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Qualitative behavior of global solutions to some nonlinear fourth order differential equations

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Abstract

We study global solutions to a fourth order semilinear ordinary differential equation. We determine sufficient conditions on the nonlinearity that ensure global continuation of the solutions. Furthermore, we discuss their qualitative behaviors such as oscillations and boundedness.
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1 Introduction

In this paper we study the equation

$$w''''(s) + kw''(s) + f(w(s)) = 0 \qquad (s \in \mathbb{R}) \tag{1}$$

where $k \in \mathbb{R}$ and f is a locally Lipschitz function.

This equation arises in several contexts. With no hope of being exhaustive, let us mention some models which lead to (1). When k is negative (1) is known as the extended Fisher-Kolmogorov equation, whereas when k is positive it is referred to as Swift-Hohenberg equation, see [23]. For $f(t) = t - t^2$, (1) arises in the dynamic phase-space analogy of a nonlinearly supported elastic strut [17]. In [1] the existence of even homoclinics to $w \equiv 0$ was proved whenever $k \leq 0$. When $f(t) = t^3 - t$, (1) serves as a model of pattern formation in many physical, chemical or biological systems, see [7, 8] and references therein. The slightly different nonlinearity $f(t) = t - t^3 + t^5$ was used by Peletier [24] in order to investigate localization and spreading of deformation of a strut confined by an elastic foundation.

When $k \in (0, 2)$ and $f(t) = (t+1)^+ - 1$ equation (1) describes traveling waves in a suspension bridge according to the model suggested by McKenna-Walter [21]. Later, Chen-McKenna [11] use the smoothed nonlinearity $f(t) = e^t - 1$ instead of $(t+1)^+ - 1$, see also [19] for the existence of ground states for fourth order wave equations with other nonlinearities. When $k = -4$, equation (1) with the very same nonlinearity $f(t) = e^t - 1$ arises from a suitable transformation of the biharmonic pde

$$\Delta^2 u + e^u = \frac{1}{|x|^4} \qquad \text{in } \mathbb{R}^4 \setminus \{0\}, \tag{2}$$

namely a fourth order coercive nonautonomous version of the celebrated Gelfand problem [15, 18] which reads $-\Delta u = e^u$. The noncoercive equation $\Delta^2 u = e^u$ has recently attracted much interest both in $\mathbb{R}^4$ (see [5, 10, 20]) and in $\mathbb{R}^n$ for $n \geq 5$ (see [2, 3, 4, 5, 6, 12, 13]). Here, we deal with the coercive case (2) in the largest space dimension ($n = 4$) for which the nonlinearity $u \mapsto e^u$ is subcritical. Then not only the singular

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function $x \mapsto -4\log|x|$ is a fundamental solution of Δ^2, but it is also a solution to (2). For other values of n and k further biharmonic equations arise, see Section 2.5 for the details.

All the just mentioned nonlinearities were considered in [25] in order to study the existence of homoclinics. Last but not least, we mention the important book by Peletier-Troy [23] where one can find many other physical models, a survey of existing results, and further references. It is clear that the parameter k plays a crucial role in the behavior of solutions to (1). Different cases are analyzed in [23].

The purpose of the present paper is to contribute to a better understanding of the qualitative properties of solutions to (1) when the nonlinearity f satisfies

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}), \quad f(t)\,t > 0 \quad \text{for every } t \in \mathbb{R} \setminus \{0\}. \tag{3}$$

Further assumptions on f will be needed in the sequel.

We first show that, under a fairly weak additional assumption on f, local solutions to (1) are global (Theorem 1) and that, if this assumption is violated, finite time blow up may occur only with wide oscillations (Theorem 2). Then we study the qualitative behavior of global solutions, namely their oscillations and boundedness, see Sections 2.2 and 2.3. In view of (3), the only stationary solution to (1) is $w \equiv 0$. Its stability properties are studied in Section 3.2. The stability analysis appears very delicate and is still unclear for large values of k. Homoclinic solutions to $w \equiv 0$ may exist only if the stable and unstable manifold at 0 are nonempty; we give the state of art and a couple of new results in Section 2.4.

This paper is organized as follows. In next section we state our main results which are divided in three groups; we first discuss whether local solutions to (1) are global, then we study their asymptotic behavior, finally we discuss the existence of homoclinics. In Section 2.5 we show that some nonlinear biharmonic pde's may be studied by means of (1). In Section 3 we define several energy functions which will be used throughout the paper and we discuss the stability of a 4×4 system of nonlinear first order equations equivalent to (1). The remaining part of the paper is devoted to the proofs of the results.

2 Main results

2.1 Existence of global solutions

In this section we establish whether any solution to (1) is global. To this end, we need a further assumption on f. We will require one of the following conditions

$$\limsup_{t\to+\infty} \frac{f(t)}{t} < +\infty \qquad \text{or} \qquad \limsup_{t\to-\infty} \frac{f(t)}{t} < +\infty. \tag{4}$$

Under (3) and one of the two above assumptions we will show that local solutions to (1) may be continued to the whole real line. Before stating the precise result, let us make a few comments on (4).

Roughly speaking, (4) states that f is "one-sided at most linear". Of course, (4) does not cover functions f (satisfying (3)) with uncontrolled behaviors at both $\pm\infty$ such as

$$f(t) = t^3\,, \qquad f(t) = \frac{t}{1+t^2} + t^3(1+\sin t). \tag{5}$$

Nevertheless, if we exclude these cases, assumption (4) appears general enough to include all the interesting models satisfying (3). In particular, (4) is satisfied if f is either concave or convex.

Let us now turn to assumption (3). The next examples show that if it is violated global continuation may fail.

Examples. If $f(t) = -24t^5 - 2kt^3$ then $w(s) = \frac{1}{s}$ is a local solution to (1) which blows up in finite time. While (4) certainly holds at both $\pm\infty$, if $k \geq 0$ this function f has the "wrong" sign when compared to (3). If $k < 0$, then (3) holds locally, namely $f(t)t > 0$ for all $t \neq 0$ only in a neighborhood of 0. This example, however, has a nonlinearity f satisfying $f'(0) = 0$, a case which is often excluded in our statements below, see (15) and Remark 22. In order to have a positive derivative at $t = 0$, take $f(t) = 24(t - t^5)$ which also satisfies (3) locally; for this nonlinearity, the finite time blow up function $w(s) = \tan s$ solves (1) for $k = -20$.

We now state our global continuation result

Theorem 1. *Let $k \in \mathbb{R}$ and assume that f satisfies* (3) *and* (4)*. Then, any local solution to* (1) *exists for all $s \in \mathbb{R}$.*

As a by-product of our proof (see Lemma 23 below) we infer that, under the sole assumption (3), the only way that finite time blow up can occur is with wide oscillations of the solution.

Theorem 2. *Let $k \in \mathbb{R}$ and assume that f satisfies* (3)*. If a local solution w to* (1) *blows up at some finite $R \in \mathbb{R}$, then*

$$\liminf_{s \to R} w(s) = -\infty \qquad \textit{and} \qquad \limsup_{s \to R} w(s) = +\infty\,.$$

These wide oscillations are somehow related to uncontrolled behaviors of f such as (5). At this point, we suggest

Problem 3. Prove or disprove Theorem 1 under the sole assumption (3).

Finally, as an immediate consequence of Theorem 1, we may exclude the existence of large solutions in bounded intervals. More precisely, for any $R > 0$ the (autonomous) problem

$$\begin{cases} w''''(s) + kw''(s) + f(w(s)) = 0 & s \in (-R, R) \\ \lim\limits_{s \to \pm R} w(s) = \infty \end{cases}$$

admits no solution if f satisfies (3) and (4). Clearly, we also have nonexistence of solutions on intervals $(-\infty, R)$ or $(R, +\infty)$ which blow up at some finite R. This will be used when dealing with radial solutions to biharmonic equations, see Corollary 16 below.

2.2 Qualitative behavior of global solutions

We study here the behavior of global solutions to (1) as $s \to \pm\infty$. If $k \geq 0$, then solutions to (1) have oscillations.

Theorem 4. *Let $k \geq 0$ and f satisfy* (3)*. If w is a global solution to* (1)*, then*

$$\liminf_{s \to +\infty} w(s) \leq 0 \leq \limsup_{s \to +\infty} w(s)\,, \tag{6}$$

so that if $\lim\limits_{s \to +\infty} w(s)$ exists then

$$\lim_{s \to +\infty} w(s) = 0\,. \tag{7}$$

Furthermore, if $w \not\equiv 0$ then $w(s)$ changes sign infinitely many times as $s \to +\infty$. Similar statements hold for $s \to -\infty$.

The next result shows that a similar phenomenon may not occur when $k < 0$.

Theorem 5. *Let $k < 0$ and assume that f satisfies* (3) *and*

$$\sup_{t \in \mathbb{R}} f(t) = M < +\infty. \tag{8}$$

Then there exists a global solution w of (1) *which is eventually positive, increasing, and convex as $s \to +\infty$; in particular,*

$$\lim_{s\to+\infty} w(s) = +\infty\,. \tag{9}$$

If, instead of (8)*, f satisfies*

$$\inf_{t\in\mathbb{R}} f(t) = -M > -\infty, \tag{10}$$

then there exists a global solution w of (1) *which is eventually negative, decreasing, and concave as $s \to +\infty$; in particular,*

$$\lim_{s\to+\infty} w(s) = -\infty\,.$$

Similarly, there exist solutions having the above mentioned behaviors as $s \to -\infty$.

We point out that (8) (respectively (10)) implies the first (respectively the second) condition in (4) so that Theorem 1 states that all the solutions are global. We also emphasize that assumption (8) is essential in the previous statement. The next statement shows that if it is violated, then the solution is bounded from above.

Theorem 6. *Let $k < 0$ and f satisfy* (3), (10) *and*

$$\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty\,. \tag{11}$$

Then any solution w of (1) *is global and*

$$\sup_{s\in\mathbb{R}} w(s) < +\infty \quad \textit{and} \quad \inf_{s\in\mathbb{R}} w(s) = -\infty\,. \tag{12}$$

On the contrary, suppose that (10) *is replaced by* (8) *and* (11) *holds as $t \to -\infty$. Then any solution w of* (1) *is global and*

$$\sup_{s\in\mathbb{R}} w(s) = +\infty \quad \textit{and} \quad \inf_{s\in\mathbb{R}} w(s) > -\infty\,.$$

Remark 7. Conditions (10)-(11) are somehow necessary since if we drop them it may happen that there exists a solution w of (1) which does not satisfy (12). For example, we may consider the linear problem (1) with $k = -2$ and $f(t) = t$, i.e.

$$w'''' - 2w'' + w = 0\,. \tag{13}$$

Then $w(s) = e^s$ is a solution of (13) which does not satisfy (12).

2.3 Further properties of global solutions when $k \leq 0$

We first state a criterion to recognize the behavior at infinity of global solutions.

Theorem 8. *Suppose $k < 0$ and that f satisfies* (3)*. Let $w(s)$ be a global solution to* (1) *and let*

$$H(s) = w'(s)w''(s) - w(s)w'''(s) - kw(s)w'(s). \tag{14}$$

Then $H(s)$ is nondecreasing and the following alternative holds.

(i) If $H(s)$ is bounded as $s \to +\infty$, then $H(s) \leq 0$ for all s and

$$\lim_{s\to+\infty} H(s) = \lim_{s\to+\infty} w(s) = 0.$$

(ii) If $H(s_0) > 0$ at some point s_0, then both $H(s)$ and $w(s)$ are unbounded as $s \to +\infty$.

