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FREE GRADIENT DISCONTINUITY AND IMAGE INPAINTING

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Abstract. We introduce and study a formulation of inpainting problem for 2-dimensional images which are locally damaged. This formulation is based on the regularization of the solution of a second order variational problem with Dirichlet boundary condition. A variational approximation algorithm is proposed.

Contents

1. Introduction

In image restoration the term inpainting denotes the process of filling in the missing information over subdomains where a given image is damaged: these domains may correspond to scratches in a camera picture, occlusion by objects, blotches in an old movie film or aging of canvas and colors in a painting ([4], [26], [27], [33], [40]).
Minimization of Blake & Zisserman functional is a variational approach to segmentation and denoising in image analysis which deals with free discontinuity, free gradient discontinuity and second derivatives. This second order functional was introduced to overcome the over-segmentation of steep gradients (ramp effect) and other drawbacks which occur in lower order models as in case of Mumford & Shah functional ([38], [39]). We refer to [7], [13], [14], [15], [17], [19], [37], [38] for motivation and analysis of variational approach to image segmentation and digital image processing.
In this paper we face the inpainting problem for a monochromatic image with a variational approach: solving a Dirichlet type problem for the main part of Blake & Zisserman functional. A similar problem was studied in [22] with the aim of finding a segmentation of a given noisy image.

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Previously the Mumford & Shah model has been proposed by several authors for the inpainting problem, but some inconvenient has been detected in this approach (see [27] and [33]). In the Mumford & Shah model ([30], [39]), the preferable edge curves are those which have the shortest length, therefore it favours straight edges and it produces the emerging of artificial corners. In the Blake & Zisserman model, the presence of second derivatives smooths such corners (see Figure 3).
About minimization of the Blake & Zisserman functional under Neumann boundary condition we refer to [12], [13], [14], [16], [18]. For a description of the rich list of differential, integral and geometric extremality conditions we refer to [19]. The results of the paper [22] were deeply exploited in [20], [21] and [23] to study fine properties of local minimizers of Blake & Zisserman functional under Neuman boundary condition, particularly about their singular set related to optimal segmentation; in the present paper they are applied to the derivation and study of a variational algorithm for image inpainting.
In general uniqueness of minimizers for functionals of this kind fails due to lack of convexity. We refer to [5] for explicit examples of multiplicity. Nevertheless in the 1-D formulation, uniqueness of minimizer is a generic property with respect to admissible data: in [6] is proven that for a G_δ (countable intersection of dense open sets) set of admissible data the minimizer is unique. Hence the whole picture is coherent with the presence of instable patterns, each of them corresponding to a bifurcation of optimal segmentation under variation of parameters related to contrast threshold, "luminance sensitivity", resistance to noise, crease detection, double edge detection.

In this paper we propose two different second order functionals E and F aiming to image inpainting, respectively in the cases of complete or partial loss of information in a small subregion.
First we focus on the functional E, which is defined as follows:

$$E(K_0, K_1, v) = \int_{\Omega\setminus(K_0\cup K_1)} |D^2 v|^2 \, d\mathbf{x} + \alpha\mathcal{H}^1\left(K_0 \cap \overline{\Omega}\right) + \beta\mathcal{H}^1\left((K_1 \setminus K_0) \cap \overline{\Omega}\right) . \tag{1.1}$$

To face the inpainting problem we look for minimizers of E among admissible triplets (K_0, K_1, v), say triplets fulfilling

$$\begin{cases} K_0\,,\ K_1 \text{ Borel subsets of } \mathbb{R}^2, \qquad K_0 \cup K_1 \text{ closed}, \\ v \in C^2\left(\widetilde{\Omega} \setminus (K_0 \cup K_1)\right),\ v \text{ approximately continuous in } \widetilde{\Omega} \setminus K_0, \\ v = w \text{ a.e. in } \widetilde{\Omega} \setminus \overline{\Omega}\,, \end{cases} \tag{1.2}$$

where Ω, $\widetilde{\Omega}$ are open sets with $\Omega \subset\subset \widetilde{\Omega} \subset\subset \mathbb{R}^2$, Ω is Lipschitz and w is a given function in $\widetilde{\Omega} \setminus \Omega$.
The raw image under processing is damaged due to the presence of blotches in the set $\overline{\Omega}$: the noiseless brightness intensity w of the image is known in $\widetilde{\Omega} \setminus \Omega$ while is completely lost in the possibly disconnected set Ω.
If (K_0, K_1, u) is a minimizing triplet of E, then u provides the inpainted restoration of the whole image, and $K_0 \cup K_1$ can be interpreted as an optimal segmentation of the restored image: the three elements of a minimizing triplet (K_0, K_1, u) play respectively the role of edges, creases and smoothly varying intensity in the region $\widetilde{\Omega}\setminus(K_0 \cup K_1)$ for the segmented image.
In the simplified case when the image is smooth where damage does not occur, our result reads as follows.

Theorem 1.1. *Let* α, β, Ω, $\widetilde{\Omega}$ *and* w *be s.t.*

$$0 < \beta \leq \alpha \leq 2\beta, \tag{1.3}$$

$$\Omega \subset\subset \widetilde{\Omega} \subset\subset \mathbb{R}^2\,, \tag{1.4}$$

$$\Omega \textit{ is an open set with piecewise } C^2 \textit{ boundary } \partial\Omega\,,\ \widetilde{\Omega} \textit{ is an open set,} \tag{1.5}$$

$$w \textit{ has a } C^2(\widetilde{\Omega}) \textit{ extension which fulfils } \ D^2 w \in L^\infty(\widetilde{\Omega}\,)\,. \tag{1.6}$$

Then there exists a triplet (C_0, C_1, u) *which minimizes the functional*

$$E(K_0, K_1, v) = \int_{\Omega\setminus(K_0\cup K_1)} \left|D^2 v\right|^2\, d\mathbf{x} + \alpha\mathcal{H}^1\left(K_0 \cap \overline{\Omega}\right) + \beta\mathcal{H}^1\left((K_1 \setminus K_0) \cap \overline{\Omega}\right)$$

among admissible triplets (K_0, K_1, v) *as in* (1.2)*, with* $E(C_0, C_1, u) < +\infty$.
Moreover any minimizing triplet (K_0, K_1, v) *fulfils:*

$$K_0 \cap \overline{\Omega} \textit{ and } K_1 \cap \overline{\Omega} \textit{ are } (\mathcal{H}^1, 1) \textit{ rectifiable sets,} \tag{1.7}$$

$$\mathcal{H}^1(K_0 \cap \overline{\Omega}) \;=\; \mathcal{H}^1(\overline{S_v})\,, \quad \mathcal{H}^1(K_1 \cap \overline{\Omega}) \;=\; \mathcal{H}^1(\overline{S_{\nabla v}} \setminus S_v)\,, \tag{1.8}$$

$$\left\{ \begin{array}{c} v \in GSBV^2(\widetilde{\Omega}),\ \textit{hence} \\ v \textit{ and } \nabla v \textit{ have well defined two-sided traces, finite } \mathcal{H}^1 \textit{ a.e. on } K_0 \cup K_1\,, \end{array} \right. \tag{1.9}$$

where S_v *and* $S_{\nabla v}$ *respectively denote the singular sets of* v *and* ∇v.

