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Remainder terms in a higher order Sobolev inequality

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Abstract

For higher order Hilbertian Sobolev spaces, we improve the embedding inequality for the critical L^p-space by adding a remainder term with a suitable weak norm.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be any domain and for an integer m consider the space $\mathcal{D}^{m,2}(\Omega)$, namely the completion of the space of real-valued C^∞-functions with compact support in Ω with respect to the norm

$$\|u\| = \left(\int_\Omega (-\Delta)^m u \cdot u\right)^{1/2} = \begin{cases} |\Delta^{m/2} u|_2 & \text{if } m \text{ is even,} \\ |\nabla \Delta^{(m-1)/2} u|_2 & \text{if } m \text{ is odd,} \end{cases} \tag{1.1}$$

where $|u|_p$ denotes the L^p-norm of a function $u \in L^p(\Omega)$. We assume that $m < \frac{N}{2}$, then the so-called critical Sobolev exponent $2^* = 2N/(N-2m)$ is well-defined and the following inequality holds

$$S\,|u|_{2^*}^2 \le \|u\|^2 \qquad \text{for all } u \in \mathcal{D}^{m,2}(\Omega). \tag{1.2}$$

It is known [12, 14] that the best constant

$$S = \inf_{\substack{u \in \mathcal{D}^{m,2}(\Omega) \\ u \neq 0}} \frac{\|u\|^2}{|u|_{2^*}^2}$$

in inequality (1.2) does not depend on the domain Ω, and that S is attained if and only if $\Omega = \mathbb{R}^N$ and

$$u \in \mathcal{M} := \{cU_{\lambda,y} : c \in \mathbb{R} \setminus \{0\}, y \in \mathbb{R}^N, \lambda > 0\} \tag{1.3}$$

where

$$U_{\lambda,y} \in \mathcal{D}^{m,2}(\mathbb{R}^N), \qquad U_{\lambda,y}(x) := \lambda U(\lambda^{\frac{2}{N-2m}}(x-y)),$$

and $U \in \mathcal{D}^{m,2}$ is given by $U(x) = (1+|x|^2)^{-\frac{N-2m}{2}}$. In the sequel we will also write U_λ in place of $U_{\lambda,0}$. The minimization property of the functions $U_{\lambda,y}$ implies that they satisfy the equation

$$(-\Delta)^m U_{\lambda,y} = \tau_m |U_{\lambda,y}|^{2^*-2} U_{\lambda,y} \qquad \text{with } \tau_m = \frac{\|U_{\lambda,y}\|^2}{|U_{\lambda,y}|_{2^*}^{2^*}} = 2^{2m}\frac{\Gamma(\frac{N}{2}+m)}{\Gamma(\frac{N}{2}-m)}. \tag{1.4}$$

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In the present paper, we are interested in bounded domains $\Omega \subset \mathbb{R}^N$. In this case, the space $\mathcal{D}^{m,2}(\Omega)$ is usually denoted by $H_0^m(\Omega)$ and we stick to this notation. Since S is not attained when Ω is bounded, it is natural to wonder if some lower bounds exist for the remainder term $\|u\|^2 - S|u|_{2^*}^2$ whenever $u \in H_0^m(\Omega)$. Generalizing a result of Brezis-Lieb [4] for the first order case $m = 1$, Gazzola-Grunau [7] proved that for any bounded domain $\Omega \subset \mathbb{R}^N$ there exists $C = C(\Omega, m) > 0$ such that

$$\|u\|^2 - S|u|_{2^*}^2 \geq C|u|_w^2 \qquad \text{for all } u \in H_0^m(\Omega) \tag{1.5}$$

where $|u|_w$ denotes the weak $L^{2^*/2}$-norm (see [11]) defined by

$$|u|_w = \sup_{\substack{A \subset \Omega \\ |A|>0}} |A|^{-\frac{2m}{N}} \int_A |u|.$$

The space $H_0^m(\Omega)$ is of interest for the study of boundary value problems for the polyharmonic operator $(-\Delta)^m$ complemented with Dirichlet boundary conditions $u = u_\nu = \cdots = \frac{\partial^{m-1}}{\partial \nu^{m-1}} u = 0$ on $\partial\Omega$. If these boundary conditions are replaced by Navier boundary conditions $u = \Delta u = \Delta^2 u = \ldots \Delta^{m-1} u = 0$ on $\partial\Omega$, one is led to consider the space

$$H_\theta^m(\Omega) = \left\{ u \in H^m(\Omega) \ : \ \Delta^j u = 0 \text{ for } 0 \leq j < \frac{m}{2} \right\}$$

