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On an isoperimetric inequality for capacity conjectured by Pólya and Szegö

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Abstract

We study a conjecture by Pólya and Szegö on the approximation of the electrostatic capacity of convex bodies in terms of their surface measure. We prove that a "local version" of this conjecture holds true and we give some results which bring further evidence to its global validity.

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1 Introduction

The electrostatic capacity of a bounded set $\Omega \subset \mathbb{R}^3$ is defined by

$$\mathrm{Cap}(\Omega) = \frac{1}{4\pi} \inf \left\{ \int_{\mathbb{R}^3} |\nabla u|^2 \ : \ u \in \mathcal{C}_c^\infty(\mathbb{R}^3),\, u = 1 \text{ in } \Omega \right\} . \tag{1}$$

It can be equivalently obtained through the asymptotic expansion

$$u_\Omega(x) = \mathrm{Cap}(\Omega)|x|^{-1} + O(|x|^{-2}) \qquad \text{for } |x| \to +\infty ,$$

where u_Ω is the *electrostatic potential* of Ω, namely the unique function which solves the Euler-Lagrange equation for problem (1):

$$\Delta u_\Omega = 0 \quad \text{in } \mathbb{R}^3 \setminus \Omega , \qquad u_\Omega = 1 \quad \text{on } \partial\Omega , \qquad \lim_{|x|\to\infty} u_\Omega(x) = 0 . \tag{2}$$

For general Ω, this exterior problem is quite delicate to solve, even numerically, so that it is difficult to compute the exact value of $\mathrm{Cap}(\Omega)$. Nevertheless, since capacity appears in many physical phenomena, when the exact value is missing it is of great interest to find some estimates of it (see *e.g.* the pioneering works [7, 13, 14, 15, 16]). In this spirit, one may consider the following ratio:

$$\mathcal{E}(\Omega) = \frac{\mathrm{Cap}(\Omega)}{\sqrt{\frac{S(\Omega)}{4\pi}}} , \tag{3}$$

where $S(\Omega)$ denotes the surface area of Ω, namely the 2-dimensional Hausdorff measure $\mathcal{H}^2$ of its boundary $\partial\Omega$; in case Ω is a planar set, we understand that $S(\Omega) = 2\mathcal{H}^2(\Omega)$. Thus $\mathcal{E}(\Omega)$ is well-defined if and only if $\mathcal{H}^2(\Omega) > 0$. The denominator in (3) is also known as *Russell capacity* of Ω. Indeed, as already noticed about one century ago by Russell [17] (see also [1]), for simple geometries Ω where calculations may be performed explicitly, it gives a good approximation of $\mathrm{Cap}(\Omega)$. Clearly, for a fixed Ω, the closer $\mathcal{E}(\Omega)$ is to 1, the better the Russell capacity approximates the true capacity.

The approximation is perfect for balls, as the normalization constant at the denominator ensures that the value of $\mathcal{E}$ on balls is 1 (notice that $\mathcal{E}$ is invariant under dilations, as both numerator and denominator are 1-homogeneous). If one wants to get a uniform estimate on the approximation error when Ω varies, one has to study a shape optimization problem, consisting in the minimization of $\mathcal{E}(\Omega)$ over some class of admissible sets. It is immediate to realize that, in order to keep the infimum of $\mathcal{E}$ strictly positive, the *convexity* constraint is irremissible: indeed, a nonconvex set contained into a fixed sphere may have an arbitrarily large surface area, whereas by monotonicity its capacity remains bounded from above. No further constraint is a priori needed. Thus, setting

$$\mathcal{K} := \{\Omega \subset \mathbb{R}^3 \ : \ \Omega \text{ bounded and convex}, \mathcal{H}^2(\Omega) > 0\}$$

one comes to the optimization problem

$$\inf_{\mathcal{K}} \mathcal{E}(\Omega) \ . \tag{4}$$

Let us point out that the analogous maximization problem is not of interest since, by considering a sequence of thinning prolate ellipsoids, one can see that $\sup_{\mathcal{K}} \mathcal{E}(\Omega) = +\infty$ [4, Section 4.1]. On the contrary, the minimization in (4) is a long-standing open problem. More than half a century ago, Pólya and Szegö [16, §I.1.18] made the following:

Conjecture. Let D be a 2-dimensional disk. Then

$$\mathcal{E}(\Omega) \geq \inf_{\mathcal{K}} \mathcal{E} = \mathcal{E}(D) = \frac{2\sqrt{2}}{\pi} \approx 0.9 \ .$$

Moreover, $\mathcal{E}(\Omega) = \inf_{\mathcal{K}} \mathcal{E}$ if and only if Ω is a disk.