The main result of this paper is Theorem 4.1 in Section 4. Its statement is quite technical, but it is a more useful tool than Theorem 1.1, since it deals with free discontinuity and free gradient discontinuity in $\widetilde{\Omega}\setminus\Omega$ of the given raw image w to be processed together with some additional noisy information, denoted by g, in a Borel subset U with

$$U \subset\subset \Omega \subset\subset \widetilde{\Omega}. \tag{1.10}$$

Theorem 4.1 refers to the other functional proposed in this paper. Such functional is labeled by F and deals with the noisy part by summing a fidelity term $\displaystyle\int_{\Omega\setminus U} |v-g|^2\, d\mathbf{x}$ to the functional E. Precisely, we introduce the functional

$$F(K_0, K_1, v) = E(K_0, K_1, v) + \mu\ \int_{\Omega\setminus U} |v-g|^2\ d\mathbf{x} \tag{1.11}$$

to be minimized among triplets (K_0, K_1, v) verifying (1.2). We apply direct methods of Calculus of Variations to functional (1.11) by proving the partial regularity for solutions of a weak version $\mathcal{F}$ of (1.11), which is introduced in (2.3) of Section 2.

We emphasize that if (K_0, K_1, v) is a minimizing triplet of F than v fulfils the Euler equations

$$\Delta^2 v + \mu(v-g) = 0 \qquad \text{in } \ \Omega \setminus (\overline{U} \cup K_0 \cup K_1), \tag{1.12}$$

$$\Delta^2 v = 0 \qquad \text{in } \ U \setminus (K_0 \cup K_1) \tag{1.13}$$

together with many kind of integral and geometric relationship as like as minimizing triplet of Blake & Zisserman functional for image segmentation (see [19], [23]).
To achieve the existence of minimizing triplets of F, inspired by the seminal papers of De Giorgi and Ambrosio [28] and [29], we introduce a relaxed functional: the weak Blake & Zisserman functional for inpainting $\mathcal{F}(v)$ (see (2.3)). The idea is to deal with a simpler

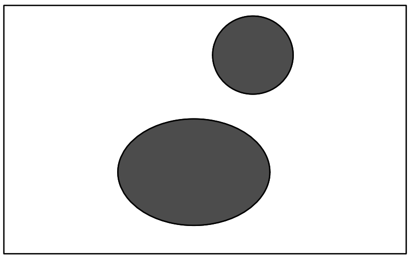

FIGURE 1. Theorem 1.1: the image domain is the rectangle $\widetilde{\Omega}$, the blotches $\Omega \subset\subset \widetilde{\Omega}$ with complete loss of information are the black region $\overline{\Omega}$.

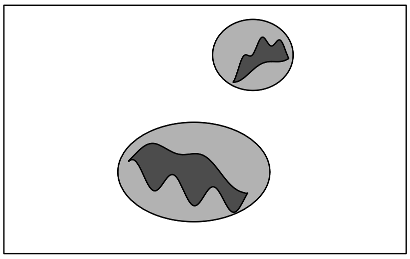

FIGURE 2. Theorem 4.1: the image domain is the rectangle $\widetilde{\Omega}$, the blotches $\Omega \subset\subset \widetilde{\Omega}$ with some loss of information, complete loss of information in the black region U, the partially damaged image is given in the gray region $\Omega \setminus U$.

object, just depending on the function v, and then to recover the set of jumps K_0 and creases $K_1 \setminus K_0$ by taking respectively the discontinuity set $\overline{S_v}$ and $\overline{S_{\nabla v}} \setminus S_v$. The functional class where we set the problem is given by second order generalized functions with special bounded variation: say $GSBV^2(\widetilde{\Omega})$ (for the formal definition see (2.1) and (2.2)). The class $GSBV^2(\widetilde{\Omega})$ is the right functional setting, more appropriate than $BH(\widetilde{\Omega})$ (bounded hessian functions whose second derivatives are Radon measure). Indeed BH functions in two variables are continuous with integrable gradient; on the other hand BH contains too much irregular functions: for instance the choice of a raw image as a primitive of the Cantor-Vitali function leads to $\inf F = 0$.
In this framework compactness and lower semicontinuity Theorems 8 and 10 of [13] give the existence of minimizers for the relaxed functional $\mathcal{F}(v)$. The results of Theorem 4.1 are achieved by showing partial regularity of the obtained weak solution with penalized Dirichlet datum (Theorem 2.1). The novelty here consists in the regularization at the boundary for a free gradient discontinuity problem with Dirichlet datum (in the set $\partial\Omega$) or transmission condition (in the set ∂U). For a concise summary of these steps see the proof of Theorem 4.1.
In Section 2 is introduced the weak formulation. In Section 3 are collected several estimates in the space $GSBV^2$. In section 4 is stated and proved the main result. In Section 5 we present the variational approximation of the functional F: functionals $\mathcal{G}_h$ defined by (5.1). A numerical scheme, the convergence analysis and its implementation are contained in forthcoming papers [8], [9].

2. Weak Blake & Zisserman functional for image inpainting

We denote by $B_\varrho(\mathbf{x})$ the open ball $\{\, \mathbf{y} \in \mathbb{R}^2 ;\ |\mathbf{y}-\mathbf{x}| < \varrho \,\}$, and set $B_\varrho = B_\varrho(\mathbf{0})$, $B_\varrho^+ = B_\varrho \cap \{(x,y):\ y>0\}$, $B_\varrho^- = B_\varrho \cap \{(x,y):\ y<0\}$. We denote by χ_V the characteristic function of V for any $V \subset \mathbb{R}^2$, by $\mathcal{H}^1(V)$ its 1-dimensional Hausdorff measure and by $|V|$ its Lebesgue outer measure.

For any Borel function $v:\Omega \to \mathbb{R}$ and $\mathbf{x} \in \Omega$, $z \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty,+\infty\}$, we set $z = \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$ (*approximate limit* of v at $\mathbf{x}$) if, for every $g \in C^0(\overline{\mathbb{R}})$,

$$g(z) \;=\; \lim_{\varrho\to 0} \fint_{B_\varrho(\mathbf{0})} g(v(\mathbf{x}+\boldsymbol{\xi}))\, d\boldsymbol{\xi}\,;$$

the function $\widetilde{v}(\mathbf{x}) \;=\; \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$ is called *representative* of v; the *singular set* of v is $S_v = \{\mathbf{x} \in \Omega : \nexists z \ s.t.\ \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y}) = z\}$.

A Borel function $v:\Omega \to \mathbb{R}$ is *approximately continuous at* $\mathbf{x} \in \Omega$ iff $v(\mathbf{x}) = \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$.

