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Polyharmonic boundary value problems

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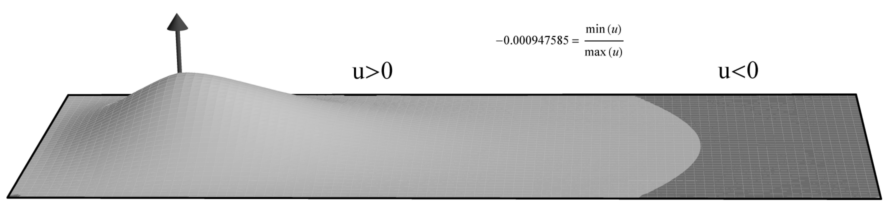
Polyharmonic boundary value problems

A monograph on positivity preserving and nonlinear higher order elliptic equations in bounded domains



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Dedicated to our wives Chiara, Brigitte and Barbara.

The cover figure displays the solution of $\Delta^2 u = f$ in a rectangle with homogeneous Dirichlet boundary condition for a nonnegative function f with its support concentrated near a point on the left hand side. The dark part shows the region where $u < 0$.

Preface

Linear elliptic equations arise in several models describing various phenomena in the applied sciences, the most famous being the second order stationary heat equation or, equivalently, the membrane equation. For this intensively well-studied linear problem there are two main lines of results. The first line consists of existence and regularity results. Usually the solution exists and "gains two orders of differentiation" with respect to the source term. The second line contains comparison type results, namely the property that a positive source term implies that the solution is positive under suitable side constraints such as homogeneous Dirichlet boundary conditions. This property is often also called positivity preserving or, simply, maximum principle. These kinds of results hold for general second order elliptic problems, see the books by Gilbarg-Trudinger [197] and Protter-Weinberger [346]. For linear higher order elliptic problems the existence and regularity type results remain, as one may say, in their full generality whereas comparison type results may fail. Here and in the sequel "higher order" means order at least four.

Most interesting models, however, are nonlinear. By now, the theory of second order elliptic problems is quite well developed for semilinear, quasilinear and even for some fully nonlinear problems. If one looks closely at the tools being used in the proofs, then one finds that many results benefit in some way from the positivity preserving property. Techniques based on Harnack's inequality, De Giorgi-Nash-Moser's iteration, viscosity solutions etc., all use suitable versions of a maximum principle. This is a crucial distinction from higher order problems for which there is no obvious positivity preserving property. A further crucial tool related to the maximum principle and intensively used for second order problems is the truncation method, introduced by Stampacchia. This method is helpful in regularity theory, in properties of first order Sobolev spaces and in several geometric arguments, such as the moving planes technique which proves symmetry of solutions by reflection. Also the truncation (or reflection) method fails for higher order problems. For instance, the modulus of a function belonging to a second order Sobolev space may not belong to the same space. The failure of maximum principles and of truncation methods, one could say, are the main reasons why the theory of nonlinear higher order elliptic equations is by far less developed than the theory of analogous second order equations. On the other hand, in view of many applications and increasing interest especially in the last twenty years, one should try to develop new tools suitable for higher order problems involving polyharmonic operators.

The simple example of the two functions $x \mapsto \pm|x|^2$ shows that already for the biharmonic operator the standard maximum principle fails. Nevertheless, taking also boundary conditions into account could yield comparison or positivity preserving properties and indeed, in certain special situations, such behaviour can be observed. It is one goal of the present exposition to describe situations where positivity preserving properties hold true or fail, respectively, and to explain how we have tackled the main difficulties related to the lack of a general comparison principle. In the present book we also show that in many higher order problems positivity preserving "almost" occurs. By this we mean that the solution to a problem inherits the sign of the data, except for some small contribution. By the experience from the present work, we hope that suitable techniques may be developed in order to obtain results quite analogous to the second order situation. Many recent higher order results give support to this hope.

A further goal of the present book is to collect some of those problems, where the authors were particularly involved, and to explain by which new methods one can replace second order techniques. In particular, to overcome the failure of the maximum principle and of the truncation method several ad hoc ideas will be introduced.

Let us now explain in some detail the subjects we address within this book.

Linear higher order elliptic problems

The polyharmonic operator $(-\Delta)^m$ is the prototype of an elliptic operator L of order $2m$, but with respect to linear questions, much more general operators can be considered. A general theory for boundary value problems for linear elliptic operators L of order $2m$ was developed by Agmon-Douglis-Nirenberg [4, 5, 6, 148]. Although the material is quite technical, it turns out that the Schauder theory as well as the L^p-theory can be developed to a large extent analogously to second order equations. The only exception are maximum modulus estimates which, for linear higher order problems, are much more restrictive than for second order problems. We provide a summary of the main results which hopefully will prove to be sufficiently wide to be useful for anybody who needs to refer to linear estimates or existence results.

The main properties of higher – at least second – order Sobolev spaces will be recalled. Since more orders of differentiation are involved, several different equivalent norms are available in these spaces. A crucial role in the choice of the norm is played by the regularity of the boundary. For the second order Dirichlet problem for the Poisson equation a nonsmooth boundary leads to technical difficulties but, due to the maximum principle, there is an inherent stability so that, when approximating nonsmooth domains by smooth domains, one recovers most of the features for domains with smooth boundary, see [46]. For Neumann boundary conditions the situation is more complicated in domains with rather wild boundaries, although even for polygonal boundaries they do not show spectacular changes. For higher order boundary value problems some peculiar phenomena occur. For instance, the so-called Babuška and Sapondžyan paradoxes [28, 357] forces one to be very careful in the choice of the norm in second order Sobolev spaces since some boundary value problems strongly depend on the regularity of the boundary. This phenomenon and its consequences will be studied in some detail.

Positivity in higher order elliptic problems

As long as existence and regularity results are concerned, the theory of linear higher order problems is already quite well developed as explained above. This is no longer true as soon as qualitative properties of the solution related to the source term are investigated. For instance, if we consider the clamped plate equation

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

the "simplest question" seems to find out whether the positivity of the datum implies the positivity of the solution, Or, physically speaking,

does upwards pushing of a clamped plate yield upwards bending?

Equivalently, one may ask whether the corresponding Green function G is positive. In some special cases, the answer is "yes", while it is "no" in general. However, in numerical experiments, it appears very difficult to display the negative part and heuristically, one feels that the negative part of G – if present at all – is small in a suitable sense compared with the "dominating" positive part. We discuss not only the cases where one has positive Green functions and develop a perturbation theory of positivity, but we shall also discuss systematically under which conditions one may expect the negative part of the Green function to be small. We expect such smallness results to have some impact on future developments in the theory of nonlinear higher order elliptic boundary value problems.

Boundary conditions

For second order elliptic equations one usually extensively studies the case of Dirichlet boundary conditions because other boundary conditions do not exhibit too different behaviours. For the biharmonic equation $\Delta^2 u = f$ in a bounded domain of $\mathbb{R}^n$ it is not at all obvious which boundary condition would serve as a role model. Then a good approach is to focus on some boundary conditions that describe physically relevant situations. We consider a simplified energy functional and derive its Euler-Lagrange equation including the corresponding natural boundary conditions. We start with the linearised model for the beam. From a physical point of view, as long as the fourth order planar equation is considered, the most interesting seem to be not only the Dirichlet boundary conditions but also the Navier or Steklov boundary conditions. The Dirichlet conditions correspond to the clamped plate model whereas Navier and Steklov conditions correspond to the hinged plate model, either by neglecting or considering the contribution of the curvature of the boundary. Each one of these boundary conditions requires the unknown function to vanish on the boundary, the difference being on the second boundary condition. These three boundary conditions have their own features and none of them may be thought to play the model role. We discuss all of them and emphasise their own peculiarities with respect to the comparison principles, to their variational formulation and to solvability of related nonlinear problems.

Eigenvalue problems

For second order problems, such as the Dirichlet problem for the Laplace operator, one has not only the existence of infinitely many eigenvalues but also the simplicity and the one sign property of the first eigenfunction. For the biharmonic Dirichlet problem, this property is true in a ball but it is false in general. Again, a crucial role is played by the sign of the corresponding Green function. Concerning the isoperimetric properties of the first eigenvalue of the Dirichlet-Laplacian, the Faber-Krahn [162, 253, 254] result states that, among domains having the same finite volume it attains its minimum when the domain is a ball. A similar result was conjectured to hold for the biharmonic operator under homogeneous Dirichlet boundary conditions by Lord Rayleigh [350] in 1894. Although this statement has been proved only in domains of dimensions $n = 2,3$, it is the common feeling that it should be true in any dimension. The minimisation of the first Steklov eigenvalue appears to be less obvious. And, indeed, we will see that a Faber-Krahn type result does not hold in this case.

Semilinear equations

Among nonlinear problems for higher order elliptic equations one may just mention models for thin elastic plates, stationary surface diffusion flow, the Paneitz-Branson equation and the Willmore equation as frequently studied. In membrane biophysics the Willmore equation is also known as Helfrich model [227]. Moreover, several results concerning semilinear equations with power type nonlinear sources are also extremely useful in order to understand interesting phenomena in functional analysis such as the failure of compactness in the critical Sobolev embedding and in related inequalities.

One further motivation to study nonlinear higher order elliptic reaction-diffusion type equations like

$$(-\Delta)^m u = f(u) \qquad (*)$$

in bounded domains is to understand whether the results available in the simplest case $m = 1$ can also be proved for any m, or whether the results for $m = 1$ are special, in particular as far as positivity and the use of maximum principles are concerned. The differential equation $(*)$ is complemented with suitable boundary conditions. As already mentioned above, if $m = n = 2$, equation $(*)$ may be considered as a nonlinear plate equation for plates subject to nonlinear feedback forces, one may think e.g. of suspension bridges. In this case, $(*)$ may also be interpreted as a reaction-diffusion equation, where the diffusion operator Δ^2 refers to (linearised) surface diffusion.

The first part of Chapter 7 is devoted to the proof of symmetry results for positive solutions to $(*)$ in the ball under Dirichlet boundary conditions. As already mentioned, truncation and reflection methods do not apply to higher order problems so that a suitable generalisation of the moving planes technique is needed here.

Equation $(*)$ deserves a particular attention when $f(u)$ has a power-type behaviour. In this case, a crucial role is played by the critical power $s = (n+2m)/(n-2m)$ which corresponds to the critical (Sobolev) exponent which appears whenever $n > 2m$. Indeed, subcritical problems in bounded domains enjoy compactness properties as a consequence of the Rellich-Kondrachov embedding theorem. But compactness is lacking when the critical growth is attained and by means of Pohožaev-type identities, this gives rise to many interesting phenomena. The existence theory can be developed similarly to the second order case $m = 1$ while it becomes immediately quite difficult to prove positivity or nonexistence of certain solutions. Nonexistence phenomena are related to so-called critical dimensions introduced by Pucci-Serrin [347, 348]. They formulated an interesting conjecture concerning these critical dimensions. We give a proof of a relaxed form of it in Chapter 7. We also give a functional analytic interpretation of these nonexistence results, which is reflected in the possibility of adding L^2–remainder terms in Sobolev inequalities with critical exponent and optimal constants. Moreover, the influence of topological and geometrical properties of Ω on the solvability of the equation is investigated. Also applications to conformal geometry, such as the Paneitz-Branson equation, involve the critical Sobolev exponent since the corresponding semilinear equation enjoys a conformal covariance property. In this context a key role is played by a fourth order curvature invariant, the so-called Q-curvature. Our book does not aim at giving an overview of this rapidly developing subject. For this purpose we refer to the monographs of Chang [89] and Druet-Hebey-Robert [149]. We want to put a spot on some special aspects of such kind of equations. First, we consider the question whether in suitable domains in euclidean space it is possible to change the euclidean background metric conformally into a metric which has strictly positive constant Q-curvature, while at the same time, certain geometric quantities vanish on the boundary. Secondly, we study a phenomenon of nonuniqueness of complete metrics in hyperbolic space, all being conformal to the Poincaré-metric and all having the same constant Q-curvature. This result is in strict contrast with the corresponding problem involving constant negative scalar curvature.

We conclude the discussion of semilinear elliptic problems with some observations on fourth order problems with supercritical growth. Corresponding second order results heavily rely on the use of maximum principles and constructions of many refined auxiliary functions having some sub- or supersolution property. Such techniques are not available at all for the fourth order problems. In symmetric situations, however, they could be replaced by different tools so that many of the results being well established for second order equations do indeed carry over to the fourth order ones.

A Dirichlet problem for Willmore surfaces of revolution

A frame invariant modeling of elastic deformations of surfaces like thin plates or biological membranes gives rise to variational integrals involving curvature and area terms. A special case is the Willmore functional

$$\int_{\Gamma} \mathtt{H}^2 \, d\omega,$$

which up to a boundary term is conformally invariant. Here $\mathtt{H}$ denotes the mean curvature of the surface Γ in $\mathbb{R}^3$. Critical points of this functional are called Willmore surfaces, the corresponding Euler-Lagrange equation is the so-called Willmore equation. It is quasilinear, of fourth order and elliptic. While a number of beautiful results have been recently found for closed surfaces, see e.g. [35, 156, 262, 263, 264, 371], only little is known so far about boundary value problems since the difficulties mentioned earlier being typical for fourth order problems due to a lack of maximum principles add here to the difficulty that the ellipticity of the equation is not uniform. The latter reflects the geometric nature of the equation and gives rise e.g. to the problem that minimising sequences for the Willmore functional are in general not bounded in the Sobolev space H^2. In this book we confine ourselves to a very special situation, namely the Dirichlet problem for symmetric Willmore surfaces of revolution. Here, by means of some refined geometric constructions, we succeed in considering minimising sequences of the Willmore functional subject to Dirichlet boundary conditions and with suitable additional C^1-properties thereby gaining weak H^2- and strong C^1-compactness. We expect the theory of boundary value problems for Willmore surfaces to develop rapidly and consider this chapter as one contribution to outline directions of possible future research in quasilinear geometric fourth order equations.

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And last but not least, we thank the referees for their helpful and encouraging comments.

Contents

Chapter 1
Models of higher order

The goal of this chapter is to explain in some detail which models and equations are considered in this book and to provide some background information and comments on the interplay between the various problems. Our motivation arises on the one hand from equations in continuum mechanics, biophysics or differential geometry and on the other hand from basic questions in the theory of partial differential equations.

In Section 1.1, after providing a few historical and bibliographical facts, we recall the derivation of several linear boundary value problems for the plate equation. In Section 1.8 we come back to this issue of modeling thin elastic plates where the full nonlinear differential geometric expressions are taken into account. As a particular case we concentrate on the Willmore functional, which models the pure bending energy in terms of the squared mean curvature of the elastic surface. The other sections are mainly devoted to outlining the contents of the present book. In Sections 1.2-1.4 we introduce some basic and still partially open questions concerning qualitative properties of solutions of various linear boundary value problems for the linear plate equation and related eigenvalue problems. Particular emphasis is laid on positivity and – more generally – "almost positivity" issues. A significant part of the present book is devoted to semilinear problems involving the biharmonic or polyharmonic operator as principal part. Section 1.5 gives some geometric background and motivation, while in Sections 1.6 and 1.7 semilinear problems are put into a context of contributing to a theory of nonlinear higher order problems.

1.1 Classical problems from elasticity

Around 1800 the physicist Chladni was touring Europe and showing, among other things, the nodal line patterns of vibrating plates. Jacob Bernoulli II tried to model these vibrations by the fourth order operator $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ [54]. His model was not accepted, since it is not rotationally symmetric and it failed to reproduce the nodal line patterns of Chladni. The first use of Δ^2 for the modeling of an elastic plate is attributed to a correction of Lagrange of a manuscript by Sophie Germain from 1811.

For historical details we refer to [79, 249, 324, 397]. For a more elaborate history of the biharmonic problem and the relation with elasticity from an engineering point of view one may consult a survey of Meleshko [299]. This last paper also contains a large bibliography so far as the mechanical engineers are interested. Mathematically interesting questions came up around 1900 when Almansi [8, 9], Boggio [62, 63] and Hadamard [221, 222] addressed existence and positivity questions.

In order to have physically meaningful and mathematically well-posed problems the plate equation $\Delta^2 u = f$ has to be complemented with prescribing a suitable set of boundary data. The most commonly studied boundary value problems for second order elliptic equations are named Dirichlet, Neumann and Robin. These three types appear since they have a physical meaning. For fourth order differential equations such as the plate equation the variety of possible boundary conditions is much larger. We will shortly address some of those that are physically relevant. Most of this book will be focussed on the so-called clamped case which is again referred to by the name of Dirichlet. An early derivation of appropriate boundary conditions can be found in a paper by Friedrichs [173]. See also [58, 141]. The following derivation is taken from [387].

1.1.1 The static loading of a slender beam

If $u(x)$ denotes the deviation from the equilibrium of the idealised one-dimensional beam at the point x and $p(x)$ is the density of the lateral load at x, then the elastic energy stored in the bending beam due to the deformation consists of terms that can be described by bending and by stretching. This stretching occurs when the horizontal position of the beam is fixed at both endpoints. Assuming that the elastic force is proportional to the increase of length, the potential energy density for the beam fixed at height 0 at the endpoints a and b would be

$$J_{st}(u) = \int_a^b \left(\sqrt{1+u'(x)^2} - 1 \right) dx.$$

For a string one neglects the bending and, by adding a force density p, one finds

$$J(u) = \int_a^b \left(\sqrt{1+u'(x)^2} - 1 - p(x)u(x) \right) dx.$$

For a thin beam one assumes that the energy density stored by bending the beam is proportional to the square of the curvature:

$$J_{sb}(u) = \int_a^b \frac{u''(x)^2}{\left(1+u'(x)^2\right)^3} \sqrt{1+u'(x)^2}\, dx. \tag{1.1}$$

Formula (1.1) for J_{sb} highlights the curvature and the arclength. A two-dimensional analogue of this functional is the Willmore functional, which is discussed below in Section 1.8. Note that the functional J_{sb} does not include a term that corresponds to an increase in the length of the beam which would occur if the ends are fixed and the beam would bend. That is, the function in $H^2 \cap H_0^1(a,b)$ minimising $J_{sb}(u) - \int_a^b pu \ dx$ should be an approximation for the so-called supported beam which is free to move in horizontal directions at its endpoints.

For small deformations of a beam an approximation that takes care of stretching, bending and a force density would be

$$J(u) = \int_a^b \left(\tfrac{1}{2} u''(x)^2 + \tfrac{c}{2} u'(x)^2 - p(x)u(x) \right) dx,$$

where $c > 0$ represents the initial tension of the beam which is also fixed horizontally at the endpoints.

The linear Euler-Lagrange equation that arises from this situation contains both second and fourth order terms:

$$u'''' - cu'' = p. \tag{1.2}$$

If one lets the beam move freely at the boundary points (and in the case of zero initial tension), one arrives at the simplest fourth order equation $u'''' = p$. This differential equation may be complemented with several boundary conditions.

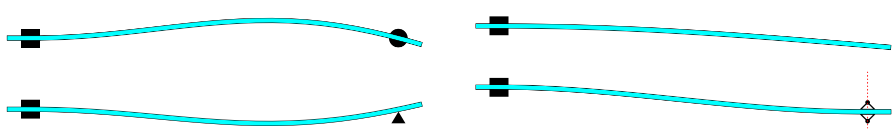

Fig. 1.1 The depicted boundary condition for the left endpoints of these four beams is "clamped". The boundary conditions for the right endpoints are respectively "hinged" and "simply supported" on the left; on the right one finds "free" and one that allows vertical displacement but fixes the derivative by a sliding mechanism.

The mathematical formulation that corresponds to the boundary conditions in Figure 1.1 are as follows:

- Clamped: $u(a) = 0 = u'(a)$, also known as homogeneous Dirichlet boundary conditions.
- Hinged: $u(b) = 0 = u''(b)$, also known as homogeneous Navier boundary conditions. This is not a real hinged situation since the vertical position is fixed but the beam is allowed to slide in the hinge itself.
- Simply supported: $\max(u(b), 0)\, u'''(b) = 0 = u''(b)$. In applications, when the force is directed downwards, this boundary condition simplifies to the hinged one $u(b) = 0 = u''(b)$. However, when upward forces are present it might happen that $u(b) > 0$ and then the natural boundary condition $u'''(b) = 0$ appears.

- Free: $u'''(b) = 0 = u''(b)$.
- Free vertical sliding but with fixed derivative: $u'(b) = u'''(b) = 0$.

The second and third order derivatives appear as natural boundary conditions by the derivation of the strong Euler-Lagrange equations.

If the beam would be moving in an elastic medium, then, again for small deviations one adds a further term to J and finds

$$J(u) = \int_a^b \left(\tfrac{1}{2}(u'')^2 + \tfrac{\gamma}{2}\, u^2 - pu\right) dx.$$

This leads to the Euler-Lagrange equation $u'''' + \gamma u = p$.

Also a suspension bridge may be seen as a beam of given length L, with hinged ends and whose downward deflection is measured by a function $u(x,t)$ subject to three forces. These forces can be summarised as the stays holding the bridge up as nonlinear springs with spring constant k, the constant weight per unit length of the bridge W pushing it down, and the external forcing term $f(x,t)$. This leads to the equation

$$\begin{cases} u_{tt} + \gamma u_{xxxx} = -ku^+ + W + f(x,t), \\ u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \end{cases} \tag{1.3}$$

where γ is a physical constant depending on the beam, Young's modulus, and the second moment of inertia. The model leading to (1.3) is taken from the survey papers [270, 295].

The famous collapse of the Tacoma Narrows Bridge, see [16, 61], was the consequence of a torsional oscillation. McKenna [295, p. 106] explains this fact as follows.

> *A large vertical motion had built up, there was a small push in the torsional direction to break symmetry, the instability occurred, and small aerodynamic torsional periodic forces were sufficient to maintain the large periodic torsional motions.*

For this reason, a major role is played by *travelling waves*. If one neglects the effect of external forces and normalises all the constants, then (1.3) becomes

$$u_{tt} + u_{xxxx} = -u^+ + 1\,. \tag{1.4}$$

In order to find travelling waves, one seeks solutions of (1.4) for $(x,t) \in \mathbb{R}^2$ of the kind $u(x,t) = 1 + y(x - ct)$ where $c > 0$ denotes the speed of propagation. Hence, the function y satisfies the fourth order ordinary differential equation

$$y'''' + c^2 y'' + (y+1)^+ - 1 = 0 \qquad \text{in } \mathbb{R}\,.$$

This is a nonlinear version of (1.2). We refer to the papers [270, 271, 295, 297, 298] and references therein for variants of these equations and for a number of results and open problems related to suspension bridges.

1.1.2 The Kirchhoff-Love model for a thin plate

As for the beam we assume that the plate, the vertical projection of which is the planar region $\Omega \subset \mathbb{R}^2$, is free to move horizontally at the boundary. Then a simple model for the elastic energy is

$$J(u) = \int_\Omega \left(\tfrac{1}{2} (\Delta u)^2 + (1-\sigma) \left(u_{xy}^2 - u_{xx} u_{yy} \right) - f\, u \right) dxdy, \tag{1.5}$$

where f is the external vertical load. Again u is the deflection of the plate in vertical direction and, as above for the beam, first order derivatives are left out which indicates that the plate is free to move horizontally.

This modern variational formulation appears already in [173], while a discussion for a boundary value problem for a thin elastic plate in a somehow old fashioned notation is made already by Kirchhoff [249]. See also the two papers of Birman [57, 58], the books by Mikhlin [303, §30], Destuynder-Salaun [141], Ciarlet [102], or the article [103] for the clamped case.

In (1.5) σ is the Poisson ratio, which is defined by $\sigma = \frac{\lambda}{2(\lambda+\mu)}$ with the so-called Lamé constants λ, μ that depend on the material. For physical reasons it holds that $\mu > 0$ and *usually* $\lambda \geq 0$ so that $0 \leq \sigma < \frac{1}{2}$. Moreover, it *always* holds true that $\sigma > -1$ although some exotic materials have a negative Poisson ratio, see [265]. For metals the value σ lies around 0.3 (see [280, p. 105]). One should observe that for $\sigma > -1$, the quadratic part of the functional (1.5) is always positive.

For small deformations the terms in (1.5) are taken as approximations being purely quadratic with respect to the second derivatives of u of respectively twice the squared mean curvature and the Gaussian curvature supplied with the factor $\sigma - 1$. For those small deformations one finds

$$\begin{aligned} \tfrac{1}{2} (\Delta u)^2 + (1-\sigma) \left(u_{xy}^2 - u_{xx} u_{yy} \right) &\approx \tfrac{1}{2} (\kappa_1 + \kappa_2)^2 - (1-\sigma)\, \kappa_1 \kappa_2 \\ &= \tfrac{1}{2} \kappa_1^2 + \sigma \kappa_1 \kappa_2 + \tfrac{1}{2} \kappa_2^2, \end{aligned}$$

where κ_1, κ_2 are the principal curvatures of the graph of u. Variational integrals avoiding such approximations and involving the original expressions for the mean and the Gaussian curvature are considered in Section 1.8 and lead as a special case to the Willmore functional.