which may also be endowed with the norm (1.1). Clearly, whenever $m \geq 2$, the space $H_\theta^m(\Omega)$ is strictly larger than $H_0^m(\Omega)$. Nevertheless, it has been shown in [8] (see also previous work in [9, 15]) that the Sobolev inequality (1.2) holds with the same optimal constant S also for functions in $H_\theta^m(\Omega)$. Whenever $m \geq 2$, this fact does not follow by a trivial extension argument, as is most easily seen in the special case $m = 2$. Indeed, in this case any extension of a function in $H_\theta^2(\Omega)$ with nontrivial outer normal derivative u_ν on $\partial\Omega$ to a function in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ increases the norm $\|\cdot\|$ if $\mathbb{R}^N \setminus \overline{\Omega} \neq \varnothing$. We also point out that the optimal constant changes for subcritical embeddings, namely embeddings in L^p with $p < 2^*$, see [5]. In this paper we prove a remainder term estimate of type (1.5) for functions $u \in H_\theta^m(\Omega)$. We note that the proof of (1.5) in [7] does not carry over to functions in this larger space since one cannot trivially extend functions in $H_\theta^m(\Omega)$ to functions in $H_\theta^m(B)$ where B is a ball containing Ω; moreover, a further nontrivial radial extension outside this larger ball B was needed in [7] and this extension seems not to be possible in $H_\theta^m(\Omega)$ even if Ω is itself a ball. The following is the main result of the present paper.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ a bounded domain with $\partial\Omega$ of class C^m. Then there exists a constant $C = C(\Omega, m) > 0$ such that*

$$\|u\|^2 - S|u|_{2^*}^2 \geq C|u|_w \qquad \textit{for all } u \in H_\theta^m(\Omega).$$

The exponent of the weak norm is sharp. Indeed, using functions of the form ψU_λ as test functions with $0 \in \Omega$, large λ and a cut off function ψ, it is easily seen that an estimate of this type cannot hold for $q > 2*/2$. For expansions of different norms of ψU_λ as $\lambda \to \infty$, see [6, 9, 10]. On the other hand, Theorem 1.1 implies that for all $q \in [1, 2^*/2)$ there exists a constant $C_q = C_q(n, \Omega) > 0$ such that

$$\|u\|^2 \geq S|u|_{2^*}^2 + C_q|u|_q^2 \qquad \text{for all } u \in H_\theta^m(\Omega).$$

Our proof of Theorem 1.1 is based on the following tools. First, we use Talenti's comparison principle [13] to reduce the problem to radial positive functions in a ball. Second, we apply the extension map constructed in the recent paper [8] in order to pass to radial functions in $\mathcal{D}^{m,2}(\mathbb{R}^N)$. Finally, we use a remainder term estimate proved in [2]. In Section 2 below we collect and discuss these tools, and in Section 3 we complete the proof of Theorem 1.1.

2 Preliminaries

In the following, for the sake of clarity we will sometimes specify the domain of integration in the norms we use, that is, we write $|\cdot|_{p,\Omega}$, $\|\cdot\|_\Omega$ and $|\cdot|_{w,\Omega}$. We denote by B the unit ball in $\mathbb{R}^N$, by $e_N = |B|$ its measure and by $f^* \in L^2(B)$ the spherical rearrangement of $f \in L^2(\Omega)$ when $|\Omega| = |B|$. Here we use the definition of f^* given in [13, p. 701], so the superlevel sets $\{x \in B : f^*(x) > t\}$ are concentric balls centered at zero with the same measure as $\{x \in \Omega : |f(x)| > t\}$. With this definition, $f^* = |f|^*$ is always a nonnegative and radially decreasing function - even if f is sign changing.
The first crucial tool for the proof of Theorem 1.1 is the following comparison principle due to Talenti [13, Theorem 1].

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a C^m-smooth bounded domain such that $|\Omega| = |B| = e_N$. Let $m = 2k$ be an even number. Let $g \in L^2(\Omega)$ and let $u \in H^m_\theta(\Omega)$ be the unique strong solution to*

$$\begin{cases} (-\Delta)^k u = g & \text{in } \Omega, \\ \Delta^j u = 0 & \text{on } \partial\Omega, \qquad j = 0, \dots, k-1. \end{cases}$$

Let $g^* \in L^2(B)$ and $u^* \in H^1_0(B)$ denote respectively the spherical rearrangements of g and u, and let $v \in H^m_\theta(B)$ be the unique strong solution to

$$\begin{cases} (-\Delta)^k v = g^* & \text{in } B, \\ \Delta^j v = 0 & \text{on } \partial B, \qquad j = 0, \dots, k-1. \end{cases} \tag{2.1}$$

Then, $v \geq u^*$ a.e. in B.