By referring to [2], [14], [19], [35]: Dv denotes the distributional gradient of v, $\nabla v(\mathbf{x})$ denotes the approximate gradient of v, say v is approximately differentiable at x if there exists a vector $\nabla v(x) \in \mathbb{R}^2$ (the approximate gradient of v at x) such that

$$\operatorname*{aplim}_{y\to x} \frac{|v(y)-\widetilde{v}(x)-\nabla v(x)\cdot(y-x)|}{|y-x|} = 0,$$

and $SBV(\Omega)$ denotes the De Giorgi class of functions $v \in BV(\Omega)$ such that

$$\int_\Omega |Dv| = \int_\Omega |\nabla v|\, d\mathbf{x} + \int_{S_v} |v^+ - v^-|\, d\mathcal{H}^1,$$

where for $\mathcal{H}^1$ almost all $x \in S_u$ there exist $\nu(x) \in \partial B_1$, $v_+(x) \in \mathbb{R}$, $v_-(x) \in \mathbb{R}$ with $v_+(x) > v_-(x)$ such that

$$\lim_{\varrho\to 0} \varrho^{-n} \int_{\{y\in B_\varrho; y\cdot\nu(x)>0\}} |v(x+y)-v_+(x)|\, dy = 0,$$

$$\lim_{\varrho\to 0} \varrho^{-n} \int_{\{y\in B_\varrho; y\cdot\nu(x)<0\}} |v(x+y)-v_-(x)|\, dy = 0.$$

Moreover:

$$SBV_{loc}(\Omega) \;:=\; \{v \in SBV(\Omega');\ \forall\, \Omega' \subset\subset \Omega\}\,,$$

$$GSBV(\Omega) \;:=\; \big\{v:\Omega\to\mathbb{R} \text{ Borel function}; -k \vee v \wedge k \in SBV_{loc}(\Omega)\ \forall k \in \mathbb{N}\big\}. \tag{2.1}$$

$$GSBV^2(\Omega) \;:=\; \big\{\, v \in GSBV(\Omega),\ \nabla v \in \big(GSBV(\Omega)\big)^2 \big\}. \tag{2.2}$$

In order to study the functional F by direct methods in Calculus of Variations, we introduce the weak Blake & Zisserman functional for inpainting $\mathcal{F}$, which is similar to the one introduced in [13] for image segmentation, but with a fidelity term which acts only on a portion of the domain:

$$\mathcal{F}(v) = \int_\Omega |\nabla^2 v|^2\, d\mathbf{x} + \alpha\mathcal{H}^1\left(S_v \cap \overline{\Omega}\right) + \beta\mathcal{H}^1\left((S_{\nabla v}\setminus S_v)\cap\overline{\Omega}\right) + \mu \int_{\Omega\setminus U} |v-g|^2\, d\mathbf{x}\,. \tag{2.3}$$

We emphasize that $\mathcal{F}$ is still a non convex functional, but has the advantage of depending only on the function v and the sets $K_0,\ K_1$ are recovered by the singular sets S_v and $S_{\nabla v}$.

About the functional defined by (2.3) we will often use the short notation $\mathcal{F}$; nevertheless, whenever required by clearness of exposition about interchange of various ingredients (function, parameters, Dirichlet datum, domain A) we will use several different (self-explaining) notation:

$$\mathcal{F}(v),\ \mathcal{F}_{g\,w}(v),\ \mathcal{F}_{g\,w}(v,\mu,\alpha,\beta,A)\,.$$

Theorem 2.1. *(***Minimizers of weak Blake & Zisserman functional $\mathcal{F}$ for inpainting***)*
Assume (1.3), (1.4), (1.5),

$$\mu>0,\ U\subset\Omega \textit{ is an open set, } g\in L^2(\Omega\setminus U), \tag{2.4}$$

$$w\in C^2\left(\widetilde{\Omega}\setminus\overline{(S_w\cup S_{\nabla w})}\right),\quad w \textit{ approximately continuous in } \widetilde{\Omega}\setminus S_w\,, \tag{2.5}$$

$$\mathcal{F}(w)\ <\ +\infty \tag{2.6}$$

$$\mathcal{H}^1\left(\overline{(S_w\cup S_{\nabla w})}\ \setminus(S_w\cup S_{\nabla w})\right)\ =\ 0\,, \tag{2.7}$$

$$\mathcal{H}^1\left(\overline{(S_w\cup S_{\nabla w})}\cap\partial\Omega\right)\ =\ 0\qquad(\textit{ or } \overline{(S_w\cup S_{\nabla w})}\cap\partial\Omega \textit{ finite})\,. \tag{2.8}$$

Set

$$X(\widetilde{\Omega})\ \stackrel{\text{def}}{=}\ \left\{v\in\ GSBV^2(\widetilde{\Omega})\ \textit{s.t. } v=w\ \textit{a.e. in } \widetilde{\Omega}\setminus\Omega\right\}\,. \tag{2.9}$$

Then there exists u minimizing the functional $\mathcal{F}$ in $X(\widetilde{\Omega})$ with finite energy.

Proof. Obviously $\mathcal{F}(v)\geq 0\ \ \forall v\in X(\widetilde{\Omega})\,.$
Assumptions (2.4)-(2.8), the interpolation Theorem 6 in [13] and Lemma 2.3 in [30] entail $w\in X(\widetilde{\Omega})$ and $\inf_{v\in X(\widetilde{\Omega})}\mathcal{F}(v)<+\infty\,.$
Let $v_h\in X$ be a minimizing sequence for $\mathcal{F}$. By Theorem 8 in [13] there is v_∞ in $X(\widetilde{\Omega})$ and a subsequence s.t., without relabeling, $v_h\to v_\infty$ a.e. in $\widetilde{\Omega}$.
The properties $v_h=w$ in $\widetilde{\Omega}\setminus\Omega$ entail $v_\infty=w$ in $\widetilde{\Omega}\setminus\Omega$. By Theorem 10 in [13]:

$$\mathcal{F}(v_\infty)\ \leq\ \liminf_h\ \mathcal{F}(v_h)\,,$$

hence $\mathcal{F}(v_\infty)=\inf_{v\in X(\widetilde{\Omega})}\mathcal{F}(v)\,.$ □

3. Truncation, Poincaré inequalities and compactness in $GSBV^2$

In the present section we list the key tools in the regularity theory for the minimizers in $GSBV^2$ of the functional $\mathcal{F}$.
We state a Poincaré-type inequality in the class $GSBV$ which was proven in [14] allowing surgical truncations of non integrable functions of several variables and we refine its statement with the aim of taming blow-up at boundary points in case of functions vanishing in a sector of positive measure. About this inequality we emphasize that $v\in GSBV^2(\Omega)$ does not even entail that either v or ∇v belong to $L^1_{loc}(\Omega)$.

Let B be an open ball in $\mathbb{R}^2$. For every measurable function $v : B \to \mathbb{R}$ we define the least median $m_*(v, B)$ of v in B as follows (see [24])

$$m_*(v, B) = \inf \left\{ t \in \mathbb{R};\ |\{v < t\} \cap B| \geq \frac{1}{2}|B| \right\}.$$

We remark that $m_*(\cdot, B)$ is a non linear operator and in general it has no relationship with the averaged integral $\int_B \cdot\, dy \,/\, |B|$.