Which are the appropriate boundary conditions? For the clamped and hinged boundary condition the natural settings, that is the Hilbert spaces for these two situations, are respectively $H = H_0^2(\Omega)$ and $H = H^2 \cap H_0^1(\Omega)$. Minimising the energy functional leads to the weak Euler-Lagrange equation $\langle dJ(u), v \rangle = 0$, that is

$$\int_\Omega \left(\Delta u \Delta v + (1-\sigma) \left(2u_{xy} v_{xy} - u_{xx} v_{yy} - u_{yy} v_{xx} \right) - f\, v \right) dxdy = 0 \tag{1.6}$$

for all $v \in H$. Let us assume both that minimisers u lie in $H^4(\Omega)$ and that the exterior normal $\nu = (\nu_1, \nu_2)$ and the corresponding tangential $\tau = (\tau_1, \tau_2) = (-\nu_2, \nu_1)$ are well-defined. Then an integration by parts of (1.6) leads to

$$
\begin{aligned}
0 = &\int_{\Omega} \left(\Delta^2 u - f\right) \, v \, dxdy + \int_{\partial\Omega} \left(\frac{\partial}{\partial \nu}\Delta u\right) \, v \, ds \\
&+ (1-\sigma) \int_{\partial\Omega} \left(\left(\nu_1^2 - \nu_2^2\right) u_{xy} - \nu_1 \nu_2 \left(u_{xx} - u_{yy}\right) \right) \frac{\partial}{\partial \tau} v \, ds \\
&+ \int_{\partial\Omega} \left(\Delta u + (1-\sigma) \left(2\nu_1 \nu_2 u_{xy} - \nu_2^2 u_{xx} - \nu_1^2 u_{yy} \right) \right) \frac{\partial}{\partial \nu} v \, ds. \qquad (1.7)
\end{aligned}
$$

- Following [141] let us split the boundary $\partial\Omega$ in a clamped part Γ_0, a hinged part Γ_1 and a free part $\Gamma_2 = \partial\Omega \setminus (\Gamma_0 \cup \Gamma_1)$, which are all assumed to be smooth. Moreover, to keep our derivation simple, we assume that Γ_2 has empty relative boundary in $\partial\Omega$, i.e. it is a union of connected components of $\partial\Omega$.
On Γ_0 one has $u = u_\nu = 0$. The type of boundary conditions on Γ_0 are generally referred to as homogeneous Dirichlet.
On Γ_1 one has $u = 0$ and may rewrite the second boundary condition that appears from (1.7) as

$$
\begin{aligned}
&\Delta u + (1-\sigma) \left(2u_{xy} \nu_1 \nu_2 - u_{xx} \nu_2^2 - u_{yy} \nu_1^2 \right) \\
&= \sigma \Delta u + (1-\sigma) \left(2u_{xy} \nu_1 \nu_2 + u_{xx} \nu_1^2 + u_{yy} \nu_2^2 \right) \\
&= \sigma \Delta u + (1-\sigma) u_{\nu\nu} = \sigma \left(u_{\nu\nu} + \kappa u_\nu \right) + (1-\sigma) u_{\nu\nu} \\
&= u_{\nu\nu} + \sigma \kappa u_\nu = \Delta u - (1-\sigma) \kappa u_\nu. \qquad (1.8)
\end{aligned}
$$

Here κ is the curvature of the boundary. We use the sign convention that $\kappa \geq 0$ for convex boundary parts and $\kappa \leq 0$ for concave boundary parts.
On Γ_2, which we recall to have empty relative boundary in $\partial\Omega$, an integration by parts along the boundary shows

$$
\begin{aligned}
&\int_{\Gamma_2} \left(\frac{\partial}{\partial \nu}\Delta u\right) \, v \, ds + (1-\sigma) \int_{\Gamma_2} \left(\left(\nu_1^2 - \nu_2^2\right) u_{xy} - \nu_1 \nu_2 \left(u_{xx} - u_{yy}\right) \right) \frac{\partial}{\partial \tau} v \, ds \\
&\qquad = - \int_{\Gamma_2} (1-\sigma) \left(u_{\tau\tau\nu} + \frac{\partial}{\partial \nu}\Delta u \right) \, v \, ds.
\end{aligned}
$$

Summarising, on domains with smooth $\Gamma_0, \Gamma_1, \Gamma_2$ one finds the following boundary value problem:

$$
\begin{cases}
\Delta^2 u = f & \text{in } \Omega, \\
u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_0, \\
u = \Delta u - (1-\sigma) \kappa \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1, \\
\sigma \Delta u + (1-\sigma) u_{\nu\nu} = (1-\sigma) u_{\tau\tau\nu} + \frac{\partial}{\partial \nu}\Delta u = 0 & \text{on } \Gamma_2.
\end{cases}
$$

The differential equation $\Delta^2 u = f$ is called the Kirchhoff-Love model for the vertical deflection of a thin elastic plate.
- The clamped plate equation, i.e. the pure Dirichlet case when $\partial\Omega = \Gamma_0$, is as follows:

$$
\begin{cases}
\Delta^2 u = f & \text{in } \Omega, \\
u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega.
\end{cases} \qquad (1.9)
$$

Notice that σ does not play any role for clamped boundary conditions. In this case, after an integration by parts like in (1.7), the elastic energy (1.5) becomes

$$J(u) = \int_\Omega \left(\tfrac{1}{2} (\Delta u)^2 - f\, u \right) dx$$

and this functional has to be minimised over the space $H_0^2(\Omega)$.

- The physically relevant boundary value problem for the pure hinged case when $\partial\Omega = \Gamma_1$ reads as

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\,\kappa \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

  See [141, II.18 on p. 42]. These boundary conditions are named after Steklov due the first appearance in [379]. In this case, with an integration by parts like in (1.7) and arguing as in (1.8), the elastic energy (1.5) becomes

$$J(u) = \int_\Omega \left(\tfrac{1}{2} (\Delta u)^2 - f\, u \right) dx - \frac{1-\sigma}{2} \int_{\partial\Omega} \kappa\, u_\nu^2 \, d\omega; \tag{1.11}$$

  for details see the proof of Corollary 5.23. This functional has to be minimised over the space $H^2 \cap H_0^1(\Omega)$.

- On straight boundary parts $\kappa = 0$ holds and the second boundary condition in (1.10) simplifies to $\Delta u = 0$ on $\partial\Omega$. The corresponding boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

  is in general referred to as the one with homogeneous Navier boundary conditions, see [141, II.15 on p. 41]. On polygonal domains one might naively expect that (1.10) simplifies to (1.12). Unless $\sigma = 1$ this is an erroneous conclusion and instead of $\kappa \frac{\partial u}{\partial \nu}$ one should introduce a Dirac-δ-type contribution at the corners. See Section 2.7 and [293].

1.1.3 Decomposition into second order systems

Note that the combination of the boundary conditions in (1.12) or (1.10) allows for rewriting these fourth order problems as a second order system

$$\begin{cases} -\Delta u = w \ \text{ and } \ -\Delta w = f & \text{in } \Omega, \\ \quad u = 0 \ \text{ and } \quad w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

respectively

$$\begin{cases} -\Delta u = w \ \text{ and } \ -\Delta w = f & \text{in } \Omega, \\ \quad u = 0 \ \text{ and } \ w = -(1-\sigma)\,\kappa \frac{\partial u}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \tag{1.14}$$

The boundary value problems in (1.13) can be solved consecutively. Indeed, for smooth domains the solution u coincides with the minimiser in $H^2 \cap H_0^1(\Omega)$ of

$$J(u) = \int_\Omega \left(\tfrac{1}{2} (\Delta u)^2 - f\, u \right) dx. \tag{1.15}$$

For domains with corners this is not necessarily true. For a reentrant corner a phenomenon may occur that was first noticed by Sapondžyan, see Section 1.4.1 and Example 2.33.

A splitting into a system of two consecutively solvable second order boundary value problems is not possible for (1.14). Nevertheless, for convex domains we have $\kappa \geq 0$ and this fact turns (1.14) into a cooperative second order system for which some of the techniques for second order equations apply. "Cooperative" means that the coupling supports the sign properties of the single equations. Cooperative systems of second order boundary value problems are well-studied in the literature and will not be addressed in this monograph.

A more intricate situation occurs for the clamped case where a similar approach to split the fourth order problem into a system of second order equations results in

$$\begin{cases} -\Delta u = w & \text{and} \quad -\Delta w = f \quad \text{in } \Omega, \\ u = \frac{\partial}{\partial \nu} u = 0 & \text{and} \qquad — \qquad \text{on } \partial\Omega. \end{cases} \tag{1.16}$$

For most questions such a splitting has not yet appeared to be very helpful. The first boundary value problem has too many boundary conditions, the second one none at all. Techniques for second order equations, however, can be used e.g. in numerical approximations, when the problem is put as follows. Find stationary points $(u,w) \in H_0^1(\Omega) \times H^1(\Omega)$ of

$$F(u,w) = \int_\Omega \left(\nabla u \cdot \nabla w - f\, u - \frac{1}{2} w^2 \right) dx. \tag{1.17}$$

The weak Euler-Lagrange equation becomes

$$\langle dF(u,w), (\varphi, \psi) \rangle = \int_\Omega (\nabla u \cdot \nabla \psi + \nabla \varphi \cdot \nabla w - f\, \varphi - w\, \psi)\, dx = 0 \tag{1.18}$$

for all $(\varphi, \psi) \in H_0^1(\Omega) \times H^1(\Omega)$. Assuming $u, w \in H^2(\Omega)$, an integration by parts gives

$$\int_{\partial\Omega} \frac{\partial}{\partial \nu} u\, \psi\, d\omega + \int_\Omega (-\Delta u - w)\ \psi\, dx + \int_\Omega (-\Delta w - f)\ \varphi\, dx = 0.$$

Testing with $(\varphi, \psi) \in H_0^1(\Omega) \times H^1(\Omega)$ we find $u \in H_0^2(\Omega)$, $-\Delta u = w$ and $-\Delta w = f$, thereby recovering (1.16) as Euler-Lagrange-equation for the functional F in (1.17).

The formulation in (1.18) can be used to construct approximate solutions using piecewise linear finite elements instead of the $C^{1,1}$ elements that are necessary for functionals containing second order derivatives. For smooth domains one may show that the stationary points of (1.15) and (1.17) coincide. For nonsmooth domains similar phenomena like the Babuška paradox might appear, which is described below in Section 1.4.2, see also Section 2.7.

1.2 The Boggio-Hadamard conjecture for a clamped plate

Since maximum principles do not only allow for proving nice results on geometric properties of solutions of *second* order elliptic problems but are also extremely important technical tools in this field, one might wonder in how far such results still hold in higher order boundary value problems. First of all it is an obvious remark that a general maximum principle can no longer be true. The biharmonic functions $x \mapsto \pm|x|^2$ have a strict global minimum or maximum respectively in any domain containing the origin. On the other hand, it may be reasonable to ask for positivity preserving properties of *boundary value problems*, i.e. whether positive data yield positive solutions. In physical terms this question may be rephrased as follows:

Does upwards pushing of a plate yield upwards bending?

The answer, of course, depends on the model considered and on the imposed boundary conditions. For instance, in the Dirichlet problem for the plate equation

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.19}$$

there is – at least no obvious way – to take advantage of second order comparison principles and in this sense, it may be considered as the prototype of a "real" fourth order boundary value problem. On the other hand, the plate equation complemented with Navier boundary conditions (1.12) can be written as a system of two second order boundary value problems and enjoys a sort of comparison principle. In particular, under these conditions it is obvious that $f \geq 0$ implies that $u \geq 0$. However, when adding lower order perturbations, the case of a so-called noncooperative coupling may occur and this simple argument breaks down. In this case, the positivity issue becomes quite involved also under Navier boundary conditions, see e.g. [309].

A significant part of the present book will be devoted to discussing the following mathematical question.

What remains true of: "$f \geq 0$ in the clamped plate boundary value problem (1.19) implies positivity of the solution $u \geq 0$"?

In view of the representation formula

$$u(x) = \int_B G_{\Delta^2,\Omega}(x,y) f(y)\, dy,$$

one equivalently may wonder whether the corresponding Green function is positive or even strictly positive, i.e. $G_{\Delta^2,\Omega} > 0$? Lauricella ([268], 1896) found an explicit formula for $G_{\Delta^2,\Omega}$ in the special case of the unit disk $\Omega = B := B_1(0) \subset \mathbb{R}^2$. Boggio ([63, p. 126], 1905) generalised this formula to the Dirichlet problem for any polyharmonic operator $(-\Delta)^m$ in any ball in any $\mathbb{R}^n$ and found a particularly elegant expression for the Green function, see Lemma 2.27. In case of the biharmonic operator in the two-dimensional disk $B \subset \mathbb{R}^2$, this formula reads:

$$G_{\Delta^2,B}(x,y) = \frac{1}{8\pi}\,|x-y|^2 \int_1^{\left||x|y-\frac{x}{|x|}\right|/|x-y|} \frac{(v^2-1)}{v}\,dv > 0. \tag{1.20}$$

Positivity is here quite obvious since

$$\left||x|y-\frac{x}{|x|}\right|^2 - |x-y|^2 = (1-|x|^2)(1-|y|^2) > 0.$$

Almansi ([8], 1899) found an explicit construction for solving $\Delta^2 u = 0$ with prescribed boundary data for u and u_ν on domains $\Omega \subset \mathbb{R}^2$ with $\Omega = p(B)$ and $p : B \to \Omega$ being a conformal polynomial mapping. Probably inspired by Almansi's result and supported by physically plausible behaviour of plates, Boggio conjectured (see [221, 222]) that for the clamped plate boundary value problem (1.19), the Green function is always positive.

In 1908, Hadamard [222] already knew that this conjecture fails e.g. in annuli with small inner radius (see also [316]). He writes that Boggio had mentioned to him that the conjecture was meant for simply connected domains. In [222] he also writes:

> *Malgré l'absence de démonstration rigoureuse, l'exactitude de cette proposition ne paraît pas douteuse pour les aires convexes.*

Accordingly the conjecture of Boggio and Hadamard may be formulated as follows:

> *The Green function $G_{\Delta^2,\Omega}$ for the clamped plate boundary value problem on convex domains is positive.*

Using the explicit formula from [8] for the "limaçons de Pascal", see Figure 1.2, Hadamard in [222] even claimed to have proven positivity of the Green function $G_{\Delta^2,\Omega}$ when Ω is such a limaçon.

However, after 1949 numerous counterexamples ([107, 108, 150, 176, 252, 278, 326, 367, 370, 389]) disproved the positivity conjecture of Boggio and Hadamard. The first result in this direction came by Duffin ([150, 152]), who showed that the Green function changes sign on a long rectangle. A most striking example was found by Garabedian. He could show change of sign of the Green function in ellipses with ratio of half axes ≈ 1.6 ([176], [177, p. 275]). For an elementary proof of a slightly weaker result see [370]. Hedenmalm, Jakobsson and Shimorin [226] mention that sign change occurs already in ellipses with ratio of half axes ≈ 1.2. Nakai and Sario [317] give a construction how to extend Garabedian's example also to higher dimensions. Sign change is also proven by Coffman-Duffin [108] in any bounded domain containing a corner, the angle of which is not too large. Their arguments are based on previous results by Osher and Seif [326, 367] and cover, in particular, squares. This means that neither in arbitrarily smooth uniformly convex nor in rather symmetric domains the Green function needs to be positive. Moreover, in [120] it has been proved that Hadamard's claim for the limaçons is not correct. Limaçons are a one-parameter family with circle and cardioid as extreme cases. For domains close enough to the cardioid, the Green function is no longer positive. Surprisingly, the extreme case for positivity is not convex. Hence convexity is neither sufficient nor necessary for a positive Green function. One should observe that in one dimension any bounded interval is a ball and so, one always has positivity there thanks to Boggio's formula.

For the history of the Boggio-Hadamard conjecture one may also see Maz'ya's and Shaposhnikova's biography [294] of Hadamard.

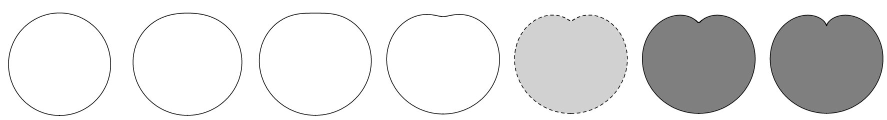

Fig. 1.2 Limaçons vary from circle to cardioid. The fifth limaçon from the left is critical for a positive Green function.

Despite the fact that the Green function is usually sign changing, it is very hard to find real world experiments where loss of positivity preserving can be observed. Moreover, in all numerical experiments in smooth domains, it is very difficult to display the negative part and heuristically, one feels that the negative part of $G_{\Delta^2,\Omega}$ – if present at all – is small in a suitable sense compared with the "dominating" positive part. We refine the Boggio-Hadamard conjecture as follows:

> In arbitrary domains $\Omega \subset \mathbb{R}^n$, the negative part of the biharmonic Green's function $G_{\Delta^2,\Omega}$ is small relative to the singular positive part. In the investigation of nonlinear problems, the negative part is technically disturbing but it does not give rise to any substantial additional assumption in order to have existence, regularity, etc. when compared with analogous second order problems.

The present book may be considered as a first contribution to the discussion of this conjecture and Chapters 5 and 6 are devoted to it. Chapter 4 provides the necessary kernel estimates. Let us mention some of those results which we have obtained so far to give support to this conjecture. For any smooth domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ we show that there exists a constant $C = C(\Omega)$ such that for the biharmonic Green's function $G_{\Delta^2,\Omega}$ under Dirichlet boundary conditions one has the following estimate from below:

$$G_{\Delta^2,\Omega}(x,y) \geq -C\,\mathrm{dist}(x,\partial\Omega)^2\,\mathrm{dist}(y,\partial\Omega)^2.$$

This means that although in general, $G_{\Delta^2,\Omega}$ has a nontrivial negative part, this behaves completely regular and is in this respect not affected by the singularity of the Green's function. Qualitatively, only its positive part is affected by its singularity. See Theorem 6.24 and the subsequent remarks. Moreover, in Theorems 6.3 and 6.29 we show that positivity in the Dirichlet problem for the biharmonic operator does hold true not only in balls but also in smooth domains which are close to balls in a suitably strong sense. Although being a perturbation result it *is not* just a consequence of continuous dependence on data. The problem in proving positivity for Green's functions consists in gaining uniformity when their singularities approach the boundaries.

Finally, in Section 5.4 positivity issues for the biharmonic operator under Steklov boundary conditions are addressed. With respect to positivity it may be considered, at least in some cases, to be intermediate between Dirichlet conditions on the one hand and Navier boundary conditions on the other hand, see Theorems 5.26 and 5.27.

1.3 The first eigenvalue

It is well-known that for general second order elliptic Dirichlet problems the eigenfunction φ_1 that corresponds to the first eigenvalue is of one sign. In case of the Laplacian such a result can be proven directly sticking to the variational characterisation of the first eigenvalue

$$\Lambda_{1,1} := \min_{v \in H_0^1 \setminus \{0\}} \frac{\int |\nabla v|^2 \, dx}{\int |v|^2 \, dx} = \frac{\int |\nabla \varphi_1|^2 \, dx}{\int |\varphi_1|^2 \, dx}$$

by comparing $|\varphi_1|$ with φ_1. For quite general and even non-selfadjoint second order Dirichlet problems the same result is proven by using more abstract results such as the Kreĭn-Rutman theorem. The first approach uses the truncation method and so, a version of the maximum principle, while the Kreĭn-Rutman theorem requires the presence of a comparison principle. A simple alternative is provided by the dual cone method of Moreau [311]. This approach, which is explained in Section 3.1.2, is on one hand restricted to a symmetric setting in a Hilbert space but on the other hand, can also be applied in semilinear problems.

Considering $\Omega \mapsto \Lambda_{1,1}(\Omega)$ in dependence of the domains Ω being subject to having all the same volume as the unit ball $B \subset \mathbb{R}^n$ one may wonder whether this map is minimised for $\Omega = B$. Indeed, this was proved by Faber-Krahn [162, 253, 254] and, moreover, balls of radius 1 are the only minimisers.

1.3.1 The Dirichlet eigenvalue problem

Whenever the biharmonic operator under Dirichlet boundary conditions has a strictly positive Green's function, the first eigenvalue $\Lambda_{2,1}$ is simple and the corresponding first eigenfunction is of fixed sign, see Section 3.1.3. Related to the first eigenvalue is a question posed by Lord Rayleigh in 1894 in his celebrated monograph [350]. He studied the vibration of (planar) plates and conjectured that among domains of given area, *when the edges are clamped, the form of gravest pitch is doubtless the circle*, see [350, p. 382]. This corresponds to saying that

$$\Lambda_{2,1}(B) \leq \Lambda_{2,1}(\Omega) \qquad \text{whenever } |\Omega| = \pi \tag{1.21}$$

for planar domains ($n = 2$). Szegö [388] assumed that in any domain the first eigenfunction for the clamped plate has always a fixed sign and proved that this hypothesis would imply the isoperimetric inequality (1.21). The assumption that the first eigenfunction is of fixed sign, however, is not true as Duffin pointed out. In [152], where he explains some counterexamples, he referred to this assumption as Szegö's conjecture on the clamped plate. Details of these counterexamples can be found in [153, 154, 155].

Subsequently, concerning Rayleigh's conjecture, Mohr [310] showed in 1975 that if among all domains of given area there exists a smooth minimiser for $\Lambda_{2,1}$ then the domain is a disk. However, he left open the question of existence. In 1981, Talenti [392] extended Szegö's result in two directions. He showed that the statement remains true under the weaker assumption that the nodal set of the first eigenfunction φ_1 of (3.1) is empty or is included in $\{x \in \Omega;\, \nabla\varphi_1 = 0\}$. This result holds in any space dimension $n \geq 2$. Moreover, for general domains, instead of (1.21) he showed that

$$C_n\Lambda_{2,1}(B) \leq \Lambda_{2,1}(\Omega) \qquad \text{whenever } |\Omega| = e_n$$

where $0.5 < C_n < 1$ is a constant depending on the dimension n. These constants were increased by Ashbaugh-Laugesen [24] who also showed that $C_n \to 1$ as $n \to \infty$.

A complete proof of Rayleigh's conjecture was finally obtained one century later than the conjecture itself in a celebrated paper by Nadirashvili [315]. This result was immediately extended by Ashbaugh-Benguria [22] to the case of domains in $\mathbb{R}^3$.

More results about the positivity of the first eigenfunction in general domains and a proof of Rayleigh's conjecture can be found in Chapter 3.

1.3.2 An eigenvalue problem for a buckled plate

In 1910, Th. von Kármán [403] described the large deflections and stresses produced in a thin elastic plate subject to compressive forces along its edge by means of a system of two fourth order elliptic quasilinear equations. For a derivation of this model from three dimensional elasticity one may also see [174] and references therein. An interesting phenomenon associated with this nonlinear model is the appearance of "buckling", namely the plate may deflect out of its plane when these forces reach a certain magnitude. We also refer to more recent work in [48, 101].

The linearisation of the von Kármán equations for an elastic plate over planar domains $\Omega \subset \mathbb{R}^2$ under pressure leads to the following eigenvalue problem

$$\begin{cases} \Delta^2 u = -\mu \Delta u & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\kappa u_\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.22}$$

Miersemann [301] studied this eigenvalue problem and he was one of the first to apply the dual cone setting of Moreau [311] to a fourth order boundary value problem. He could show that on convex $C^{2,\gamma}$-domains the first eigenvalue for (1.22) is simple and that the corresponding eigenfunction is of fixed sign. The setting introduced by Moreau will be also most convenient for a number of nonlinear problems as we shall outline in Chapters 3 and 7, see in particular Sections 7.2.3 and 7.3.

We also consider the Dirichlet eigenvalue problem

$$\begin{cases} \Delta^2 u = -\mu \Delta u & \text{in } \Omega, \\ u = u_\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.23}$$

related to (1.22) and where the least eigenvalue $\mu_1(\Omega)$ represents the buckling load of a clamped plate. Inspired by Rayleigh's conjecture (1.21), Pólya-Szegö [343, Note F] conjectured that

$$\mu_1(B) \le \mu_1(\Omega) \qquad \text{whenever } |\Omega| = \pi \tag{1.24}$$

for any bounded planar domain $\Omega \subset \mathbb{R}^2$. And again, using rearrangement techniques they proved (1.24) under the assumption that the solution u to (1.23) is positive, see [343, 388]. Unfortunately, as for the clamped plate eigenvalue, this property fails in general, for instance in the square $(0,1)^2$, see Wieners [412]. Without imposing this sign assumption on the first eigenfunction, Ashbaugh-Laugesen [24] proved the bound $\gamma\mu_1(B) \le \mu_1(\Omega)$ whenever $|\Omega| = \pi$ for $\gamma = 0.78\dots$ which is, of course, much weaker than (1.24).

A complete proof of (1.24) is not yet known. A quite well established strategy which could be used to prove (1.24) involves shape derivatives, see e.g. [228]. It mainly consists in three steps.