Moreover, a similar alternative holds with $+\infty$ replaced by $-\infty$ and all inequalities reversed.

As far as only the *sign* of k is concerned, the properties of (1) do not depend on the particular function f considered, provided (3) holds. But if also the *value* of k is concerned, then we need to normalize f in a suitable sense. We will assume the further condition

$$f \text{ is differentiable at } t = 0 \text{ and } f'(0) = 1. \tag{15}$$

Condition (15) is not restrictive. If f is a function such that $f'(0) = A > 0$ and w is a solution of (1) then $z(s) = w(s/\sqrt[4]{A})$ solves the new equation

$$z''''(s) + \frac{k}{\sqrt{A}}\, z''(s) + \widetilde{f}(z(s)) = 0$$

where $\widetilde{f}(t) = \frac{1}{A}\, f(t)$ and $\widetilde{f}'(0) = 1$. Of course, if $f'(0) = 0$ this trick is no longer available and we refer to Remark 22 for some comments on this case.

Next, we study possible oscillations of global solutions.

Theorem 9. *Assume that f satisfies* (3)*.*

(i) If $k \leq -2$ and f also satisfies one of the following

$$f(t) \geq t \quad \textit{near } t = 0 \quad \textit{or} \quad f(t) \leq t \quad \textit{near } t = 0 \tag{16}$$

then any global solution w to (1) *such that $\lim_{s\to+\infty} w(s) = 0$ is of one sign as $s \to +\infty$. Moreover, a similar statement holds with $+\infty$ replaced by $-\infty$.*

(ii) If $-2 < k < 0$ and f also satisfies (15) *and*

$$\liminf_{|t|\to+\infty} \frac{f(t)}{t} > k^2\,, \tag{17}$$

then any global nontrivial solution w to (1) *changes sign infinitely many times both as $s \to \pm\infty$.*

Then we study the behavior of the solution between two local extrema.

Theorem 10. *Let $k \leq 0$ and f satisfy* (3)*. Assume that w is a nontrivial solution to equation* (1) *having a local maximum at some s_1, a local minimum at some $s_2 > s_1$ and $w'(s) \leq 0$ for $s \in [s_1, s_2]$. Then at least one of the two following facts occurs:*

(i) there exists $\tau_2 > s_2$ such that $w(\tau_2) = w(s_1)$ and

$$w'(s) > 0, \quad w''(s) > 0, \quad w'''(s) \geq 0 \quad \forall s \in (s_2, \tau_2]; \tag{18}$$

(ii) there exists $\tau_1 < s_1$ such that $w(\tau_1) = w(s_2)$ and

$$w'(s) > 0, \quad w''(s) < 0, \quad w'''(s) \geq 0 \quad \forall s \in [\tau_1, s_1). \tag{19}$$

By modifying slightly the proof, one sees that the statement of Theorem 10 may be reversed by assuming that $s_2 < s_1$. In particular, Theorem 10 shows that if $k \leq 0$ then one of the following facts occurs:

1) w has at most one local extremum (which is then a global extremum);

2) w has at least two local extrema and for every couple of consecutive extrema at least one of them is overcome in the subsequent monotone branch of the solution.

2.4 Homoclinics

Important global solutions are the so-called *homoclinics*. These are nontrivial solutions w such that

$$\lim_{s\to\pm\infty} w(s) = 0.$$

In literature, one may find different definitions which also involve the derivatives of the solutions, see e.g. Section 5.2 in [23]. In fact, these definitions are equivalent, see for example the general statement [14, Proposition 1], where Sobolev embedding and classical Schauder estimates are exploited. In the Appendix we give an elementary proof of

Proposition 11. *Let* $k \in \mathbb{R}$ *and let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $f(0) = 0$*. Let* w *be a global solution to* (1) *such that*

$$\lim_{s\to+\infty} w(s) = 0\ .$$

Then

$$\lim_{s\to+\infty} w'(s) = \lim_{s\to+\infty} w''(s) = \lim_{s\to+\infty} w'''(s) = \lim_{s\to+\infty} w''''(s) = 0\ .$$

The same result also holds with $-\infty$ *in place of* $+\infty$*.*

The following statement is known, see [23, Section 3.2 and Theorem 10.1.1]:

Proposition 12. *Let* $k \le 0$ *and* f *satisfy* (3)*. Then, equation* (1) *has no nontrivial bounded solutions. In particular, equation* (1) *has no homoclinic solutions.*

In Remark 19 of Section 3.1 we give a simple proof of the second statement in Proposition 12. In fact, under an additional assumption on f, we may exclude the existence of homoclinics also for some positive values of k.

Theorem 13. *Let* $k > 0$*, the following statements hold.*

(i) If $k \le 2$ *and* f *satisfies*

$$\frac{f(t)}{t} \ge 1 \qquad \forall t \ne 0\,, \tag{20}$$

then equation (1) *has no homoclinic solutions.*

(ii) If f *satisfies* (3) *and* (15)*, if* w *is a homoclinic solution to* (1)*, and if* $\{s_m\}_{m\ge1}$ *denotes the increasing sequence of zeroes of* w *as* $s \to +\infty$*, then*

$$\liminf_{m\to+\infty}\ (s_{m+1} - s_m) \ge \frac{\pi\sqrt{k+\sqrt{k^2+12}}}{\sqrt{6}}. \tag{21}$$

A similar statement holds as $s \to -\infty$*.*

(iii) If $k < 2$ *and* f *satisfies* (3) *and* (15)*, then any homoclinic solution* w *to* (1) *satisfies* $w \in H^2(\mathbb{R})$*.*

Existence of homoclinics to (1) is a tricky problem (see [22, Problem 6.2]) which goes somehow beyond our scopes. For completeness, we recall which are the known results putting them in the framework of the present paper. Here and in the sequel we denote

$$F(t) := \int_0^t f(\tau)\, d\tau\ .$$

In particular, by (3) we see that $F(t) > 0$ for all $t \ne 0$. In [25], by using a mountain-pass procedure, the authors prove

Proposition 14. *Assume that f satisfies* (3), (15) *and*

$$\lim_{t\to-\infty}\frac{F(t)}{t^2}=0\,. \tag{22}$$

Then there exists a homoclinic solution to (1) *for almost every $k\in(0,2)$.*

An alternative approach consists in studying a suitable constrained minimization problem. In [24] the author proves

Proposition 15. *Let f be three times differentiable at* 0 *and satisfy* (3) *and* (15)*. Assume furthermore that*

$$f''(0)=0\,,\quad f'''(0)<0\,,\quad \textit{and}\quad \liminf_{|t|\to+\infty}\frac{F(t)}{t^2}>0\,.$$

Then, for every $\lambda>0$ there exists $k_\lambda\in(0,2)$ and $w_\lambda\in H^2(\mathbb{R})$ such that $\int_{\mathbb{R}}|w_\lambda'(s)|^2\,ds=\lambda$ and w_λ is a homoclinic solution to (1) *with $k=k_\lambda$.*

The proof in [24, Sections 2 and 7] is performed when f has the form

$$f(t)=t-t^3+\alpha\,t^5,\quad \alpha\geq\frac{3}{16}$$

but it extends to more general f as stated in Proposition 15. Note that if $\alpha>1/4$ this f satisfies (3). See [24, Figure 1.3] for a plot of k_λ as function of λ and for related numerical experiments.

Finally we mention that, when $f(t)=e^t-1$, a multiplicity result for homoclinics is obtained in [9] by means of a computer-assisted proof.

2.5 Biharmonic Gelfand-type problems

In this section, we give an interpretation of our results in terms of suitable biharmonic pde's. Let $k\in\mathbb{R}$ and consider the equation

$$\Delta^2u-2(n-4)\frac{x\cdot\nabla\Delta u}{|x|^2}+(n^2-6n+12+k)\frac{\Delta u}{|x|^2}-(n-2)\left[(n-2)^2+k\right]\frac{x\cdot\nabla u}{|x|^4}+e^u=\frac{1}{|x|^4} \tag{23}$$

where $x\in\mathbb{R}^n\setminus\{0\}$ ($n\geq2$). For any $k\in\mathbb{R}$, (23) admits an explicit global radial solution which is given by $\overline{u}(x)=-4\log|x|$. To see this, one may write (23) in its radial form, that is

$$u''''(r)+6\,\frac{u'''(r)}{r}+(7+k)\,\frac{u''(r)}{r^2}+(1+k)\,\frac{u'(r)}{r^3}+e^{u(r)}=\frac{1}{r^4}\,,$$

where $r=|x|\in(0,+\infty)$. Then, with the change of variables

$$s=\log r\qquad w(s):=u(e^s)+4s\qquad s\in\mathbb{R}\,,$$

one finds that $w=w(s)$ solves (1) with $f(t)=e^t-1$ and the singular solution $\overline{u}(x)=-4\log|x|$ to (23) corresponds to the trivial solution $w\equiv0$.

For problem (23), Theorem 1 reads

Corollary 16. *Let $k\in\mathbb{R}$ and B_R be the ball in $\mathbb{R}^n$ ($n\geq2$) with radius $0<R<+\infty$ and center the origin. Then, any radial solution to* (23) *in $B_R\setminus\{0\}$ admits a radial extension to $\mathbb{R}^n\setminus\{0\}$. In particular, equation* (23) *in $B_R\setminus\{0\}$ subject to the boundary condition*

$$\lim_{|x|\to R}u(x)=\infty\,,$$

admits no radial solution.

On the other hand, Propositions 11 and 12 read

Corollary 17. *Let* $k \le 0$ *and let* u *be a radial solution to* (23)*. If*

$$\lim_{|x|\to 0}(u(x)+4\log|x|) = 0 = \lim_{|x|\to+\infty}(u(x)+4\log|x|)\,,$$

then $u(x) \equiv -4\log|x|$.

Let us consider some meaningful values of n and k in (23).

If $n = 4$, (23) becomes

$$\Delta^2 u + (4+k)\left(\frac{\Delta u}{|x|^2} - 2\,\frac{x\cdot\nabla u}{|x|^4}\right) + e^u = \frac{1}{|x|^4}\,, \qquad x \in \mathbb{R}^4\setminus\{0\}.$$

Hence, if furthermore $k = -4$ we get equation (2).