For any measurable set $E \subset B$ we have $m_*(v\chi_{B\setminus E} + m_*(v, B)\chi_E, B) = m_*(v, B)$. For every $v \in GSBV(B)$ and $a \in \mathbb{R}$ with $(2\gamma_2\mathcal{H}^1(S_v))^2 \leq a \leq \frac{1}{2}|B|$, we set

$$\tau'(v, a, B) = \inf \{t \in \mathbb{R};\ |\{v < t\} \cap B|\ \geq\ a\},$$

$$\tau''(v, a, B) = \inf \{t \in \mathbb{R};\ |\{v \geq t\} \cap B|\ \leq\ a\},$$

here γ_2 is the isoperimetric constant relative to the balls of $\mathbb{R}^2$, i.e.

$$\min\{|E \cap B|^{\frac{1}{2}}, |B \setminus E|^{\frac{1}{2}}\} \leq \gamma_2 P(E, B) \qquad \text{for every measurable set } E,$$

and $P(E, B)$ denotes the perimeter of E in B : $P(E, B) = \int_B |D\chi_E|$.

For $\eta \geq 0$ we define the truncation operator

$$T(v, a, \eta) = (\tau'(v, a, B) - \eta) \vee v \wedge (\tau''(v, a, B) + \eta). \tag{3.1}$$

Then

$$T(T(v, a, \eta), a, \eta) = T(v, a, \eta), \quad |\nabla T(v, a, \eta)| \leq |\nabla v| \text{ a.e. on } B, \tag{3.2}$$

$$m_*(T(v, a, \eta), B) = m_*(v, B), \quad T(\lambda v, a, \lambda\eta) = \lambda T(v, a, \eta) \quad \forall\, \lambda > 0, \tag{3.3}$$

$$|\{v \neq T(v, a, \eta)\}| \leq 2a. \tag{3.4}$$

The operators m_* and T are defined component-wise in case of vector-valued v.

For any given function in $GSBV$, we define an affine polynomial correction such that both median and gradient median vanish.

Let $B_r(\mathbf{x}) \subset \Omega$ and $v \in GSBV(B_r(\mathbf{x}))$; for every $\mathbf{y} \in \mathbb{R}^2$ we set

$$(M_{\mathbf{x},r}\, v)(\mathbf{y}) = m_*(\nabla v, B_r(\mathbf{x})) \cdot (\mathbf{y} - \mathbf{x}) \tag{3.5}$$

$$(\mathcal{P}_{\mathbf{x},r}\, v)(\mathbf{y}) = (M_{\mathbf{x},r}\, v)(\mathbf{y}) + m_*(v - M_{\mathbf{x},r}\, v, B_r(\mathbf{x})). \tag{3.6}$$

Since $m_*(v - c, B_r(\mathbf{x})) = m_*(v, B_r(\mathbf{x})) - c$ for every $c \in \mathbb{R}$ and $\nabla(\mathcal{P}_{\mathbf{x},r}\, v) = \nabla(M_{\mathbf{x},r}\, v) = m_*(\nabla v, B_r(\mathbf{x}))$ then we have $\mathcal{P}_{\mathbf{x},r}\,(v - \mathcal{P}_{\mathbf{x},r}\, v) = 0$, say

$$m_*(v - \mathcal{P}_{\mathbf{x},r}\, v, B_r(\mathbf{x})) = 0, \qquad m_*(\nabla(v - \mathcal{P}_{\mathbf{x},r}\, v), B_r(\mathbf{x})) = \mathbf{0}.$$

We notice that there are v such that $m_*(v, B_r(\mathbf{x})) \neq m_*(\mathcal{P}_{\mathbf{x},r}\, v, B_r(\mathbf{x}))$, take e.g. $v(x, y) = (x^2 - x)H(-x) - \frac{x}{2}H(x)$, where H is the Heaviside function.

Theorem 3.1. *(***Poincaré inequality for** $GSBV$ **functions in a ball***)*
Let $B \subset \mathbb{R}^2$ be an open ball, $v \in GSBV(B)$ and $a \in \mathbb{R}$ with

$$\left(2\gamma_2\mathcal{H}^1(S_v)\right)^2 \leq a \leq \frac{1}{2}|B|, \tag{3.7}$$

let $\eta \geq 0$ and $T(v, a, \eta)$ as in (3.1). Then

$$\int_B |D\, T(v, a, \eta)| \leq 2|B|^{\frac{1}{2}} \left(\int_B |\nabla T(v, a, \eta)|^2\, dy \right)^{\frac{1}{2}} + 2\eta\mathcal{H}^1(S_v). \tag{3.8}$$

We have also, for every $s \geq 2$,

$$\int_B |T(v,a,\eta) - m_*(v,B)|^s dy \leq 2^{s-1} (\gamma_2 s)^s \left(\int_B |\nabla T(v,a,0)|^2 \, dy \right)^{\frac{s}{2}} |B| + (2\eta)^s a \,. \tag{3.9}$$

Proof. See [14], Theorem 4.1. □

Theorem 3.2. *(***Classical Poincaré inequality in** *BV)*
For any $\mathbf{x} \in \mathbb{R}^2$, $r > 0$, and $0 < \vartheta < 1$ there is K_ϑ such that

$$\|v\|_{L^2(B_r(\mathbf{x}))} \;\leq\; K_\vartheta \int_{B_r(\mathbf{x})} |Dv| \qquad \forall v \in BV(B_r(\mathbf{x})) \quad s.t. \tag{3.10}$$

$$|\{\mathbf{y} \in B_r(\mathbf{x}) \;:\; v(\mathbf{y}) = 0\}\,| \,/\, |B_r(\mathbf{x})| \;\geq\; \vartheta \,. \tag{3.11}$$

Proof. See [34], Theorem 5.6.1(iii). □

Theorem 3.3. *(***Poincaré inequality for** *GSBV* **functions vanishing in a sector***)*
Let $B \subset \mathbb{R}^2$ be an open ball, $v \in GSBV(B)$ s.t. (3.11) holds true and $a \in \mathbb{R}$ with

$$\big(2\gamma_2 \mathcal{H}^1(S_v)\big)^2 \leq a \leq \frac{1}{2}|B|, \tag{3.12}$$

let $\eta \geq 0$ and $T(v,a,\eta)$ as in (3.1). Then

$$\int_B |D\,T(v,a,\eta)| \leq 2|B|^{\frac{1}{2}} \left(\int_B |\nabla T(v,a,\eta)|^2 \, dy \right)^{\frac{1}{2}} + 2\eta \mathcal{H}^1(S_v). \tag{3.13}$$

We have also, for every $s \geq 2$,

$$\int_B |T(v,a,\eta)|^s dy \leq 2^{s-1} (K_\vartheta s)^s \left(\int_B |\nabla T(v,a,0)|^2 \, dy \right)^{\frac{s}{2}} |B| + (2\eta)^s a \,. \tag{3.14}$$

Proof. Similar to the proof of Theorem 4.1 in [14] except for the use of Theorem 3.2 instead of Poincaré inequality (4.12) in [14], since we do not need to force vanishing of least median of v. □

Theorems 3.1 and 3.3 have been used for estimating also first derivatives of functions $v \in GSBV^2(B)$, as in the following theorem.