1. In a suitable class of domains, prove the existence of a minimiser Ω_o for the map $\Omega \mapsto \mu_1(\Omega)$.
2. Prove that $\partial\Omega_o$ is smooth, for instance $\partial\Omega_o \in C^{2,\gamma}$, in order to be able to compute the derivative of $\Omega \mapsto \mu_1(\Omega)$ and to impose that it vanishes when $\Omega = \Omega_o$.
3. Exploit the just obtained stationarity condition, which usually gives an overdetermined condition on $\partial\Omega_o$, to prove that Ω_o is a ball.

In Section 3.2 we show how Item 1 has been successfully settled by Ashbaugh-Bucur [23] and how Item 3 has been achieved by Weinberger-Willms [415], see also [244, Proposition 4.4]. Therefore, for a complete proof of (1.24), "only" Item 2 is missing!

1.3.3 A Steklov eigenvalue problem

Usually, eigenvalue problems arise when one studies oscillation modes in the respective time dependent problem in order to have a physically well motivated theory and representation of solutions.

However, in what follows, a most natural motivation for considering a further eigenvalue problem comes from a seemingly quite different mathematical question. We explain how L^2-estimates for the Dirichlet problem for harmonic functions link with the Steklov eigenvalue problem for biharmonic functions.

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and consider the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.25}$$

where $g \in L^2(\partial\Omega)$. It is well-known that (1.25) admits a unique solution $u \in H^{1/2}(\Omega) \subset L^2(\Omega)$, see e.g. [275, Remarque 7.2, p. 202] and also [237, 238] for an extension to nonsmooth domains. One is then interested in a priori estimates, namely in determining the sharp constant C_Ω such that

$$\|u\|_{L^2(\Omega)} \le C_\Omega \|g\|_{L^2(\partial\Omega)}.$$

By Fichera's principle of duality [170] (see also Section 3.3.2) one sees that C_Ω coincides with the inverse of the first Steklov eigenvalue $\delta_1 = \delta_1(\Omega)$, namely the smallest constant a such that the problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = \Delta u - a u_\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.26}$$

admits a nontrivial solution. Notice that the "true" eigenvalue problem for the hinged plate equation should include the curvature in the second boundary condition, see (1.8). The map $\Omega \mapsto \delta_1(\Omega)$ has several surprising properties which we establish in Section 3.3.2. By rescaling, one sees that $\delta_1(k\Omega) = k^{-1}\delta_1(\Omega)$ for any bounded domain Ω and any $k > 0$ so that $\delta_1(k\Omega) \to 0$ as $k \to \infty$. One is then led to seek domains which minimise δ_1 under suitable constraints, the most natural one being the *volume constraint*. Smith [373] stated that, analogously to the Faber-Krahn result [162, 253, 254], the minimiser for δ_1 should exist and be a ball, at least for planar domains. But, as noticed by Kuttler and Sigillito, the argument in [373] contains a gap. In the "Note added in proof" in [374, p. 111], Smith writes:

> *Although the result is probably true, a correct proof has not yet been found.*

A few years later, Kuttler [258] proved that a (planar) square has a first Steklov eigenvalue $\delta_1(\Omega)$ which is strictly smaller than the one of the disk having the same measure. The estimate by Kuttler was subsequently improved in [165]. Therefore, it is *not true* that $\delta_1(\Omega^*) \leq \delta_1(\Omega)$ where Ω^* denotes the spherical rearrangement of Ω. For this reason, Kuttler [258] suggested a different minimisation problem with a *perimeter constraint*; in [258, Formula (11)] he conjectures that a planar disk minimises δ_1 among all domains having fixed perimeter. He provides numerical evidence that on rectangles his conjecture seems true, see also [259, 261]. In Theorem 3.24 we show that also this conjecture is false and that an optimal shape for δ_1 does not exist under a perimeter constraint in any space dimension $n \geq 2$. In fact, under such a constraint, the infimum of δ_1 is zero.

The spectrum of (1.26) has a nice application in functional analysis. In Section 3.3.1 we show that the closure of the space spanned by the Steklov eigenfunctions is the orthogonal complement of $H^2_0(\Omega)$ in $H^2 \cap H^1_0(\Omega)$.

1.4 Paradoxes for the hinged plate

The most common domains for plate problems that appear in engineering are polygonal ones. On the straight boundary parts of a polygonal domain the hinged boundary conditions (1.10) lead to Navier boundary conditions (1.12). Without taking care of a possible singularity due to "$\kappa = \infty$" in the corners it would mean that the solution no longer depends on the Poisson ratio σ. Sapondžyan [357] noticed that the solution one obtains by solving (1.12) iteratively does not necessarily have a bounded energy. Babuška noticed in [28] that the difference between (1.10) and (1.12) would mean that by approximating a curvilinear domain by polygons, as is done in most finite elements methods, the approximating solutions would not converge to the solution on the curvilinear domain.

Although both paradoxes are usually referred to by the name Babuška, they do cover different phenomena as we will explain in more detail.

1.4.1 Sapondžyan's paradox by concave corners

One might expect that the problem that appeared in these paradoxes is due to a boundary condition not being well-defined in corners. Indeed, the curvature that appears in the boundary condition is singular and apparently leads to a δ-distribution type contribution. By adding appropriate extra terms in the corners there is some hope to find the real solution. The situation for reentrant corners can be 'worse'. Due to Kondratiev [65, 251], Maz'ya et al. [288, 289], Grisvard [199] and many others, it is well-known that corners may lead to a loss of regularity. It is less known that a corner may lead to multiple solutions, that is, the solution depends crucially on the space that one chooses.

An example where two different solutions appear naturally from two straightforward settings goes as follows. Both fourth order boundary value problems, hinged or Steklov (1.10) as well as Navier (1.12) boundary conditions, allow a reformulation as a coupled system, see (1.14) and (1.13), respectively. In the latter case, one tends to solve by an iteration of the Green operator for the second order Poisson problem. This approach works fine for bounded smooth domains, but whenever the domain has a nonconvex corner, one does not necessarily get the solution one is looking for. Indeed, for the fourth order problem the natural setting for a weak solution to the Navier boundary value problem would be $H^2 \cap H_0^1(\Omega)$. The second Navier boundary condition $\Delta u = 0$ would follow naturally on smooth boundary parts from the weak formulation where u satisfies

$$\int_\Omega (\Delta u \Delta \varphi - f\varphi)\, dx = 0 \text{ for all } \varphi \in H^2 \cap H_0^1(\Omega). \tag{1.27}$$

However, for the system in (1.13) the natural setting is that one looks for function pairs $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$. In [320] it is shown that for domains with a reentrant corner both problems have a unique solution but the solutions u_1 to (1.13) and u_2 to (1.27) are different. Indeed, there exist a constant c_f and a nontrivial biharmonic function b that satisfies (1.13) with zero Navier boundary condition except in the corner such that $u_1 = u_2 + c_f b$. The related problem for domains with edges is considered in [319]. We refer to Section 2.7 for more details and an explicit example.

1.4.2 The Babuška paradox

In the original Babuška or polygon-circle paradox one considers problem (1.10) for $f = 1$ and when $\Omega = P_m \subset B$ ($m \geq 3$) is the interior of the regular polygon with corners $e^{2k\pi i/m}$ for $k \in \mathbb{N}$, namely

$$\begin{cases} \Delta^2 u = 1 & \text{in } P_m, \\ u = \Delta u = 0 & \text{on } \partial P_m. \end{cases}$$

If u_m denotes the solution of this problem extended by 0 in $B \setminus P_m$, it can be shown that the sequence (u_m) converges uniformly to

$$u_\infty(x) := \frac{3}{64} - \frac{1}{16}|x|^2 + \frac{1}{64}|x|^4$$

which *is not* the solution to the “limit problem” (where $\kappa = 1$), namely

$$\begin{cases} \Delta^2 u = 1 & \text{in } B, \\ u = \Delta u - (1-\sigma)\,\kappa \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B \end{cases}$$

unless $\sigma = 1$, see Figure 1.3.

For more details on this Babuška paradox see Section 2.7.

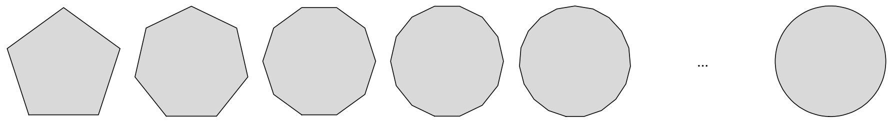

Fig. 1.3 The Babuška or polygon-circle paradox. On polygonal domains (1.10)=(1.12); on curvilinear domains (1.10)≠(1.12). Approximating curvilinear domains by polygonal ones does not give the correct limit solution to the hinged plate problem.

1.5 Paneitz-Branson type equations

Let $(\mathscr{M},g)$ be an n–dimensional Riemannian manifold with $n>4$. The conformal Laplacian is frequently studied and well understood and one may be interested in higher order analogues. Again, the biharmonic case is particularly interesting. The metric g is subject to a conformal change $g_u := u^{\frac{4}{n-4}}g$, $u>0$, and one wonders about the existence of a fourth order differential operator enjoying a conformal covariance property such that for all $\varphi \in C^\infty(\mathscr{M})$ one has

$$(P_4^n)_u(\varphi) = u^{-\frac{n+4}{n-4}}(P_4^n)(u\varphi).$$

Here, P_4^n denotes the desired operator with respect to the background metric g, while $(P_4^n)_u$ refers to the conformal metric g_u. Indeed, Paneitz [329, 330] and Branson [66, 67] found the following conformal covariant fourth order elliptic operator

$$P_4^n := \Delta^2 - \sum_{i,j=1}^{n} \nabla^i \left(\frac{(n-2)^2+4}{2(n-1)(n-2)} R g_{ij} - \frac{4}{n-2} R_{ij} \right) \nabla^j + \frac{n-4}{2} Q_4^n$$

on $\mathscr{M}$, where $\Delta = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right)$ denotes the Laplace-Beltrami operator with respect to g in local coordinates, R_{ij} the Ricci-tensor and R the scalar curvature. Moreover, $\nabla^j\varphi = \sum_{k=1}^n g^{jk}\partial_k\varphi$ gives the gradient of a function and

$$\sum_{i=1}^{n} \nabla^i Z_i = \sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \partial_i\left(\sqrt{g} g^{ij} Z_j\right)$$

the divergence of a covector field. A key role is played by the following fourth order curvature invariant

$$Q_4^n := -\frac{2}{(n-2)^2}|(R_{ij})|^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R^2 - \frac{1}{2(n-1)}\Delta R,$$

the so-called Q-curvature. Here $|(R_{ij})|^2 = \sum_{i,j,k,\ell}^n g^{ij}g^{k\ell}R_{ik}R_{j\ell}$. The transformation of the corresponding Q_4^n-curvature under this conformal change of metrics is governed by the Paneitz equation

$$P_4^n u = \frac{n-4}{2}(Q_4^n)_u u^{\frac{n+4}{n-4}}. \tag{1.28}$$

In analogy to the second order Yamabe problem (for an overview see [381, Section III.4]), obvious questions here concern the existence of conformal metrics with constant or prescribed Q-curvature. Huge work has so far been done by research groups around Chang-Yang-Gursky and Hebey, as well as many others. For a survey and references see the books by Chang [89] and by Druet-Hebey-Robert [149]. Difficult problems arise from ensuring the positivity requirement of the conformal factor $u > 0$ and from the necessity to know about the kernel of the Paneitz operator. These problems have only been solved partly yet.

In order to explain the geometrical importance of the Q-curvature, we assume now for a moment that the manifold $(\mathscr{M}, g)$ is four-dimensional. Then, the Paneitz operator is defined by

$$P_4^4 := \Delta^2 - \sum_{i,j=1}^{4} \nabla^i \left(\frac{2}{3} R g_{ij} - 2R_{ij} \right) \nabla^j$$

in such a way that under the conformal change of metrics $g_u = e^{2u} g$ one has

$$(P_4^4)_u(\varphi) = e^{-4u} P_4^4(\varphi).$$

In order to achieve a prescribed Q-curvature on the four-dimensional manifold $(\mathscr{M}, g_u)$, one has to find u solving

$$P_4^4 u + 2Q_4^4 = 2Q e^{4u},$$

where Q_4^4 is the curvature invariant

$$12Q_4^4 = -\Delta R + R^2 - 3|(R_{ij})|^2.$$

In this situation, one has the following Gauss-Bonnet-formula

$$\int_{\mathscr{M}} \left(Q + \frac{1}{8} |W|^2 \right) dS = 4\pi^2 \chi(\mathscr{M}),$$

where W is the Weyl tensor and $\chi(\mathscr{M})$ is the Euler characteristic. Since $\chi(\mathscr{M})$ is a topological and $|W|^2 dS$ is a pointwise conformal invariant, this shows that $\int_{\mathscr{M}} Q\, dS$ is a conformal invariant, which governs e.g. the existence of conformal Ricci positive metrics (see e.g. Chang-Gursky-Yang [90, 91]) and eigenvalue estimates for Dirac operators (see Guofang Wang [407]). All these facts show that the Q-curvature in the context of fourth order conformally covariant operators takes a role quite analogous to the scalar curvature with respect to second order operators.

Getting back to the general case $n > 4$, let us outline what we are going to prove in the present book. We do not aim at giving an overview – not even of parts – of the theory of Paneitz operators but at giving a spot on some aspects of this issue. Namely, in Section 7.9 we address the question whether in specific bounded smooth domains $\Omega \subset \mathbb{R}^n$ $(n > 4)$ there exists a metric $g_u = u^{4/(n-4)}(\delta_{ij})$ being conformal to the flat euclidean metric and subject to certain homogeneous boundary conditions such that it has strictly positive constant Q-curvature. In view of the nonexistence results in Section 7.5.1 one expects that for generic domains the corresponding boundary value problems do not have a positive solution. Hence, in geometrically or topologically simple domains, such a conformal metric does in general not exist. Nevertheless, the boundary value problems have nontrivial solutions in topologically or specific geometrically complicated domains (see Section 7.9). For the Navier problem, i.e. $u = \Delta u = 0$ on $\partial\Omega$, one can also show positivity of u so that it may be considered as a conformal factor and one has such a nontrivial conformal metric as described above. Under Dirichlet boundary conditions, which could be interpreted as vanishing of length and normal curvature of the conformal metric on $\partial\Omega$, the positivity question has so far to be left open. The same difficulty prevents Esposito and Robert [161] from solving the Q-curvature analogue of the Yamabe problem.

In Section 7.10 the starting point is the hyperbolic ball $B = B_1(0) \subset \mathbb{R}^n$ which is equipped with the Poincaré metric $g_{ij} = 4\delta_{ij}/(1-|x|^2)^2$. This metric has constant Q-curvature $Q \equiv \frac{1}{8}n(n^2-4)$ and we address the question, whether there are further conformal metrics $g_u = u^{4/(n-4)}g$ having the same constant Q-curvature such that the resulting manifold is complete. Somehow surprisingly there exists infinitely many such metrics and even infinitely many among them have negative scalar curvature. This high degree of nonuniqueness is in sharp contrast with the corresponding question for the scalar curvature. There is no further conformal complete metric having the same constant negative curvature as g, see [279].

1.6 Critical growth polyharmonic model problems

The prototype to be studied is the semilinear polyharmonic eigenvalue problem

$$\begin{cases} (-\Delta)^m u = \lambda u + |u|^{s-1}u, & u \not\equiv 0 \text{ in } \Omega, \\ D^\alpha u|_{\partial\Omega} = 0 & \text{for } |\alpha| \le m-1. \end{cases} \tag{1.29}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $n > 2m$, $\lambda \in \mathbb{R}$; $s = (n+2m)/(n-2m)$ is the critical Sobolev exponent. If $m = 2$ and $\lambda = 0$ we are back in the situation discussed in the previous section with a euclidean background metric. The existence theory for (1.29) can be developed similarly to the second order case $m = 1$ while it becomes immediately quite difficult or even impossible to prove positivity or nonexistence of certain solutions. In particular, thanks to a Pohožaev identity [339, 340] one can exclude the existence of solutions to (1.29) in starshaped domains whenever $\lambda < 0$ but as far as the limit case $\lambda = 0$ is considered, things change dramatically in the two situations where $m = 1$ and $m \ge 2$. With a suitable application of the unique continuation principle (see e.g. [247, 345]), one can exclude when $m = 1$ the existence of any solution to (1.29) in starshaped domains even for $\lambda = 0$. In order to apply the same principle to (1.29) when $m \ge 2$ one would need to know the boundary behaviour of more derivatives than those already included in the Dirichlet boundary conditions and provided by the Pohožaev identity. Therefore, when $m \geq 2$ one can try to prove nonexistence of *positive* solutions in *strictly* starshaped domains, see Theorem 7.33 for the case $m = 2$. Unfortunately, this result is not satisfactory since positivity is not ensured in general domains, see also the discussion in the next section. So far, only in balls a more satisfactory discussion can be given. We refer to Section 7.5.1 for an up-to-date state of the art.

A first natural question is then to find out whether the nonexistence result for $\lambda = 0$ really depends on the geometry of the domain and starshapedness is not just a technical assumption. The answer is positive. For instance, problem (1.29) with $m = 2$ and $\lambda = 0$ admits a solution in domains with small holes and in some contractible non-starshaped domains, see Section 7.9. A second natural question then arises. Do the nonexistence results also depend on the boundary conditions considered? It is known that (1.29) admits no *positive* solution if $m = 2$, $\lambda = 0$, Ω is starshaped and *Navier* boundary conditions are considered, see [307, 398] and also Section 7.6. Moreover, in Section 7.7 we address the same problem under Steklov boundary conditions when $m = 2$ and Ω is a ball. We find all the values of the boundary parameter a in (1.26) for which the critical growth equation in (1.29) admits a positive solution.

Problem (1.29) in the case $m = 1$ has been studied extensively by Brezis-Nirenberg [72] who also discovered an interesting phenomenon when Ω is the unit ball. There exists a positive radial solution to (1.29) for every $\lambda \in (0, \Lambda_{1,1})$ if $n \geq 4$ and for every $\lambda \in (\frac{1}{4}\Lambda_{1,1}, \Lambda_{1,1})$ if $n = 3$. Moreover, they could show that in the latter case problem (1.29) has no nontrivial radial solution if $\lambda \leq \frac{1}{4}\Lambda_{1,1}$. Here and in the sequel $\Lambda_{m,1}$ denotes the first eigenvalue of $(-\Delta)^m$ in B under homogeneous Dirichlet boundary conditions.

Pucci and Serrin [348] raised the question in which way this critical behaviour of certain dimensions depends on the order $2m$ of the semilinear polyharmonic eigenvalue problem (1.29). They introduced the name critical dimensions.

Definition 1.1. Let $\Omega \subset \mathbb{R}^n$ be a ball. The dimension n is called *critical* if there is a positive bound $\Lambda > 0$ such that a necessary condition for the existence of a nontrivial radial solution to (1.29) is $\lambda > \Lambda$.

Pucci and Serrin [348] showed that for any m the dimension $n = 2m+1$ is critical and, moreover, that $n = 5, 6, 7$ are critical in the fourth order problem, $m = 2$. They suggested

Conjecture 1.2 (Pucci–Serrin).
The critical dimensions are precisely $n = 2m+1, \dots, 4m-1$.

In Section 7.5.2 we prove a weakened version of this conjecture. This nonexistence phenomenon has a functional analytic interpretation, which is reflected in the possibility of adding L^2–remainder terms in Sobolev inequalities with critical exponent and optimal constants in any bounded domain Ω, see Section 7.8.

1.7 Qualitative properties of solutions to semilinear problems

Radial symmetry of positive solutions to suitable semilinear higher order Dirichlet problems in the ball is obtained thanks to a suitable implementation of the moving planes procedure, see Section 7.1.2. One of the crucial steps in the moving planes procedure consists in comparing the solution u in a segment of the ball with its reflection u^r across the hyperplane which bounds the segment, see e.g. [195, Lemma 2.2]. For second order problems the comparison follows from suitable versions of the maximum principle since $u^r \geq u$ holds a priori on the boundary of this segment. This information however is not enough for higher order problems, and therefore the classical moving planes method fails. We employ a different technique to carry out the moving planes mechanism, using the integral representation of u in terms of the Green function of the polyharmonic operator $(-\Delta)^m$ in B under Dirichlet boundary conditions.

As repeatedly emphasised, linear higher order boundary value problems in general do not enjoy a positivity preserving property. This feature may also be observed in nonlinear problems. Let us illustrate this situation for the subcritical model problem corresponding to (1.29), namely

$$\begin{cases} (-\Delta)^m u = \lambda u + |u|^{p-1}u, & u \not\equiv 0 \text{ in } \Omega, \\ D^\alpha u|_{\partial\Omega} = 0 & \text{for } |\alpha| \leq m-1, \end{cases} \tag{1.30}$$

where $1 < p < s$. Thanks to some compactness, which is not available for (1.29), one may find a nontrivial solution to (1.30) as a suitable constrained minimum provided that $\lambda < \Lambda_{m,1}$. If $m = 1$ one can easily prove that such a minimum is positive just by replacing it with its modulus and by applying the maximum principle. This procedure fails in general if $m \geq 2$, even if Ω is a ball. This problem is discussed in detail in Section 7.2.

Bifurcation branches of solutions to nonlinear problems depending on some parameter λ are often quite complicated to be figured out. The case where only positive solutions are considered is much simpler. This situation is well illustrated by the so-called (second order) Gelfand problem [194, 239] where the nonlinearity is of exponential type, namely λe^u. A similar behaviour can be observed for the "approximate problem" where the nonlinearity is $\lambda(1+u)^p$. For this power-type nonlinearity, the bifurcation branch for the second order problem appears particularly interesting in the supercritical case $p > \frac{n+2}{n-2}$. In order to find out whether a similar behaviour can also be observed in higher order problems, one has to face the possible lack of positivity of the solution. As already discussed in Section 1.2 this can be overcome so far only in some particular situations, such as the case where Ω is a ball. In Section 7.11 we carefully study the branch of solutions to this biharmonic supercritical growth problem with the help of a suitable Lyapunov functional. Our study also takes advantage of the radial symmetry of positive solutions in the ball.

1.8 Willmore surfaces

At the beginning of this chapter the modeling of thin elastic plates was explained in some detail. There, curvature expressions were somehow "linearised" in order to have a purely quadratic behaviour of the leading terms of the energy functionals. This simplification results in linear Euler-Lagrange equations, which are justified for small deviations from a horizontal equilibrium shape. As soon as large deflections occur or a coordinate system is chosen in such a way that the equilibrium shape is not the *x*-*y*-plane, one has to stick to the frame invariant modeling of the bending energy in terms of differential geometric curvature expressions. When compared with the "linearised" energy integral (1.5) in Section 1.1, the integral

$$\int_{\Gamma} \left(\alpha + \beta(\mathtt{H} - \mathtt{H}_0)^2 - \gamma \mathtt{K}\right) d\omega \tag{1.31}$$

with suitable constants $\alpha, \beta, \gamma, \mathtt{H}_0$ may serve as a more realistic model for the bending and stretching energy of a thin elastic plate, which is described by a two-dimensional manifold $\Gamma \subset \mathbb{R}^3$. Here, $\mathtt{H}$ denotes its mean and $\mathtt{K}$ its Gaussian curvature. According to [324], α is related to the surface tension, β and γ are elastic moduli, while one may think of $\mathtt{H}_0$ as some preferred "intrinsic" curvature due to particular properties of the material under consideration. Physically reasonable assumptions on the coefficients are $\alpha \geq 0$, $0 \leq \gamma \leq \beta$, $\beta\gamma\mathtt{H}_0^2 \leq \alpha(\beta - \gamma)$, which ensure the functional to be positive definite. For modeling aspects and a thorough explanation of the meaning of each term we refer again to the survey article [324] by Nitsche. A discussion of the full model (1.31), however, seems to be out of reach at the moment, and for this reason one usually confines the investigation to the most important and dominant term, i.e. the contribution of $\mathtt{H}^2$.

Given a smooth immersed surface Γ, the Willmore functional is defined by

$$W(\Gamma) := \int_{\Gamma} \mathtt{H}^2 d\omega.$$

Apart from its meaning as a model for the elastic energy of thin shells or biological membranes, it is also of great geometric interest, see e.g [413, 414]. Furthermore, it is used in image processing for problems of surface restoration and image inpainting, see e.g. [105] and references therein. In these applications one is usually concerned with minima, or more generally with critical points of the Willmore functional. It is well-known that the corresponding surface Γ has to satisfy the Willmore equation

$$\Delta_{\Gamma}\mathtt{H} + 2\mathtt{H}(\mathtt{H}^2 - \mathtt{K}) = 0 \qquad \text{on } \Gamma, \tag{1.32}$$

where Δ_{Γ} denotes the Laplace-Beltrami operator on Γ with respect to the induced metric. A solution of (1.32) is called a Willmore surface. An additional difficulty here arises from the fact that Δ_{Γ} depends on the unknown surface so that the equation is quasilinear. Moreover, the ellipticity is not uniform which, in the variational framework, is reflected by the fact that minimising sequences may in general be unbounded in H^2. A difficult step is to pass to suitable minimising sequences enjoying sufficient compactness in H^2 and C^1.