If $n = 2$, (23) corresponds to the equation

$$\Delta^2 u + 4\,\frac{x\cdot\nabla\Delta u}{|x|^2} + (4+k)\,\frac{\Delta u}{|x|^2} + e^u = \frac{1}{|x|^4}\,, \qquad x \in \mathbb{R}^2\setminus\{0\}.$$

Thus, by taking $k = -4$, the equation reduces to

$$\Delta^2 u + 4\,\frac{x\cdot\nabla\Delta u}{|x|^2} + e^u = \frac{1}{|x|^4}\,, \qquad x \in \mathbb{R}^2\setminus\{0\}.$$

Finally, for any $n \ge 2$, taking $k = -(n^2-6n+12) \in (-\infty,-3]$ we have

$$\Delta^2 u - 2(n-4)\left[\frac{x\cdot\nabla\Delta u}{|x|^2} + (n-2)\frac{x\cdot\nabla u}{|x|^4}\right] + e^u = \frac{1}{|x|^4}\,, \qquad x \in \mathbb{R}^n\setminus\{0\},$$

while taking $k = -(n-2)^2 \in (-\infty,0]$ leads to

$$\Delta^2 u - 2(n-4)\,\left[\frac{x\cdot\nabla\Delta u}{|x|^2} + \frac{\Delta u}{|x|^2}\right] + e^u = \frac{1}{|x|^4}\,, \qquad x \in \mathbb{R}^n\setminus\{0\}.$$

3 Two useful tools

3.1 Energy functions

To equation (1) we associate the *energy function*

$$E(s) := \frac{1}{2}\,w''(s)^2 - \frac{k}{2}\,w'(s)^2 - F(w(s)), \tag{24}$$

for any $s \in \mathbb{R}$. Then we prove

Lemma 18. *Let* $w = w(s)$ *be a solution to* (1) *and let* s_1 *and* s_2 *be two critical points for* w*, namely* $w'(s_1) = w'(s_2) = 0$*. Then* $E(s_1) = E(s_2)$*.*

Proof. By differentiating we obtain

$$E'(s) = w'''(s)w''(s) - kw''(s)w'(s) - f(w(s))w'(s). \tag{25}$$

Hence, if $s_2 > s_1$, an integration by parts yields

$$\begin{aligned} E(s_2) - E(s_1) &= \int_{s_1}^{s_2} E'(s)\,ds \\ &= \int_{s_1}^{s_2} \Big(w'''(s)w''(s) - kw''(s)w'(s) - f(w(s))w'(s) \Big)\,ds \\ &= -\int_{s_1}^{s_2} \Big(w''''(s) + kw''(s) + f(w(s)) \Big) w'(s)\,ds = 0, \end{aligned}$$

where we used $w'(s_1) = w'(s_2) = 0$ and (1). □

More generally, by using (1) we may rewrite (25) as

$$E'(s) = w'''(s)w''(s) + w''''(s)w'(s),$$

so that, for any $s_1 < s_2$ we obtain

$$E(s_2) - E(s_1) = w'''(s_2)w'(s_2) - w'''(s_1)w'(s_1). \tag{26}$$

To equation (1) we may also associate a different *energy function*

$$\mathcal{E}(s) := \frac{1}{2}\,w''(s)^2 - \frac{k}{2}\,w'(s)^2 - w'(s)w'''(s) - F(w(s)) = E(s) - w'(s)w'''(s). \tag{27}$$

Then, if w solves (1), there holds

$$\mathcal{E}'(s) = w'''(s)w''(s) + w''''(s)w'(s) - (w'(s)w'''(s))' = 0 \quad \Longrightarrow \quad \mathcal{E}(s) = C\,, \tag{28}$$

for some $C \in \mathbb{R}$. Therefore, if one is interested in homoclinics, then $\mathcal{E}(s) = 0$ for all $s \in \mathbb{R}$.

Finally, a third useful energy function is available. Define

$$H(s) := w'(s)w''(s) - w(s)w'''(s) - kw(s)w'(s) \tag{29}$$

and its antiderivative

$$G(s) := w'(s)^2 - w(s)w''(s) - \frac{k}{2}\,w(s)^2. \tag{30}$$

A short computation gives

$$H'(s) = w''(s)^2 - kw'(s)^2 + w(s)f(w(s)). \tag{31}$$

Also this energy function will be used in the sequel. Here, we just make the following

Remark 19. If $k \leq 0$ and (3) holds, by (31) we infer that $H'(s) \geq 0$ so that H is nondecreasing and G is convex. If w is a homoclinic solution then $\lim_{s\to\pm\infty} H(s) = 0$, see Proposition 11. Hence, $H = H' \equiv 0$ and we have a simple proof of the fact that no homoclinic solution exists for (1).

3.2 A corresponding system

In some situations, it may be useful to transform the fourth order ode (1) into a *first order system* of four equations. Let $w = w(s)$ be a solution to (1) and put

$$Y(s) = (y_1(s), y_2(s), y_3(s), y_4(s)) = (w(s), w'(s), w''(s), w'''(s))$$

so that (1) may be rewritten as a system

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_4 \\ y_4' = -ky_3 - f(y_1). \end{cases} \tag{32}$$

If we define $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\Phi(y_1, y_2, y_3, y_4) = (y_2, y_3, y_4, -ky_3 - f(y_1))$$

then any solution $Y(s) = (y_1(s), y_2(s), y_3(s), y_4(s))$ of (32) may be rewritten as

$$Y'(s) = \Phi(Y(s)). \tag{33}$$

In view of (3), $f(s)$ admits a unique zero at $s = 0$. Therefore, the dynamical system (32) admits a unique stationary point which is $O = (0, 0, 0, 0)$. This point corresponds to the solution $w \equiv 0$ to (1). We now study the stability of O.

If we assume (15), then the linearized problem at O for (32) reads

$$Y'(s) = AY(s) , \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -k & 0 \end{pmatrix}.$$

The eigenvalues λ of A satisfy the equation $\lambda^4 + k\lambda^2 + 1 = 0$ and therefore also

$$\lambda^2 = \frac{-k \pm \sqrt{k^2 - 4}}{2}.$$

Hence, if $k < -2$ the eigenvalues of A are all real and are given by

$$\lambda \in \left\{ \pm\sqrt{\frac{|k| + \sqrt{k^2 - 4}}{2}}, \pm\sqrt{\frac{|k| - \sqrt{k^2 - 4}}{2}} \right\}. \tag{34}$$

If $-2 < k < 2$, the eigenvalues of A are

$$\lambda \in \left\{ \pm\frac{\sqrt{2 - k}}{2} \pm i\,\frac{\sqrt{2 + k}}{2}, \pm\frac{\sqrt{2 - k}}{2} \mp i\,\frac{\sqrt{2 + k}}{2} \right\}. \tag{35}$$

If $k > 2$, the eigenvalues of A are given by

$$\lambda \in \left\{ \pm i\sqrt{\frac{k + \sqrt{k^2 - 4}}{2}}, \pm i\sqrt{\frac{k - \sqrt{k^2 - 4}}{2}} \right\}.$$

If $k = -2$ the eigenvalues of A are $\lambda \in \{\pm 1\}$, they both have multiplicity 2, and the corresponding eigenvectors are $v_+ = (1, 1, 1, 1)$ and $v_- = (1, -1, 1, -1)$.

If $k = 2$ the eigenvalues of A are $\lambda \in \{\pm i\}$ (with multiplicity 2) and the corresponding eigenvectors are $v = (1, i, -1, -i)$ and $\overline{v} = (1, -i, -1, i)$.

Summarizing, we have

Proposition 20. *Assume* (3) *and* (15). *For any* $k \in \mathbb{R}$, (32) *has a unique stationary point* $O = (0,0,0,0)$ *which satisfies*

(i) *if* $k < -2$, *O has a 2-dimensional stable manifold and a 2-dimensional unstable manifold, both not oscillating near O;*

(ii) *if* $k = -2$, *O has a 2-dimensional stable manifold (tangent to* v_- *near O) and a 2-dimensional unstable manifold (tangent to* v_+ *near O);*

(iii) *if* $-2 < k < 2$, *O has a 2-dimensional stable manifold and a 2-dimensional unstable manifold, both having locally the form of a spiral near O;*

(iv) *if* $k = 2$, *the linearized problem at O has 2 (opposite) double purely imaginary eigenvalues;*

(v) *if* $k > 2$, *the linearized problem at O has 4 purely imaginary eigenvalues.*

As shown in [16, Exercise 3.1 p.216], when the linearized problem admits purely imaginary eigenvalues there is no direct way to deduce the stability properties of a stationary point. A possible way to proceed is to evaluate the sign of

$$\frac{1}{2}\frac{d}{ds}\|Y(s)\|^2 = y_1(s)y_2(s) + y_2(s)y_3(s) + (1-k)y_3(s)y_4(s) - y_4(s)f(y_1(s)).$$

But this seems a task out of reach.

Problem 21. Study the stability of the origin O for the dynamical system (32) in the case $k \geq 2$. Note that if $k > 0$ and w solves (1), then the function $z(s) = w(s/\sqrt{k})$ solves

$$z''''(s) + z''(s) + \frac{1}{k^2}f(z(s)) = 0 \qquad (s \in \mathbb{R}).$$

Therefore, as $k \to +\infty$ equation (1) may be seen as a perturbation of the equation $z''''(s) + z''(s) = 0$. Similarly, if $k \to -\infty$ equation (1) may be seen as a perturbation of the equation $z''''(s) - z''(s) = 0$.

A slightly different way to tackle the problem is to use a first integral, related to the energy function $\mathcal{E}$ in (27), defined by

$$J : \mathbb{R}^4 \to \mathbb{R}\,, \qquad J(y) = \frac{y_3^2}{2} - k\frac{y_2^2}{2} - y_2y_4 - F(y_1).$$

In order to prove that J is indeed a first integral, one needs to show that $\nabla J(y) \perp \Phi(y)$ for all $y \in \mathbb{R}^4$ and this follows by noticing that $\nabla J(y) \cdot \Phi(y) = 0$. Therefore, any orbit of (32) is contained in a surface level of J. In particular, for the stability of O, we are interested in the behavior of the surface at level 0:

$$S_0 = \{y \in \mathbb{R}^4 : \ J(y) = 0\}.$$

Since $O \in S_0$ we need to compute

$$\nabla J(O) = 0\,, \qquad D^2J(O) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -k & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

where $D^2J(O)$ denotes the Hessian matrix of J at O (recall (15)). The eigenvalues μ of this (symmetric) matrix are all real and

$$\mu \in \left\{\pm 1, \frac{-k \pm \sqrt{k^2+4}}{2}\right\}.$$

So, for any $k \in \mathbb{R}$ the Hessian matrix $D^2J(O)$ admits 2 positive and 2 negative eigenvalues and if $k \neq 0$ they all have multiplicity 1.

Remark 22. If instead of (15), we assume that $f'(0) = 0$, then the eigenvalues of the linearized matrix A become $\lambda = 0$ (double) and $\lambda = \pm\sqrt{-k}$. Therefore, the stability analysis of O only depends on the sign of k and appears more delicate. Moreover, the eigenvalues of the Hessian matrix $D^2J(0)$ are 0, 1, $\frac{-k\pm\sqrt{k^2+4}}{2}$.

4 Proof of Theorem 1

Let w be a local solution to (1) and let (ρ, R) be the maximal interval of continuation for w with $-\infty \le \rho < R \le +\infty$. We claim that $\rho = -\infty$ and $R = +\infty$. Since the function $t \mapsto w(-t)$ is also a solution of (1) it is sufficient to prove that $R = +\infty$. Since (1) is an autonomous equation, up to a translation we may assume that $R > 0$.

The next lemma states that a one-sided boundedness is enough to ensure global continuation.

Lemma 23. *Assume that f satisfies* (3) *and let w be a solution to* (1) *in a maximal interval of continuation* $(0, R)$. *The following implications hold*

$$\exists C \in \mathbb{R}\,, \quad w(s) \le C \quad \forall s \in (0, R) \implies R = +\infty,$$

$$\exists C \in \mathbb{R}\,, \quad w(s) \ge C \quad \forall s \in (0, R) \implies R = +\infty.$$

Proof. It is enough to prove that

$$w \quad \text{and} \quad w'' \quad \text{are bounded in } (0, R). \tag{36}$$

Indeed, from (1) and (36) we deduce that w'''' is also bounded in $(0, R)$ and, if $R < +\infty$, all the derivatives of w remain bounded so that the solution can be continued beyond R. But (36) can be further simplified. If we know that

$$w \quad \text{is bounded in } (0, R)\ , \tag{37}$$

then by setting $v(s) := w''(s) + kw(s)$ we see that v'' is bounded in $(0, R)$. Hence, if $R < +\infty$, also v is bounded and since we assume (37), we obtain (36). Therefore, the proof is complete if we show (37).