As we shall see, Proposition 2.1 enables us to reduce the proof of Theorem 1.1 to the case where $\Omega = B$ and to the subspace of H^m_θ of radially symmetric and decreasing functions, which we denote by $R^m_\theta(B)$.

The second tool needed in the proof of Theorem 1.1 ia an extension argument taken from [8] which we now explain in some detail. Consider first the case where m is even, namely $m = 2k$ for some $k \geq 1$. For any $g : [0, \infty) \to \mathbb{R}$ with appropriate integrability conditions, we define

$$(\mathcal{G}g)(r) := \int_r^\infty \int_0^\rho \left(\frac{s}{\rho}\right)^{N-1} g(s) ds d\rho.$$

If g goes to 0 fast enough for $r \to \infty$ (e.g. like $r^{-\gamma}$ with $\gamma > 2$), then an integration by parts gives

$$(\mathcal{G}g)(r) = \frac{1}{N-2} r^{2-N} \int_0^r s^{N-1} g(s) ds + \frac{1}{N-2} \int_r^\infty s g(s) ds, \tag{2.2}$$

and

$$-\Delta (\mathcal{G}g)(|x|) = g(|x|) \text{ for } x \in \mathbb{R}^N.$$

Moreover, we denote by $\mathcal{G}^k$ the k-th iteration of the operator $\mathcal{G}$. With these notations we recall a result by Gazzola-Grunau-Sweers [8]:

Proposition 2.2. *Let $m = 2k$ and let $u \in R^m_\theta(B) \setminus \{0\}$. Let $w(r) = \left(\mathcal{G}^k f\right)(r)$ for*

$$f(r) = \begin{cases} (-\Delta)^k u(r) & \text{if } r \leq 1, \\ 0 & \text{if } r > 1, \end{cases}$$

then $w \in \mathcal{D}^{m,2}(\mathbb{R}^N)$, $\|w\|_{\mathbb{R}^N} = \|u\|_B$, and $|w|_{2^*,\mathbb{R}^N} > |u|_{2^*,B}$.

In particular, if $m = 2$ the extension of a radial function $u = u(r)$ in $R^2_\theta(B)$ is given by

$$w(r) = \begin{cases} u(r) + \dfrac{1}{N-2}|u'(1)| & \text{if } r \in (0,1), \\ \dfrac{r^{N-2}}{N-2}|u'(1)| & \text{if } r \in [1,\infty). \end{cases}$$

Proposition 2.2 also enables us to treat the case of odd m, namely $m = 2k+1$ for some $k \geq 1$. Since $H^{2k+1}_\theta(B) \subset H^{2k}_\theta(B)$, by Proposition 2.2 we know that any $u \in R^{2k+1}_\theta(B) \setminus \{0\}$ allows to define an entire function w such that

$$w > u \text{ in } B\,, \quad \Delta^k(w-u) = 0 \text{ in } B\,, \quad \Delta^k w = 0 \text{ in } \mathbb{R}^N \setminus B.$$

In particular, this implies that also

$$\nabla(\Delta^k(w-u)) = 0 \text{ in } B\,, \quad \nabla(\Delta^k w) = 0 \text{ in } \mathbb{R}^N \setminus B. \tag{2.3}$$

The construction for the $2k$-case also enables us to conclude that $w \in C^{2k-1}(\mathbb{R}^N)$, a regularity which is not enough to obtain $w \in \mathcal{D}^{2k+1,2}(\mathbb{R}^n)$, here we need one more degree of regularity. This is obtained by recalling the extra boundary condition that appears by going from $H^{2k}_\theta(B)$ to $H^{2k+1}_\theta(B)$, namely $\Delta^k u = 0$ on ∂B, and that $\Delta^k w = 0$ in $\mathbb{R}^N \setminus B$.

Next, we recall a result by Bartsch, Weth and Willem [2]:

Proposition 2.3. *There exists a constant $\alpha > 0$ such that*

$$\|u\|^2 - S|u|^2_{2^*} \geq \alpha \text{dist}(u,\mathcal{M})^2 \qquad \textit{for all } u \in \mathcal{D}^{m,2}(\mathbb{R}^N).$$

Here $\text{dist}(u,\mathcal{M}) = \inf\{\|u-v\| : v \in \mathcal{M}\}$ *is the distance of u from $\mathcal{M}$ in $\mathcal{D}^{m,2}(\mathbb{R}^N)$.*

For $m = 1$ this result is due to Bianchi and Egnell [3], solving a problem posed by Brezis and Lieb [4].