In the past years a lot of very interesting work has been done, mainly on closed Willmore surfaces, see e.g. [35, 60, 156, 262, 263, 264, 287, 355, 361, 371, 372]. For instance, one knows about minimisers of the Willmore energy of prescribed genus and about global existence and convergence of the Willmore flow to the sphere under explicit smallness assumptions which, by means of counterexamples, have been proved to be sharp.

The situation changes if one considers boundary value problems. Except for small data results, our knowledge is still somehow limited, see e.g. [50, 115, 116, 138, 360] and references therein.

Possible boundary value problems for the linear plate equation were discussed in Section 1.1 above to some extent. In the nonlinear context here, one could discuss the same issue, but now considering the geometric terms instead of their linearisations. For details again we refer to [324]. Here we will be concerned with a Dirichlet problem for Willmore surfaces where, in some particularly symmetric situations, results are available. These are not just small data results or application of linear theory combined with the implicit function theorem. Let us mention an important recent contribution by Schätzle [360]. He proved a general result concerning existence of branched Willmore immersions in $\mathbb{S}^n$ with boundary which satisfy Dirichlet boundary conditions. Assuming the boundary data to obey some explicit geometrically motivated smallness condition these immersions can even be shown to be embedded. By working in $\mathbb{S}^n$, some compactness problems could be overcome; on the other hand, when pulling pack these immersions to $\mathbb{R}^n$ it cannot be excluded that they contain the point ∞. Moreover, in general, the existence of branch points cannot be ruled out, and due to the generality of the approach, it seems to us that only little topological information about the solutions can be extracted from the existence proof. We think that it is quite interesting to identify situations where it is possible to work with a priori bounded minimising sequences or where solutions with additional properties like e.g. being a graph or enjoying certain symmetry properties can be found. In view of the lack of general comparison principles and of the highly nonlinear character of (1.32) this is a rather difficult task. In order to outline directions of future research we think that it is a good strategy to investigate first relatively special situations which e.g. enjoy some symmetry.

This is exactly the subject of Section 8. We restrict ourselves to surfaces of revolution satisfying Dirichlet boundary conditions. In this class we can find minimising sequences enjoying sufficient compactness properties thereby constructing a classical solution where a number of additional qualitative properties are obtained. While the underlying differential equation is one-dimensional the geometry is already two-dimensional. The interplay between mean and Gaussian curvature in (1.32) already causes great difficulties.

Chapter 2
Linear problems

Linear polyharmonic problems and their features are essential in order to achieve the main tasks of this monograph, namely the study of positivity and nonlinear problems. With no hope of being exhaustive, in this chapter we outline the main tools and results, which will be needed subsequently. We start by introducing higher order Sobolev spaces and relevant boundary conditions for polyharmonic problems. Then using a suitable Hilbert space, we show solvability of a wide class of boundary value problems. The subsequent part of the chapter is devoted to regularity results and a priori estimates both in Schauder and L^p setting, including also maximum modulus estimates. These regularity results are particularly meaningful when writing explicitly the solution of the boundary value problem in terms of the data by means of a suitable kernel. Focusing on the Dirichlet problem for the polyharmonic operator, we introduce Green's functions and the fundamental formula by Boggio in balls. We conclude with a study of a biharmonic problem in nonsmooth domains explaining two paradoxes which are important in particular when approximating solutions numerically.

2.1 Polyharmonic operators

Unless otherwise specified, throughout this monograph Ω denotes a bounded domain (open and connected) of $\mathbb{R}^n$ ($n \geq 2$). The smoothness assumptions on the boundary $\partial\Omega$ will be made precise in each situation considered. However, we shall always assume that $\partial\Omega$ is Lipschitzian so that the tangent hyperplane and the unit outward normal $\nu = \nu(x)$ are well-defined for a.e. $x \in \partial\Omega$, where a.e. means here with respect to the $(n-1)$-dimensional Hausdorff measure. When it is clear from the context, in the sequel we omit writing "a.e."

The Laplacian Δu of a smooth function $u : \Omega \to \mathbb{R}$ is the trace of its Hessian matrix, namely

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \ .$$

We are interested in iterations of the Laplace operator, namely *polyharmonic operators* defined inductively by

$$\Delta^m u = \Delta(\Delta^{m-1}u) \qquad \text{for } m = 2,3,\dots .$$

Arguing by induction on m, it is straightforward to verify that

$$\Delta^m u = \sum_{\ell_1+\dots+\ell_n=m} \frac{m!}{\ell_1!\dots\ell_n!}\,\frac{\partial^{2m}u}{\partial x_1^{2\ell_1}\dots\partial x_n^{2\ell_n}} .$$

The polyharmonic operator Δ^m may also be seen in an abstract way through the polynomial $L_m : \mathbb{R}^n \to \mathbb{R}$ defined by

$$L_m(\xi) = \sum_{\ell_1+\dots+\ell_n=m} \frac{m!}{\ell_1!\dots\ell_n!}\left(\prod_{i=1}^{n}\xi_i^{2\ell_i}\right) = |\xi|^{2m} \text{ for } \xi \in \mathbb{R}^n.$$

Formally, $\Delta^m = L_m(\nabla)$. In particular, this shows that $L_m(\xi) > 0$ for all $\xi \neq 0$ so that Δ^m is an *elliptic operator*, see [5, p. 625] or [275, p. 121]. Ellipticity is a property of the principal part (containing the highest order partial derivatives) of the differential operator.

In this chapter, we study linear differential elliptic operators of the kind

$$u \mapsto Au = (-\Delta)^m u + \mathscr{A}(x;D)u, \tag{2.1}$$

where

$$\mathscr{A} : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \dots \times \mathbb{R}^{n^{2m-1}} \to \mathbb{R}$$

is a linear operator containing all the lower order partial derivatives of the function u. The coefficients of the derivatives are measurable functions of x in Ω. For elliptic differential operators A of the form (2.1) and under suitable assumptions on f, we shall consider solutions $u = u(x)$ to the equation

$$(-\Delta)^m u + \mathscr{A}(x;D)u = f \qquad \text{in } \Omega, \tag{2.2}$$

which satisfy some boundary conditions on $\partial\Omega$. We discuss the class of "admissible" boundary conditions in Section 2.3. What we mean by *solution* to (2.2) will be made clear in each situation considered.

Finally, let us mention that our statements also hold if we replace $(-\Delta)^m$ with the m-th power of any other second order elliptic operator L; for instance, in Section 6.1 we consider powers of

$$Lu = -\sum_{i,j=1}^{2} \tilde{a}_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{with the matrix } \{\tilde{a}_{ij}\} \text{ being positive definite,}$$

or

$$Lu = -\frac{2}{|\nabla h|^2}\Delta u \quad \text{with} \nabla h \neq 0.$$

2.2 Higher order Sobolev spaces

Before introducing the boundary conditions to be associated to (2.2), we briefly recall the definition and basic properties of higher order Sobolev spaces and of their embedding into L^q spaces. In particular, we need to define the traces in order to give some meaning to the boundary conditions. We restrict our attention to those statements which will be frequently used in this book. Except in this section, Ω is assumed to be bounded throughout the whole Chapter 2.

2.2.1 *Definitions and basic properties*

Given a domain $\Omega \subset \mathbb{R}^n$, $\|\,.\,\|_{L^p}$ denotes the standard $L^p(\Omega)$-norm for $1 \le p \le \infty$. For all $m \in \mathbb{N}^+$ let us define the norm

$$u \mapsto \mathbf{N}(u) := \left(\sum_{k=0}^{m} \|D^k u\|_{L^p}^p \right)^{1/p}, \tag{2.3}$$

where $D^0 u = u$,

$$D^k u \cdot D^k v = \sum_{i_1,\dots,i_k=1}^{n} \frac{\partial^k u}{\partial x_{i_1} \dots \partial x_{i_k}} \frac{\partial^k v}{\partial x_{i_1} \dots \partial x_{i_k}} \text{ and } |D^k u| = \left(D^k u \cdot D^k u \right)^{1/2}.$$

Note that we will specify the domain Ω in $\|\,.\,\|_{L^p}$ only when it is not clear from the context. Next, we define the space

$$W^{m,p}(\Omega) := \overline{\{u \in C^m(\Omega);\, \mathbf{N}(u) < \infty\}}^{\mathbf{N}},$$

that is, the completion with respect to the norm (2.3). Alternatively, $W^{m,p}(\Omega)$ may be defined as the subspace of $L^p(\Omega)$ of functions having generalised derivatives up to order m in $L^p(\Omega)$, see [300].

If $\Omega \neq \mathbb{R}^n$ and its boundary $\partial\Omega$ is smooth, then a function $u \in W^{m,p}(\Omega)$ admits some traces on $\partial\Omega$ where, for our purposes, it is enough to restrict the attention to the case $p \in (1,\infty)$. More precisely, if ν denotes the unit outer normal to $\partial\Omega$, then for any $u \in C^m(\overline{\Omega})$ and any $j = 0,\dots,m$ we define the traces

$$\gamma_j u := \frac{\partial^j u}{\partial \nu^j}\Big|_{\partial\Omega}. \tag{2.4}$$

By [275, Théorème 8.3], these linear operators may be extended continuously to the larger space $W^{m,p}(\Omega)$. We set

$$W^{m-j-1/p,p}(\partial\Omega) := \gamma_j[W^{m,p}(\Omega)] \text{ for } j = 0,\dots,m-1. \tag{2.5}$$

In particular, $W^{1/p',p}(\partial\Omega) = \gamma_{m-1}[W^{m,p}(\Omega)]$, where p' is the conjugate of p (that is, $p + p' = pp'$). We also put

$$\begin{aligned}\gamma_m[W^{m,p}(\Omega)] &= W^{-1/p,p}(\partial\Omega) := [W^{1/p,p'}(\partial\Omega)]' \\ &= \text{the dual space of } W^{1/p,p'}(\partial\Omega),\end{aligned} \tag{2.6}$$

so that (2.5) makes sense for all $j = 0,\dots,m$. With an abuse of notation, in the sequel we simply write u (respectively $\frac{\partial^j u}{\partial \nu^j}$) instead of $\gamma_0 u$ (respectively $\gamma_j u$ for $j = 1,...,m$).

When $p = 2$, we put $H^m(\Omega) := W^{m,2}(\Omega)$. Moreover, when $p = 2$ and $m \geq 1$ we write $H^{m-1/2}(\partial\Omega) = W^{m-1/2,2}(\partial\Omega)$ and

$$H^{-m+\frac{1}{2}}(\partial\Omega) = [H^{m-\frac{1}{2}}(\partial\Omega)]' = \text{the dual space of } H^{m-\frac{1}{2}}(\partial\Omega). \tag{2.7}$$

The space $H^m(\Omega)$ becomes a Hilbert space when endowed with the scalar product

$$(u,v) \mapsto \sum_{k=0}^{m} \int_\Omega D^k u \cdot D^k v\, dx \qquad \text{for all } u,v \in H^m(\Omega).$$

In some cases one may simplify the just defined norms and scalar products. As a first step, we mention that thanks to interpolation theory, see [1, Theorem 4.14], one can neglect intermediate derivatives in (2.3). More precisely, $W^{m,p}(\Omega)$ is a Banach space also when endowed with the following norm, which is equivalent to (2.3):

$$\|u\|_{W^{m,p}} = \left(\|u\|_{L^p}^p + \|D^m u\|_{L^p}^p\right)^{1/p} \qquad \text{for all } u \in W^{m,p}(\Omega), \tag{2.8}$$

whereas $H^m(\Omega)$ is a Hilbert space also with the scalar product

$$(u,v)_{H^m} := \int_\Omega (uv + D^m u \cdot D^m v)\, dx \qquad \text{for all } u,v \in H^m(\Omega).$$

Of particular interest is the closed subspace of $W^{m,p}$ defined as the intersection of the kernels of the trace operators in (2.4), that is for any bounded domain Ω we consider

$$W_0^{m,p}(\Omega) := \bigcap_{j=0}^{m-1} \ker \gamma_j\, .$$

Moreover, for bounded domains Ω and for $1 < p < \infty$, if p' is the conjugate of p we write

$$W^{-m,p'}(\Omega) := [W_0^{m,p}(\Omega)]' = \text{the dual space of } W_0^{m,p}(\Omega) \tag{2.9}$$

and, for $p = 2$,

$$H^{-m}(\Omega) := [H_0^m(\Omega)]' = [W_0^{m,2}(\Omega)]'.$$

Consider the bilinear form

$$(u,v)_{H_0^m} := \begin{cases} \displaystyle\int_\Omega \Delta^k u \, \Delta^k v \, dx & \text{if } m = 2k, \\ \displaystyle\int_\Omega \nabla(\Delta^k u) \cdot \nabla(\Delta^k v) \, dx & \text{if } m = 2k+1, \end{cases} \tag{2.10}$$

and the corresponding norm

$$\|u\|_{H_0^m} := \begin{cases} \|\Delta^k u\|_{L^2} & \text{if } m = 2k, \\ \|\nabla(\Delta^k u)\|_{L^2} & \text{if } m = 2k+1. \end{cases} \tag{2.11}$$

For general $p \in (1,\infty)$, one has the choice of taking the L^p-version of (2.11) or the equivalent norm

$$\|u\|_{W_0^{m,p}} := \|D^m u\|_{L^p}.$$

Thanks to these norms, one may define the above spaces in a different way.

Theorem 2.1. *If $\Omega \subset \mathbb{R}^n$ is a bounded domain, then*

$$\begin{aligned} W_0^{m,p}(\Omega) &= \text{ the closure of } C_c^\infty(\Omega) \text{ with respect to the norm } \|\,.\,\|_{W^{m,p}} \\ &= \text{ the closure of } C_c^\infty(\Omega) \text{ with respect to the norm } \|\,.\,\|_{W_0^{m,p}}. \end{aligned}$$

Theorem 2.1 follows by combining interpolation inequalities (see [1, Theorem 4.14]) with the classical Poincaré inequality $\|\nabla u\|_{L^p} \geq c\|u\|_{L^p}$ for all $u \in W_0^{1,p}(\Omega)$.

If Ω is unbounded, including the case where $\Omega = \mathbb{R}^n$, we define

$$\begin{aligned} \|u\|_{\mathscr{D}^{m,p}(\Omega)} &:= \|D^m u\|_{L^p(\Omega)}, \\ \mathscr{D}^{m,p}(\Omega) &:= \text{ the closure of } C_c^\infty(\Omega) \text{ with respect to the norm } \|\,.\,\|_{\mathscr{D}^{m,p}}, \end{aligned}$$

and, again, let $W_0^{m,p}(\Omega)$ denote the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\,.\,\|_{W^{m,p}}$. In this unbounded case, a similar result as in Theorem 2.1 is no longer true since although $W_0^{m,p}(\Omega) \subset \mathscr{D}^{m,p}(\Omega)$, the converse inclusion fails. For instance, if $\Omega = \mathbb{R}^n$, then $W_0^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$, whereas the function $u(x) = (1+|x|^2)^{(1-n)/4}$ belongs to $\mathscr{D}^{1,2}(\mathbb{R}^n)$ but not to $H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$.

Theorem 2.1 states that, when Ω is bounded, the space $H_0^m(\Omega)$ is a Hilbert space when endowed with the scalar product (2.10). The striking fact is that not only all lower order derivatives (including the derivative of order 0!) are neglected but also that some of the highest order derivatives are dropped. This fact has a simple explanation since

$$(u,v)_{H_0^m} = \int_\Omega D^m u \cdot D^m v \, dx \qquad \text{for all } u,v \in H_0^m(\Omega). \tag{2.12}$$

One can verify (2.12) by using a density argument, namely for all $u,v \in C_c^\infty(\Omega)$. And with this restriction, one can integrate by parts several times in order to obtain (2.12). The bilinear form (2.10) also defines a scalar product on the space $\mathscr{D}^{m,2}(\Omega)$ whenever Ω is an unbounded domain. We summarise all these facts in

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain. Then the bilinear form*

$$(u,v) \mapsto \begin{cases} \int_\Omega \Delta^k u\, \Delta^k v\, dx & \text{if } m = 2k, \\ \int_\Omega \nabla(\Delta^k u) \cdot \nabla(\Delta^k v)\, dx & \text{if } m = 2k+1, \end{cases} \tag{2.13}$$

defines a scalar product on $H^m_0(\Omega)$ (respectively $\mathscr{D}^{m,2}(\Omega)$) if Ω is bounded (respectively unbounded). If Ω is bounded, then this scalar product induces a norm equivalent to (2.3).

2.2.2 Embedding theorems

Consider first the case of unbounded domains.

Theorem 2.3. *Let $m \in \mathbb{N}^+$, $1 \le p < \infty$, with $n > mp$. Assume that $\Omega \subset \mathbb{R}^n$ is an unbounded domain with uniformly Lipschitzian boundary $\partial\Omega$, then:*

1. *$\mathscr{D}^{m,p}(\Omega) \subset L^{np/(n-mp)}(\Omega)$;*
2. *$W^{m,p}(\Omega) \subset L^q(\Omega)$ for all $p \le q \le \frac{np}{n-mp}$.*

On the other hand, in bounded domains subcritical embeddings become compact.

Theorem 2.4 (Rellich-Kondrachov). *Let $m \in \mathbb{N}^+$, $1 \le p < \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitzian domain, then for any $1 \le q < \frac{np}{n-mp}$ there exists a compact embedding $W^{m,p}(\Omega) \subset L^q(\Omega)$. Here we make the convention that $\frac{np}{n-mp} = +\infty$ if $n \le mp$.*

Remark 2.5. The optimal constants of the compact embeddings in Theorem 2.4 are attained on functions solving corresponding Euler-Lagrange equations. We refer to Section 7.2 for a discussion of these problems where, for simplicity, we restrict again our attention to the case $m = 2$.

In fact, if $n < mp$, Theorem 2.4 may be improved by the following statement.

Theorem 2.6. *Let $m \in \mathbb{N}^+$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitzian boundary. Assume that there exists $k \in \mathbb{N}$ such that $n < (m-k)p$. Then*

$$W^{m,p}(\Omega) \subset C^{k,\gamma}(\overline{\Omega}) \qquad \text{for all } \gamma \in \left(0, m-k-\frac{n}{p}\right] \cap (0,1)$$

with compact embedding if $\gamma < m - k - \frac{n}{p}$.

The statements of Theorems 2.4 and 2.6 also hold if we replace $W^{m,p}(\Omega)$ with its proper subspace $W^{m,p}_0(\Omega)$. In this case, no regularity assumption on the boundary

$\partial\Omega$ is needed. Let us also mention that there is a simple way to remember the embeddings in Theorem 2.6. It is based on the so-called *regularity index*, see [11, Section 8.7]. In n-dimensional bounded domains Ω, the regularity index for $W^{m,p}(\Omega)$ is $m-n/p$ whereas for $C^{k,\gamma}(\overline{\Omega})$ it is $k+\gamma$. A Sobolev space is embedded into any other space with a smaller regularity index. For instance, $W^{m,p}(\Omega) \subset W^{\mu,q}(\Omega)$ provided $m-n/p \geq \mu - n/q$ (and $m \geq \mu$). Also $W^{m,p}(\Omega) \subset C^{k,\gamma}(\overline{\Omega})$ whenever $m-n/p \geq k+\gamma$ and $\gamma \in (0,1)$, which is precisely the statement in Theorem 2.6. A similar rule is also available for trace operators, namely if $m-n/p \geq \mu-(n-1)/q$ (and $m > \mu$) then the trace operator on $W^{m,p}(\Omega)$ is continuous into $W^{\mu,q}(\partial\Omega)$.

We conclude this section with the multiplicative properties of functions in Sobolev spaces.

Theorem 2.7. *Assume that $\Omega \subset \mathbb{R}^n$ is a Lipschitzian domain. Let $m \in \mathbb{N}^+$ and $p \in [1,\infty)$ be such that $mp > n$. Then $W^{m,p}(\Omega)$ is a commutative Banach algebra.*

Remark 2.8. Theorem 2.7 can be generalised by considering multiplications of functions in possibly different Sobolev spaces. For instance, if $m_1, m_2 \in \mathbb{N}^+$ and $\mu = \min\{m_1, m_2, m_1+m_2-[\frac{n}{2}]-1\}$, then $H^{m_1}(\Omega)H^{m_2}(\Omega) \subset H^{\mu}(\Omega)$.

We postpone further properties of the Hilbertian critical embedding, that is, $H^m \subset L^{2n/(n-2m)}$ with $n > 2m$, to Sections 7.3 and 7.8. The reasons are both that we need further tools and that these properties have a natural application to nonexistence results for semilinear polyharmonic equations at critical growth.

2.3 Boundary conditions

For the rest of Chapter 2, we assume the domain Ω to be bounded. Under suitable assumptions on $\partial\Omega$, to equation (2.2) we may associate m boundary conditions. These conditions will be expressed by linear differential operators $B_j(x;D)$, namely

$$B_j(x;D)u = h_j \text{ for } j = 1,\dots,m \text{ on } \partial\Omega, \tag{2.14}$$

where the functions h_j belong to suitable functional spaces. Each B_j has a maximal order of derivatives $m_j \in \mathbb{N}$ and the coefficients of the derivatives are sufficiently smooth functions on $\partial\Omega$. The regularity assumptions on these coefficients and on $\partial\Omega$ will be made precise in each statement.

For the problems considered in this monograph, it always appears that

$$m_j \leq 2m-1 \qquad \text{for all } j = 1,\dots,m. \tag{2.15}$$

Therefore, we shall always assume that (2.15) holds, although some of our statements remain true under less restrictive assumptions. The meaning of (2.14) will remain unclear until the precise definition of solution to (2.2) will be given; in most cases, they should be seen as traces, namely satisfied in a generalised sense given by the operators (2.4).

The choice of the B_j's is not completely free, we need to impose a certain algebraic constraint, the so-called *complementing condition*. For any j, let B_j' denote the highest order part of B_j which is precisely of order m_j, then for equations (2.2) which have the polyharmonic operator as principal part, we have the following

Definition 2.9. For every point $x \in \partial\Omega$, let $\nu(x)$ denote the normal unit vector. We say that the *complementing condition* holds for (2.14) if, for any nontrivial tangential vector $\tau(x)$, the polynomials in t $B_j'(x;\tau+t\nu)$ are linearly independent modulo the polynomial $(t-i|\tau|)^m$.

As explained in [5, Section 10], the complementing condition is crucial in order to obtain a priori estimates for solutions to (2.2)-(2.14) and, in turn, existence and uniqueness results.

Clearly, the solvability of (2.2)-(2.14) depends on the assumptions made on $\mathscr{A}$, f, B_j and h_j. We are here interested in *structural assumptions*, namely properties of the problem and not of its data.

Assumptions on the homogeneous problem. If we assume that $f=0$ in Ω and that $h_j=0$ on $\partial\Omega$ for all $j=1,\dots,m$, then (2.2)-(2.14) admits the trivial solution $u=0$, in whatever sense this is intended. The natural question is then to find out whether this is the only solution. The answer depends on the structure of the problem. In fact, for any "reasonable" $\mathscr{A}$ and B_j's there exists a discrete set $\Sigma \subset \mathbb{R}$ such that, if $\sigma \notin \Sigma$, then the problem

$$\begin{cases} (-\Delta)^m u + \sigma\mathscr{A}(x;D)u = 0 & \text{in } \Omega, \\ B_j(x;D)u = 0 & \text{with } j=1,\dots,m \text{ on } \partial\Omega, \end{cases} \tag{2.16}$$

only admits the trivial solution. If $\sigma \in \Sigma$, then the solutions of (2.16) form a nontrivial linear space; if $\mathscr{A}$ and the B_j's are well-behaved (in the sense specified below) this space has finite dimension. Therefore, we shall assume that

$$\text{the associated homogeneous problem only admits the trivial solution } u=0. \tag{2.17}$$

Assumption (2.17) is a structural assumption which only depends on $\mathscr{A}$ and the B_j's. Thanks to the Fredholm alternative (see e.g. [69, Theorem VI.6]), we know that if (2.17) fails, then for any possible choice of the data f and h_j problem (2.2)-(2.14) fails to have either existence or uniqueness of the solution.

Assumptions on $\mathscr{A}$. Assume that $\mathscr{A}$ has the following form

$$\mathscr{A}(x;D)u = \sum_{|\beta|\le 2m-1} a_\beta(x)D^\beta u\,, \qquad a_\beta \in C^{|\beta|}(\overline{\Omega}). \tag{2.18}$$

Actually, for some of our results, less regularity is needed on the coefficients a_β but we will not go deep into this. We just mention that, for instance, if $n \ge 5$ then in order to obtain existence of a weak solution (according to Theorem 2.16 below) it is enough to assume $a_0 \in L^{n/4}(\Omega)$.

Assumptions on the boundary conditions. Assume that, according to Definition 2.9,

$$\text{the linear boundary operators } B_j\text{'s satisfy the complementing condition.} \tag{2.19}$$

We now discuss the main boundary conditions considered in this monograph.

Dirichlet boundary conditions. In this case, $B_j(x,D)u = B'_j(x,D)u = \frac{\partial^{j-1}u}{\partial \nu^{j-1}}$ for $j = 1,\dots,m$ so that $m_j = j-1$ and (2.14) become

$$u = h_1,\ \dots,\ \frac{\partial^{m-1}u}{\partial \nu^{m-1}} = h_m \qquad \text{on } \partial\Omega. \tag{2.20}$$

Hence, $B'_j(x;\tau + t\nu) = t^{j-1}$ and, as mentioned in [5, p. 627], the complementing condition is satisfied for (2.20).