In what follows we denote by $C_i \in \mathbb{R}$ suitable constants. Assume that $w(s) \le C$ for all $s \in (0, R)$ and, for contradiction, that $R < +\infty$. Then, by (3), we have

$$v''(s) = w''''(s) + kw''(s) = -f(w(s)) \ge C_1 \qquad \forall s \in (0, R).$$

By integrating twice we get $v(s) \ge C_2$ in $(0, R)$.

If $k \ge 0$, this gives

$$w''(s) = v(s) - kw(s) \ge C_2 - kC$$

so that $w(s)$ is also bounded from below and (37) follows.

If $k < 0$, the lower bound on v yields

$$\begin{aligned} w(s) &= w(0)\cosh(\sqrt{-k}s) + \frac{w'(0)}{\sqrt{-k}}\sinh(\sqrt{-k}s) + \frac{1}{\sqrt{-k}}\int_0^s \sinh[\sqrt{-k}(s-t)]v(t)\,dt \\ &\ge w(0)\cosh(\sqrt{-k}s) + \frac{w'(0)}{\sqrt{-k}}\sinh(\sqrt{-k}s) + \frac{C_2}{\sqrt{-k}}\int_0^s \sinh[\sqrt{-k}(s-t)]\,dt \end{aligned}$$

so that $w(s)$ is also bounded from below and (37) follows for any $k \in \mathbb{R}$.

Assume now that $w(s) \ge C$ for all $s \in (0, R)$. Then, by repeating the above argument and reversing all the inequalities, we obtain an upper bound for $w(s)$. The proof is so complete. □

We now recall some well-known Poincaré inequalities. If $a < b$ (both finite!), then

$$\|u\|_2 \le (b-a)\|u'\|_2 \le (b-a)^2\|u''\|_2\ , \quad \max_{s\in[a,b]} |u(s)| \le \sqrt{b-a}\|u'\|_2 \qquad \forall u \in H^2 \cap H^1_0(a, b) \tag{38}$$

where $\|\cdot\|_2$ denotes the $L^2(a,b)$-norm.

Proof when (4) holds at $+\infty$. In this case, we know that

$$\exists M>0 \quad \text{such that} \quad tf(t)+F(t)\le 1+Mt^2 \qquad \forall t\ge 0. \tag{39}$$

Assume for contradiction that $R<+\infty$. Then Lemma 23 states that there exists an increasing sequence $\{\sigma_m\}$ such that

$$\lim_{m\to+\infty}\sigma_m=R, \quad w(s)\ge 0 \text{ for } s\in\bigcup_{\ell=0}^{\infty}[\sigma_{2\ell},\sigma_{2\ell+1}], \quad w(s)\le 0 \text{ for } s\in\bigcup_{\ell=0}^{\infty}[\sigma_{2\ell+1},\sigma_{2\ell+2}].$$

Clearly, we may choose σ_0 sufficiently close to R in such a way that

$$1-|k|(R-\sigma_0)^2-2M(R-\sigma_0)^4\ge\frac{1}{2}. \tag{40}$$

Multiply (1) by $w(s)$ and integrate by parts over $(\sigma_{2\ell},\sigma_{2\ell+1})$ for some $\ell\in\mathbb{N}$ to obtain

$$\int_{\sigma_{2\ell}}^{\sigma_{2\ell+1}} w'(s)w'''(s)\,ds+k\int_{\sigma_{2\ell}}^{\sigma_{2\ell+1}} w'(s)^2\,ds=\int_{\sigma_{2\ell}}^{\sigma_{2\ell+1}} f(w(s))w(s)\,ds. \tag{41}$$

By (27)-(28) we know that there exists $C\in\mathbb{R}$ (independent of ℓ!) such that

$$w'(s)w'''(s)=\frac{1}{2}\,w''(s)^2-\frac{k}{2}\,w'(s)^2-F(w(s))-C \qquad \forall s\in(\sigma_0,R).$$

Inserting this into (41), we infer that

$$\int_{\sigma_{2\ell}}^{\sigma_{2\ell+1}}[f(w(s))w(s)+F(w(s))]\,ds+C(\sigma_{2\ell+1}-\sigma_{2\ell})=\frac{1}{2}\int_{\sigma_{2\ell}}^{\sigma_{2\ell+1}} w''(s)^2\,ds+\frac{k}{2}\int_{\sigma_{2\ell}}^{\sigma_{2\ell+1}} w'(s)^2\,ds.$$

We estimate both sides of this identity by means of (38) and (39) and obtain

$$(C+1)(\sigma_{2\ell+1}-\sigma_{2\ell})+M\|w\|_2^2\ge\frac{1}{2}\left(\frac{1}{(\sigma_{2\ell+1}-\sigma_{2\ell})^2}-|k|\right)\|w'\|_2^2$$

where the L^2-norms are over the interval $(\sigma_{2\ell},\sigma_{2\ell+1})$. Using again (38) we then get

$$(C+1)(\sigma_{2\ell+1}-\sigma_{2\ell})\ge\frac{1}{2}\left(\frac{1}{(\sigma_{2\ell+1}-\sigma_{2\ell})^2}-|k|-2M(\sigma_{2\ell+1}-\sigma_{2\ell})^2\right)\|w'\|_2^2.$$

Since $\sigma_{2\ell+1}-\sigma_{2\ell}<R-\sigma_0$, by (40) and the last estimate we obtain

$$\|w'\|_2^2\le 4(C+1)(\sigma_{2\ell+1}-\sigma_{2\ell})^3\le 4(C+1)(R-\sigma_0)^3.$$

By applying once more (38) we finally obtain

$$\max_{s\in[\sigma_{2\ell},\sigma_{2\ell+1}]}|w(s)|=\max_{s\in[\sigma_{2\ell},\sigma_{2\ell+1}]}w(s)\le\gamma$$

for a suitable $\gamma>0$ independent of ℓ. Therefore, $w(s)$ is bounded from above on its region of positivity in (σ_0,R). Hence, by Lemma 23 it remains bounded and $R=+\infty$.

Proof when (4) holds at $-\infty$. It follows exactly the same steps as the previous case, we just have to consider the intervals $(\sigma_{2\ell+1},\sigma_{2\ell+2})$ instead of $(\sigma_{2\ell},\sigma_{2\ell+1})$. On these intervals we have $w(s)\le 0$ and this can be managed as above, provided we replace (39) with

$$\exists M>0 \quad \text{such that} \quad tf(t)+F(t)\le 1+Mt^2 \qquad \forall t\le 0.$$

5 Proof of Theorem 4

It suffices to prove the statement for $s \to +\infty$. Indeed, once this is done, to obtain the same statement for $s \to -\infty$ it is enough to remark that the equation (1) is invariant under the change of variables $s \mapsto -s$.

We first prove the weaker statement (6).

Lemma 24. *Let $k \geq 0$ and f satisfy* (3)*. If w is a global solution to* (1)*, then* (6) *holds.*

Proof. Assume for contradiction that

$$\liminf_{s \to +\infty} w(s) \in (0, +\infty] \,. \tag{42}$$

Then there exist $\sigma \in \mathbb{R}$ and $\gamma > 0$ such that $f(w(s)) \geq \gamma$ for all $s \geq \sigma$. Hence,

$$[w''(s) + kw(s)]'' = w''''(s) + kw''(s) \leq -\gamma \qquad \forall s \geq \sigma \,.$$

This negative upper bound for the second derivative of the map $s \mapsto w''(s) + kw(s)$ shows that

$$\lim_{s \to +\infty} \Big(w''(s) + kw(s) \Big) = -\infty \,.$$

By (42), this readily implies that $w''(s) \to -\infty$ as $s \to +\infty$, which contradicts (42).

If we assume now that

$$\limsup_{s \to +\infty} w(s) \in [-\infty, 0) \,,$$

in a similar way we conclude that $w''(s) \to +\infty$ as $s \to +\infty$, thereby reaching a contradiction. □

Next we show that, if $k \geq 0$, then global solutions to (1) cannot maintain the same sign. First we deal with the case $k > 0$:

Lemma 25. *Let $k > 0$ and f satisfy* (3)*. If w is a global nontrivial solution to* (1)*, then $w(s)$ changes sign infinitely many times as $s \to +\infty$.*

Proof. Assume for contradiction that $w(s)$ is eventually nonnegative. Since (1) is autonomous, up to a translation, this corresponds to say that

$$w(s) \geq 0 \quad \forall s \geq 0 \,. \tag{43}$$

If (43) occurs, then by (1) we deduce

$$[w'''(s) + kw'(s)]' = w''''(s) + kw''(s) = -f(w(s)) \leq 0 \quad \forall s \geq 0$$

so that $s \mapsto w'''(s) + kw'(s)$ is nonincreasing and two cases may occur: either its limit for $s \to +\infty$ is strictly negative or it is nonnegative. Recalling again that (1) is autonomous, this gives the following alternatives:

$$(i) \quad w'''(s) + kw'(s) < 0 \quad \forall s \geq 0 \,, \qquad (ii) \quad w'''(s) + kw'(s) \geq 0 \quad \forall s \geq 0 \,.$$

The proof will be complete if we show that both cases (i) and (ii) lead to a contradiction.

Case (i) cannot occur. Indeed, if it occurs then

$$[w''(s) + kw(s)]' = w'''(s) + kw'(s) \to -K \in [-\infty, 0)$$

proving that $w''(s) + kw(s) \to -\infty$. In turn, by (43), this shows that $w''(s) \to -\infty$ and contradicts (43).

Case (ii) cannot occur. This case is more delicate. If it occurs, then

$$[w''(s) + kw(s)]' = w'''(s) + kw'(s) \geq 0$$

so that

$$s \mapsto w''(s) + kw(s) \quad \text{is nondecreasing} \tag{44}$$

and the following limit exists

$$\lim_{s\to+\infty}[w''(s) + kw(s)] = \ell \in (-\infty, +\infty] \,. \tag{45}$$

We first rule out the case where

$$\ell \leq 0 \,. \tag{46}$$

Indeed, if (46) holds, then from (44) we infer that $w''(s) + kw(s) \leq 0$ for $s \geq 0$ so that, by (43), also $w''(s) \leq 0$ for $s \geq 0$. Since $w(s)$ is nonnegative and concave (but not identically zero), it attains a strictly positive limit. By (45)-(46), this shows that $s \mapsto w''(s)$ has a strictly negative limit, contradicting (43).