Theorem 3.4. ***(Compactness and lower semicontinuity for** $GSBV^2$ **functions vanishing in a set of positive Lebesgue measure)***
Assume $B_r(\mathbf{x}) \subset \mathbb{R}^2$, $u_h \in GSBV^2(B_r(\mathbf{x}))$, $0 < \vartheta < 1$

$$|\{\mathbf{y} \in B_r(\mathbf{x}) \;:\; u_h(\mathbf{y}) = 0\}| \;/\; |B_r(\mathbf{x})| \;\geq\; \vartheta \,, \tag{3.15}$$

$$\sup_h \int_{B_r(\mathbf{x})} |\nabla^2 u_h|^2 \, d\mathbf{y} \;<\; +\infty \tag{3.16}$$

and

$$\lim_h \, L_h \;=\; 0 \,, \qquad \textit{where} \quad L_h = \mathcal{H}^1(S_{u_h} \cup S_{\nabla u_h}) \,. \tag{3.17}$$

Then there are a positive constant c (dependent on the left-hand side of (3.16)), $u_\infty \in W^{2,2}(B_r(\mathbf{x}))$ and a sequence $z_h \in GSBV^2(B_r(\mathbf{x}))$ (whose construction is given by (3.25)-(3.30)) s.t., up to a finite number of indices,

$$|\,\{z_h \neq u_h\}\,| \leq \; c\, {L_h}^2 \tag{3.18}$$

$$P(\{z_h \neq u_h\},\, B_r(\mathbf{x})) \;\leq\; c\, L_h \tag{3.19}$$

and there is a subsequence z_{h_k} *such that*

$$\lim_k \; z_{h_k} \;=\; u_\infty \qquad \textit{strongly in } L^p(B_r(\mathbf{x}))\,,\ \forall p \geq 1\,, \tag{3.20}$$

$$\lim_k \; \nabla\, z_{h_k} \;=\; Du_\infty \qquad \textit{strongly in } L^p(B_r(\mathbf{x}))\,,\ \forall p \geq 1\,, \tag{3.21}$$

$$\int_{B_r(\mathbf{x})} |D^2 u_\infty|^2\, d\mathbf{y} \;\leq\; \liminf_k \int_{B_r(\mathbf{x})} |\nabla^2 z_{h_k}|^2\, d\mathbf{y} \;\leq\; \liminf_k \int_{B_r(\mathbf{x})} |\nabla^2 u_{h_k}|^2\, d\mathbf{y}\,, \tag{3.22}$$

$$\lim_k \; u_{h_k} \;=\; u_\infty \qquad \textit{a.e. in } B_r(\mathbf{x})\,, \tag{3.23}$$

$$\lim_k \; \nabla\, u_{h_k} \;=\; Du_\infty \qquad \textit{a.e. in } B_r(\mathbf{x})\,. \tag{3.24}$$

Proof. The proof can be achieved by the same procedure exploited in the proof of Theorem 4.3 in [14], except for the fact that we can avoid forcing least median of u_h and ∇u_h to vanish since we can use Theorem 3.3 for functions vanishing in a sector instead of Poincaré inequality in *GSBV* given by Theorem 4.1 in [14].
The construction of the extracted sequence z_h is described in the following.
By setting $a_h = 4{\gamma_2}^2 {L_h}^2$ we have $a_h \leq |B_r|/2$ for large h. Hence there is c dependent on the left-hand side of (3.16) and there are $\eta_h^k \in (0,1)\,, h\, \in \mathbb{N},\ k = 1, 2$, s.t.

$$\big|\, \{\, T(\nabla_k\, u_h, a_h, \eta_h^k) \;\neq\; \nabla_k\, u_h \,\}\, \big| \;\leq\; c\, {L_h}^2 \tag{3.25}$$

$$P\left(\{\, T(\nabla_k\, u_h, a_h, \eta_h^k) \;\neq\; \nabla_k\, u_h \,\},\; B_r\,\right) \;\leq\; c\left(L_h + \mathcal{H}^1(S_{\nabla_k\, u_h})\right) \tag{3.26}$$

Referring to definition (3.1) of truncating operator T, we set

$$E_h \;=\; \bigcup_{k=1,2} \{\, \mathbf{y} \in B_r \;:\; T(\nabla_k\, u_h, a_h, \eta_h^k) \;\neq\; \nabla_k\, u_h \,\} \tag{3.27}$$

$$\xi_h \;=\; u_h\; \chi_{B_r \setminus E_h} \tag{3.28}$$

$$b_h \;=\; 4\, {K_\vartheta}^2\, \left(\mathcal{H}^1(S_{\xi_h} \cup S_{\nabla \xi_h})\right)^2 \;\leq\; \frac{1}{2}\, |B_r| \tag{3.29}$$

$$z_h \;=\; T(\xi_h, b_h, \eta_h) \tag{3.30}$$

□

4. Strong Blake & Zisserman functional for image inpainting

In this Section we state and prove our main result about image inpainting via Blake & Zisserman functional.

Theorem 4.1. *(***Minimizers of strong Blake & Zisserman functional** F **for inpainting***)*
Assume α, β, μ, g, U, Ω, $\widetilde{\Omega}$ *and* w *fulfil*

$$0<\beta\le\alpha\le 2\beta,\ \ \mu>0,\ \ g\in L^\infty(\Omega\setminus U), \tag{4.1}$$

$$U\subset\subset\Omega\subset\subset\widetilde{\Omega}\subset\subset\mathbb{R}^2\,, \tag{4.2}$$

$$\Omega \textit{ and } U \textit{ open sets with piecewise } C^2 \textit{ boundary, } \widetilde{\Omega} \textit{ open set,} \tag{4.3}$$

$$T_0, T_1 \textit{ Borel sets}\,,\ T_0\cup T_1 \textit{ closed subset of } \mathbb{R}^2\,,\ \mathcal{H}^1\left((T_0\cup T_1)\cap\widetilde{\Omega}\right)<+\infty, \tag{4.4}$$

$$(T_0\cup T_1)\cap\partial\Omega \ \textit{ is a finite set}\,, \tag{4.5}$$

$$w\in C^2\left(\widetilde{\Omega}\setminus(T_0\cup T_1)\right)\,,\quad w \textit{ approximately continuous in } \widetilde{\Omega}\setminus T_0\,, \tag{4.6}$$

$$\begin{cases} D^2w\in L^2(\widetilde{\Omega}\setminus(T_0\cup T_1)\,),\ D^2w\in L^\infty\big(A\setminus(T_0\cup T_1)\big) \\ \ \textit{with } A \textit{ open set s.t. } \partial\Omega\subset A\subset\widetilde{\Omega}\,, \\ \exists C>0\ :\ \|w\|_{L^\infty},\|\nabla w\|_{L^\infty},\|\nabla^2 w\|_{L^\infty}\le C \ \textit{ in } A\,, \\ \mathrm{Lip}(\gamma')\le C \ \textit{ with } \gamma \textit{ arc-length parametrization of } \partial\Omega\,, \\ \exists\bar\varrho>0\ :\ \mathcal{H}^1\big(\partial\Omega\cap B_\varrho(\mathbf{x})\big)<C\varrho \quad \forall\mathbf{x}\in\partial\Omega\,,\ \forall\varrho\le\bar\varrho\,, \end{cases} \tag{4.7}$$

$$\begin{array}{l} \textit{there is no triplet } (\mathfrak{T}_0,\mathfrak{T}_1,\omega) \textit{ fulfilling:} \\ (4.4),(4.6),\ \ \omega=\mathrm{aplim}\,w \textit{ in } \widetilde{\Omega}\setminus\mathfrak{T}_0, \textit{ and } (\mathfrak{T}_0\cup\mathfrak{T}_1)\underset{\neq}{\subset}(T_0\cup T_1)\,. \end{array} \tag{4.8}$$