Navier boundary conditions. In this case, $B_j(x,D)u = B'_j(x,D)u = \Delta^{j-1}u$ for $j = 1,...,m$ so that $m_j = 2(j-1)$ and (2.14) become

$$u = h_1,\ \dots,\ \Delta^{m-1}u = h_m \qquad \text{on } \partial\Omega. \tag{2.21}$$

Under these conditions, if $\mathscr{A}$ has a suitable form then (2.2) may be written as a system of m Poisson equations, each one of the unknown functions satisfying Dirichlet boundary conditions. Therefore, the complementing condition follows by the theory of elliptic systems [6].

Mixed Dirichlet-Navier boundary conditions. We make use of these conditions in Section 5.2. They are a suitable combination of (2.20)-(2.21). For instance, if m is odd, they read $B_j(x,D)u = \frac{\partial^{j-1}u}{\partial \nu^{j-1}}$ for $j = 1,...,m-1$ and $B_m(x,D)u = \Delta^{(m-1)/2}u$. Again, the complementing condition is satisfied.

Steklov boundary conditions. We consider these conditions only for the biharmonic operator. Let $a \in C^0(\partial\Omega)$ and to the equation $\Delta^2 u = f$ in Ω we associate the boundary operators $B_1(x,D)u = u$ and $B_2(x,D)u = \Delta u - a\frac{\partial u}{\partial \nu}$. Then (2.14) become

$$u = h_1 \qquad \text{and} \qquad \Delta u - a\frac{\partial u}{\partial \nu} = h_2 \qquad \text{on } \partial\Omega. \tag{2.22}$$

Since B'_j (for $j = 1,2$) is the same as for (2.21), also (2.22) satisfy the complementing condition.

More generally, Hörmander [230] characterises all the sets of boundary operators B_j which satisfy the complementing condition.

We conclude this section by giving an example of boundary conditions which do not satisfy the complementing condition. Consider the fourth order problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ \Delta u = 0 & \text{on } \partial\Omega, \\ \frac{\partial(\Delta u)}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.23}$$

For any unit vector τ tangential to $\partial\Omega$ we have $B_1(\tau+t\nu) = B_1'(\tau+t\nu) = t^2+1$ and $B_2(\tau+t\nu) = B_2'(\tau+t\nu) = t^3+t$. These polynomials *are not* linearly independent modulo $(t-i)^2$ so that the complementing condition is not satisfied. Note also that any harmonic function solves (2.23) so that the space of solutions *does not* have finite dimension. In particular, if we take any point $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$, the fundamental solution u_0 of $-\Delta$ having pole in x_0 (namely, $u_0(x) = \log|x-x_0|$ if $n=2$ and $u_0(x) = |x-x_0|^{2-n}$ if $n \geq 3$) solves (2.23). This shows that it is not possible to obtain uniform a priori bounds in any norm. Indeed, as x_0 approaches the boundary $\partial\Omega$ it is clear that (for instance!) the H^1-norm of the solution cannot be bounded uniformly in terms of its L^2-norm.

2.4 Hilbert space theory

2.4.1 Normal boundary conditions and Green's formula

In this section we study the solvability of the polyharmonic equation

$$(-\Delta)^m u + \sum_{\circ} D^\beta \left[a_{\beta,\mu}(x) D^\mu u \right] = f \qquad \text{in } \Omega \tag{2.24}$$

complemented with the linear boundary conditions

$$\sum_{|\alpha| \leq m_j} b_{j,\alpha}(x) D^\alpha u = h_j \quad \text{on } \partial\Omega \text{ with } j = 1, \dots, m, \tag{2.25}$$

where $m_j \leq 2m-1$, see (2.15), and $\sum_{\circ}$ means summation over all multi-indices β and μ such that

$$|\beta| \leq m, \qquad |\mu| \leq m, \qquad |\beta| + |\mu| \leq 2m-1. \tag{2.26}$$

With the notations of (2.2) and (2.14), we have

$$\mathscr{A}(x;D)u = \sum_{\circ} D^\beta [a_{\beta,\mu}(x) D^\mu u], \qquad B_j(x;D)u = \sum_{|\alpha| \leq m_j} b_{j,\alpha}(x) D^\alpha u.$$

Assume that

$$a_{\beta,\mu} \in C^{|\beta|}(\overline{\Omega}) \qquad \text{for all } \beta \text{ and } \mu \text{ satisfying (2.26).} \tag{2.27}$$

To the linear differential operator A defined by

$$Au := (-\Delta)^m u + \sum_{\circ} D^\beta [a_{\beta,\mu}(x) D^\mu u] \tag{2.28}$$

we associate the bilinear form

$$\Psi(u,v) = (u,v) + \sum_{\circ} (-1)^{|\beta|} \int_\Omega a_{\beta,\mu}(x) D^\mu u D^\beta v \, dx \text{ for all } u,v \in H^m(\Omega), \tag{2.29}$$

where $(\,.\,,.\,)$ is defined in (2.13). Formally, Ψ is obtained by integrating by parts $\int Au\,v$ and by neglecting the boundary integrals. We point out that, in view of (2.27), $\Psi(u,v)$ is well-defined for all $u,v \in H^m(\Omega)$.

Let us recall that m_j denotes the highest order derivatives of u appearing in B_j. With no loss of generality, we may assume that the boundary conditions (2.25) are ordered for increasing m_j's so that

$$m_j \le m_{j+1} \qquad \text{for all } j = 1,...,m-1. \tag{2.30}$$

Moreover, we assume that the coefficients in (2.25) satisfy

$$b_{j,\alpha} \in C^{2m-m_j}(\overline{\Omega}) \qquad \text{for all } j = 1,...,m \text{ and } |\alpha| \le m_j; \tag{2.31}$$

by this, we mean that the functions $b_{j,\alpha}$ are restrictions to the boundary $\partial\Omega$ of functions in $C^{2m-m_j}(\overline{\Omega})$.

We also need to define well-behaved systems of boundary operators.

Definition 2.10. Let $k \in \mathbb{N}^+$. We say that the boundary value operators $\{F_j(x;D)\}_{j=1}^k$ satisfying (2.30) form a *normal system* on $\partial\Omega$ if $m_i < m_j$ whenever $i < j$ and if $F_j(x;D)$ contains the term $\partial^{m_j}/\partial\nu^{m_j}$ with a coefficient different from 0 on $\partial\Omega$. Moreover, we say that $\{F_j(x;D)\}_{j=1}^k$ is a *Dirichlet system* if, in addition to the above conditions, we have $m_j = j-1$ for $j = 1,...,k$; the number k is then called the *order* of the Dirichlet system.

Remark 2.11. The assumption "F_j contains the term $\partial^{m_j}/\partial\nu^{m_j}$ with a coefficient different from 0 on $\partial\Omega$" requires some explanations since it may happen that the term $\partial^{m_j}/\partial\nu^{m_j}$ does not appear explicitly in F_j. One should then rewrite the boundary conditions on $\partial\Omega$ in local coordinates; the system of coordinates should contain the $n-1$ tangential directions and the normal direction ν. Then the assumption is that in this new system of coordinates the term $\partial^{m_j}/\partial\nu^{m_j}$ indeed appears with a coefficient different from 0. For instance, imagine that $m_j = 2$ and that Δu represents the terms of order 2 in F_j; it is known that if $\partial\Omega$ and u are smooth, then $\Delta u = \frac{\partial^2 u}{\partial\nu^2} + (n-1)H\frac{\partial u}{\partial\nu} + \Delta_\tau u$ on $\partial\Omega$, where H denotes the mean curvature at the boundary and $\Delta_\tau u$ denotes the tangential Laplacian of u. Therefore, any boundary operator which contains Δ as principal part satisfies this condition.

It is clear that if a normal system of boundary value operators $\{F_j(x;D)\}_{j=1}^k$ is such that $m_k = k-1$, then it is a Dirichlet system.

Proposition 2.12. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Let $k \in \mathbb{N}^+$ and assume that the boundary value operators $\{B_j(x;D)\}_{j=1}^k$ form a normal system on $\partial\Omega$. If $m \geq m_k$, then there exists a (non-unique) system $\{S_j(x;D)\}_{j=k+1}^m$ such that $\{B_1,...,B_k,S_{k+1},...,S_m\}$ forms a Dirichlet system of order m. Here, all the boundary operators are supposed to have smooth coefficients.*

We can now give a suitable version of Green's formula.

Theorem 2.13. *Let*

$$\partial\Omega \in C^{2m,1} \tag{2.32}$$

and suppose that the differential operator A in (2.28) has coefficients satisfying (2.27). Assume also that $\{F_j(x;D)\}_{j=1}^m$ forms a Dirichlet system of order m (so that $m_j = j-1$) with coefficients satisfying (2.31). Then there exists a normal system of boundary operators $\{\Phi_j(x;D)\}_{j=1}^m$ with coefficients satisfying (2.31) (and with Φ_j of order $2m-j$) such that

$$\Psi(u,v) = \int_\Omega Au\,v\,dx + \sum_{j=1}^m \int_{\partial\Omega} \Phi_j(x;D)u\,F_j(x;D)v\,d\omega \quad \textit{for all } u,v \in H^{2m}(\Omega).$$

The operators $\{\Phi_j(x;D)\}_{j=1}^m$ given by Theorem 2.13 are called *Green adjoint boundary operators* of $\{F_j(x;D)\}_{j=1}^m$.

2.4.2 Homogeneous boundary value problems

In this section we study the solvability of (2.24) in the case of vanishing boundary data h_j in conditions (2.25), namely

$$\begin{cases} (-\Delta)^m w + \sum_\circ D^\beta \left(a_{\beta,\mu}(x) D^\mu w\right) = g & \text{in } \Omega, \\ B_j(x;D)w = 0 \quad\quad \text{for } j=1,...,m & \text{on } \partial\Omega. \end{cases} \tag{2.33}$$

The solvability of (2.33) is studied in the framework of Hilbertian Sobolev spaces. To this end, let us explain what is meant by a Hilbert triple.

Definition 2.14. Let V and H be Hilbert spaces such that $V \subset H$ with injective, dense and continuous embedding. Let V' denote the dual space of V; a scheme of this type (namely $V \subset H \subset V'$) is called a *Hilbert triple*.

For a Hilbert triple $V \subset H \subset V'$ also the embedding $H \subset V'$ is necessarily injective, dense and continuous, see [416, Theorem 17.1]. Notice also that, although there exists the Riesz isomorphism between V and V' (see [69, Theorem V.5]), we will represent functionals from V' with the scalar product in H and not with the scalar product in V.

We proceed in several steps in order to simplify problem (2.33) and to give the correct assumptions for its solvability.

Introduction of a suitable Hilbert triple. Divide the boundary operators in (2.33) into two classes. If $m_j < m$ we say that the boundary operator $B_j(x;D)$ is *stable* while if $m_j \geq m$ we say that it is *natural*. Assume that there are p stable boundary operators, with p being an integer between 0 and m. If $p = 0$ all the boundary operators are natural, whereas if $p = m$ all boundary operators are stable. In view of (2.30) the stable operators correspond to indices $j \leq p$. Then we define the space

$$V := \{v \in H^m(\Omega);\ B_j(x,D)v = 0 \text{ on } \partial\Omega \text{ for } j = 1,...,p\}. \tag{2.34}$$

Clearly, if $p = 0$ we have $V = H^m(\Omega)$ while if $p = m$ we have $V = H_0^m(\Omega)$ (provided the assumption (2.36) below holds). In particular, in the case of Dirichlet boundary conditions (2.20) we have

$$V = H_0^m(\Omega),$$

in the case of Navier boundary conditions (2.21) we have

$$V = H_\vartheta^m(\Omega) := \left\{v \in H^m(\Omega);\ \Delta^j v = 0 \text{ on } \partial\Omega \text{ for } j < \frac{m}{2}\right\}, \tag{2.35}$$

in the case of Steklov boundary conditions (2.22) we have

$$V = H^2 \cap H_0^1(\Omega) = H_\vartheta^2(\Omega).$$

In any case, the space V is well-defined since each B_j contains trace operators of maximal order $m_j < m$. Moreover, V is a closed subspace of $H^m(\Omega)$ which satisfies $H_0^m(\Omega) \subset V \subset H^m(\Omega)$ with continuous embedding. Therefore, V inherits the scalar product and the Hilbert space structure from $H^m(\Omega)$. If we put $H = L^2(\Omega)$, then $V \subset H \subset V'$ forms a Hilbert triple with compact embeddings.

Assumptions on the boundary operators. Assume that

$$\{B_j(x;D)\}_{j=1}^m \quad \text{forms a normal system} \tag{2.36}$$

and that the orders of the B_j's satisfy

$$m_i + m_j \neq 2m - 1 \quad \text{for all } i, j = 1,...,m. \tag{2.37}$$

This assumption is needed since we are not free to choose the orders of the B_j's. For every $k = 0,...,m-1$ there must be exactly one m_j in the set $\{k, 2m-k-1\}$.

Let p denote the number of stable boundary operators. In view of (2.30) we know that these operators are precisely $\{B_j\}_{j=1}^p$ and, of course, they also form a normal system of boundary operators. By Proposition 2.12, there exists a family of normal operators $\{S_j\}_{j=p+1}^m$ such that $\{B_1,...,B_p,S_{p+1},...,S_m\}$ forms a Dirichlet system of order m. We relabel this system and define

$$\{F_j\}_{j=1}^m \equiv \{B_1,...,B_p,S_{p+1},...,S_m\} \tag{2.38}$$

the re-ordered system in such a way that the order of F_j equals $j-1$. The indices $j=1,\dots,m$ are so divided into two subsets J_1 and J_2 according to the following rule: $j\in J_1$ if $F_j\in\{B_j\}_{j=1}^{p}$ whereas $j\in J_2$ if $F_j\in\{S_j\}_{j=p+1}^{m}$.

Let $\{\Phi_j\}_{j=1}^{m}$ denote the Green adjoint boundary operators of $\{F_j\}_{j=1}^{m}$ according to Theorem 2.13. We finally assume that the S_j's and the Φ_j's may be chosen in a such a way that

$$\{B_j\}_{j=p+1}^{m}\subset\{\Phi_j\}_{j=1}^{m}\,. \tag{2.39}$$

The condition in (2.39) is quite delicate since it requires the construction of the S_j's and the Φ_j's before being checked. Note that if $p=m$ (Dirichlet boundary conditions) or $p=0$, then (2.37) and (2.39) are automatically fulfilled.

Assumption on g. Assume that

$$g\in V'\,. \tag{2.40}$$

If $V=H_0^m(\Omega)$, then $V'=H^{-m}(\Omega)$ and V' has a fairly simple representation, see [416, Theorem 17.6]. If $V=H^m(\Omega)$, then elements of V' have a more difficult characterisation, see [416, Theorem 17.5]. In all the other cases, V' has even more complicated forms but we always have $[H^m(\Omega)]'\subset V'\subset H^{-m}(\Omega)$ with continuous embeddings.

Coercivity of the bilinear form. In order to ensure solvability of (2.42) we need a crucial assumption on the bilinear form Ψ. By (2.27) we know that there exists $c_1>0$ such that $\Psi(u,v)\le c_1\|u\|_{H^m(\Omega)}\|v\|_{H^m(\Omega)}$ for all $u,v\in H^m(\Omega)$. Assume that there exists $c_2\in(0,c_1)$ such that

$$\Psi(u,u)\ge c_2\|u\|^2_{H^m(\Omega)}\qquad\text{for all } u\in V. \tag{2.41}$$

In fact, (2.41) is nothing else but a strengthened ellipticity assumption for the operator A; it gives a quadratic lower bound behaviour for Ψ (in terms of the H^m norm) but only on the subspace V. One is then interested in finding sufficient conditions which ensure that (2.41) holds. The most general such condition is due to Agmon [3] and is quite technical to state; since it is beyond the scope of this book, we will not discuss it here. We just limit ourselves to verify (2.41) in some simple cases. If $Au=(-\Delta)^m u$ for some $m\ge 2$ then $\Psi(u,v)=(u,v)_{H_0^m}$ and (2.41) holds with $c_1=c_2=1$ and $V=H_0^m(\Omega)$; hence, Dirichlet boundary conditions (2.20) are allowed with $Au=(-\Delta)^m u$. If $Au=\Delta^2 u$ then $\Psi(u,v)=(u,v)_{H_0^2}$ and (2.41) holds again with $c_1=c_2=1$ but now for both the cases $V=H_0^2(\Omega)$ and $V=H^2\cap H_0^1(\Omega)$ so that Dirichlet (2.20) and Navier (2.21) boundary conditions are allowed, see also Theorem 2.31 below. As we shall see in Section 3.3.1 and in Theorem 5.22, if $Au=\Delta^2 u$ and $V=H^2\cap H_0^1(\Omega)$, also Steklov boundary conditions (2.22) are allowed but now with the bilinear form $\Psi(u,v)=(u,v)_{H_0^2}-\int_{\partial\Omega} a u_\nu v_\nu\, d\omega$ provided a satisfies suitable assumptions which ensure (2.17).

Finally, we say that $w\in V$ is a weak solution to (2.33) if

$$\Psi(w,\varphi)=\langle g,\varphi\rangle\qquad\text{for all } \varphi\in V. \tag{2.42}$$

Thanks to the Lax-Milgram theorem we may now state the existence and uniqueness result for weak solutions to the homogeneous problem (2.33).

Theorem 2.15. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying (2.32). Assume that:*

- *the operator A in (2.28) and the bilinear form Ψ in (2.29) satisfy (2.27) and (2.41);*
- *the operators B_j satisfy (2.30), (2.31), (2.36), (2.37), (2.39);*
- *g satisfies (2.40).*

Then problem (2.42) admits a unique weak solution $w \in V$; moreover, there exists a constant $C = C(\Omega, m, \mathscr{A}, B_j) > 0$ independent of g, such that

$$\|w\|_{H^m(\Omega)} \le C\|g\|_{V'}.$$

To conclude, let us highlight the existing connection between (2.42) and (2.33). It is clear that any solution $w \in H^{2m}(\Omega)$ to (2.33) is also a solution to (2.42). On the other hand, any $w \in V$ satisfying (2.42) automatically satisfies the stable boundary conditions since these are contained in the definition of V. We show that if g and w are smooth then w also satisfies the natural boundary conditions and solves (2.33). To see this, let $\{F_j\}$ be as in (2.38) and let $\{\Phi_j\}$ denote the normal system of boundary operators associated to $\{F_j\}$ through Theorem 2.13. Then if we assume that $g \in L^2(\Omega)$ and $w \in V \cap H^{2m}(\Omega)$, Theorem 2.13 combined with (2.42) gives

$$\int_\Omega Aw\,\varphi\,dx + \sum_{j=1}^m \int_{\partial\Omega} \Phi_j(x;D)w\,F_j(x;D)\varphi\,d\omega = \int_\Omega g\,\varphi\,dx \tag{2.43}$$

for all $\varphi \in V$. Taking arbitrary $\varphi \in C_c^\infty(\Omega)$ in (2.43) shows that $Aw = g$ a.e. in Ω so that the equation in (2.33) is satisfied (recall the definition of A in (2.28)). Once this is established, (2.43) yields

$$\sum_{j\in J_2} \int_{\partial\Omega} \Phi_j(x;D)w\,F_j(x;D)\varphi\,d\omega = \sum_{j=1}^m \int_{\partial\Omega} \Phi_j(x;D)w\,F_j(x;D)\varphi\,d\omega = 0 \tag{2.44}$$

for all $\varphi \in V$, where the first equality is a consequence of the fact that $\varphi \in V$, namely $F_j\varphi = 0$ on $\partial\Omega$ for all $j \in J_1$. Again by arbitrariness of $\varphi \in V$, (2.44) shows that

$$\Phi_j(x;D)w = 0 \quad \text{on } \partial\Omega \text{ for all } j \in J_2.$$

By assumptions (2.37) and (2.39) we know that $\Phi_j = B_{2m-m_j-1}$ for all $j \in J_2$, therefore the latter is equivalent to

$$B_j(x;D)w = 0 \quad \text{on } \partial\Omega \text{ for all } j = p+1,\dots,m$$

and w also satisfies the natural boundary conditions in (2.33).

2.4.3 Inhomogeneous boundary value problems

In this section we study weak solvability of (2.24)-(2.25) without assuming that the boundary data h_j vanish. After requiring suitable regularity on the data h_j, we explain what is meant by weak solution and we reduce the inhomogeneous problem to an homogeneous one.

Regularity assumptions on the data. Let V be as in (2.34) and assume that

$$f \in V' . \tag{2.45}$$

Weak solutions to (2.24)-(2.25) will be sought in a suitable convex subset of $H^m(\Omega)$. According to Theorem 8.3 in Chapter 1 in [275], it is then necessary to assume that

$$h_j \in H^{m-m_j-\frac{1}{2}}(\partial\Omega) \quad \text{for all } j = 1,...,m. \tag{2.46}$$

We have $m - m_j - \frac{1}{2} > 0$ for all $m_j < m$, namely for all $j = 1,...,p$ where p is the number of stable boundary operators. If $j = p+1,...,m$, we have $m - m_j - \frac{1}{2} < 0$ and we recall the definition in (2.7).

If we assume (2.31), (2.32), (2.36) and (2.46), we may apply [416, Theorem 14.1] to infer that

$$\text{there exists } v \in H^m(\Omega) \text{ such that } B_j(x;D)v = h_j \text{ on } \partial\Omega \tag{2.47}$$

for all $j = 1,...,p$. Then consider the set

$$K := \{w \in H^m(\Omega);\, w - v \in V\}\ ;$$

it is straightforward to verify that K is a closed convex nonempty subset of $H^m(\Omega)$. If $p = 0$, then no v needs to be determined by (2.47) and K becomes the whole space $V = H^m(\Omega)$. Let us define the (ordered) family of boundary operators $\{F_j\}_{j=1}^m$ as in (2.38) and let J_1 and J_2 denote the subsets defined there. We say that $u \in K$ is a weak solution to (2.24)-(2.25) if

$$\Psi(u,\varphi) = \langle f,\varphi\rangle + \sum_{j\in J_2} \langle h_{2m-m_j-1}, F_j(x;D)\varphi\rangle_{\partial\Omega,j} \ \text{ for all } \varphi \in V, \tag{2.48}$$

where Ψ is defined in (2.29), $\langle\,.\,,\,.\,\rangle$ denotes the duality between V' and V and $\langle\,.\,,\,.\,\rangle_{\partial\Omega,j}$ denotes the duality between $H^{m-m_j-\frac{1}{2}}(\partial\Omega)$ and $H^{-m+m_j+\frac{1}{2}}(\partial\Omega)$.

Reduction to an homogeneous boundary value problem. Let $v \in H^m(\Omega)$ be defined by (2.47) and let $u \in K$ be a weak solution to (2.24)-(2.25), according to (2.48). Subtract $\Psi(v,\varphi)$ from the equations in (2.48) to obtain

$$\Psi(u-v,\varphi) = \langle f,\varphi\rangle + \sum_{j\in J_2} \langle h_{2m-m_j-1}, F_j(x;D)\varphi\rangle_{\partial\Omega,j} - \Psi(v,\varphi)$$

for all $\varphi \in V$. By (2.45), the linear functional g defined by

$$g : \varphi \mapsto \langle f, \varphi\rangle + \sum_{j\in J_2} \langle h_{2m-m_j-1}, F_j(x;D)\varphi\rangle_{\partial\Omega,j} - \Psi(v,\varphi) \quad \text{for } \varphi \in V$$

is continuous on V so that (2.40) holds. Now put $w := u - v$; then $w \in V$ satisfies (2.42). Therefore, we shall proceed as follows. We first determine a function v as in (2.47), then we solve problem (2.33) (whose variational formulation is (2.42)) and find $w \in V$. Putting $u = v + w$ we obtain a solution $u \in K$ to (2.24)-(2.25) (whose variational formulation is (2.48)).