Next, we rule out the case where

$$\ell \in (0, +\infty) \,. \tag{47}$$

For contradiction, assume (47). According to (43) and Lemma 24, two subcases may occur:

$$\text{either} \quad \lim_{s\to+\infty} w(s) = 0 \quad \text{or} \quad 0 = \liminf_{s\to+\infty} w(s) < \limsup_{s\to+\infty} w(s). \tag{48}$$

The first situation in (48) may be excluded by noticing that, together with (45) and (47), it yields $w''(s) \to \ell > 0$ which implies $w(s) \to +\infty$ and contradicts $w(s) \to 0$. The second situation in (48) implies that there exist two divergent sequences $\{s_m^j\}_{j\in\mathbb{N}}$ and $\{s_M^j\}_{j\in\mathbb{N}}$ of local minima and local maxima for w such that

$$w'(s) \geq 0 \quad \forall s \in [s_m^j, s_M^j] \,, \qquad w'(s) \leq 0 \quad \forall s \in [s_M^j, s_m^{j+1}] \tag{49}$$

for all $j \in \mathbb{N}$. Multiplying (45) by $w'(s)$ gives

$$w''(s)w'(s) + kw'(s)w(s) = (\ell + o(1))w'(s) \qquad \text{as } s \to +\infty$$

which, integrated over $[s_m^j, s_M^j]$, yields

$$\frac{k}{2}[w(s_M^j)^2 - w(s_m^j)^2] = (\ell + o(1))[w(s_M^j) - w(s_m^j)] \qquad \text{as } j \to +\infty$$

and, finally,

$$\frac{k}{2}[w(s_M^j) + w(s_m^j)] = \ell + o(1) \qquad \text{as } j \to +\infty \,. \tag{50}$$

By (43) and Lemma 24, we infer that

$$\liminf_{j\to+\infty} w(s_m^j) = \liminf_{s\to+\infty} w(s) = 0.$$

From now on, we consider a subsequence of $\{s_m^j\}_{j\in\mathbb{N}}$ (which we still denote in the same fashion) such that

$$\lim_{j\to+\infty} w(s_m^j) = 0. \tag{51}$$

Consider also the corresponding sequence $\{s_M^j\}_{j\in\mathbb{N}}$ defined by (49). Inserting these sequences into (50) proves that

$$\lim_{j\to+\infty} w(s_M^j) = \frac{2\ell}{k} \,. \tag{52}$$

By inserting (51) and (52) into (45) we deduce that

$$\lim_{j\to+\infty} w''(s_m^j) = \ell \,, \qquad \lim_{j\to+\infty} w''(s_M^j) = -\ell \,. \tag{53}$$

Using the energy function defined in (24) and applying Lemma 18, we obtain

$$\frac{w''(s_m^j)^2}{2} - F(w(s_m^j)) = E(s_m^j) = E(s_M^j) = \frac{w''(s_M^j)^2}{2} - F(w(s_M^j))$$

which, by letting $j \to +\infty$ and taking into account (51)-(52)-(53), yields

$$\frac{\ell^2}{2} - F(0) = \frac{\ell^2}{2} - F\left(\frac{2\ell}{k}\right).$$

Recalling that $F(t) = F(0) = 0$ if and only if $t = 0$, this implies that $\ell = 0$ and contradicts (47). Therefore, also the second situation in (48) leads to a contradiction. This rules out (47).

Since both (46) and (47) are ruled out, it remains to consider the case where

$$\ell = +\infty. \tag{54}$$

Again, we have the two subcases (48). If $w(s) \to 0$, then (45) and (54) give a contradiction. If we have oscillations, then we still have (51)-(52) and the first of (53) (with $\ell = +\infty$) so that $E(s_m^j) \to +\infty$ as $j \to +\infty$. But since $j \mapsto E(s_m^j)$ is constant in view of Lemma 18, this gives again a contradiction.

We have so shown that (43) cannot occur since in any case (and subcase) we reach a contradiction. In a completely similar way (by changing all the signs involved) one can show that also

$$w(s) \le 0 \quad \forall s \ge 0$$

cannot occur. This completes the proof. □

We conclude with the case $k = 0$:

Lemma 26. *Let $k = 0$ and f satisfy* (3). *If w is a global nontrivial solution to* (1)*, then $w(s)$ changes sign infinitely many times as $s \to +\infty$.*

Proof. Assume for contradiction that (43) holds. According to Lemma 24, the two subcases in (48) may occur. Assume first that $\lim_{s\to+\infty} w(s) = 0$. By Proposition 11 we know that

$$\lim_{s\to+\infty} w'(s) = \lim_{s\to+\infty} w''(s) = \lim_{s\to+\infty} w'''(s) = 0\,. \tag{55}$$

On the other hand, by (43) and (3), we get that

$$w''''(s) = -f(w(s)) \le 0 \qquad \forall s \ge 0.$$

This implies that $s \mapsto w'''(s)$ is nonincreasing and, in turn, by (55) that

$$w'''(s) \ge 0 \qquad \forall s \ge 0.$$

The same argument iterated leads to

$$w''(s) \le 0 \qquad \text{and} \qquad w'(s) \ge 0 \qquad \forall s \ge 0$$

and finally to

$$w(s) \le 0 \qquad \forall s \ge 0.$$

Together with (43), this gives the contradiction $w \equiv 0$.

Now we consider the second situation in (48). Let $\{s_M^j\}_{j\in\mathbb{N}}$ (resp. $\{s_m^j\}_{j\in\mathbb{N}}$) denote the increasing divergent sequence of local maxima (resp. minima) of w. By what observed above and by (43), we know that the map $s \mapsto w''(s)$ is concave and, in turn that $\lim_{s\to+\infty} w''(s)$ exists. Since $w''(s_m^j) \geq 0$ and $w''(s_M^i) \leq 0$ we infer that

$$\lim_{s\to+\infty} w''(s) = 0. \tag{56}$$

On the other hand, by Lemma 24 and (48) we infer that, up to a subsequence,

$$\lim_{j\to+\infty} w(s_m^j) = 0 \quad \text{and} \quad \lim_{i\to+\infty} w(s_M^i) = \delta \in (0, +\infty].$$

Using the energy function defined in (24) and applying Lemma 18, we obtain

$$\frac{w''(s_m^j)^2}{2} - F(w(s_m^j)) = E(s_m^j) = E(s_M^i) = \frac{w''(s_M^i)^2}{2} - F(w(s_M^i))$$

which, by letting $i, j \to +\infty$, and using (56) yields $F(\delta) = F(0)$. Since $\delta > 0$, this gives a contradiction.

By reversing all signs, one obtains that it cannot eventually be $w(s) \leq 0$. □

6 Proof of Theorem 5

Assume first that (8) holds and consider a solution w of (1) satisfying the following initial conditions

$$w(0) = 0, \qquad w'(0) = 0, \qquad w''(0) > \frac{M}{|k|} > 0, \qquad w'''(0) = 0 \tag{57}$$

where M is as in (8). Since $k < 0$, (1) and (57) imply that $w''''(0) = -kw''(0) > M$. Define

$$\overline{s} := \sup\{s > 0 : w''''(\sigma) > 0 \text{ for all } \sigma \in (0, s)\} \in (0, +\infty] .$$

By (8) and Theorem 1 we know that w is defined on the whole real line and we claim that $\overline{s} = +\infty$. Assume by contradiction that $\overline{s} < +\infty$, then

$$w''''(\overline{s}) = 0 . \tag{58}$$

Since w''' is increasing in $(0, \overline{s}]$ and $w'''(0) = 0$, then w''' is positive in $(0, \overline{s}]$. Hence w'' is increasing in $(0, \overline{s}]$ and since $w''(0) > 0$, also w'' is positive in $(0, \overline{s}]$. In turn, w' is increasing in $(0, \overline{s}]$ and since $w'(0) = 0$, we infer that w' is positive in $(0, \overline{s}]$. This finally shows that w is increasing in $(0, \overline{s}]$ and since $w(0) = 0$ we infer that w is positive in $(0, \overline{s}]$. Therefore, by (1) and (57)

$$w''''(\overline{s}) = |k|w''(\overline{s}) - f(w(\overline{s})) \geq |k|w''(0) - M > 0$$

in contradiction with (58). This proves that $\overline{s} = +\infty$. In particular by the above iterative scheme, we have that w is positive, increasing, and convex in $(0, +\infty)$. This shows that (9) holds and completes the proof of the first part of the theorem.

When (10) holds, the same proof can be repeated by assuming in (57) that $w''(0) < -M/|k| < 0$ and reversing all signs.

7 Proof of Theorem 6

We first mention that the statement about the infimum follows by Proposition 12 once we prove the statement about the supremum. Hence, thanks to the change of variables $s \mapsto -s$, it is sufficient to prove that $\limsup\limits_{s\to+\infty} w(s) < +\infty$. To this end, we prove the next lemma and then we proceed in several steps.

Lemma 27. *Suppose $k < 0$ and that f satisfies* (3). *Let w be a global, eventually of one sign solution of* (1). *Then the limit*

$$L_w = \lim_{s\to+\infty} w(s) \tag{59}$$

exists and $L_w \in \{0, \pm\infty\}$. A similar statement holds true as $s \to -\infty$.

Proof. We prove the lemma as $s \to +\infty$, (1) being invariant under the change of variables $s \mapsto -s$. Assume $w(s) \geq 0$ eventually, the other case being similar. We claim that also w'' has eventually the same sign. If not, consider an interval (a, b) where $w(s) \geq 0$, $w''(s) < 0$ and $w''(a) = w''(b) = 0$. Then by multiplying (1) by w'' and integrating we obtain

$$-\int_a^b w'''(s)^2\,ds + k\int_a^b w''(s)^2\,ds = -\int_a^b w''(s)f(w(s))\,ds \geq 0. \tag{60}$$

Since the left hand side is the sum of non-positive terms, we get $w''' \equiv 0$ in (a, b); this makes w'' constant in (a, b) and hence $w'' \equiv 0$ in (a, b), being $w''(a) = w''(b) = 0$, a contradiction.

Since w'' is eventually of one sign, the limit L_w defined by (59) exists. Suppose L_w is nonzero and finite for the sake of contradiction. Then by (1) we have

$$\lim_{s\to+\infty}[w''(s) + kw(s)]'' = -\lim_{s\to+\infty} f(w(s)) = -f(L_w) < 0,$$

so $w''(s) + kw(s) \to -\infty$ as $s \to +\infty$. On the other hand, $w(s)$ has a finite limit by assumption, so also $w''(s) \to -\infty$ as $s \to +\infty$, a contradiction. □

Step 1. We prove that if

$$\lim_{s\to+\infty} w(s) = +\infty \tag{61}$$

holds then, up to a translation, we have

$$w(s) > 0 \qquad w'(s) > 0 \qquad v(s) < 0 \qquad v'(s) < 0 \qquad \text{for all } s > 0\,, \tag{62}$$

where $v(s) = w''(s) + kw(s)$.

Assume that there exists a global solution w of (1) such that (61) holds. Since equation (1) is autonomous, it suffices to prove that the four inequalities in (62) hold eventually. The first one is trivial in view of (61). Since $v''(s) = -f(w(s))$, then by (61), (11) and two integrations, it follows that

$$\lim_{s\to+\infty} v'(s) = -\infty \qquad \text{and} \qquad \lim_{s\to+\infty}[w''(s) + kw(s)] = \lim_{s\to+\infty} v(s) = -\infty$$

so that the last two inequalities in (62) are proved.

Proceeding as in the proof of Lemma 27 one can show that w'' has eventually the same sign and hence w' is eventually monotonic. Since $\lim_{s\to+\infty} w(s) = +\infty$, w' is eventually positive.

Step 2. We prove that if

$$\text{there exists } C > 0 \text{ such that for any } \overline{s} > 0 \text{ there exists } s_0 > \overline{s} \text{ such that } w'(s_0) \leq Cw(s_0) \tag{63}$$

then (61) does not hold.