Then there exists a triplet (C_0,C_1,u) *which minimizes the functional*

$$F(K_0,K_1,v)=E(K_0,K_1,v)+\mu\int_{\Omega\setminus U}|v-g|^2\,d\mathbf{x}$$

$$=\int_{\Omega\setminus(K_0\cup K_1)}|D^2v|^2\,d\mathbf{x}+\alpha\mathcal{H}^1\left(K_0\cap\overline{\Omega}\right)+\beta\mathcal{H}^1\left((K_1\setminus K_0)\cap\overline{\Omega}\right)+\mu\int_{\Omega\setminus U}|v-g|^2\,d\mathbf{x}$$

among admissible triplets (K_0,K_1,v) *as in* (1.2)*, with* $F(C_0,C_1,u)<+\infty$.
Moreover any minimizing triplet (K_0,K_1,v) *fulfils:*

$$K_0\cap\overline{\Omega} \textit{ and } K_1\cap\overline{\Omega} \textit{ are } (\mathcal{H}^1,1) \textit{ rectifiable sets,} \tag{4.9}$$

$$\mathcal{H}^1(K_0\cap\overline{\Omega})\,=\,\mathcal{H}^1(\overline{S_v})\,,\quad \mathcal{H}^1(K_1\cap\overline{\Omega})\,=\,\mathcal{H}^1(\overline{S_{\nabla v}}\setminus S_v)\,, \tag{4.10}$$

$$\begin{cases} v\in GSBV^2(\widetilde{\Omega}),\ \textit{hence} \\ v \textit{ and } \nabla v \textit{ have well defined two-sided traces, finite } \mathcal{H}^1 \textit{ a.e. on } K_0\cup K_1\,, \end{cases} \tag{4.11}$$

where S_v *and* $S_{\nabla v}$ *respectively denote the singular sets of* v *and* ∇v.

Before proving Theorem 4.1 we state:

- a decay estimate in L^2-norm of second derivatives for bi-harmonic functions in a half-disk which vanish together with normal derivative on the diameter (Theorem 4.2);
- a blow-up property for a sequence of local minimizers at Dirichlet boundary points (Theorem 4.3);
- a decay estimate of the functional $\mathcal{F}$ at points $\mathbf{x}$ where the quotient $\varrho^{-1}\mathcal{F}(u, B_\varrho(\mathbf{x}))$ is smaller than a suitable threshold $\varepsilon_1 > 0$ (Theorem 4.4).

The following Theorems 4.2, 4.3, 4.4 are proven in [22].

Theorem 4.2. ***(L^2-hessian decay for bi-harmonic functions in half-disk which vanish together with normal derivative along diameter)***
Set $B_1^+ = B_1(\mathbf{0}) \cap \{(x,y) \in \mathbb{R}^2 : y > 0\} \subset \mathbb{R}^2$, $\Gamma = B_1(\mathbf{0}) \cap \{(x,y) \in \mathbb{R}^2 : y = 0\}$.
Assume $z \in H^2(B_1^+)$, $\Delta^2 z = 0$ *on* B_1^+, $z = \partial z/\partial y = 0$ *on* Γ.
Then

$$\|D^2 z\|^2_{L^2(B_\varrho^+)} \leq \varrho^2 \|D^2 z\|^2_{L^2(B_1^+)} \qquad \forall \varrho \leq 1 . \tag{4.12}$$

Moreover there exists an unique extension Z of z in whole B_1 such that $\Delta^2 Z \equiv 0$ and both z, Z have the following expansion in polar coordinates, which is strongly convergent in $L^2(B_1)$ and strongly convergent in $H^2(B_1^+)$:

$$Z(x,y) = \sum_{k=0}^{\infty} \big(a_k \cos(k\vartheta) + b_k \sin(k\vartheta) + (\alpha_k \cos(k\vartheta) + \beta_k \sin(k\vartheta))\, r^2 \big)\, r^k . \tag{4.13}$$

Theorem 4.3. ***(Blow-up of the functional $\mathcal{F}$ at $\partial\Omega$)***
Assume (4.1)-(4.6). We focus a generic point where $\partial\Omega$ is C^2 (it is not restrictive to assume that such point is $\mathbf{0}$) and

$$\left\{\begin{array}{c} \mathbf{0} \in \partial\Omega\,,\ B_r(\mathbf{0}) \subset \widetilde{\Omega},\ \psi_h \in C^2(-r,r)\,, \psi_h(0) = 0, \\ \psi_h'(0) = 0\ \ Lip\,(\psi_h') \leq 1\,, \psi_h \to 0 \text{ in } W^{2,\infty}(-r,r)\,, \\ \omega_h \in C^2(B_r) \text{ with } \omega_h \to \omega_\infty \equiv 0 \text{ in } W^{2,\infty}(B_r(\mathbf{0})) \\ \\ B^{\psi_h +} \overset{\text{def}}{=} B_r(\mathbf{0}) \cap \{y > \psi_h(x)\}\,, \quad B^{\psi_h -} \overset{\text{def}}{=} B_r(\mathbf{0}) \cap \{y < \psi_h(x)\}\,, \\ \\ B_\varrho^\tau = \{\mathbf{x} = (x,y) : |\mathbf{x}| < \varrho,\ y > \tau)\} \ \mathit{for}\ 0 < \tau < \varrho < r\,. \end{array}\right. \tag{4.14}$$

$\gamma_h \in L^\infty(\widetilde{\Omega})$, *let* α_h, β_h, μ_h, *three sequences of positive numbers with* $\beta_h \leq \alpha_h$, *and let* $v_\infty \in H^2(B_r(\mathbf{0}))$ *s.t.* $v_\infty \equiv 0$ *in* $B_r^-(\mathbf{0})$.
Assume $v_h \in GSBV^2(\widetilde{\Omega})$, $v_h = \omega_h$ *a.e. in* $B^{\psi_h -}$ *and*

(i) v_h *are* Ω *local minimizers of* $\mathcal{F}_{\gamma_h\,\omega_h}(\,\cdot\,, \mu_h, \alpha_h, \beta_h, B_r(\mathbf{0}))$,
(ii) $\lim_h \mathcal{H}^1\left((S_{v_h} \cup S_{\nabla v_h}) \cap B_r(\mathbf{0})\right) = 0$,
(iii) $\exists\ \lim_h \mathcal{F}_{\gamma_h\,\omega_h}(v_h, \mu_h, \alpha_h, \beta_h, \overline{B_\varrho^\tau}) \overset{\text{def}}{=} \delta(\varrho,\tau) \leq 1$
for a.e. $\varrho, \tau \in (0,r)$ *with* $\tau < \varrho$, *and set* $\delta(\varrho,\tau) = 0$ *if* $\varrho < \tau$.
(iv) $\lim_h v_h = v_\infty$ *a.e. in* $B_r(\mathbf{0})$,
(v) $\lim_h \mu_h = 0$, $\lim_h \mu_h \|\gamma_h\|^2_{L^2(B_r(\mathbf{0}))} = 0$.