With these arguments, Theorem 2.15 immediately gives

Theorem 2.16. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying (2.32). Assume that:*

- *the operator A in (2.28) and the bilinear form Ψ in (2.29) satisfy (2.27) and (2.41);*
- *the operators B_j satisfy (2.30), (2.31), (2.36), (2.37), (2.39);*
- *f satisfies (2.45) and the h_j's satisfy (2.46).*

Then problem (2.48) admits a unique weak solution $u \in K$; moreover, there exists a constant $C = C(\Omega, m, \mathscr{A}, B_j) > 0$ independent of f and of the h_j's, such that

$$\|u\|_{H^m(\Omega)} \le C \left(\|f\|_{V'} + \sum_{j=1}^{p} \|h_j\|_{H^{m-m_j-\frac{1}{2}}(\partial\Omega)} \right).$$

As for the homogeneous problem, let us explain the link between weak and strong solutions. Again, any strong solution $u \in H^{2m}(\Omega)$ to (2.24)-(2.25) certainly satisfies (2.48); note that a strong solution may exist only if

$$h_j \in H^{2m-m_j-\frac{1}{2}}(\partial\Omega) \text{ for } j = 1,\dots,m \text{ and } f \in L^2(\Omega). \tag{2.49}$$

Conversely, assume that (2.49) holds and let $u \in K \cap H^{2m}(\Omega)$ be a solution to (2.48). Let $\{F_j\}$ be as in (2.38) and let $\{\Phi_j\}$ denote the normal system of boundary operators associated to $\{F_j\}$ through Theorem 2.13. Then (2.48) gives

$$\begin{aligned} &\int_\Omega Au\,\varphi\,dx + \sum_{j=1}^{m} \int_{\partial\Omega} \Phi_j(x;D)u\,F_j(x;D)\varphi\,d\omega \\ &= \int_\Omega f\,\varphi\,dx + \sum_{j\in J_2} \int_{\partial\Omega} h_{2m-m_j-1} F_j(x;D)\varphi\,d\omega \quad \text{for all } \varphi \in V. \end{aligned} \tag{2.50}$$

Taking arbitrary $\varphi \in C_c^\infty(\Omega)$ in (2.50) shows that $Au = f$ a.e. in Ω so that (2.24) is satisfied. Once this is established, (2.50) yields

$$\sum_{j\in J_2} \int_{\partial\Omega} \Phi_j(x;D)u\,F_j(x;D)\varphi\,d\omega = \sum_{j\in J_2} \int_{\partial\Omega} h_{2m-m_j-1}\,F_j(x;D)\varphi\,d\omega$$

for all $\varphi \in V$. Then the same arguments used after (2.44) show that

$$B_j(x;D)u = h_j \quad \text{on } \partial\Omega \text{ for all } j = p+1,\dots,m,$$

which proves that u also satisfies the natural boundary conditions in (2.25).

Remark 2.17. Although (2.17) and the complementing condition (see Definition 2.9) do not explicitly appear in Theorem 2.16, they are hidden in the assumptions. The coercivity assumption (2.41) ensures that (2.17) is satisfied, see Theorem 2.15. On the other hand, assumptions (2.36) and (2.39) ensure that the complementing condition holds, see [275, Section 2.4].

If the boundary $\partial\Omega$ and the data f and h_j are more regular, elliptic theory applies and also the solution u given in Theorem 2.16 is more regular, see the next section.

2.5 Regularity results and a priori estimates

2.5.1 Schauder theory

Here we consider classical solutions to (2.2)-(2.14). To do so, we need the Schauder theory and a good knowledge of Hölder continuity.

First fix an integer ℓ such that $\max\{m_j\} \leq \ell \leq 2m$. Then slightly modify the problem and consider the equation

$$(-\Delta)^m u + \sum_{*} D^\beta \Big[a_{\beta,\mu}(x) D^\mu u \Big] = \sum_{|\beta| \leq 2m-\ell} D^\beta f_\beta \qquad \text{in } \Omega, \tag{2.51}$$

complemented with the boundary conditions

$$\sum_{|\alpha| \leq m_j} b_{j,\alpha}(x) D^\alpha u = h_j \quad \text{on } \partial\Omega \text{ with } j = 1, \dots, m, \tag{2.52}$$

where $\sum_{*}$ means summation over all multi-indices β and μ such that

$$|\beta| \leq 2m - \ell\,, \qquad |\mu| \leq \ell\,, \qquad |\beta| + |\mu| \leq 2m - 1\,. \tag{2.53}$$

With the notations of (2.2) and (2.14), we have now

$$\mathscr{A}(x;D)u = \sum_{*} D^\beta \Big[a_{\beta,\mu}(x) D^\mu u \Big]\,, \qquad B_j(x;D)u = \sum_{|\alpha| \leq m_j} b_{j,\alpha}(x) D^\alpha u\,. \tag{2.54}$$

Fix a second integer $k \geq \ell$ and put $\overline{\ell} = \max\{2m, k\}$. Then assume that for some $0 < \gamma < 1$ we have

$$\begin{cases} a_{\beta,\mu} \in C^{k-\ell,\gamma}(\overline{\Omega}) & \text{for all } \beta,\mu \text{ satisfying (2.53)}, \\ f_\beta \in C^{k-\ell,\gamma}(\overline{\Omega}) & \text{for all } |\beta| \le 2m-\ell, \\ b_{j,\alpha} \in C^{k-m_j,\gamma}(\partial\Omega) & \text{for all } j=1,\dots,m \text{ and } |\alpha| \le m_j, \\ h_j \in C^{k-m_j,\gamma}(\partial\Omega) & \text{for all } j=1,\dots,m. \end{cases} \tag{2.55}$$

Note that under assumptions (2.55), problem (2.51)-(2.52) needs not to make sense in a classical way. Therefore, we first need to introduce a different kind of solution.

Definition 2.18. We say that $u \in C^{k,\gamma}(\overline{\Omega})$ is a *mild solution* to (2.51)-(2.52) if

$$\begin{aligned} \int_\Omega u(-\Delta)^m \varphi\, dx + \sum_* (-1)^{|\beta|} \int_\Omega a_{\beta,\mu}(x) D^\mu u D^\beta \varphi\, dx \\ = \sum_{|\beta| \le 2m-\ell} (-1)^{|\beta|} \int_\Omega f_\beta(x) D^\beta \varphi\, dx \end{aligned}$$

for all $\varphi \in C_c^\infty(\Omega)$ and if u satisfies pointwise the boundary conditions in (2.52).

Hence, for any mild solution the boundary conditions (2.52) are well-defined since $k \ge \ell \ge m_j$ for all j.

We are now ready to state

Theorem 2.19. *Let $k \ge \ell \in [\max\{m_j\}, 2m] \cap \mathbb{N}$ and $\overline{\ell} = \max\{2m,k\}$. Assume that (2.55) holds and that $\mathscr{A}$ and the B_j's satisfy (2.54)-(2.19). Assume (2.15) and (2.17). Assume moreover that $\partial\Omega \in C^{\overline{\ell},\gamma}$. Then (2.51)-(2.52) admits a unique mild solution $u \in C^{k,\gamma}(\overline{\Omega})$. Moreover, there exists a constant $C = C(\Omega,k,m,a_{\beta,\mu},b_{j,\alpha}) > 0$ independent of the f_β's and of the h_j's, such that the following a priori estimate holds*

$$\|u\|_{C^{k,\gamma}(\overline{\Omega})} \le C \left(\sum_{|\beta| \le 2m-\ell} \|f_\beta\|_{C^{k-\ell,\gamma}(\overline{\Omega})} + \sum_{j=1}^m \|h_j\|_{C^{k-m_j,\gamma}(\partial\Omega)} \right).$$

The constant C depends on Ω only through its measure $|\Omega|$ and the $C^{\overline{\ell},\gamma}$-norms of the local maps which define the boundary $\partial\Omega$. If $k \ge 2m$ then the solution u is classical.

Finally, if (2.17) is dropped, then for any solution u to (2.51)-(2.52) one has the following local variant of the estimate

$$\begin{aligned} \|u\|_{C^{k,\gamma}(\overline{\Omega}\cap B_R(x_0))} \le C \Bigg(\sum_{|\beta| \le 2m-\ell} \|f_\beta\|_{C^{k-\ell,\gamma}(\overline{\Omega}\cap B_{2R}(x_0))} \\ + \sum_{j=1}^m \|h_j\|_{C^{k-m_j,\gamma}(\partial\Omega\cap B_{2R}(x_0))} + \|u\|_{L^1(\Omega\cap B_{2R}(x_0))} \Bigg) \end{aligned}$$

for any $R > 0$ and any $x_0 \in \Omega$. Here, C also depends on R.

Roughly speaking, equation (2.51) says that $2m$ derivatives of the solution u belong to $C^{k-2m,\gamma}(\overline{\Omega})$; if $k \geq 2m$ this has an obvious meaning while if $k < 2m$ this should be intended in a generalised sense. In any case, Theorem 2.19 states that the solution gains $2m$ derivatives on the datum $\sum_\beta D^\beta f_\beta$.

2.5.2 L^p-theory

In this section we give an existence result for (2.2)-(2.14) in the framework of L^p spaces. Under suitable assumptions on the parameters involved in the problem, we show that the solution has at least $2m$ derivatives in $L^p(\Omega)$. In this case, the equation (2.2) is satisfied a.e. in Ω and we say that u is a *strong solution*.

The following statement should also be seen as a regularity complement to Theorem 2.16.

Theorem 2.20. *Let $1 < p < \infty$ and take an integer $k \geq 2m$. Assume that $\partial\Omega \in C^k$ and that*

$$\begin{cases} a_\beta \in C^{k-2m}(\overline{\Omega}) & \text{for all } |\beta| \leq 2m-1, \\ b_{j,\alpha} \in C^{k-m_j}(\partial\Omega) & \text{for all } j = 1,\dots,m,\ |\alpha| \leq m_j. \end{cases} \tag{2.56}$$

Assume also that (2.15), (2.17) hold and that $\mathscr{A}$ and the B_j's satisfy (2.18)-(2.19). Then for all $f \in W^{k-2m,p}(\Omega)$ and all $h_j \in W^{k-m_j-\frac{1}{p},p}(\partial\Omega)$ with $j = 1,\dots,m$, the problem (2.2)-(2.25) admits a unique strong solution $u \in W^{k,p}(\Omega)$. Moreover, there exists a constant $C = C(\Omega,k,m,\mathscr{A},B_j) > 0$ independent of f and of the h_j's, such that the following a priori estimate holds

$$\|u\|_{W^{k,p}(\Omega)} \leq C \left(\|f\|_{W^{k-2m,p}(\Omega)} + \sum_{j=1}^{m} \|h_j\|_{W^{k-m_j-\frac{1}{p},p}(\partial\Omega)} \right) .$$

The constant C depends on Ω only through its measure $|\Omega|$ and the C^k-norms of the local maps which define the boundary $\partial\Omega$. If $k > 2m + \frac{n}{p}$ then u is a classical solution.

Finally, if (2.17) is dropped, then for any solution u to (2.2)-(2.25) one has the following local variant of the estimate

$$\begin{aligned} \|u\|_{W^{k,p}(\Omega\cap B_R(x_0))} \leq C \Big(& \|f\|_{W^{k-2m,p}(\Omega\cap B_{2R}(x_0))} \\ & + \sum_{j=1}^{m} \|h_j\|_{W^{k-m_j-\frac{1}{p},p}(\partial\Omega\cap B_{2R}(x_0))} + \|u\|_{L^1(\Omega\cap B_{2R}(x_0))} \Big) \end{aligned}$$

for any $R > 0$ and any $x_0 \in \Omega$. Here, C also depends on R.

The proof of this general result is quite involved, especially if $p \neq 2$. It requires the representation of the solution u in terms of the fundamental solution and the Calderon-Zygmund theory [83] on singular integrals in L^p.

In the case of Dirichlet boundary conditions Theorem 2.20 reads

Corollary 2.21. *Let $1 < p < \infty$ and take an integer $k \geq 2m$. Assume that $\partial\Omega \in C^k$ and that (2.56) holds. Assume moreover that (2.17) holds and that $\mathscr{A}$ satisfies (2.18). Then for all $f \in W^{k-2m,p}(\Omega)$ equation (2.2) admits a unique strong solution $u \in W^{k,p} \cap W_0^{m,p}(\Omega)$; moreover, there exists a constant $C = C(\Omega, k, m, \mathscr{A}) > 0$ independent of f, such that*

$$\|u\|_{W^{k,p}(\Omega)} \leq C\|f\|_{W^{k-2m,p}(\Omega)}.$$

For equations in variational form such as (2.51), L^p-estimates are available under weaker regularity assumptions. For our purposes we just consider the following special situation.

Theorem 2.22. *For $\partial\Omega \in C^2$, $p \in (1,\infty)$, and $f \in L^p(\Omega)$ there exists a unique solution $u \in W_0^{2,p}(\Omega)$ of*

$$\begin{cases} \Delta^2 u = \nabla^2 f & \text{in } \Omega, \\ u = u_\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where ∇^2 means any second derivative. Moreover, the following L^p-estimate holds

$$\|u\|_{W^{2,p}(\Omega)} \leq C\,\|f\|_{L^p(\Omega)}$$

with $C = C(p,\Omega) > 0$.

For Steklov boundary conditions (2.22) associated to the biharmonic operator, Theorem 2.20 reads as follows.

Corollary 2.23. *Let $1 < p < \infty$ and take an integer $k \geq 4$. Assume that $\partial\Omega \in C^k$ and $a \in C^{k-2}(\partial\Omega)$, then there exists $C = C(k, p, \alpha, \Omega) > 0$ such that*

$$\|u\|_{W^{k,p}(\Omega)} \leq$$

$$\leq C\left(\|u\|_{L^p(\Omega)} + \left\|\Delta^2 u\right\|_{W^{k-4,p}(\Omega)} + \|u\|_{W^{k-\frac{1}{p},p}(\partial\Omega)} + \|\Delta u - a u_\nu\|_{W^{k-2-\frac{1}{p},p}(\partial\Omega)}\right)$$

for all $u \in W^{k,p}(\Omega)$. The same statement holds for any $k \geq 2$, provided the norms in the right hand side are suitably interpreted, see (2.5), (2.6), and (2.9).

Remark 2.24. In the estimates of Theorems 2.19 and 2.20 and of Corollaries 2.21 and 2.23, the constants depend in an indirect and nonconstructive way on the particular differential and boundary operators. As soon as one puts (for instance) the L^1-norm of the solution on the right hand side, the constants become explicit and depend only on bounds for the data (k,m, domain, and coefficients) of the problem. This kind of uniformity will be needed in the proof of positivity for Green's functions in perturbed domains, see Section 6.5. There we have uniformly coercive problems which yield an explicit estimate for some lower order norms, so that L^p or Schauder estimates depending on the specific operator would be useless.

2.5.3 The Miranda-Agmon maximum modulus estimates

We start by recalling that it is in general false that $\Delta u \in C^0$ implies $u \in C^2$ even if u satisfies homogeneous Dirichlet boundary conditions. Therefore, this lack of regularity is a local problem, irrespective of how smooth the boundary data are. To see why the implication fails, consider the function

$$u(x_1,x_2) = \begin{cases} x_1x_2 \log|\log(x_1^2+x_2^2)| & \text{if } (x_1,x_2) \neq (0,0), \\ 0 & \text{if } (x_1,x_2) = (0,0), \end{cases}$$

which is well-defined for $|x| < 1$. Some computations show that u solves the problem

$$\begin{cases} -\Delta u = f & \text{in } B_r(0), \\ u = 0 & \text{on } \partial B_r(0), \end{cases}$$

where $r = 1/\sqrt{e}$ and

$$f(x_1,x_2) = \begin{cases} \frac{4x_1x_2(1-2\log(x_1^2+x_2^2))}{(x_1^2+x_2^2)\log^2(x_1^2+x_2^2)} & \text{if } (x_1,x_2) \neq (0,0), \\ 0 & \text{if } (x_1,x_2) = (0,0),. \end{cases}$$

One can check that $f \in C^0(\overline{B_r(0)})$. On the other hand, for $(x_1,x_2) \neq (0,0)$ we have

$$\begin{aligned} &u_{x_1x_2}(x_1,x_2) \\ &= \log|\log(x_1^2+x_2^2)| + \frac{2(x_1^4+x_2^4)}{(x_1^2+x_2^2)^2\log(x_1^2+x_2^2)} - \frac{4x_1^2x_2^2}{(x_1^2+x_2^2)^2\log^2(x_1^2+x_2^2)} \end{aligned}$$

which is unbounded for $(x_1,x_2) \to (0,0)$. Therefore, $u \notin C^2(B_r(0))$.

This example shows that a version of Theorem 2.19 in the framework of spaces C^k of continuously differentiable functions is not available. On the other hand, the well-known Poisson integral formula shows that for continuous Dirichlet boundary data any harmonic function in the ball B is of class $C^0(\overline{B})$, see [197, Theorem 2.6]. In other words, the solution inherits continuity properties from its trace. We state below the corresponding result for polyharmonic equations in a particular situation which is, however, general enough for our purposes. We consider boundary conditions (2.14) with *constant coefficients* and the problem

$$\begin{cases} (-\Delta)^m u + \mathscr{A}(x;D)u = f & \text{in } \Omega \\ B_j(D)u = \sum_{|\alpha|\le m_j} b_{j,\alpha} D^\alpha u = h_j & \text{on } \partial\Omega \text{ with } j = 1,\dots,m \end{cases} \tag{2.57}$$

for some constants $b_{j,\alpha} \in \mathbb{R} \setminus \{0\}$. Then we have the following a priori estimates for the maximum modulus of solutions and some of their derivatives.

Theorem 2.25. *Assume (2.15), (2.17) and that $\mathscr{A}$ and the B_j's satisfy (2.18)-(2.19). Assume also that $\partial\Omega \in C^{2m}$ and let $\mu = \max_j m_j$. Finally, assume that $f \in C^0(\overline{\Omega})$*

and that $h_j \in C^{\mu-m_j}(\partial\Omega)$ for any $j = 1,\dots,m$. Then (2.57) admits a unique strong solution $u \in C^{\mu}(\overline{\Omega}) \cap W^{2m,p}_{loc}(\Omega)$ for any $p \in (1,\infty)$. Moreover, there exists $C > 0$ independent of f, h_j such that

$$\max_{0\le k\le\mu} \|D^k u\|_{L^\infty} \le C \left(\sum_{j=1}^{m} \|h_j\|_{C^{\mu-m_j}(\partial\Omega)} + \|f\|_{L^\infty} + \|u\|_{L^1} \right).$$

Proof. We split problem (2.57) into the two subproblems

$$\begin{cases} (-\Delta)^m v + \mathscr{A}(x;D)v = f & \text{in } \Omega, \\ B_j(D)v = 0 & \text{on } \partial\Omega,\ j = 1,\dots,m, \end{cases} \tag{2.58}$$

$$\begin{cases} (-\Delta)^m w + \mathscr{A}(x;D)w = 0 & \text{in } \Omega, \\ B_j(D)w = h_j & \text{on } \partial\Omega,\ j = 1,\dots,m. \end{cases} \tag{2.59}$$

Since $f \in C^0(\overline{\Omega}) \subset L^p(\Omega)$ for any $p \ge 1$, by Theorem 2.20 (with $k = 2m$) we know that there exists a unique solution $v \in W^{2m,p}(\Omega)$ to (2.58). By Theorem 2.6 we infer that $v \in C^{2m-1,\gamma}(\overline{\Omega})$ for all $\gamma \in (0,1)$. Moreover, there exist constants $c_1, c_2 > 0$ such that

$$\|v\|_{C^{2m-1,\gamma}(\overline{\Omega})} \le c_1 \|f\|_{L^\infty} + c_2 \|v\|_{L^1} ,$$

see again Theorem 2.20.

On the other hand, by generalising the Miranda-Agmon procedure [4, 304, 305] one shows that (2.59) admits a unique solution $w \in C^{\mu}(\overline{\Omega})$ satisfying

$$\max_{0\le k\le\mu} \|D^k w\|_{L^\infty} \le c_3 \sum_{j=1}^{m} \|h_j\|_{C^{m_j}(\partial\Omega)} + c_4 \|w\|_{L^1} \tag{2.60}$$

for some $c_3, c_4 > 0$. This procedure consists in constructing a suitable approximate solution w_0 to (2.59). To this end one uses the explicit Poisson kernels which solve a related boundary value problem in the half space. These Poisson kernels are determined in [5] and, since (2.19) holds, it makes no difference to consider the Dirichlet problem as in [4, 304, 305] or the general boundary value problem in (2.59). Once this approximate solution w_0 is constructed, one shows that it satisfies (2.60) with $c_4 = 0$. Then one uses again L^p elliptic estimates from Theorem 2.20 and embedding arguments in order to show that the solution w to (2.59) satisfies (2.60).

Once the solutions v and w to (2.58) and (2.59) are obtained, the solution u of (2.57) is determined by adding, $u = v + w$, so that also the estimate of the C^{μ}-norm follows. $\square$

2.6 Green's function and Boggio's formula

The regularity results of the previous sections are somehow directly visible when writing explicitly the solution of the boundary value problem in terms of the data by means of a suitable kernel. Let us focus on the polyharmonic analogue of the clamped plate boundary value problem

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ D^\alpha u|_{\partial\Omega} = 0 & \text{for } |\alpha| \le m-1. \end{cases} \tag{2.61}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, f a datum in a suitable functional space and u denotes the unknown solution.

In order to give an explicit formula for solving (2.61), the first step is to define the fundamental solution of the polyharmonic operator $(-\Delta)^m$ in $\mathbb{R}^n$. We put

$$F_{m,n}(x) = \begin{cases} \frac{2\Gamma(n/2-m)}{ne_n 4^m \Gamma(n/2)(m-1)!} |x|^{2m-n} & \text{if } n > 2m \text{ or } n \text{ is odd}, \\ \frac{(-1)^{m-n/2}}{ne_n 4^{m-1}\Gamma(n/2)(m-n/2)!(m-1)!} |x|^{2m-n} (-\log|x|) & \text{if } n \le 2m \text{ is even}, \end{cases}$$

so that, in distributional sense

$$(-\Delta)^m F_{m,n} = \delta_0, \tag{2.62}$$

where δ_0 is the Dirac mass at the origin. Of course, one may add any m-polyharmonic function to $F_{m,n}$ and still find (2.62). For $n > 2m$ there is a unique fundamental solution when one adds the "boundary condition"

$$\lim_{|x|\to\infty} F_{m,n}(x) = 0. \tag{2.63}$$

For $n \le 2m$ no fundamental solution satisfies (2.63) and there does not seem to be a natural restriction to fix a unique fundamental solution.

Thanks to the fundamental solution, we may introduce the notion of Green function.

Definition 2.26. A Green function for (2.61) is a function $(x,y) \mapsto G(x,y) : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R} \cup \{\infty\}$ satisfying:

1. $x \mapsto G(x,y) - F_{m,n}(x-y) \in C^{2m}(\Omega) \cap C^{m-1}(\overline{\Omega})$ for all $y \in \Omega$ if defined suitably for $x = y$;
2. $(-\Delta_x)^m (G(x,y) - F_{m,n}(x-y)) = 0$ for all $(x,y) \in \Omega^2$ if defined suitably for $x = y$;
3. $D_x^\alpha G(x,y) = 0$ for all $(x,y) \in \partial\Omega \times \Omega$ and $|\alpha| \le m-1$.

Formally, the Green function enables one to write the unique solution to (2.61) as

$$u(x) = \int_\Omega G(x,y) f(y) dy. \tag{2.64}$$

Provided f belongs to a suitable functional space, this formula makes sense and gives the solution u.

Clearly, the exact form of the Green function G is not easily determined. However, as we already mentioned in Section 1.2, Boggio [63, p. 126] could explicitly calculate the Green function $G_{m,n} := G_{(-\Delta)^m,B}$ for problem (2.61) when Ω is the unit ball in $\mathbb{R}^n$.

Lemma 2.27. *The Green function for the Dirichlet problem (2.61) with $\Omega = B$ is positive and given by*

$$G_{m,n}(x,y) = k_{m,n}\,|x-y|^{2m-n} \int_1^{\left||x|y-\frac{x}{|x|}\right| / |x-y|} (v^2-1)^{m-1} v^{1-n}\,dv. \tag{2.65}$$

The positive constants $k_{m,n}$ are defined by

$$k_{m,n} = \frac{1}{ne_n 4^{m-1}\left((m-1)!\right)^2}, \qquad e_n = \frac{\pi^{n/2}}{\Gamma\left(1+n/2\right)}.$$

Remark 2.28. If $n > 2m$, then by applying the Cayley transform one finds for the half space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 > 0\}$

$$G_{(-\Delta)^m,\mathbb{R}^n_+}(x,y) = k_{m,n}\,|x-y|^{2m-n} \int_1^{|x^*-y|/|x-y|} (v^2-1)^{m-1} v^{1-n}\,dv, \tag{2.66}$$

where $x,y \in \mathbb{R}^n_+$, $\quad x^* = (-x_1, x_2, \ldots, x_n)$. We also emphasise that the assumption $n > 2m$ is required in this half space $\mathbb{R}^n_+$ in order to have uniqueness of the corresponding Green function. When $n \le 2m$ one may achieve uniqueness in some cases by adding restrictions such as upper bounds for its growth at infinity (see Remark 6.28 for the case $m = 2$ and $n = 3,4$) . Alternatively, one may just *impose* that the Green function in the half space is the Cayley transform of its counterpart in the ball and hence given by (2.66).

2.7 The space $H^2 \cap H_0^1$ and the Sapondžyan-Babuška paradoxes

In this section, we consider in some detail the space $H^2 \cap H_0^1$ which is in some sense “intermediate” between H^2 and H_0^2. This space is also related to both the homogeneous Navier (2.21) and Steklov (2.22) boundary conditions, see the discussion following (2.34). The norm to be used in this space strongly depends on the smoothness of $\partial\Omega$. It was assumed in Theorem 2.20 that $\partial\Omega \in C^4$. We first show that this assumption may be relaxed in some cases. On the other hand, if it is “too relaxed” then uniqueness, regularity or continuous dependence may fail, leading to some apparent paradoxes. We also point out that the regularity of the boundary plays an important role in the definition of the first Steklov eigenvalue, see Section 3.3.2.

