb{R}^n\setminus B_{R_\alpha}} |\Delta\varphi|^2 dx \geq 0 \end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \overline{B_{R_\alpha}})$, which proves (14) for any compact set K containing the ball B_{R_α}. □

We now study the stability of the separatrix.

Lemma 10. *Let $3 \le n \le 4$ and let $\alpha \in \mathbb{R}$. Assume that $\beta_0 = \beta_0(\alpha)$ and $u_{\alpha,\beta}$ are as in Theorem 2. Then u_{α,β_0} is stable outside suitable compact sets.*

Proof. Let $V_n(r)$ be as defined in Lemma 9. Since by Theorem 4 and Corollary 1, we have that for $3 \le n \le 4$, $e^{u_{\alpha,\beta_0}}(r) = o(V_n(r))$ as $r \to +\infty$, the proof follows arguing as in the proof of Lemma 9. □

When $n \ge 5$, we prove

Lemma 11. *Let $5 \le n \le 12$ and $\alpha \in \mathbb{R}$. Assume that $\beta_0 = \beta_0(\alpha)$ and u_{α,β_0} are as in Theorem 2, then the solution u_{α,β_0} is unstable outside every compact set.*

Proof. By (11), we have that for every $m > 0$, there exists $R_m > 0$ such that

$$u_{\alpha,\beta_0}(r) > \log\left(\frac{8(n-2)(n-4)}{r^4}\right) - \frac{1}{m},$$

whenever $r \ge R_m$. By contradiction, assume that u_{α,β_0} is stable outside a compact set K, we can always choose R_m so large that $K \subset \overline{B_{R_m}}$, where B_{R_m} is the ball of radius R_m and center the origin. Then, by (14) we deduce

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B_{R_m}} |\Delta\varphi|^2\,dx - e^{-1/m}\,8(n-2)(n-4)\int_{\mathbb{R}^n\setminus B_{R_m}} \frac{\varphi^2}{|x|^4}\,dx\\ &\ge \int_{\mathbb{R}^n\setminus B_{R_m}} |\Delta\varphi|^2\,dx - \int_{\mathbb{R}^n\setminus B_{R_m}} e^{u_{\alpha,\beta_0}}\varphi^2\,dx \ge 0\end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \overline{B_{R_m}})$. Since $8(n-2)(n-4) > \frac{n^2(n-4)^2}{16}$, for $5 \le n \le 12$, we can choose m so large that

$$e^{-1/m}\,8(n-2)(n-4) > \frac{n^2(n-4)^2}{16},$$

contradicting the optimality of the constant in (17). □

Lemma 12. *Let $n \ge 13$ and $\alpha \in \mathbb{R}$. Assume that $\beta_0 = \beta_0(\alpha)$ and $u_{\alpha,\beta}$ are as in Theorem 2. Then*

$$u_{\alpha,\beta_0}(r) < \log\left(\frac{8(n-2)(n-4)}{r^4}\right) \quad \forall\, r \in (0,+\infty)\,. \tag{36}$$

Furthermore, for every $\beta \le \beta_0$ we have that $u_{\alpha,\beta}$ is stable.

Proof. By Lemma 1 the solutions to (5) are ordered. Then $u_{\alpha,\beta}(r) \le u_{\alpha,\beta_0}(r)$ for all $r > 0$ and for every $\alpha \in \mathbb{R}$ and $\beta \le \beta_0$. Hence, it suffices to prove the statement for $\beta = \beta_0$.

For shortness we write $u = u_{\alpha,\beta_0}$. By performing the change of variable

$$w(s) = u(e^s) + 4s\,, \quad s = \log r \in (-\infty,+\infty)\,,$$

the equation in (5) becomes

$$P_n(\partial_s)w(s) = e^{w(s)} - 8(n-2)(n-4)\,,$$

where $\partial_s = \frac{d}{ds}$ and P_n is the polynomial $P_n(\mu) := \mu(\mu-2)(\mu+n-2)(\mu+n-4)$ for any $n \ge 13$.

We follow the idea of [26, Proposition 9]. Putting $v(s) = w(s) - \log 8(n-2)(n-4)$, we deduce

$$P_n(\partial_s)v(s) = e^{w(s)} - 8(n-2)(n-4) = 8(n-2)(n-4)(e^{v(s)} - 1) \ge 8(n-2)(n-4)v(s)\,,$$

by the convexity of the exponential function. Namely, we have

$$[P_n(\partial_s) - 8(n-2)(n-4)]\,v(s) \ge 0\,.$$

Invoking the analysis performed in [6, Section 3.1], the above ODE can be factorized as follows

$$(\partial_s - \nu_4)(\partial_s - \nu_3)(\partial_s - \nu_2)(\partial_s - \nu_1)v(s) \geq 0\,,$$

where $\nu_4, \nu_3, \nu_2 < 0 < \nu_1$. Then, since from (11) and the definition of v we have

$$\lim_{s\to-\infty}(v(s) - 4s) = \alpha - \log 8(n-2)(n-4)\,, \quad \lim_{s\to-\infty} v'(s) = 4\,, \quad \lim_{s\to-\infty} v''(s) = 0 = \lim_{s\to-\infty} v'''(s)\,,$$

we deduce that

$$\lim_{s\to-\infty} e^{-\nu_2 s}v^{(i)}(s) = 0\,, \quad \lim_{s\to-\infty} e^{-\nu_3 s}v^{(i)}(s) = 0\,, \quad \lim_{s\to-\infty} e^{-\nu_4 s}v^{(i)}(s) = 0\,, \quad \text{for } i = 0, 1, 2, 3.$$

Exploiting this and integrating three times the ODE over $(-\infty, s)$, we get

$$(\partial_s - \nu_1)v(s) \geq 0\,.$$

Finally, multiplying by $e^{-\nu_1 s}$ and integrating over $(s, +\infty)$ we conclude that

$$e^{-\nu_1 s}v(s) \leq \lim_{s\to+\infty} e^{-\nu_1 s}v(s)\,. \tag{37}$$

By (11), $\lim_{s\to+\infty} v(s) = 0$ and we deduce that the right hand side in (37) is zero. Summarizing, by (37) we conclude that $v(s) \leq 0$, namely that

$$u(r) \leq \log\left(\frac{8(n-2)(n-4)}{r^4}\right).$$

To get the strict inequality one may repeat the proof of [26, Theorem 3] with minor changes.

Now we turn to the stability issue. Since by Lemma 1 the solutions are ordered, it suffices to prove stability of u_{α,β_0}. By (36) we deduce

$$\int_{\mathbb{R}^n} |\Delta\varphi|^2\,dx - \int_{\mathbb{R}^n} e^{u_{\alpha,\beta_0}}\varphi^2\,dx > \int_{\mathbb{R}^n} |\Delta\varphi|^2\,dx - 8(n-2)(n-4)\int_{\mathbb{R}^n} \frac{\varphi^2}{|x|^4}\,dx > 0 \quad \forall\,\varphi \in C_c^\infty(\mathbb{R}^n),$$

where the last inequality comes from (15) since $8(n-2)(n-4) < \frac{n^2(n-4)^2}{16}$ for $n \geq 13$. $\square$

Theorem 5 follows from Lemmas 9 and 10 whereas Theorem 6 follows from Lemmas 9, 11, and 12.

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