By contradiction, assume (63) and let (61) hold. In view of (61), (11), we may take $\overline{s} > 0$ such that

$$f(w(s)) > \theta w(s) \qquad \text{for any } s > \overline{s}\,, \tag{64}$$

where $\theta > k^2 + C|k|^{3/2}$ is fixed. Let s_0 be as in (63), we can write for any $s > s_0$

$$w(s) = w(s_0)\cosh\left[\sqrt{|k|}(s - s_0)\right] + \frac{w'(s_0)}{\sqrt{|k|}}\sinh\left[\sqrt{|k|}(s - s_0)\right] + \frac{1}{\sqrt{|k|}}\int_{s_0}^{s}\sinh[\sqrt{|k|}(s - t)]v(t)\,dt \tag{65}$$

and after two integrations by parts, using (62), (63) and (64), we obtain

$$
\begin{aligned}
0 \leq w(s) =& w(s_0)\cosh\left[\sqrt{|k|}(s-s_0)\right] + \frac{w'(s_0)}{\sqrt{|k|}}\sinh\left[\sqrt{|k|}(s-s_0)\right] + \frac{|v(s)|}{|k|} \\
&- \frac{|v(s_0)|}{|k|}\cosh\left[\sqrt{|k|}(s-s_0)\right] - \frac{|v'(s_0)|}{|k|^{3/2}}\sinh\left[(\sqrt{|k|}(s-s_0)\right] \\
&- \frac{1}{|k|^{3/2}}\int_{s_0}^{s}\sinh\left[\sqrt{|k|}(s-t)\right]f(w(t))\,dt \\
\leq& w(s_0)\cosh\left[\sqrt{|k|}(s-s_0)\right] + \frac{w'(s_0)}{\sqrt{|k|}}\sinh\left[\sqrt{|k|}(s-s_0)\right] + \frac{|v(s)|}{|k|} \\
&- \frac{\theta}{k^2}w(s_0)\left\{\cosh\left[\sqrt{|k|}(s-s_0)\right] - 1\right\} \\
\leq& \frac{|v(s)|}{|k|} + \frac{e^{-\sqrt{|k|}s_0}w(s_0)}{2k^2}\left(k^2 + C|k|^{3/2} - \theta\right)e^{\sqrt{|k|}s} + o\big(e^{\sqrt{|k|}s}\big) \quad \text{as } s \to +\infty.
\end{aligned}
$$

Since $\theta > k^2 + C|k|^{3/2}$, this yields

$$
\exists \widetilde{C}(\theta, s_0) > 0 \ \text{ such that } \ \frac{v(s)}{\sqrt{|k|}} \leq -\widetilde{C}(\theta, s_0)e^{\sqrt{|k|}s} \qquad \text{for all } s > s_0.
$$

By using this into (65) we obtain for all $s > s_0$

$$
\begin{aligned}
w(s) \leq& w(s_0)\cosh\left[\sqrt{|k|}(s-s_0)\right] + \frac{w'(s_0)}{\sqrt{|k|}}\sinh\left[\sqrt{|k|}(s-s_0)\right] \\
&- \frac{\widetilde{C}(\theta, s_0)}{2}se^{\sqrt{|k|}s} + \left(\frac{s_0\widetilde{C}(\theta, s_0)}{2} + \frac{\widetilde{C}(\theta, s_0)}{4\sqrt{|k|}}\right)e^{\sqrt{|k|}s} - \frac{\widetilde{C}(\theta, s_0)e^{2\sqrt{|k|}s_0}}{4\sqrt{|k|}}e^{-\sqrt{|k|}s} \\
=& -\frac{\widetilde{C}(\theta, s_0)}{2}se^{\sqrt{|k|}s} + o\left(se^{\sqrt{|k|}s}\right) \qquad \text{as } s \to +\infty\ .
\end{aligned}
$$

This contradiction proves that (61) does not occur whenever (63) holds.

Step 3. We prove that if

$$
\text{for any } C > 0 \text{ there exists } \overline{s} > 0 \text{ such that for any } s > \overline{s},\ w'(s) > Cw(s) \tag{66}
$$

then (61) does not hold.

Fix $C = \sqrt{|k|}$ and consider the corresponding $\overline{s}$ for which (66) holds true. Then, an integration yields

$$
w(s) \geq w(\overline{s})e^{\sqrt{|k|}(s-\overline{s})} \qquad \text{for all } s > \overline{s}\ .
$$

By (11), possibly choosing a larger $\overline{s}$, we may suppose that for some $\theta > 0$ we have

$$
v''(s) = -f(w(s)) < -\theta w(s) \leq -\theta w(\overline{s})e^{\sqrt{|k|}(s-\overline{s})} \qquad \text{for any } s > \overline{s}.
$$

After two integrations of this inequality we obtain $v(s) < -C(\theta)e^{\sqrt{|k|}s}$ for all $s > \overline{s}$. Inserting this into (65) and proceeding as in the previous case we reach a contradiction with (61). This contradiction proves that (61) does not occur even if (66) holds.

Step 4. We infer that (61) does not occur.

Indeed, assumptions (63) and (66) exhaust all possible situations.

Step 5. We prove that w is bounded from above at $+\infty$.

Suppose by contradiction that w is not bounded from above at $+\infty$. Since in Step 4, we ruled out (61), this means that

$$-\infty \le \liminf_{s\to+\infty} w(s) < \limsup_{s\to+\infty} w(s) = +\infty \, . \tag{67}$$

Hence, there exists an increasing divergent sequence $\{s_m\}_{m\in\mathbb{N}}$ of local maxima of w such that

$$\lim_{m\to+\infty} w(s_m) = +\infty \, . \tag{68}$$

By (67) and Theorem 8 (whose proof is independent of Theorem 6!) we have

$$L := \lim_{s\to+\infty} [w'(s)w''(s) - w(s)w'''(s) - kw(s)w'(s)] = +\infty \, . \tag{69}$$

By Lemma 27, if w were eventually nonnegative then w would admit a limit as $s \to +\infty$ in contradiction with (67). Therefore, w changes sign infinitely many times and hence for any $m \in \mathbb{N}$ we may define

$$z_m = \sup\{s > s_m : w > 0 \text{ in } (s_m, s)\} < +\infty \, .$$

By Lemma 18 there exists $C \in \mathbb{R}$ such that

$$\frac{1}{2} w''(s_m)^2 - F(w(s_m)) = C$$

so that by (68) and (11) it follows that

$$\lim_{m\to+\infty} w''(s_m) = -\infty \qquad \text{and} \qquad w(s_m) = o(w''(s_m)) \quad \text{as } m \to +\infty \, .$$

This shows that

$$\lim_{m\to+\infty} v(s_m) = -\infty \tag{70}$$

where, again $v := w'' + kw$. By (69) we infer that

$$\lim_{m\to+\infty} w(s_m)w'''(s_m) = -\lim_{m\to+\infty} [w'(s_m)w''(s_m) - w(s_m)w'''(s_m) - kw(s_m)w'(s_m)] = -\infty \, .$$

This proves that $w'''(s_m)$ is eventually negative. Hence, since s_m is a stationary point for w and $v'(s_m) = w'''(s_m) + kw'(s_m) = w'''(s_m)$, we infer that there exists $\overline{m} \in \mathbb{N}$ such that

$$v'(s_m) < 0 \qquad \text{for any } m > \overline{m} \, . \tag{71}$$

Since $w > 0$ in (s_m, z_m), by (1) and (3) we deduce that $v'' < 0$ in (s_m, z_m). Inequality (71) then yields

$$v'(z_m) < 0 \qquad \text{for any } m > \overline{m} \, . \tag{72}$$

Actually $v' < 0$ in (s_m, z_m) and hence by (70)

$$\lim_{m\to+\infty} v(z_m) \le \lim_{m\to+\infty} v(s_m) = -\infty \, . \tag{73}$$

By (1), (10) and (72) we have for any $m > \overline{m}$

$$v'(s) = v'(z_m) - \int_{z_m}^{s} f(w(t))\, dt \le M(s - z_m) \qquad \text{for any } s > z_m \, .$$

By integrating the latter inequality over the interval (z_m, s) we obtain

$$v(s) \le v(z_m) + M \int_{z_m}^{s} (t - z_m)\, dt = v(z_m) + \frac{M}{2}(s - z_m)^2 \qquad \text{for any } s > z_m \, . \tag{74}$$

By (73) we may fix $m > \overline{m}$ large enough such that $v(z_m) < -\frac{M}{|k|}$. Since $w(z_m) = 0$ and $w'(z_m) \le 0$, by (1), (74) and integration we obtain

$$\begin{aligned} w(s) &= \frac{w'(z_m)}{\sqrt{|k|}} \sinh[\sqrt{|k|}(s - z_m)] + \frac{1}{\sqrt{|k|}} \int_{z_m}^{s} \sinh[\sqrt{|k|}(s - t)] v(t)\, dt \\ &\le \frac{v(z_m)}{\sqrt{|k|}} \int_{z_m}^{s} \sinh[\sqrt{|k|}(s - t)]\, dt + \frac{M}{2\sqrt{|k|}} \int_{z_m}^{s} (t - z_m)^2 \sinh[\sqrt{|k|}(s - t)]\, dt \\ &= \left(\frac{v(z_m)}{|k|} + \frac{M}{k^2} \right) \left\{ \cosh[\sqrt{|k|}(s - z_m)] - 1 \right\} - \frac{M}{2|k|}(s - z_m)^2 \le 0 \qquad \text{for any } s > z_m \,. \end{aligned}$$

This shows that w is eventually negative, in contradiction with (67).

The second part of the statement can be achieved by reversing all signs in the above proof.

8 Proof of Theorem 8

We first prove the following statement which has its own independent interest.

Lemma 28. *Let $k \le 0$ and f satisfy* (3). *Assume that w is a solution to equation* (1) *such that w has two local extrema. Then on any closed interval whose bounds are two consecutive local extrema of w, the maximum of $s \mapsto |w''(s)|$ is attained in one of the extrema.*

Proof. Assume that $s_1 < s_2$ are two consecutive local extrema of w and let $\overline{s} \in [s_1, s_2]$ be the global maximum of $s \mapsto |w''(s)|$. For contradiction, assume that $\overline{s} \in (s_1, s_2)$. Then $w(\overline{s}) \in (w(s_1), w(s_2))$ (or the converse if $w(s_2) < w(s_1)$) and, by (3),

$$\max\{F(w(s_1)), F(w(s_2))\} > F(w(\overline{s}))\,.$$

Let $s_* \in \{s_1, s_2\}$ be such that $F(w(s_*)) = \max\{F(w(s_1)), F(w(s_2))\}$ so that

$$F(w(s_*)) > F(w(\overline{s})). \tag{75}$$

Since $s \mapsto w''(s)^2$ attains its maximum at $\overline{s}$ we have

$$w'''(\overline{s}) = 0 \qquad \text{and} \qquad |w''(\overline{s})| \ge |w''(s_*)|. \tag{76}$$

Using the fact that the energy function $\mathcal{E}$ defined in (27) is constant, we obtain

$$\frac{w''(s_*)^2}{2} - F(w(s_*)) = \mathcal{E}(s_*) = \mathcal{E}(\overline{s}) = \frac{w''(\overline{s})^2}{2} - \frac{k}{2}\, w'(\overline{s})^2 - F(w(\overline{s}))$$

which, recalling $k \le 0$, contradicts (75)-(76). □

Let $k < 0$ and consider the functions $H(s)$ and $G(s)$, defined in (14) and (30). Since $H(s)$ is increasing by (31), both $G(s)$ and $H(s)$ must attain a limit as $s \to +\infty$.