Let us first remark that in the case $m = 2$, Theorem 2.2 reads

Corollary 2.29. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. On the space $H_0^2(\Omega)$, the bilinear form*

$$(u,v) \mapsto (u,v)_{H_0^2} := \int_\Omega \Delta u \Delta v \, dx \qquad \text{for all } u,v \in H_0^2(\Omega)$$

defines a scalar product over $H_0^2(\Omega)$ which induces a norm equal to $\|D^2 \,.\,\|_{L^2}$ and equivalent to $(\|D^2 \,.\,\|_{L^2}^2 + \|\,.\,\|_{L^2}^2)^{1/2}$.

We now show the less obvious result that the very same scalar product may also be used in the larger space $\ker \gamma_0 = H^2 \cap H_0^1(\Omega)$ when $\partial\Omega$ is not too bad. For later use, we state this result in general (possibly nonsmooth) domains. The class of domains considered is explained in the following definition taken from [2].

Definition 2.30. We say that a bounded domain $\Omega \subset \mathbb{R}^n$ satisfies an *outer ball condition* if for each $y \in \partial\Omega$ there exists a ball $B \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B$. We say that it satisfies a *uniform outer ball condition* if the radius of the ball B can be taken independently of $y \in \partial\Omega$.

In particular, a convex domain is a Lipschitz domain which satisfies a uniform outer ball condition. We have

Theorem 2.31. *Assume that $\Omega \subset \mathbb{R}^n$ is a Lipschitz bounded domain which satisfies a uniform outer ball condition. Then the space $H^2 \cap H_0^1(\Omega)$ becomes a Hilbert space when endowed with the scalar product*

$$(u,v) \mapsto \int_\Omega \Delta u \Delta v \, dx \qquad \text{for all } u,v \in H^2 \cap H_0^1(\Omega).$$

This scalar product induces a norm equivalent to $\|\,.\,\|_{H^2}$.

Proof. Under the assumptions of the theorem Adolfsson [2] proved that there exists a constant $C > 0$ independent of u, such that

$$\|u\|_{H^2} \le C \|\Delta u\|_{L^2} \qquad \text{for all } u \in H^2 \cap H_0^1(\Omega).$$

For all $u \in H^2 \cap H_0^1(\Omega)$ we also have

$$|D^2 u|^2 = \sum_{i,j=1}^n (\partial_{ij} u)^2 \ge \sum_{i=1}^n (\partial_{ii} u)^2 \ge \frac{1}{n} |\Delta u|^2 \qquad \text{a.e. in } \Omega. \tag{2.67}$$

This shows that the two norms are equivalent. □

Remark 2.32. Let $\Omega \subset \mathbb{R}^n$ be a bounded and convex domain with smooth boundary. Consider the set

$$V := \left\{ u \in C^2(\overline{\Omega});\, u \ge 0,\ \frac{\partial u}{\partial \nu} \ge 0,\ u \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}.$$

Let W denote the closure of V with respect to the norm $\|\,.\,\|_{H^2}$. Then we have

$$\int_\Omega |D^2u|^2\,dx \le \int_\Omega |\Delta u|^2\,dx \qquad \text{for all } u \in W,$$

see [198, Theorem 2.1]. This inequality is somehow the converse of (2.67).

The assumptions on $\partial\Omega$ under which Theorem 2.31 holds are related to the so-called Sapondžyan or concave corner paradox. This paradox relies on the fact that, for some nonsmooth domains Ω, the linear Navier problem may have several different solutions, according to the functional space where they are sought.

One way of getting the existence is through a "system solution" which belongs to $H_0^1(\Omega)$ as well as its Laplacian. A second type of solution is obtained using Kondratiev's techniques in the space $H^2 \cap H_0^1(\Omega)$. Note that, since $\|.\|$ defined by $\|u\| := \|\Delta u\|_{L^2}$ is not a norm on $H^2 \cap H_0^1(\Omega)$ when the domain has a reentrant corner, one cannot directly apply the Lax-Milgram theorem. Indeed, Theorem 2.31 may not hold if a uniform outer ball condition fails. The following example appears suitable to illustrate this dichotomy in some detail.

Example 2.33. For $\alpha \in \left(\frac{1}{2}\pi, \pi\right)$ fix the domain

$$\Omega_\alpha = \left\{(r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2; 0 < r < 1 \text{ and } |\varphi| < \alpha\right\}.$$

Let $f \in L^2(\Omega_\alpha)$ and consider the homogeneous Navier problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega_\alpha,\\ u = 0 & \text{on } \partial\Omega_\alpha,\\ \Delta u = 0 & \text{on } \partial\Omega_\alpha \setminus \{0\}. \end{cases} \tag{2.68}$$

We say that u is a *system solution* to (2.68) if $u, \Delta u \in H_0^1(\Omega_\alpha)$ and

$$\begin{cases} -\Delta u = w \text{ and } -\Delta w = f & \text{in } \Omega_\alpha,\\ \quad u = 0 \quad \text{and} \quad w = 0 & \text{on } \partial\Omega_\alpha. \end{cases} \tag{2.69}$$

By applying twice the Lax-Milgram theorem in $H_0^1(\Omega_\alpha)$, this system solution, as a solution to an iterated Dirichlet Laplace problem on a bounded domain, exists for any $f \in L^2(\Omega_\alpha)$. Using [251] one finds that there also exists a solution in $H^2 \cap H_0^1(\Omega)$ of (2.68), which indeed satisfies $\Delta u = 0$ pointwise on $\partial\Omega \setminus \{0\}$. Since its second derivatives are square summable, let us call this the *energy solution*.

Next we consider a special function. For $\rho = \frac{\pi}{2\alpha}$ the function v_α defined by

$$v_\alpha(r,\varphi) = \left(r^{-\rho} - r^{\rho}\right)\cos(\rho\varphi)$$

satisfies

$$\begin{cases} -\Delta v_\alpha = 0 & \text{in } \Omega_\alpha,\\ \quad v_\alpha = 0 & \text{on } \partial\Omega_\alpha \setminus \{0\}. \end{cases}$$

Moreover, one directly checks that $v_\alpha \in L^2(\Omega_\alpha)$ for $\rho \in \left(\frac{1}{2}, 1\right)$. Then there exists a unique solution $b_\alpha \in H_0^1(\Omega_\alpha)$ of

$$\begin{cases} -\Delta b_\alpha = v_\alpha & \text{in } \Omega_\alpha, \\ \quad b_\alpha = 0 & \text{on } \partial\Omega_\alpha. \end{cases}$$

One has $\Delta b_\alpha \notin H_0^1(\Omega_\alpha)$ and may check that $b_\alpha \notin H^2(\Omega_\alpha)$. So we have found a nontrivial solution to (2.68) with $f = 0$. This b_α is neither a system solution nor an energy solution. Let u be the system solution. Then the following holds:

1. For all $c \in \mathbb{R}$ we have $u_c := u + cb_\alpha \in H_0^1(\Omega_\alpha)$ and $\Delta u_c \in L^2(\Omega_\alpha)$.
2. For all $c \in \mathbb{R}$, the function u_c satisfies (2.68). Using results in [318] one may show that in fact $u_c \in C^0(\overline{\Omega_\alpha})$ and $\Delta u_c \in C^0_{loc}(\overline{\Omega_\alpha} \setminus \{0\})$ whenever $f \in L^2(\Omega_\alpha)$.
3. One finds $\Delta u_c \in H_0^1(\Omega_\alpha)$ if and only if $c = 0$.
4. For $f \in L^2(\Omega_\alpha)$ let
$$c_\alpha(f) := -\|v_\alpha\|_{L^2}^{-2} \int_{\Omega_\alpha} v_\alpha \mathscr{G}_{-\Delta,\Omega_\alpha} f \, dx.$$
We have $u_c \in H^2 \cap H_0^1(\Omega_\alpha)$ if and only if $c = c_\alpha(f)$.
5. The energy solution to (2.68) is u_c with $c = c_\alpha(f)$. Hence the system solution is different from the energy solution whenever $c_\alpha(f) \neq 0$.

Now let f be positive. A close inspection shows, see [320], that for the system solution the H^2-regularity fails when $c_\alpha(f) \neq 0$ while positivity holds true. On the other hand, the energy solution u_c with $c = c_\alpha(f)$ has the appropriate regularity, but positivity fails when $\alpha > \frac{3}{4}\pi$ and

$$\int_{\Omega_\alpha} \left(r^{-\frac{\pi}{\alpha}} - r^{\frac{\pi}{\alpha}} \right) \sin\left(\tfrac{\pi}{\alpha}\varphi\right) \mathscr{G}_{-\Delta,\Omega_\alpha} f \, dx \neq 0.$$

For $\alpha \in (\frac{1}{2}\pi, \frac{3}{4}\pi)$ there is only numerical evidence of sign-changing energy solutions. See Figure 2.1.

We now discuss in detail another famous paradox due to Babuška, also known as the polygon-circle paradox. The starting point is a planar hinged plate Ω with a load $f \in L^2(\Omega)$. This gives rise to the Steklov problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\,\kappa \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.70}$$

where κ denotes the curvature of the boundary and σ is the Poisson ratio. Problem (2.70) is considered both in the unit disk B and in the sequence (see Figure 1.3) of inscribed regular polygons $(P_m) \subset B$ ($m \geq 3$) with corners

$$\left\{ \left(\cos\left(\frac{2k}{m}\pi\right), \sin\left(\frac{2k}{m}\pi\right) \right); k = 1, \ldots, m \right\}.$$

Since the sides of P_m are flat, the curvature vanishes there and (2.70) becomes

$$\begin{cases} \Delta^2 u = f & \text{in } P_m, \\ u = \Delta u = 0 & \text{on } \partial P_m. \end{cases} \tag{2.71}$$

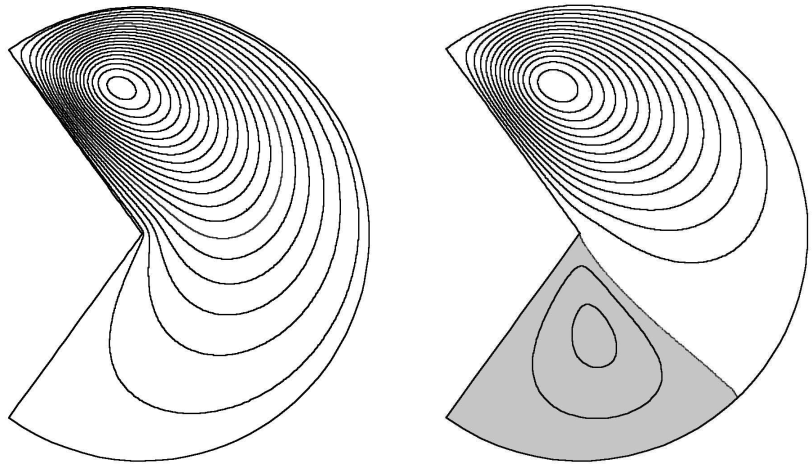

Fig. 2.1 The level lines of u and $u_c = u + cb_\alpha$ with $c = c_\alpha(f)$ for $f \geq 0$ having a small support near the left top of the domain. Grey region $= \{x : u_c(x) < 0\}$; here, a different scale is used for the level lines.

The so-called Babuška paradox shows that this argument is not correct, that is, (2.71) *is not* the right formulation of (2.70) when $\Omega = P_m$. The "infinite curvature" at the m corners cannot be neglected, one should instead consider Dirac delta-type contributions at each corner. Indeed, the next result states that the sequence of solutions to (2.71) does not converge (as $m \to \infty$) to the unique solution of (2.70) when $\Omega = B$. On the contrary, it converges to the unique solution of the following Navier problem

$$\begin{cases} \Delta^2 u = f & \text{in } B, \\ u = \Delta u = 0 & \text{on } \partial B. \end{cases} \tag{2.72}$$

More precisely, recalling the definition of system solution in (2.69), we have

Proposition 2.34. *Let $P_m \subset B$ with $m \geq 3$ be the interior of the regular polygon with corners $\left\{\left(\cos\left(\frac{2k}{m}\pi\right), \sin\left(\frac{2k}{m}\pi\right)\right); k = 1, \ldots, m\right\}$ and let $f \in L^2(B)$. Then the following holds.*

1. *There exists a unique (weak) system solution u_m of (2.71) so that $u_m, \Delta u_m \in H_0^1(P_m)$.*
2. *There exists a unique minimiser $\tilde{u}_m$ in $\left\{u \in H^2 \cap H_0^1(P_m)\right\}$ of*

$$J(u) = \int_{P_m} \left(\tfrac{1}{2}(\Delta u)^2 - fu\right) dx.$$

3. *The solution u_m satisfies*

$$u_m \in H^2 \cap C^1(\overline{P_m}) \text{ and } \Delta u_m \in H^2 \cap C^{0,\gamma}(\overline{P_m}),$$

for $\gamma \in (0,1)$ and hence $\tilde{u}_m = u_m$.

4. If we extend u_m by 0 on $B\backslash P_m$ and let u_∞ be the solution of (2.72), then

$$\lim_{m\to\infty}\|u_m - u_\infty\|_{L^\infty(B)} = 0.$$

Note that even if the identity of u_m and $\tilde{u}_m$ may not seem very surprising, Example 2.33 shows that for domains with nonconvex corners these two solutions can be different.

Proof. 1. By the Lax-Milgram theorem, for any bounded domain Ω (in particular, if $\Omega = P_m$) and any $f \in L^2(\Omega)$ the Poisson problem

$$\begin{cases} -\Delta w = f \text{ in } \Omega, \\ w = 0 \quad\quad \text{on } \partial\Omega, \end{cases} \tag{2.73}$$

has a unique (weak) solution in $H_0^1(\Omega)$. Similarly, one finds a unique solution in $H_0^1(\Omega)$ to $-\Delta u = w$ in Ω with $u = 0$ on $\partial\Omega$. We apply this fact to the case where $\Omega = P_m$.

2. The functional J is convex and coercive on $H^2 \cap H_0^1(P_m)$ in view of Theorem 2.31 since the corner points of P_m all have angles less then π (i.e. P_m satisfies a uniform outer ball condition). As the functional J is weakly lower semicontinuous and the closed unit ball in $H^2 \cap H_0^1(P_m)$ is weakly compact there exists a minimiser. The strict convexity gives uniqueness.

3. Invoking again [2] (or [240] since the P_m are convex), we know that the solution of (2.73) in P_m with source in $L^2(P_m)$ lies in $H^2(P_m)$. Hence $\Delta u_m \in H^2(P_m)$ and, by Theorem 2.6, $\Delta u_m \in C^{0,\gamma}(\overline{P_m})$ for all $\gamma \in (0,1)$. In fact, by Kondratiev [251] one finds that for a convex domain in two dimensions, with all corners having an opening angle less than or equal to $\alpha \in \left(\frac{\pi}{2}, \pi\right)$, the solution of (2.73) for $f \in L^p(\Omega)$ with $p < p_\alpha = \frac{2\alpha}{2\alpha - \pi}$ lies in $W^{2,p}(\Omega)$. Hence for each P_m one finds that $u_m \in W^{2,2+\varepsilon}(P_m)$ for $0 \le \varepsilon < \frac{4}{m-4}$. Theorem 2.6 then implies that $u_m \in C^1(\overline{P_m})$.

4. It is sufficient to prove this result for $f \ge 0$. For $r \in (0,1)$ we compare the solutions w_r of

$$\begin{cases} -\Delta w = f \text{ in } B_r, \\ w = 0 \quad\quad \text{on } \partial B_r. \end{cases}$$

Extend w_r to $B\backslash B_r$ by 0. Assuming $f \ge 0$ and $f \in L^2(B)$ one finds that for $0 < r_1 < r_2 < 1$ it holds that $w_{r_1} \le w_{r_2}$ and moreover, that

$$\lim_{s\to r}\|w_s - w_r\|_{L^\infty(B)} = 0.$$

Indeed, if $f \in L^2(B)$, then $w_s, w_r \in C^{0,\gamma}(\overline{B})$ and for $s < r$ we find that

$$\begin{cases} -\Delta\left(w_r - w_s\right) = 0 \text{ in } B_s, \\ w_r - w_s = w_r \quad\quad \text{on } \partial B_s. \end{cases}$$

Since $w_r \in C^{0,\gamma}(\overline{B})$, this yields

$$\|w_r\|_{L^\infty(B_r\backslash B_s)} \le C_f\,|r-s|^\gamma.$$

By the maximum principle

$$\|w_r - w_s\|_{L^\infty(B)} = \|w_r - w_s\|_{L^\infty(B_s)} \le \|w_r\|_{L^\infty(\partial B_s)} \le C_f\, |r-s|^\gamma .$$

Again using the maximum principle and writing u_r for the solution of (2.72) in B_r (instead of B) we find

$$u_r(x) \le u_1(x) \le u_r(x) + \tilde{C}_f\, |1-r|^\gamma . \tag{2.74}$$

Two more applications of the maximum principle result in

$$u_{r_m}(x) \le u_m(x) \le u_1(x) \tag{2.75}$$

with $r_m = \cos(\pi/m)$ since $B_{r_m} \subset P_m \subset B$. The last claim follows by combining (2.74) and (2.75). □

In order to emphasise the role played by smooth/nonsmooth boundaries in this paradox, we prove

Proposition 2.35. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^2 boundary and let $\sigma \in (-1,1)$. If $f \in L^2(\Omega)$, then there exists a unique minimiser $u_\sigma \in H^2 \cap H^1_0(\Omega)$ of*

$$J(u) = \int_\Omega \left(\tfrac{1}{2} (\Delta u)^2 + (1-\sigma) \left(u_{xy}^2 - u_{xx} u_{yy} \right) - f\, u \right) dxdy.$$

If $\partial\Omega \in C^4$, then $u_\sigma \in H^4(\Omega)$ and u_σ satisfies

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\, \kappa \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.76}$$

Proof. Assume first that $0 \le \sigma < 1$. Since the expression

$$\tfrac{1}{2} (\Delta u)^2 + (1-\sigma) \left(u_{xy}^2 - u_{xx} u_{yy} \right) = \tfrac{\sigma}{2} \left(u_{xx} + u_{yy} \right)^2 + \tfrac{1-\sigma}{2} \left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 \right)$$

is convex in the second derivatives of u, so is J. If $-1 < \sigma < 0$, following [331, Proposition 2.4] we note that

$$\begin{aligned} &\langle dJ(v_1) - dJ(v_2), v_1 - v_2 \rangle \\ &\quad = \int_\Omega \big((v_1-v_2)_{xx}^2 + (v_1-v_2)_{yy}^2 + 2\sigma (v_1-v_2)_{xx} (v_1-v_2)_{yy} \\ &\qquad\qquad + 2(1-\sigma)(v_1-v_2)_{xy}^2 \big)\, dxdy \\ &\quad \ge (1+\sigma) \int_\Omega \big((v_1-v_2)_{xx}^2 + (v_1-v_2)_{yy}^2 + 2(v_1-v_2)_{xy}^2 \big) dxdy \\ &\quad > 0 \end{aligned}$$

for all $v_1, v_2 \in H^2 \cap H^1_0(\Omega)$ ($v_1 \ne v_2$), where we used the simple inequality $2\sigma ab \ge \sigma(a^2+b^2)$. Hence, also for $-1 < \sigma < 0$, the functional J is convex.

Then existence and uniqueness of a minimiser for J are obtained as in Proposition 2.34. The minimiser u satisfies the weak Euler-Lagrange equation, that is

$$\int_\Omega \left(\Delta u \Delta \varphi + (1-\sigma)\left(2u_{xy}\varphi_{xy} - u_{xx}\varphi_{yy} - u_{yy}\varphi_{xx}\right) - f\varphi\right) dxdy = 0$$

for all $\varphi \in H^2 \cap H_0^1(\Omega)$. Regularity arguments (see Theorem 2.20) show that for $\partial\Omega \in C^4$ and $f \in L^2(\Omega)$ the minimiser lies in $H^4(\Omega)$. The integration by parts in (1.7) and (1.8) show that u_σ satisfies (2.76). $\square$

By combining Propositions 2.34 and 2.35 we may now better explain the Babuška paradox. Assume that $f \in L^2(B)$ and let $\sigma \neq 1$. If u_∞ is as in Proposition 2.34 and u_σ is as in Proposition 2.35, then

$$u_\infty \equiv u_\sigma \text{ in } B \iff \frac{\partial u_\infty}{\partial \nu} \equiv 0 \text{ on } \partial B.$$

But if $0 \neq f \geq 0$, then the maximum principle implies $-\Delta u_\infty > 0$ and $u_\infty > 0$ whereas Hopf's boundary point Lemma even gives $\frac{\partial u_\infty}{\partial \nu} < 0$ on ∂B and hence $u_\infty \neq u_\sigma$. Babuška considered the case where $f = 1$ in B. This simple source allows us to compute all the functions involved. The solution to (2.76) with $f = 1$ on B is the radially symmetric function

$$u_\sigma(x) = \frac{(5+\sigma) - (6+2\sigma)\,|x|^2 + (1+\sigma)\,|x|^4}{64\,(1+\sigma)}.$$

The limit u_∞ in Babuška's example, defined by $u_\infty(x) = \lim_{m\to\infty} u_m(x)$ equals $u_{\sigma=1}(x)$, namely

$$u_\infty(x) = \frac{3}{64} - \frac{1}{16}\,|x|^2 + \frac{1}{64}\,|x|^4,$$

see also [401, p. 499] and [135].

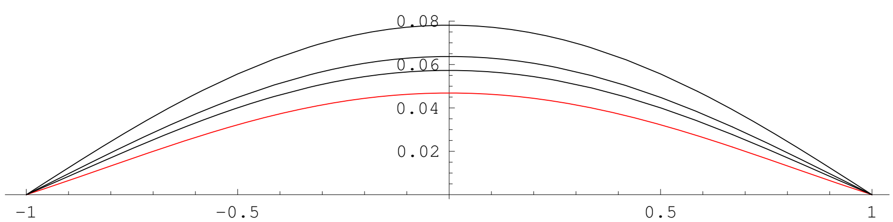

Fig. 2.2 The example of Babuška: $\Omega = B$ and $f = 1$. The solutions u_σ to (1.10) depend on σ; from top to bottom the solutions for $\sigma = 0$, .3, .5 and 1. The solution for $\sigma = 1$ is the limit of u_m from the regular m-polygon with $m \to \infty$.

2.8 Bibliographical notes

Ellipticity and the complementing condition are well explained in [5], see also Section 1 of Chapter 2 in [275]. The polynomial L_m representing the differential operator is taken from [5, Section I.1], see also Section 1.1 of Chapter 2 in [275]. For boundary conditions that do not satisfy assumption (2.15) we refer again to [5]. The complementing condition is sometimes also called Lopatinski-Shapiro condition and may be defined in an equivalent way, see [416, Section 11]. Concerning assumption (2.17), we refer to [416, Theorem 13.1] for a general statement relating ellipticity, the complementing condition, regularity results, Fredholm theory and a priori estimates.

More results about Sobolev spaces may be found in the monographs by Adams [1], Maz'ya [291] and Lions-Magenes [275]. All the embedding theorems in Section 2.2.2 may be derived from [1, Theorems 5.4 and 6.2] whereas for the scalar product in $H^m(\Omega)$ see [275, Théorème 1.1]. Theorem 2.31 is taken from [80] and Theorem 2.7 is taken from [1, Theorem 5.23].

The material from Section 2.4 is taken from Section 9 of Chapter 2 in [275] and from [416]. Theorem 2.13 is a variant of Green's formula, see [416, Theorem 14.8]. A Hilbert triple as in Definition 2.14 is a particular case of a Gelfand triple, see [416, Definition 17.1]. The coercivity condition in (2.41) is given in $H^m(\Omega)$, the framework of our setting; it is taken from [275, Definition 2.9.1]. Agmon's condition which ensures the coercivity of the bilinear form $\Psi(u,v)$ was originally stated in [3]; we also refer to Theorem 9.3 of Chapter 2 in [275] and to [416, Section 18] for equivalent formulations. Theorem 2.16 is a direct consequence of the Lax-Milgram theorem, see [416, Theorem 17.10]. Existence, uniqueness and regularity results for (2.2)-(2.14) with data in the Hilbert spaces H^s with $s \in \mathbb{R}$ (possibly also non integer and negative) are studied in full detail in [275]; in particular, we refer to Remark 7.2 of Chapter 2 in [275] for a statement including all possible cases. Theorem 2.19 is contained in [5, Theorem 9.3] whereas Theorem 2.20 is contained in [5, Theorem 15.2]. Theorem 2.22 and the extension of Corollary 2.23 to all $k \geq 2$ are justified by [5, Theorem 15.3']. Theorem 2.25 follows as a by-product of Theorems 2.6 and 2.20 on one hand and maximum modulus estimates for solutions of higher order elliptic equations on the other hand. This second tool was introduced by Miranda [304, 305] for higher order problems in the plane and subsequently generalised by Agmon [4] in any space dimension. We also refer to [377] for a simple sketch of the proof. Finally, let us mention that partial extensions and counterexamples to Theorem 2.25 in *nonsmooth* domains may be found in works by Pipher-Verchota [337, 338], Maz'ya-Rossmann [290], Mayboroda-Maz'ya [284, 285] and references therein. Lemma 2.27 is a fundamental contribution by Boggio [63] and is one of the most frequently used results in this monograph. Results on Green's functions may also be found in the monograph [21]. As for L^p-theory of higher-order elliptic operators and underlying kernel estimates one may also see the survey article [129] by Davies. Theorem 2.31 is a straightforward consequence of results by Adolfsson [2], see also [237, 238] for related results. Concerning the Sapondžyan paradox, we refer to the original paper [357] and to more recent results on "multiple" solutions in [320]. Babuška [28] noticed first that by approximating a curvilinear domain by polygons the approximating solutions would not converge to the solution on the curvilinear domain. Engineering approaches to the Babuška or polygon-circle paradox can be found in [241, 314, 354]. A mathematical approach can be found in the work by Maz'ya and Nazarov [292, 293]. These authors dealt with the paradox by a careful asymptotic analysis of the boundary layer and the contribution of the corners in this. Part of their results are based on Γ-convergence results from [422]. More recently, Davini [134] again uses Γ-convergence to find a correct approximation. He focuses on numerical methods that avoid the pitfall of this paradox. Most part of the material in Section 2.7 is taken from [387].