For the benefit of the reader, let us recall what is known at present about this conjecture.

- $\inf_{\mathcal{K}} \mathcal{E} \geq 2/\pi$ (see [16, (4), p.165]);
- the infimum $\inf_{\mathcal{K}} \mathcal{E}$ is attained (see [4]);
- the disk is the only minimizer of $\mathcal{E}$ if the class of admissible sets is restricted to the class of *planar* sets (see [15, p.14] and [16, §VII.7.3, p.157]).

The purpose of the present paper is to bring some new contributions to this challenging problem. We show that the inequality

$$\mathcal{E}(\Omega) > \mathcal{E}(D) \tag{5}$$

holds for Ω in some subclasses of $\mathcal{K}$. We distinguish our results into *local* and *global* ones.

In our results of local type we aim at proving inequality (5) for sets $\Omega \in \mathcal{K}$ which are close to D in the Hausdorff metric. We start with perturbations defined through a generic parametric family of concave functions (see Proposition 1), to deal then with perturbations obtained either by "flattening" a given graph (see Proposition 3) or via Minkowski addition (see Proposition 4). It is fascinating to compare the effect produced by different perturbations separately on surface area and capacity. For some perturbations surface area has a null derivative and capacity has a strictly positive derivative, whereas for some others surface area has a finite derivative and capacity has an infinite derivative. In any case, it turns out that the quotient $\mathcal{E}$ strictly increases as soon as the family of convex sets "detaches" from the disk. So, we arrive at the local minimality result for one-parameter perturbations stated in Theorem 5, and at the version for sequences converging to D in Hausdorff metric stated in Theorem 6.

In our results of global type, we prove inequality (5) for different classes of convex bodies, which are possibly far away from D in Hausdorff distance. First we show that (5) holds when Ω is any convex combination (through Minkowski addition) of the disk and a ball, see Theorem 7. The proof of this result is based on the Brunn-Minkowski inequality for capacity.
Then we turn our attention to the class of ellipsoids. We give numerical evidence that inequality (5) holds when Ω is a generic triaxial ellipsoid (see the plot in Figure 3), thus extending the computations made in [4] for the particular cases of prolate and oblate ellipsoids. Moreover, in Theorem 8 we prove analytically that no open triaxial ellipsoid except balls satisfies the stationarity condition for problem (4): it amounts to a Neumann condition for the electrostatic potential in terms of the mean curvature of the boundary, and thus leads to an intriguing overdetermined problem for harmonic functions on exterior domains. It was conjectured in [4] that such overdetermined exterior problem admits a solution only on the complement of a ball; in Theorem 8 we show that it does not have solution on the complement of any other ellipsoid.

The paper is organized as follows. Local results are stated in Section 2 and proved in Section 4. Global results are stated in Section 3 and proved in Section 5. Finally the Appendix is devoted to the proof of a general first variation formula for capacity, which is needed in Section 4, and may deserve an autonomous interest.

2 Local results

In this section we establish local minimality properties of D. We consider different kinds of one-parameter families of convex sets D_t, with $t \in [0,1]$. In all the situations under study, the Hausdorff distance between D_t and $D_0 = D$ will be infinitesimal as $t \to 0^+$: in fact D_t will be contained into a cylinder with height of order t. In order to determine the behaviour of the quotient energy $\mathcal{E}(D_t)$ for small $t > 0$, we need to analyze separately the behaviour of the two functions

$$C(t) := \mathrm{Cap}(D_t) \qquad \text{and} \qquad S(t) := \mathcal{H}^2(\partial D_t)\ .$$