Proof of (i). Suppose $H(s)$ is bounded as $s \to +\infty$, in which case

$$\int_0^s [w''(t)^2 - kw'(t)^2 + w(t)f(w(t))]\, dt = H(s) - H(0) < \infty.$$

Letting $s \to +\infty$, we deduce that $w', w'' \in L^2(0, \infty)$ so that $w' \in H^1(0, \infty)$ and

$$\lim_{s \to +\infty} w'(s) = 0. \tag{77}$$

Case 1. Suppose $w(s)$ changes sign infinitely many times as $s \to +\infty$. Then there exists a divergent sequence $\{s_j\}_{j\in\mathbb{N}}$ of roots of $w(s)$ such that $w(s)$ has one sign on $[s_j, s_{j+1}]$ for each $j \in \mathbb{N}$. Let t_j be any of the global extrema of $w(s)$ on $[s_j, s_{j+1}]$. Since G admits a limit, by (77) we know that

$$\lim_{s\to+\infty} G(s) = \lim_{j\to+\infty} G(s_j) = \lim_{j\to+\infty} w'(s_j)^2 = 0.$$

Therefore,

$$\lim_{j\to+\infty} G(t_j) = \lim_{j\to+\infty} \left[w(t_j)w''(t_j) + \frac{k}{2}\, w(t_j)^2 \right] = 0.$$

Since $w(t_j)w''(t_j) \le 0$ and $k < 0$, this proves that $w(t_j) \to 0$ as $j \to \infty$, hence $w(s) \to 0$ as $s \to +\infty$. In view of Proposition 11, this also implies

$$\lim_{s\to+\infty} H(s) = \lim_{s\to+\infty} [w'(s)w''(s) - w(s)w'''(s) - kw(s)w'(s)] = 0, \tag{78}$$

so $H(s)$ is increasing towards zero and the result follows.

Case 2. Suppose $w(s)$ is eventually of one sign. In what follows, we assume $w(s) \ge 0$ eventually, the other case being similar. In this case, the map $s \mapsto w'''(s) + kw'(s)$ is eventually decreasing because its derivative equals $-f(w(s)) \le 0$. Recalling (77), we deduce that the limit

$$\lim_{s\to+\infty} [w'''(s) + kw'(s)] = \lim_{s\to+\infty} w'''(s)$$

exists and it is equal to zero. Being eventually decreasing to zero, $w'''(s) + kw'(s)$ is eventually nonnegative, so the limit

$$\ell_1 = \lim_{s\to+\infty} [w''(s) + kw(s)] \in (-\infty, +\infty]$$

exists as well. Since $w(s) \ge 0$ eventually by assumption, Lemma 27 also ensures the existence of

$$\ell_2 = \lim_{s\to+\infty} [-kw(s)] \in \{0, +\infty\}.$$

Were $\ell_2 = +\infty$, we would have $\lim_{s\to+\infty} w''(s) = \ell_1 + \ell_2 = +\infty$, contrary to (77). Hence, $\ell_2 = 0$ and the result follows as in Case 1.

Proof of (ii)**.** If $H(s_0) > 0$ at some point s_0, then $H(s)$ is unbounded as $s \to +\infty$ by part (i). Suppose $w(s)$ is bounded as $s \to +\infty$ for the sake of contradiction. Were $w(s)$ eventually of one sign, we would have $w(s) \to 0$ as $s \to +\infty$ by Lemma 27 and also (78) by Proposition 11, a contradiction. Thus, $w(s)$ has a divergent sequence $\{t_j\}_{j\in\mathbb{N}}$ of local extrema. According to Lemma 18, there exists a constant $C \in \mathbb{R}$ such that

$$w''(t_j)^2 = C + 2F(w(t_j))$$

for all $j \in \mathbb{N}$. Since $w(s)$ is assumed to be bounded as $s \to +\infty$, the sequence $\{w''(t_j)\}_{j\in\mathbb{N}}$ is bounded, so Lemma 28 ensures that $w''(s)$ is itself bounded. Using the inequality

$$\sup_{s\ge0} w'(s)^2 \le 4 \sup_{s\ge0} |w(s)| \cdot \sup_{s\ge0} |w''(s)|,$$

we conclude that $w'(s)$ is uniformly bounded as well. On the other hand, we have

$$\lim_{s\to+\infty} G'(s) = \lim_{s\to+\infty} H(s) = +\infty.$$

Hence,

$$\lim_{s\to+\infty} [w'(s)^2 - w(s)w''(s) - \frac{k}{2}\, w(s)^2] = \lim_{s\to+\infty} G(s) = +\infty.$$

This is absurd since w, w' and w'' are all uniformly bounded as $s \to +\infty$.

9 Proof of Theorem 9

Proof of statement (i)**.** Assume the first in (16) holds. It suffices to prove the statement as $s \to +\infty$ since the statement as $s \to -\infty$ may be obtained by observing that (1) is invariant under the change of variables $s \mapsto -s$. Let w be a global solution to (1) such that

$$\lim_{s \to +\infty} w(s) = 0.$$

By Proposition 11 we know that

$$\lim_{s \to +\infty} (w(s), w'(s), w''(s), w'''(s)) = (0, 0, 0, 0) . \tag{79}$$

By Proposition 20 the linear problem $v'''' + kv'' + v = 0$ has four real eigenvalues $\pm\lambda$, $\pm\mu$ with $\lambda \geq \mu > 0$; write

$$u(s) = (\partial_s + \lambda)(\partial_s + \mu)w(s), \qquad (\partial_s - \lambda)(\partial_s - \mu)u(s) = w(s) - f(w(s)).$$

Since $w \leq f(w)$ near $w = 0$ by (16), we have $(\partial_s - \lambda)(\partial_s - \mu)u(s) \leq 0$ for all large enough s. In particular, $s \mapsto e^{-\lambda s}(u'(s) - \mu u(s))$ is decreasing to zero, so $s \mapsto e^{-\mu s}u(s)$ is increasing to zero and

$$(\partial_s + \lambda)(\partial_s + \mu)w(s) = u(s) \leq 0$$

eventually. This makes $s \mapsto e^{\lambda s}(w'(s) + \mu w(s))$ decreasing and we consider two cases.

Case 1. If $w' + \mu w$ is negative at some s_0, then it is negative for all $s \geq s_0$. In this case, $s \mapsto e^{\mu s}w(s)$ is decreasing and attains a limit as $s \to +\infty$. If this limit is positive (or negative), then w has eventually the same sign and we are done. If this limit is zero, then $s \mapsto e^{\mu s}w(s)$ is decreasing to zero, so w is eventually positive.

Case 2. If $w' + \mu w$ is nonnegative at all points, then $s \mapsto e^{\mu s}w(s)$ is increasing and admits a limit as $s \to +\infty$. As in the previous case, one sees that w is eventually of one sign.

The proof when the other inequality holds true in (16) works similarly by reversing all inequalities.

Proof of statement (ii)**.** Suppose by contradiction that there exists $\overline{s}_1$ such that w is of one sign in $(\overline{s}_1, +\infty)$. We start supposing that w is nonnegative in $(\overline{s}_1, +\infty)$. Then by Lemma 27 we infer that w admits a limit ℓ satisfying $\ell \in \{0, +\infty\}$.

The case $\ell = 0$. If $\ell = 0$ we may apply Proposition 11 to obtain (79), namely the solution

$$Y(s) = (y_1(s), y_2(s), y_3(s), y_4(s)) = (w(s), w'(s), w''(s), w'''(s))$$

of the corresponding dynamical system (32) converges to $(0, 0, 0, 0)$ as $s \to +\infty$. But if $-2 < k < 0$, the linearized system at the origin has four complex eigenvalues with a nontrivial real part, see Proposition 20. Moreover the stable manifold at the origin is two dimensional and it is tangent to the plane

$$\mathbf{\Pi} := \{a\mathbf{x_1} + b\mathbf{x_2} : a, b \in \mathbb{R}\}$$

where

$$\mathbf{x_1} := \left(1, -\frac{\sqrt{2-k}}{2}, -\frac{k}{2}, \frac{(k+1)\sqrt{2-k}}{2}\right), \qquad \mathbf{x_2} := \left(0, \frac{\sqrt{2+k}}{2}, -\frac{\sqrt{4-k^2}}{2}, -\frac{(k-1)\sqrt{2+k}}{2}\right).$$

This means that the hyperplane in $\mathbb{R}^4$

$$\mathbf{H} := \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : y_1 = 0\}$$

and the plane $\mathbf{\Pi}$ intersect transversally. We may conclude that any trajectory of (32) which converges to the origin as $s \to +\infty$, intersects the hyperplane $\mathbf{H}$ infinitely many times or equivalently the corresponding solution w of (1) changes sign infinitely many times as $s \to +\infty$.

The case $\ell = +\infty$. By (3) and (17), we obtain

$$\lim_{s\to+\infty} v''(s) = -\lim_{s\to+\infty} f(w(s)) = -\infty\,,$$

where $v = w'' + kw$. After integration we then have

$$\lim_{s\to+\infty}[w''(s) + kw(s)] = \lim_{s\to+\infty} v(s) = -\infty\,. \tag{80}$$

On the other hand, (17) implies that there exists $\bar{t} > 0$ such that

$$f(t) > k^2 t \qquad \forall t > \bar{t}\,.$$

By (80) and the fact that $k < 0$, we obtain

$$\begin{aligned}\lim_{s\to+\infty} w''''(s) &= \lim_{s\to+\infty}[|k|w''(s) - f(w(s))] \le \lim_{s\to+\infty}[|k|w''(s) - k^2 w(s)]\\ &= \lim_{s\to+\infty} |k|\left[w''(s) + kw(s)\right] = -\infty\,.\end{aligned}$$

After integration, this yields

$$\lim_{s\to+\infty} w(s) = -\infty$$

in contradiction with $\ell = +\infty$.

If w is nonpositive in $(\bar{s}_1, +\infty)$ we proceed similarly by using the fact that

$$\liminf_{t\to-\infty} \frac{f(t)}{t} > k^2$$

in view of (17).

10 Proof of Theorem 10

Denote by $\bar{s} \in [s_1, s_2]$ the maximum of $s \mapsto |w''(s)|$ in $[s_1, s_2]$. By Lemma 28 we know that $\bar{s} \in \{s_1, s_2\}$ and we distinguish three cases.

Case $w(s_1) \le 0$. Since $w(s_2) < w(s_1) \le 0$, by (3) we know that $F(w(s_1)) < F(w(s_2))$. On the other hand, since the energy function $\mathcal{E}$ defined in (27) is constant, we obtain

$$\frac{w''(s_1)^2}{2} - F(w(s_1)) = \mathcal{E}(s_1) = \mathcal{E}(s_2) = \frac{w''(s_2)^2}{2} - F(w(s_2)).$$

These two facts show that $w''(s_2)^2 > w''(s_1)^2$ and, together with Lemma 28, prove that $\bar{s} = s_2$. In turn, this shows that $w'''(s_2) \ge 0$ because $s \mapsto w''(s)$ is increasing in a left neighborhood of s_2. Since $w(s_2) < 0$, by (1) we see that $w''''(s_2) = -kw''(s_2) - f(w(s_2)) > 0$ showing that $s \mapsto w'''(s)$ is strictly increasing, and hence positive, in a right neighborhood of s_2. This implies that $s \mapsto w''(s)$ is strictly increasing, and hence positive (recall $w''(s_2) \ge 0$), in a right neighborhood of s_2. This finally implies that $s \mapsto w'(s)$ is strictly increasing, and hence positive (recall $w'(s_2) = 0$), in a right neighborhood of s_2. All these monotonicities continue to hold as long as $w''''(s) > 0$ and, by (1), they certainly hold as long as $w(s) < 0$. Since we assumed $w(s_1) \le 0$, case (i) in Theorem 10 occurs.