We finally note that if $u \in \mathcal{D}^{m,2}(\mathbb{R}^N)$ is a function with $\text{dist}(u,\mathcal{M}) < \|u\|$, then there exists $v \in \mathcal{M}$ with $\text{dist}(u,\mathcal{M}) = \|u-v\|$ since $\mathcal{M}$ is relatively closed in $\mathcal{D}^{m,2}(\mathbb{R}^N) \setminus \{0\}$. If, in addition, u is a radial positive function, then the distance minimizing $v \in M$ can be chosen as a positive and radial function, i.e. $v = cU_\lambda$ with $c, \lambda > 0$. To see this, we note that every positive function $v \in M$ is a translation of a radially decreasing function. Therefore $v \in M$ implies $v^* \in M$, whereas by (1.4) and [1, Theorem 2.2] we have

$$\int_{\mathbb{R}^N} (-\Delta^m v)u = \tau_m \int_{\mathbb{R}^N} v^{2^*-1}u \leq \tau_m \int_{\mathbb{R}^N} (v^*)^{2^*-1}u = \int_{\mathbb{R}^N} (-\Delta^m v^*)u$$

and therefore

$$\|u-v\|^2 = \|u\|^2 + S^2|v|^2_{2^*} - 2\int_{\mathbb{R}^N} (-\Delta^m v)u \geq \|u\|^2 + S^2|v^*|^2_{2^*} - 2\int_{\mathbb{R}^N} (-\Delta^m v^*)u = \|u-v^*\|^2$$

3 Proof of Theorem 1.1

With no loss of generality we may assume that $|\Omega| = |B| = e_N$.

Assume first that m is even, $m = 2k$ for some $k \geq 1$. Take any function $u \in H^m_\theta(\Omega)$, put $g := (-\Delta)^k u$, and let $v \in H^m_\theta(B)$ the unique solution to (2.1). Then by the properties of symmetrization, see [1], we obtain both that

$$\|v\|^2_B = |\Delta^k v|^2_{2,B} = |g^*|^2_{2,B} = |g|^2_{2,\Omega} = |\Delta^k u|^2_{2,\Omega} = \|u\|^2_\Omega \tag{3.1}$$

and

$$\|u\|^2_{2^*,\Omega} = |u^*|^2_{2^*,B} \le |v|^2_{2^*,B} \tag{3.2}$$

where, for the last inequality, we used Proposition 2.1. Moreover, for any $A \subset \Omega$ such that $|A| > 0$ we have

$$|A|^{-\frac{2m}{N}} \int_A |u| = |A|^{-\frac{2m}{N}} \int_\Omega \chi_A |u|$$

where χ_A denotes the characteristic function of A. Since by [1, Theorem 2.2] we know that

$$\int_\Omega \chi_A |u| \le \int_B \chi_A^* u^*,$$

for any such A we have

$$|A|^{-\frac{2m}{N}} \int_A |u| \le |A^*|^{-\frac{2m}{N}} \int_{A^*} u^*$$

and therefore, by taking the supremum over all such A, we deduce that $|u|_{w,\Omega} \le |u^*|_{w,B}$. In turn, by Proposition 2.1, we infer that

$$|u|_{w,\Omega} \le |v|_{w,B}. \tag{3.3}$$

Putting together (3.1), (3.2), and (3.3) shows that if we can prove Theorem 1.1 in the symmetric framework where $\Omega = B$ and $u \in R^m_\theta(B)$, then we are done.
A similar conclusion is reached if m is odd, $m = 2k+1$ for some $k \ge 0$. In this case, invoking again [1], (3.1) becomes an inequality:

$$\|v\|^2_B = |\nabla\Delta^k v|^2_{2,B} = |\nabla g^*|^2_{2,B} \le |\nabla g|^2_{2,\Omega} = |\nabla\Delta^k u|^2_{2,\Omega} = \|u\|^2_\Omega,$$

which also allows to consider just the case where $\Omega = B$ and $u \in R^m_\theta(B)$.