Then, for every $\varrho \in (0,r), \tau \in (0,\varrho)$, v_∞ *minimizes the functional*

$$\int_{B_\varrho^\tau(\mathbf{0})} \left| D^2 v \right|^2 d\mathbf{x} \tag{4.15}$$

over $\{v \in H^2(B_r(\mathbf{0})) : \ v = v_\infty \ \textit{in}\ B_r(\mathbf{0})\backslash B^\tau_\varrho\,;\ \textit{in particular}\ v = 0 \ \textit{in}\ \overline{B^-_r}(\mathbf{0})\}$.
Moreover

$$\delta(\varrho,\tau) \ = \ \int_{B^\tau_\varrho(\mathbf{0})} \left| D^2 v_\infty \right|^2 d\mathbf{x} \qquad \textit{for almost all } \varrho,\tau :\ 0<\tau<\varrho<r\,. \tag{4.16}$$

In particular $\Delta^2 v_\infty = 0$ *in* $B^+_r(\mathbf{0})$, $v_\infty = 0 = \partial v_\infty/\partial y$ *in* $B_r(\mathbf{0}) \cap \{y = 0\}$, *and* $v_\infty \in C^1(B_r(\mathbf{0}))$.

Theorem 4.4. ***(Decay of the functional*** $\mathcal{F}$ ***at*** $\partial\Omega$***)*** *Assume (4.1)-(4.6). Then, for suitable* $\bar\varrho > 0$ *and* $c_0 > 0$,

$$\forall k > 2,\ \forall \eta,\sigma \in (0,1), \quad \exists \varepsilon_1 > 0,\ \exists \vartheta_1 > 0 \qquad \textit{such that} \tag{4.17}$$

for all $\varepsilon \in (0,\varepsilon_1]$, *for any* $\mathbf{x} \in \partial\Omega$ *with* $\partial\Omega \in C^2$ *near* $\mathbf{x}$, *for any* v *which is an* $\overline{\Omega}\cap B_\varrho(\mathbf{x})$ *local minimizer of* $\mathcal{F}_{g\,w}(\cdot,\mu,\alpha,\beta,\overline{\Omega}\cap B_\varrho(\mathbf{x}))$, *for any* ϱ *s.t.* $B_\varrho(\mathbf{x}) \subset \left(\widetilde{\Omega}\setminus U\right)$, $0 < \varrho \leq \left(\varepsilon^k \wedge \bar\varrho \wedge (c_0 \vee 1)^{-1}\right)$, $\int_{B_\varrho(\mathbf{x})} |g|^4 \leq \varepsilon^k$ *and*

$$\alpha\,\mathcal{H}^1\left(S_v \cap (\overline{\Omega}\cap B_\varrho(\mathbf{x}))\right) \ + \ \beta\,\mathcal{H}^1\left((S_{\nabla v}\setminus S_v)\cap(\overline{\Omega}\cap B_\varrho(\mathbf{x}))\right) \ < \ \varepsilon\,\varrho\,, \tag{4.18}$$

we have

$$\begin{aligned} &\mathcal{F}_{g\,w}(v, B_{\eta\varrho}(\mathbf{x})) \\ &\qquad \leq \ \eta^{2-\sigma}\max\left\{\mathcal{F}_{g\,w}(v,B_\varrho(\mathbf{x}))\ ,\ \varrho^2\,\vartheta_1\ \left(\left(\mathrm{Lip}(\varphi')\right)^2 + \left(\mathrm{Lip}(Dw)\right)^2\right)\right\}\,. \end{aligned} \tag{4.19}$$

Proof of Theorem 4.1. The proof is achieved via direct methods by performing several steps which entail the partial regularity for a minimizer u of $\mathcal{F}$ (weak Blake & Zisserman functional for inpainting introduced in Section 2 by (2.3)). This is done following a scheme similar to the one used in [22], but here we have to deal also with the transmission condition at ∂U since the fidelity term $\mu\int|u-g|^2$ acts in $\Omega\setminus U$ and not in U.
A concise summary of these steps is given in the following.
The regularity is proven at points which have vanishing 1-dimensional density of $\mathcal{F}$, by performing:

(1) the same procedure of [14] at points in $\Omega\setminus\overline{U}$;
(2) the same procedure of [14] at points in U;
(3) the proof of partial regularity at points of $\partial\Omega$ via
   - blow-up at points of $\partial\Omega$ (Theorem 4.3) taking into account the two parameters describing the lunulae B^τ_ϱ (see the last line in (4.14));
   - suitable joining along lunulae filling half-disk in order to take into account Dirichlet condition at $\partial\Omega$;
   - a decay estimate of the weak functional evaluated at local minimizers (Theorem 4.4).
(4) the proof of partial regularity at points close to $\partial\Omega$ via
   - blow-up at points close to $\partial\Omega$ (Theorem 5.1 in [14])
   - L^2 hessian decay for bi-harmonic functions in a portion of a disk (Theorem 3.4 and Figure 1 in [21]) taking into account the two parameters describing the lunulae;
   - a decay estimate of the weak functional evaluated at local minimizers (Theorem 3.8 in [21]);
(5) the proof of partial regularity at points of ∂U via
   - blow-up at points of ∂U (see [25]);

- standard joining along disks;
- the decay estimate of the weak functional evaluated at local minimizers which follows by the previous blow-up.

By summing up the blow-up argument in all the previous cases, we can show that if a sequence of local minimizers has vanishing length of jumps and creases, then a subsequence (which is provided by Theorem 3.4), converges to a bi-harmonic function in the whole disk in cases (1), (2), (4) and (5), and in a half-disk in case (3). If the decay property is false we can construct a sequence of local minimizers which contradicts the previous statement, thanks to the classical estimates of the hessian in cases (1), (2), (4) and (5), while achieving the contradiction in case (3) is more difficult.

The usual approach to regularity at Dirichlet boundary points requires a smooth extension with suitable estimates of the blown-up solution: this method is satisfactory for first order problems since in that case one can exploit the extension of a harmonic function. Performing regularity analysis at Dirichlet boundary points in case (3) requires a smooth extension with suitable estimates of the blown-up solution. The extension of bi-harmonic functions is quite different from extension of an harmonic function vanishing at the diameter, the last one is based on classical Schwarz reflection principle and doubles L^2 norm of the gradient in the whole disk: this doubling property was exploited in [11] to prove decay property for local minimizers of Mumford & Shah functional with Dirichlet boundary condition (see also [36]); unfortunately bi-harmonic extension lacks this doubling property. We overcome this difficulty by a new tool, precisely an L^2 decay estimate of hessian for a bi-harmonic function in a half-disk vanishing together with its normal derivative on the diameter (Theorem 4.2): proving this decay requires a careful application (as in [22]) of Duffin extension formula [32] and Almansi decomposition [1], since the bi-harmonic extension in the whole disk may increase a lot the L^2 norm of the hessian in the complementary half-disk.