Case $w(s_2) \ge 0$. By (3) we know that $F(w(s_1)) > F(w(s_2))$ and since $\mathcal{E}$ is constant, we infer that $w''(s_1)^2 > w''(s_2)^2$ and $\bar{s} = s_1$. Since $s \mapsto w''(s)^2$ is decreasing in a right neighborhood of s_1 and $w''(s_1) \le 0$, this shows that $w'''(s_1) \ge 0$. Since $w(s_1) > 0$, by (1) we see that $w''''(s_1) < 0$ and $w''''(s)$ is negative in a left neighborhood of s_1 as long as $w(s) > 0$. Since we assumed $w(s_2) \ge 0$, case (ii) in Theorem 10 occurs.

Case $w(s_2) < 0 < w(s_1)$. In this case, we cannot establish if $\overline{s} = s_1$ or $\overline{s} = s_2$, this depends on the sign of $F(w(s_1)) - F(w(s_2))$. So, assume that

$$F(w(s_1)) \le F(w(s_2)), \tag{81}$$

the other case being similar. By the same energy argument as above, we infer that $\overline{s} = s_2$. Then we take $\tau_2 > s_2$ (to be fixed later) and enlarge the interval. With an abuse of notation, we denote once more by $\overline{s} \in [s_1, \tau_2]$ the maximum of $s \mapsto |w''(s)|$ in $[s_1, \tau_2]$. So far, we know that $\overline{s} \in \{s_1\} \cup [s_2, \tau_2]$. In fact, it can be $\overline{s} = s_1$ only if equality holds in (81); however, this case will be ruled out. Since $w(s_2) < 0$, by (1) we see that $w''''(s_2) > 0$ showing that $s \mapsto w'''(s)$ is strictly increasing and $s \mapsto w''(s)$ is strictly convex in a neighborhood of $s = s_2$. In turn, also $s \mapsto |w''(s)| = w''(s)$ is strictly convex in the same neighborhood so that it cannot attain its absolute maximum at $s = s_2$ which is in the interior of such neighborhood. We have so proved that $\overline{s} \in (s_2, \tau_2]$ and that there exists a right neighborhood of s_2 where $w'''(s) > 0$, $w''(s) > 0$, $w'(s) > 0$. In particular, we may drop the absolute value and consider the map $s \mapsto w''(s)$. We now fix $\tau_2 > s_2$ to be the first local maximum of $s \mapsto w''(s)$ on the interval $(s_2, +\infty)$. If there is no such maximum, we put $\tau_2 = +\infty$ and $s \mapsto w''(s)$ is increasing on the interval $(s_2, +\infty)$ so that w is convex and increasing and $w(s) \to +\infty$ as $s \to +\infty$; then we are done because case (i) in Theorem 10 occurs. If $\tau_2 < +\infty$, then $\overline{s} = \tau_2$, $w'''(\tau_2) = 0$, and $w''(s) > 0$ and $w'(s) > 0$ over $(s_2, \tau_2]$. In this case, we are done provided we show that $w(\tau_2) \ge w(s_1)$. For contradiction, assume that $w(s_2) < w(\tau_2) < w(s_1)$. By (3) and (81) this shows that $F(w(s_2)) > F(w(\tau_2))$. Summarizing, we have

$$F(w(s_2)) > F(w(\tau_2)), \quad w'''(\tau_2) = 0, \quad w''(s_2) < w''(\tau_2).$$

Since $k \le 0$, this contradicts the fact that the energy function $\mathcal{E}$ defined in (27) is constant:

$$\frac{w''(s_2)^2}{2} - F(w(s_2)) = \mathcal{E}(s_2) = \mathcal{E}(\tau_2) = \frac{w''(\tau_2)^2}{2} - \frac{k}{2} w'(\tau_2)^2 - F(w(\tau_2)).$$

11 Proof of Theorem 13

First, we prove (i). Assume for contradiction that w is a homoclinic solution to (1). Since $0 < k \le 2$, w changes sign infinitely many times by Theorem 4. Given any two of its roots $s_1 < s_2$, we have

$$\begin{aligned} 2\int_{s_1}^{s_2} w'(s)^2\, ds &= -2\int_{s_1}^{s_2} w(s) w''(s)\, ds \\ &= \int_{s_1}^{s_2} [w(s)^2 + w''(s)^2]\, ds - \int_{s_1}^{s_2} [w(s) + w''(s)]^2\, ds \le \int_{s_1}^{s_2} [w(s)^2 + w''(s)^2]\, ds. \end{aligned} \tag{82}$$

Let H be as in (29), then (20) and (82) ensure that

$$\begin{aligned} H(s_2) - H(s_1) &= \int_{s_1}^{s_2} [w''(s)^2 - k w'(s)^2 + w(s) f(w(s))]\, ds \\ &\ge \int_{s_1}^{s_2} [w''(s)^2 - k w'(s)^2 + w(s)^2]\, ds \\ (\bullet) &\ge (2 - k) \int_{s_1}^{s_2} w'(s)^2\, ds \ge 0. \end{aligned}$$

Hence, H is nondecreasing on the sequence of zeroes of w. In fact, there exist two roots $\overline{s}_1 < \overline{s}_2$ such that the inequality in (82) is strict since otherwise $w''(s) + w(s) = 0$ for all $s \in \mathbb{R}$, contradicting the fact that w is a homoclinic solution. In turn, if $s_1 \le \overline{s}_1 < \overline{s}_2 \le s_2$, then the inequality $(\bullet)$ becomes strict and $H(s_2) > H(s_1)$ whenever $s_1 \le \overline{s}_1 < \overline{s}_2 \le s_2$. But, since w is a homoclinic, $H(s) \to 0$ as $s \to \pm\infty$ by Proposition 11. This contradiction shows that there exist no homoclinics.

For statement (ii), we denote by $0 < s_m < s_{m+1}$ two consecutive roots of w. Since w is a homoclinic we have $\mathcal{E}(s) \equiv 0$, see (28). Hence, by using (1), we get

$$\int_{s_m}^{s_{m+1}} [w''(s)^2 + kw'(s)^2]\, ds = \int_{s_m}^{s_{m+1}} [2F(w(s)) + 2w(s)f(w(s))]\, ds. \tag{83}$$

We estimate the left hand side of (83) by solving the associated eigenvalue problem, namely

$$\min_{H^2\cap H_0^1(s_m,s_{m+1})} \frac{\int_{s_m}^{s_{m+1}} [w''(s)^2 + kw'(s)^2]\, ds}{\int_{s_m}^{s_{m+1}} w(s)^2\, ds} = \left(\frac{\pi}{s_{m+1}-s_m}\right)^4 + k\left(\frac{\pi}{s_{m+1}-s_m}\right)^2.$$

On the other hand, by (15) we know that

$$\int_{s_m}^{s_{m+1}} [2F(w(s)) + 2w(s)f(w(s))]\, ds = (3+o(1)) \int_{s_m}^{s_{m+1}} w(s)^2\, ds \qquad \text{as } m \to +\infty.$$

Summarizing, by (83) we deduce that

$$\limsup_{m\to+\infty} \left[\left(\frac{\pi}{s_{m+1}-s_m}\right)^4 + k\left(\frac{\pi}{s_{m+1}-s_m}\right)^2\right] \le 3\,,$$

by which (21) readily follows.

In order to prove statement (iii) we fix $0 < \varepsilon < 1 - k/2$ and let $\delta > 0$ be such that

$$|t| \le \delta \quad \Longrightarrow \quad \frac{f(t)}{t} \ge 1 - \varepsilon.$$

If w is a homoclinic, then there exists $s_0 > 0$ such that

$$|s| \ge s_0 \quad \Longrightarrow \quad |w(s)| \le \delta.$$

Since $0 < k < 2$, we know that w changes sign infinitely many times by Theorem 4. Using again the energy function H defined in (29) and (31) we then get

$$\begin{aligned} H(s_2) - H(s_1) &= \int_{s_1}^{s_2} [w''(s)^2 - kw'(s)^2 + w(s)f(w(s))]\, ds \\ &\ge \int_{s_1}^{s_2} [w''(s)^2 - kw'(s)^2 + (1-\varepsilon)w(s)^2]\, ds \qquad (84) \\ &\ge (1-\varepsilon)\int_{s_1}^{s_2} [(w''(s)+w(s))^2 - 2w''(s)w(s)]\, ds - k\int_{s_1}^{s_2} w'(s)^2\, ds \\ &\ge (2(1-\varepsilon)-k)\int_{s_1}^{s_2} w'(s)^2\, ds \qquad (85) \end{aligned}$$

for any two roots $s_1 < s_2$ of $w(s)$, both being in $(-\infty, -s_0)$ or in $(s_0, +\infty)$. Since w is a homoclinic, H is bounded by Proposition 11. Thus, the inequality in (85) implies that $w' \in L^2(\mathbb{R})$. Using this fact and the inequality in (84), we conclude that $w \in H^2(\mathbb{R})$.

12 Appendix: proof of Proposition 11

It is sufficient to prove the statement for $s \to +\infty$. We denote by C general constants which may vary from line to line. We also denote by $\delta_i(s)$ (for $i = 1, ..., 5$) continuous functions such that $\delta_i(s) \to 0$ as $s \to +\infty$. Assuming that $w(s) \to 0$ as $s \to +\infty$, let us rewrite (1) as

$$\Big(e^s[w'''(s) - w''(s) + (k+1)w'(s) - (k+1)w(s)]\Big)' = \delta_1(s)e^s \tag{86}$$

where $\delta_1(s) = -f(w(s)) - (k+1)w(s)$. We know that

$$\forall\varepsilon > 0 \quad \exists\sigma > 0 \quad \text{s.t.} \quad |\delta_1(s)| < \varepsilon \quad \forall s > \sigma.$$

Fix $\varepsilon > 0$ and integrate (86) over $(0, s)$ for any $s > \sigma$ to obtain

$$e^s[w'''(s) - w''(s) + (k+1)w'(s) - (k+1)w(s)] = C + \int_0^\sigma \delta_1(t)e^t\,dt + \int_\sigma^s \delta_1(t)e^t\,dt$$

and, subsequently,

$$|w'''(s) - w''(s) + (k+1)w'(s) - (k+1)w(s)| \le Ce^{-s} + \varepsilon e^{-s}\int_\sigma^s e^t\,dt = Ce^{-s} + \varepsilon \qquad \forall s > \sigma.$$

By letting $s \to +\infty$ and by arbitrariness of ε, this proves that

$$w'''(s) - w''(s) + (k+1)w'(s) - (k+1)w(s) = \delta_2(s) \tag{87}$$

for some continuous function δ_2 vanishing as $s \to +\infty$. Let us rewrite this equation as

$$\Big(e^s[w''(s) - 2w'(s) + (k+3)w(s)]\Big)' = \delta_3(s)e^s$$

where $\delta_3(s) = \delta_2(s) + (2k+4)w(s)$. Arguing as for (86), we arrive at

$$w''(s) - 2w'(s) + (k+3)w(s) = \delta_4(s) \tag{88}$$

for some continuous function δ_4 vanishing as $s \to +\infty$. We rewrite this equation as

$$\Big(e^s[w'(s) - 3w(s)]\Big)' = \delta_5(s)e^s$$

where $\delta_5(s) = \delta_4(s) - (k+6)w(s)$. By applying once more the previous scheme, this finally yields that $w'(s) - 3w(s) \to 0$ as $s \to +\infty$. Since $w(s) \to 0$, this implies $w'(s) \to 0$. In turn, these two limits and (88) imply that $w''(s) \to 0$ as $s \to +\infty$. Similarly, from (87) we obtain that $w'''(s) \to 0$, whereas from (1) we obtain $w''''(s) \to 0$ as $s \to +\infty$.

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