We now proceed by contradiction. If the assertion of Theorem 1.1 is false, then there exists a sequence of functions $u_n \in R^m_\theta(B)$ $(n \in \mathbb{N})$ such that $\|u_n\|_B = 1$ for all n and

$$\frac{1 - S|u_n|^2_{2^*,B}}{|u_n|_{w,B}} \to 0 \qquad \text{as } n \to \infty. \tag{3.4}$$

We denote by w_n the extension of u_n as given by Proposition 2.2 (if m is odd, also the remarks following Proposition 2.2 are needed). Then we know that

$$\|w_n\|_{\mathbb{R}^N} = \|u_n\|_B = 1\ , \qquad |w_n|_{2^*,\mathbb{R}^N} > |u_n|_{2^*,B}.$$

Moreover, recalling that $w_n > u_n$ in B, we also have

$$|w_n|_{w,\mathbb{R}^N} > |u_n|_{w,B}.$$

Consequently,

$$0 \le 1 - S|w_n|^2_{2^*,\mathbb{R}^N} \le 1 - S|u_n|^2_{2^*,B} \to 0$$

and therefore, by Proposition 2.3,

$$\text{dist}(w_n, \mathcal{M}) \to 0 \qquad \text{as } n \to \infty.$$

Since $\|w_n\|_{\mathbb{R}^N} = 1$ for all $n \in \mathbb{N}$, it follows by the remarks below Proposition 2.3 that there exists $c_n, \lambda_n > 0$ with $\|w_n - c_n U_{\lambda_n}\|_{\mathbb{R}^N} = \text{dist}(w_n, \mathcal{M})$, and that

$$0 < \inf_{n\in\mathbb{N}} c_n \le \sup_{n\in\mathbb{N}} c_n < \infty.$$

In case that $m = 2k$ is even, we have

$$\begin{aligned}\operatorname{dist}(w_n, \mathcal{M})^2 &= \|w_n - c_n U_{\lambda_n}\|^2_{\mathbb{R}^N} = |\Delta^k (w_n - c_n U_{\lambda_n})|^2_{2,\mathbb{R}^N} \\ &\geq |\Delta^k (w_n - c_n U_{\lambda_n})|^2_{2,\mathbb{R}^N \setminus B} = c_n |\Delta^k U_{\lambda_n}|^2_{2,\mathbb{R}^N \setminus B} \geq S\, c_n |U_{\lambda_n}|^2_{2^*,\mathbb{R}^N \setminus B}\end{aligned}$$

since $\Delta^k w_n = 0$ a.e. in $\mathbb{R}^N \setminus B$ for $n \in \mathbb{N}$. In case $m = 2k+1$ is odd, we get the same conclusion using (2.3). In both cases necessarily $\lambda_n \to \infty$ and therefore $\lambda_n \geq 1$ for all n after passing to a subsequence. This yields that

$$\begin{aligned}\frac{|U_{\lambda_n}|^{2^*}_{2^*,\mathbb{R}^N \setminus B}}{N e_N} &= \lambda_n^{2^*} \int_1^\infty \frac{r^{N-1}}{\left[1 + (\lambda_n^{\frac{2}{N-2m}} r)^2\right]^N}\, dr = \int_{\lambda_n^{\frac{2}{N-2m}}}^\infty \frac{r^{N-1}}{(1+r^2)^N}\, dr \\ &\geq 2^{-N} \int_{\lambda_n^{\frac{2}{N-2m}}}^\infty \frac{dr}{r^{N+1}} = \frac{1}{N\, 2^N\, \lambda_n^{2^*}}.\end{aligned}$$

We conclude that

$$\operatorname{dist}(w_n, \mathcal{M}) \geq \frac{C_1}{\lambda_n} \qquad \text{with } C_1 > 0 \text{ independent of } n \in \mathbb{N}.$$

On the other hand, a scaling argument shows that $|c_n U_{\lambda_n}|_{w,B} \leq \frac{c_n}{\lambda_n} |U|_{w,\mathbb{R}^N}$; we point out that scaling gives this nice estimate precisely because we deal with the weak $L^{2^*/2}$-norm. Therefore, we have

$$\begin{aligned}|u_n|_{w,B} &\leq |w_n|_{w,B} \leq |c_n U_{\lambda_n}|_{w,B} + |w_n - c_n U_{\lambda_n}|_{w,B} \\ &\leq \frac{c_n}{\lambda_n} |U|_{w,\mathbb{R}^N} + C_2 \|w_n - c_n U_{\lambda_n}\|_{\mathbb{R}^N} \leq C_3 \operatorname{dist}(w_n, \mathcal{M}).\end{aligned}$$

with constants $C_2, C_3 > 0$ independent of n. Hence, (3.4) implies that

$$\frac{\|w_n\|^2_{\mathbb{R}^N} - S |w_n|^2_{2^*,\mathbb{R}^N}}{\operatorname{dist}(w_n, \mathcal{M})^2} \to 0 \qquad \text{as } n \to \infty.$$

contrary to Proposition 2.3. This contradiction shows the claim.

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