In cases (1) and (2) we can conclude the proof as like as in the last section of [14]; in case (3) as in [22]; in case (4) as in [21]; in case (5) as in [22] but exploiting a different blow-up (see [25]) which takes into account transmission conditions at ∂U.

In all cases we deduce that $\mathcal{H}^1\Big((S_u \cup S_{\nabla u}) \cap B_\varrho\Big)$ decays faster than ϱ.

By iterating the decay estimate of the functional in smaller and smaller balls, we get

$$\mathcal{H}^1\Big(\,(\overline{S_u \cup S_{\nabla u}} \setminus S_u \cup S_{\nabla u}) \cap \widetilde{\Omega}\Big) = 0.$$

So we can define a minimizing triplet as follows:

$$K_0 = \overline{S_u}, \qquad K_1 = \overline{S_{\nabla u}} \setminus K_0, \qquad u = \widetilde{u}.$$

Hence (1.12), (1.13) hold true. □

Proof of Theorem 1.1. The statement was proven in the context of image segmentation: Theorem 2.1 of [22]. □

5. Variational approximation and numerical tests

An important problem is the one of finding effective numerical methods suitable for the determination of the solutions given in Theorem 4.1.

A variational approximation of Blake & Zisserman functional for image segmentation and denoising under Neumann boundary condition has been studied in [3] and [10]. Here we propose a variational approximation of Blake & Zisserman functional for image inpaiting under Dirichlet boundary condition, by defining a suitable sequence of elliptic functionals. All these variational approximations are obtained in the framework of the notion of Γ-convergence, introduced by De Giorgi and Franzoni in [31], whose definition is recalled below for reader's convenience.

Definition 5.1. Let (X, d) be a metric space and let $F_h, F : X \to [0, +\infty]$ be functions. We say that (F_h) Γ-converge to F if the following two conditions are satisfied:

(1) for any sequence (x_h) in X converging to x, then $\liminf_h F_h(x_h) \geq F(x)$;
(2) for any $x \in X$ there exists a sequence (x_h) converging to x such that $\limsup_h F_h(x_h) \leq F(x)$.

The importance of this notion relies on the fact that it implies the convergence of minimizers of the approximating functionals to minimizers of the limiting functional.
Coming back to the Blake & Zisserman functional, it is clear that dealing numerically with the terms $\mathcal{H}^1\left(K_0 \cap \overline{\Omega}\right)$ and $\mathcal{H}^1\left((K_1 \setminus K_0) \cap \overline{\Omega}\right)$ can be quite difficult. Moreover, for the functional (2.3) for inpainting, the argument of [3] must be suitably adapted, in order to approximate it with a sequence of (simpler) functionals not involving surface energies. This is studied in the paper [25] while the formulation and implementation of related numerical algorithms are performed in [8].
In order to obtain the variational approximation of the functional (2.3), we introduce the elliptic functionals $\mathcal{G}_h$ as follows:

$$
\left\{
\begin{array}{ll}
\mathcal{G}_h(s,\sigma,v) := & \displaystyle\int_{\widetilde{\Omega}} \Big(\sigma^2 + \eta_h\Big)|D^2 v|^2 \, d\mathbf{x} \\
& \displaystyle +(\alpha-\beta)\int_{\widetilde{\Omega}} \Big(\frac{1}{h}|Ds|^2 + h\,\frac{(s-1)^2}{4}\Big)\, d\mathbf{x} \\
& \displaystyle +\beta\int_{\widetilde{\Omega}} \Big(\frac{1}{h}|D\sigma|^2 + h\,\frac{(\sigma-1)^2}{4}\Big)\, d\mathbf{x} \\
& \displaystyle +\xi_h\int_{\widetilde{\Omega}} \Big(s^2 + \zeta_h\Big)|Dv|^2\, d\mathbf{x} + \mu\int_{\Omega\setminus U} |v-g|^2\, d\mathbf{x} \\
& \displaystyle +h\int_{\widetilde{\Omega}\setminus\Omega} |v-w|^2\, d\mathbf{x} \\
& \qquad\qquad \text{if } v \in H^2(\widetilde{\Omega}),\ s,\sigma \in H^1(\widetilde{\Omega}) \text{ and } h \in \mathbb{N}; \\
\mathcal{G}_h(s,\sigma,v) := & +\infty \text{ otherwise.}
\end{array}
\right.
\tag{5.1}
$$

Functionals $\mathcal{G}_h$ are to be minimized on triplets of functions (s, σ, v). We emphasize that the minimization acts not only on the restored image v but also on two auxiliary functions: s which is a control function for ∇v and σ which is a control function of the Hessian of v.
To understand heuristically why this approximation holds, we observe that if (s_h, σ_h, v_h) is a sequence of minimizers of $\mathcal{G}_h$, then the function s_h assumes value 1 where v is continuous and it is close to 0 in a tubular neighborhood of discontinuity set S_v of thickness $1/h$. As $h \to +\infty$, this neighborhood shrinks and then, for h large enough, s_h yields an approximate representation of contours of v.
The function σ_h, instead, assumes value 0 only in a tubular neighborhood of $S_{\nabla v}$ of thickness $1/h$. As $h \to +\infty$, this neighborhood shrinks and σ_h yields an approximate representation of creases of v. The last term in (5.1) forces v to assume the value w in

$\widetilde{\Omega} \setminus \Omega$.

We conclude by showing some pictures obtained in numerical experiments which exploit the variational approximation (5.1) of the functional (2.3): Figures 3, 4 and 5 where the inpainting algorithm removes masks or overlapping text.

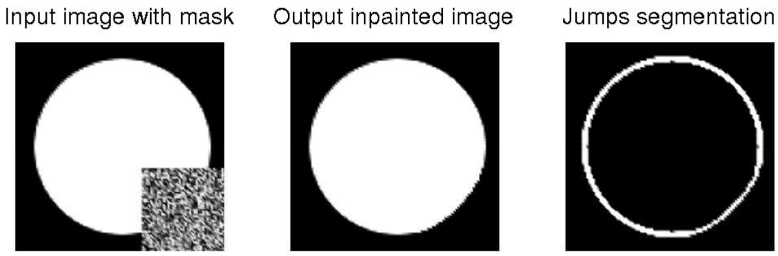

Figure 3. Inpainting of a circle without introducing artificial corners.

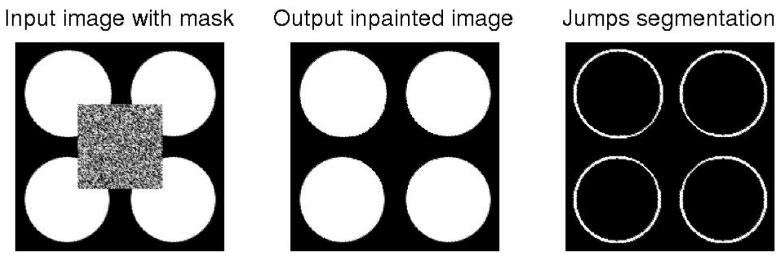

Figure 4. Inpainting of 4 circles.

Figure 5. Text removal.

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