More specifically, we are led to investigate the properties of their incremental ratios in a right neighbourhood of $t = 0$. Although they are merely *right* derivatives, we set

$$C'(0) := \lim_{t\to0^+} \frac{C(t) - C(0)}{t} \qquad \text{and} \qquad S'(0) := \lim_{t\to0^+} \frac{S(t) - S(0)}{t}\ ,$$

whenever these limits exist. For convenience, we think of D as centered at the origin and contained into the plane $x_3 = 0$, namely

$$D = \{(x_1, x_2, 0) \in \mathbb{R}^3\ :\ 0 \le r < 1\}\ ; \tag{6}$$

here and below, for any $(x_1, x_2, x_3) \in \mathbb{R}^3$, we set $r = \sqrt{x_1^2 + x_2^2}$.
We consider first the case when the D_t's are the subgraphs of a one-parameter family of concave radially symmetric functions defined on D. We call them a *parametric family* which we denote by $\phi(r;t)$, and we assume that

$$\phi \in \mathcal{C}\left([0,1]^2\right) \text{ with } t \mapsto \phi(\cdot;t) \text{ nondecreasing and } \phi(\cdot;0) \equiv 0; \tag{7}$$

$$r \mapsto \phi(r;t) \text{ nonincreasing, concave, with } \phi(0;t) = t \text{ and } \phi(1;t) = 0\ . \tag{8}$$

The next proposition is the key result from which the local minimality of D will stem. It is obtained by using as main tools the first variation formula for capacity proved in Appendix (see Theorem 15), the explicit expression for the potential of D (see Lemma 13), and careful estimates on (a regularization of) the parametric family $\phi(r;t)$.

Proposition 1. (Parametric families) *For $t \in [0,1]$, let D_t be given by*

$$D_t = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ 0 \le r < 1,\, 0 < x_3 < \phi(r;t)\} \ ,$$

where $\phi(r;t)$ is a parametric family satisfying assumptions (7) *and* (8)*. Then:*

(i) We have

$$\liminf_{t\to 0^+} \frac{C(t) - C(0)}{t} \ge \frac{2}{\pi^2}(1 - \log 2).$$

(ii) We have

$$\limsup_{t\to 0^+} \frac{S(t) - S(0)}{t} \le 2\pi \ .$$

(iii) If, for a sequence $\{t_k\}$ decreasing to zero, it holds

$$\lim_{k\to+\infty} \frac{S(t_k) - S(0)}{t_k} > 0,$$

then there exists a subsequence $\{t^l\} := \{t_{k_l}\}$ such that

$$\lim_{l\to+\infty} \frac{C(t^l) - C(0)}{t^l} = +\infty.$$

In particular, if $S'(0)$ exists and $S'(0) > 0$, then $C'(0)$ exists and $C'(0) = +\infty$.

Remark 2. One may wonder if for some parametric family $\phi(r;t)$, satisfying assumptions (7) and (8), $S'(0)$ might not exist. The answer is affirmative, and an explicit example can be constructed as follows. Let

$$\alpha(t) := t\Big(\frac{1}{2} + \frac{1}{4}\sin(\log t)\Big) \ ,$$

and define

$$\phi(r;t) := \begin{cases} t - r\,\frac{2t - \sqrt{3}\alpha}{2-\alpha} & \text{if } r \in [0, 1 - \frac{\alpha}{2}] \\ \sqrt{\alpha^2 - (r - 1 + \alpha)^2} & \text{if } r \in [1 - \frac{\alpha}{2}, 1] \ . \end{cases}$$

We leave to the reader to check that this family satisfies assumptions (7) and (8), and that the surface area increment admits the asymptotic development

$$\frac{S(t) - S(0)}{t} = 2\pi\Big(\frac{\pi}{3} - \frac{1}{2}\Big)\frac{\alpha(t)}{t} + o(1) \qquad \text{as } t \to 0^+ \ ,$$

so that by the definition of $\alpha(t)$ the derivative $S'(0)$ does not exist.

Statement (i) of Proposition 1 implies at once the local minimality of $\mathcal{E}(D_t)$ at $t = 0$, when combined with the information that $S'(0) = 0$. This occurs for instance in the case of flattening graphs considered in Proposition 3 below, when the parametric family takes the special form $\phi(r;t) = t\varphi(r)$. On the other hand, by a comparison argument, statement (iii) of Proposition 1 allows to deal with (possibly non axially-symmetric) perturbations D_t obtained from D through a Minkowski addition. Again the outcoming information on the derivatives $C'(0)$ and $S'(0)$, stated in Proposition 4 below, implies immediately the local minimality of $\mathcal{E}(D_t)$ at $t = 0$.