Chapter 3
Eigenvalue problems

For quite general second order elliptic operators one may use the maximum principle and the Kreĭn-Rutman theorem to show that the eigenfunction corresponding to the first eigenvalue has a fixed sign. It is then a natural question to ask if a similar result holds for higher order Dirichlet problems where a general maximum principle is not available. A partial answer is that a Kreĭn-Rutman type argument can still be used whenever the boundary value problem is positivity preserving. We will also explain in detail an alternative dual cone approach. Both these methods have their own advantages. The Kreĭn-Rutman approach shows under fairly weak assumptions that there exists a real eigenvalue and, somehow as a byproduct, one finds that the eigenvalue and the corresponding eigenfunction are positive. It applies in particular to non-selfadjoint settings. The dual cone decomposition only applies in a selfadjoint framework in a Hilbert space, where the existence of eigenfunctions is well-known. But in this setting it provides a very simple proof for positivity and simplicity of the first eigenfunction. A further quality of this method is that it applies also to some nonlinear situations as we shall see in Chapter 7.

We conclude this first part of the chapter with some further remarks on the connection between the positivity preserving property of the Dirichlet problem and the fixed sign property of the first eigenfunction. In particular, we show that the latter property, as well as the simplicity of the first eigenvalue, may fail.

Then we turn to the minimisation of the first Dirichlet eigenvalue of the biharmonic operator among domains of fixed measure and we show that, in dimensions $n = 2$ and $n = 3$, the ball achieves the minimum. We also consider two further eigenvalue problems, the buckling load of a clamped plate and Steklov eigenvalues. Up to some regularity to be proved, a quite hard open problem, the disk minimises the buckling load among planar domains of given measure. For the Steklov problem we first study in detail the whole spectrum and then we show that an optimal shape of given measure which minimises the first eigenvalue does not exist.

3.1 Dirichlet eigenvalues

Here we consider the eigenvalue problem

$$\begin{cases} (-\Delta)^m u = \lambda u\,,\ u \not\equiv 0 & \text{in } \Omega\,, \\ D^\alpha u|_{\partial\Omega} = 0 & \text{for } |\alpha| \le m-1 \end{cases} \tag{3.1}$$

on a given bounded domain $\Omega \subset \mathbb{R}^n$ ($n \ge 2$). The first eigenvalue of (3.1) is defined as

$$\Lambda_{m,1}(\Omega) \;=\; \min_{u \in H_0^m(\Omega)\setminus\{0\}} \frac{\|u\|^2_{H_0^m}}{\|u\|^2_{L^2}}. \tag{3.2}$$

In this section we discuss several problems related to (3.1) and to its first eigenvalue. We start by showing that the corresponding eigenfunction is of one sign whenever the problem

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega\,, \\ D^\alpha u|_{\partial\Omega} = 0 & \text{for } |\alpha| \le m-1 \end{cases} \tag{3.3}$$

is positivity preserving, see Definition 3.1 below. This statement can be obtained in two different ways, either with a somehow standard Kreĭn-Rutman type argument or with a decomposition in dual cones which we discuss in detail. Next, we discuss the positivity of the first eigenfunction and its failure in general and we end up with the minimisation of the first eigenvalue among domains of given measure.

3.1.1 A generalised Kreĭn-Rutman result

The Kreĭn-Rutman theorem, which can be considered to have its roots in Jentzsch's theorem, appears in many forms with many different and partially overlapping conditions but none of the classical versions are optimal for the solution operator of an elliptic boundary value problem. The main restriction is the necessity of having a positive cone with an open interior, see [257, Theorems 6.2 and 6.3]. As we shall see, this restriction could be removed after a profound result of de Pagter [136].

Consider the linear problem (3.3) and the following notion of positivity preserving.

Definition 3.1. We say that (3.3) has a *positivity preserving property* when the following holds for all u and f satisfying (3.3):

$$f \ge 0 \Rightarrow u \ge 0.$$

In case that a Green function exists, the positivity preserving property holds true if and only if this Green function is nonnegative. We now establish that if (3.3) is positivity preserving then a Kreĭn-Rutman result allows one to verify that the first eigenvalue for (3.1) is simple and corresponds to an eigenfunction of fixed sign.

Let us shortly introduce some terminology.

Definition 3.2. An ordered Banach space $(E, \|\,.\,\|, \geq)$ is called a *Banach lattice* if:

- the least upper bound of two elements in E lies again in E:

$$f, g \in E \text{ implies } f \vee g := \inf\{h \in E; h \geq f \text{ and } h \geq g\} \in E;$$

- the ordering of positive elements is preserved by the norm: setting $|f| = f \vee (-f)$ it holds for all $f, g \in E$ that $|f| \leq |g|$ implies $\|f\| \leq \|g\|$.

A linear subspace $A \subset E$ is called a *lattice ideal* if

$$|f| \leq |g| \text{ and } g \in A \text{ implies } f \in A.$$

We call A invariant under the operator $T : E \to E$ if $T(A) \subset A$.

We can now give a statement which improves the classical Kreĭn-Rutman theorem, see [257].

Theorem 3.3. *Let E be a Banach lattice with* $\dim(E) > 1$ *and let $T : E \to E$ be a linear operator satisfying:*

- *T is compact;*
- *T is positive: $T(\mathscr{K}) \subset \mathscr{K}$ where $\mathscr{K}$ is the positive cone in E;*
- *T is irreducible:* $\{0\}$ *and E are the only closed lattice ideals invariant under T.*

Then the spectral radius ρ of T is strictly positive and there exists $v \in \mathscr{K} \setminus \{0\}$ with $Tv = \rho v$. Moreover, the algebraic multiplicity of ρ is one, all other eigenvalues $\tilde{\rho}$ satisfy $|\tilde{\rho}| < \rho$ and no other eigenfunction is positive.

By Lemma 2.27 we know that the Green function in the ball B is positive so that problem (3.3) has a positivity preserving property whenever $\Omega = B$. In fact, Theorem 3.3 applies to any domain Ω where the corresponding Green function G_Ω is strictly positive. In this case, one takes $E = L^2(\Omega)$ or $E = \{v \in C(\overline{\Omega}); v|_{\partial\Omega} = 0\}$ and T as the solution operator for (3.3). Since for each $x \in \Omega$ the Green function $G_\Omega(x, .)$ is strictly positive on $\overline{\Omega}$ except for a set of measure 0 it follows in both settings that T is irreducible.

3.1.2 Decomposition with respect to dual cones

We state and discuss here an abstract result due to Moreau [311] about the decomposition of a Hilbert space H into dual cones; we recall that $\mathscr{K} \subset H$ is a cone if $u \in \mathscr{K}$ and $a \geq 0$ imply that $au \in \mathscr{K}$. In order to exploit the full power of this decomposition, we also establish a generalised Boggio result. This will be used in several different points of this monograph. Finally, we give a first simple application of this decomposition to a capacity problem.

It was Miersemann [301] who first observed that the dual cone decomposition could be quite helpful in the context of fourth order elliptic equations. In the next section we show how this method can be used to prove simplicity and positivity of the first eigenfunction of (3.1) in a ball. Moreover, this decomposition will turn out to be quite useful in a number of semilinear problems considered in this monograph.

Theorem 3.4. *Let H be a Hilbert space with scalar product $(.,.)_H$. Let $\mathscr{K} \subset H$ be a closed convex nonempty cone. Let $\mathscr{K}^*$ be its dual cone, namely*

$$\mathscr{K}^* = \{w \in H;\ (w,v)_H \le 0 \textit{ for all } v \in \mathscr{K}\}.$$

Then for any $u \in H$ there exists a unique $(u_1,u_2) \in \mathscr{K} \times \mathscr{K}^*$ such that

$$u = u_1 + u_2\ , \qquad (u_1,u_2)_H = 0. \tag{3.4}$$

In particular, $\|u\|_H^2 = \|u_1\|_H^2 + \|u_2\|_H^2$.

Moreover, if we decompose arbitrary $u,v \in H$ according to (3.4), namely $u = u_1 + u_2$ and $v = v_1 + v_2$, then we have that

$$\|u-v\|_H^2 \ge \|u_1 - v_1\|_H^2 + \|u_2 - v_2\|_H^2.$$

In particular, the projection onto $\mathscr{K}$ is Lipschitz continuous with constant 1.

Proof. For a given $u \in H$, we prove separately existence and uniqueness of a decomposition satisfying (3.4).

Existence. Let u_1 be the projection of u onto $\mathscr{K}$ defined by

$$\|u - u_1\| = \min_{v \in \mathscr{K}} \|u - v\|$$

and let $u_2 := u - u_1$. Then for all $t \ge 0$ and $v \in \mathscr{K}$ one has

$$\|u-u_1\|_H^2 \le \|u-(u_1+tv)\|_H^2 = \|u-u_1\|_H^2 - 2t(u-u_1,v)_H + t^2\|v\|_H^2$$

so that

$$2t(u_2,v)_H \le t^2\|v\|_H^2. \tag{3.5}$$

Dividing by $t > 0$ and letting $t \searrow 0$, (3.5) yields $(u_2,v)_H \le 0$ for all $v \in \mathscr{K}$ so that $u_2 \in \mathscr{K}^*$. Choosing $v = u_1$ also allows for taking $t \in [-1,0)$ in (3.5), so that dividing by $t < 0$ and letting $t \nearrow 0$ yields $(u_2,u_1)_H \ge 0$ which, combined with the just proved converse inequality, proves that $(u_2,u_1)_H = 0$.

Lipschitz continuity. From the two inequalities $(u_1,v_2)_H \le 0$ and $(v_1,u_2)_H \le 0$ and by orthogonality, we obtain

$$
\begin{aligned}
\|u-v\|_H^2 &= (u_1+u_2-v_1-v_2, u_1+u_2-v_1-v_2)_H \\
&= ((u_1-v_1)+(u_2-v_2), (u_1-v_1)+(u_2-v_2))_H \\
&= \|u_1-v_1\|_H^2 + \|u_2-v_2\|_H^2 + 2(u_1-v_1, u_2-v_2)_H \\
&= \|u_1-v_1\|_H^2 + \|u_2-v_2\|_H^2 - 2(u_1,v_2)_H - 2(v_1,u_2)_H \\
&\geq \|u_1-v_1\|_H^2 + \|u_2-v_2\|_H^2
\end{aligned}
$$

and Lipschitz continuity follows.

Uniqueness. It follows from the Lipschitz continuity by taking $u = v$. □

Remark 3.5. In the context of an abstract Hilbert space it is quite easy to gain an imagination of the projection u_1 of a general element u onto $\mathscr{K}$. However, in the concrete context of function spaces it is difficult to really see how u_1 arises from u and $\mathscr{K}$. Here, a different point of view is helpful: $u_2 := u - u_1 \in H$ is characterised by minimising $\|\,.\,\|_H$ subject to the constraint that $u - u_2 \in \mathscr{K}$. In the framework of the function space $H_0^2(\Omega)$ equipped with the scalar product $(u,v)_{H_0^2(\Omega)} = \int_\Omega \Delta u\, \Delta v$ and the cone $\mathscr{K} \subset H_0^2(\Omega)$ of nonnegative functions this means that u_2 has minimal (quadratic) elastic energy $\int_\Omega (\Delta u_2)^2$ among all H_0^2-functions subject to the constraint that $u - u_2 \geq 0$, i.e. $u_2 \leq u$. This means that one seeks u_2 as the solution of an obstacle problem, see [248]. See Figure 3.1 for an example of a dual cone decomposition in

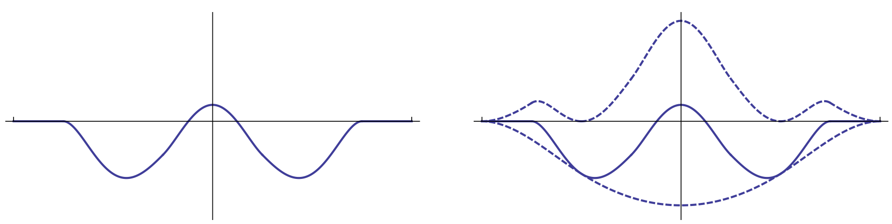

Fig. 3.1 Dual cone decomposition (right) in H_0^2 of the function displayed on the left.

H_0^2. We refer to [68] for further explanations and for some explicit examples of the dual cone decomposition.

Note also that the Lipschitz continuity property stated in Theorem 3.4 strongly depends on the norm considered. To see this, consider the special case $H = H_0^1(-1,1)$ with $(u,v)_H = \int_{-1}^1 u'v'$ and let $\mathscr{K} = \{v \in H : v \geq 0 \text{ a.e.}\}$. For any $\varepsilon \in (0,1)$ let

$$
u^\varepsilon(x) := \begin{cases} \dfrac{|x|}{\varepsilon} - 1 & \text{if } |x| \leq \varepsilon, \\ 0 & \text{otherwise.} \end{cases}
$$

Then

$$
u_2^\varepsilon(x) = |x| - 1, \qquad u_1^\varepsilon(x) = u^\varepsilon(x) - u_2^\varepsilon(x) = \begin{cases} \dfrac{|x|}{\varepsilon} - |x| & \text{if } |x| \leq \varepsilon, \\ 1 - |x| & \text{otherwise.} \end{cases}
$$

Therefore, $u^\varepsilon \to 0$ in $L^2(-1,1)$ as $\varepsilon \searrow 0$, while $u_1^\varepsilon \to 1-|x|$ and $u_2^\varepsilon \equiv |x|-1$. This shows that the decomposition in H_0^1 is not continuous with respect to the L^2-norm.

Let us now explain how we are planning to use the decomposition in Theorem 3.4. We will take H as a Hilbertian functional space (L^2, H^2, $\mathscr{D}^{2,2}\dots$) and

$$\mathscr{K} = \{u \in H;\ u \geq 0 \text{ a.e.}\}.$$

If $H = L^2(\Omega)$, then $\mathscr{K}^* = -\mathscr{K}$ and Theorem 3.4 simply yields

$$u = u^+ - u^- \qquad \text{for all } u \in L^2(\Omega).$$

If $H = H_0^1(\Omega)$, in order to characterise $\mathscr{K}^*$ we seek $v \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v\, dx \leq 0 \qquad \text{for all } u \in \mathscr{K}.$$

This means that v is weakly subharmonic (formally, $\int_\Omega u\Delta v \geq 0$) and therefore

$$\mathscr{K}^* = \{v \in H_0^1(\Omega);\ v \text{ is weakly subharmonic}\} \subsetneq -\mathscr{K}.$$

Note that although $\int_\Omega \nabla u^+ \nabla u^- = 0$, the decomposition obtained here is different from $u = u^+ - u^-$.

In higher order Sobolev spaces the decomposition $u = u^+ - u^-$ is no longer admissible because if $u \in H^m$ ($m \geq 2$) then, in general, $u^+, u^- \notin H^m$. In some situations the decomposition into dual cones may substitute the decomposition into positive and negative part. In order to facilitate and strengthen the application of Theorem 3.4 to higher order Sobolev spaces, we generalise Boggio's principle (Lemma 2.27) to weakly subpolyharmonic functions in suitable domains. Let us consider again $\mathscr{K} = \{v \in H_0^m(\Omega);\ v \geq 0 \text{ a.e. in } \Omega\}$ (or $v \in \mathscr{D}^{m,2}(\Omega)$ if Ω is unbounded and $n > 2m$), then

$$\mathscr{K}^* = \left\{ w \in H_0^m(\Omega);\ (w,v)_{H_0^m} \leq 0 \text{ for all } v \in \mathscr{K} \right\}.$$

Hence, $\mathscr{K}^* = \{v \in H_0^m(\Omega);\ (-\Delta)^m v \leq 0 \text{ weakly}\}$. In some cases, we have that $\mathscr{K}^* \subset -\mathscr{K}$.

Proposition 3.6. *Assume that either $\Omega = B_R$ (a ball of radius R), or $\Omega = \mathbb{R}^n_+$, or $\Omega = \mathbb{R}^n$; if Ω is unbounded, we also assume that $n > 2m$. Assume that $w \in L^2(\Omega)$ is a weak subsolution of the polyharmonic Dirichlet problem, namely*

$$\int_\Omega w(-\Delta)^m u\, dx \leq 0 \qquad \textit{for all } u \in \mathscr{K} \cap H^{2m} \cap H_0^m(\Omega)\,;$$

then

$$\textit{either } w \equiv 0 \textit{ or } w < 0 \textit{ a.e. in } \Omega. \tag{3.6}$$

In particular, (3.6) holds for all $w \in \mathscr{K}^*$.

Proof. We only prove the result in the case where $\Omega = B$ (the unit ball), the remaining cases being similar. Assuming $n > 2m$, for the half space it suffices to use (2.66) instead of (2.65) whereas for the whole space one takes the fundamental solution of $(-\Delta)^m$.

Take any $\varphi \in \mathscr{K} \cap C_c^\infty(B)$ and let v_φ be the unique (classical) solution of

$$\begin{cases} (-\Delta)^m v_\varphi = \varphi & \text{in } B, \\ D^\alpha v_\varphi = 0 & \text{on } \partial B \qquad \text{for } |\alpha| \le m-1. \end{cases}$$

Then by the classical Boggio's principle (Lemma 2.27) we infer that $v_\varphi \in \mathscr{K}$. Hence, v_φ is a possible test function for all such φ and therefore

$$\int_B w\varphi\,dx = \int_B w(-\Delta)^m v_\varphi\,dx \le 0 \qquad \text{for all } \varphi \in \mathscr{K} \cap C_c^\infty(B).$$

This shows that $w \le 0$ a.e. in B. Assume that $w \not< 0$ a.e. in B and let ϕ denote the characteristic function of the set $\{x \in B;\, w(x) = 0\}$ so that $\phi \ge 0$, $\phi \not\equiv 0$. Let v_0 be the unique (a.e.) solution of the problem

$$\begin{cases} (-\Delta)^m v_0 = \phi & \text{in } B, \\ D^\alpha v_0 = 0 & \text{on } \partial B \qquad \text{for } |\alpha| \le m-1. \end{cases}$$

Then by Corollary 2.21 and Theorem 2.6, we know that

$$v_0 \in \left(\bigcap_{q \ge 1} W^{2m,q}(B) \right) \subset C^{2m-1}(\overline{B})$$

and again by Boggio's principle we have $v_0 > 0$ in B. One also reads from Boggio's formula (2.65) that $(-\frac{\partial}{\partial \nu})^m v_0 > 0$ on ∂B, see Theorem 5.7 below. We infer that for all $v \in C^{2m}(\overline{B}) \cap H_0^m(B)$ there exist $t_1 \le 0 \le t_0$ such that $v + t_0 v_0 \ge 0$ and $v + t_1 v_0 \le 0$ in B. This, combined with the fact that

$$\int_B w(-\Delta)^m v_0\,dx = \int_{\{w=0\}} w\,dx = 0,$$

enables us to show that both

$$0 \le \int_B w(-\Delta)^m (v + t_0 v_0)\,dx = \int_B w(-\Delta)^m v\,dx,$$

$$0 \ge \int_B w(-\Delta)^m (v + t_1 v_0)\,dx = \int_B w(-\Delta)^m v\,dx.$$

Hence, we have for all $v \in C^{2m}(\overline{B}) \cap H_0^m(B)$

$$\int_B w(-\Delta)^m v\,dx = 0.$$

We need to show that $C^{2m}(\overline{B})\cap H_0^m(B)$ is dense in $H^{2m}\cap H_0^m(B)$. For this purpose, take any function $U\in H^{2m}\cap H_0^m(B)$ and put $f:=(-\Delta^m)U$. We approximate f in $L^2(B)$ by $C^\infty(\overline{B})$–functions f_k and solve $(-\Delta^m)U_k=f_k$ in B under homogeneous Dirichlet boundary conditions. We then even have $U_k\in C^\infty(\overline{B})$, and by L^2–theory (see Corollary 2.21) it holds that $\|U_k-U\|_{H^{2m}}\to 0$ as $k\to\infty$.

By the previous statement we may now conclude that

$$\int_B w(-\Delta)^m v\,dx=0 \qquad \text{for all } v\in H^{2m}\cap H_0^m(B).$$

Since $w\in L^2(B)$, we may take as $v\in H^{2m}\cap H_0^m(B)$ the solution of $(-\Delta)^m v=w$ under homogeneous Dirichlet boundary conditions. This finally yields $w\equiv 0$. □

We conclude this section with a first simple application of Theorem 3.4. We show the positivity of the potential in the second order capacity problem. Given a bounded domain $\Omega\subset\mathbb{R}^n$ $(n>4)$ we define its second order capacity as

$$\operatorname{cap}(\Omega)=\inf\left\{\int_{\mathbb{R}^n}|\Delta u|^2\,dx;\ u\in\mathscr{D}^{2,2}(\mathbb{R}^n),\ u\geq 1 \text{ a.e. in } \Omega\right\}.$$

Using Theorem 3.4 we can show that the potential (the minimiser) $\overline{u}$ is nonnegative. Let $H=\mathscr{D}^{2,2}(\mathbb{R}^n)$ and let $\mathscr{K}=\{u\in H:\ u\geq 0 \text{ a.e. in } \mathbb{R}^n\}$. If $\overline{u}$ is sign-changing, let $\overline{u}=u_1+u_2$ with $u_1\in\mathscr{K}$ and $u_2\in\mathscr{K}^*\setminus\{0\}$ be its decomposition according to Theorem 3.4. Then by Proposition 3.6 we know that $u_2\leq 0$. Hence, $u_1\geq 1$ in Ω so that u_1 is an admissible function. Moreover,

$$\int_{\mathbb{R}^n}|\Delta\overline{u}|^2\,dx=\int_{\mathbb{R}^n}|\Delta u_1|^2\,dx+\int_{\mathbb{R}^n}|\Delta u_2|^2\,dx>\int_{\mathbb{R}^n}|\Delta u_1|^2\,dx.$$

This contradicts the minimality of $\overline{u}$ among admissible functions.

3.1.3 Positivity of the first eigenfunction

In this section we study positivity of the first eigenfunction of (3.1) by means of the just explained dual cone decomposition. As already mentioned in Section 3.1.1, whenever we have a positivity preserving solution operator, a Kreĭn-Rutman result yields a positive first eigenfunction with the uniqueness properties stated in Theorem 3.3. In our special self-adjoint situation, the dual cone decomposition yields a direct and simpler proof. Moreover, this strategy can also be exploited for semilinear problems, see e.g. Lemma 7.22 and Theorem 7.58.

Theorem 3.7. *If $\Omega=B\subset\mathbb{R}^n$, then the first eigenvalue*

$$\Lambda_{m,1} \;=\; \inf_{H_0^m(B)\setminus\{0\}} \frac{\|u\|^2_{H_0^m}}{\|u\|^2_{L^2}}. \tag{3.7}$$

of (3.1) is simple and the corresponding first eigenfunction u is of one sign.

Proof. Let $H = H_0^m(B)$ and let $\mathscr{K} = \{u \in H : u \geq 0 \text{ a.e. in } B\}$. For contradiction, assume that u changes sign. Then according to Theorem 3.4, we decompose $u = u_1 + u_2$, $u_1 \in \mathscr{K} \setminus \{0\}$, $u_2 \in \mathscr{K}^* \setminus \{0\}$. By Proposition 3.6, we have $u_2 < 0$ a.e. in B. Replacing u with the positive function $u_1 - u_2$ would strictly increase the L^2-norm while by orthogonality we have $\|u_1 + u_2\|_H^2 = \|u_1 - u_2\|_H^2$. The ratio would strictly decrease, a contradiction.

Since a minimiser $u \geq 0$ solves $(-\Delta)^m u = \Lambda_{m,1} u \geq 0$ we have $u > 0$ by Proposition 3.6. By contradiction, assume now that (3.7) admits two linearly independent positive minimisers u and v. Then $w = u + \alpha v$ (for a suitable $\alpha < 0$) is a sign-changing minimiser, contradiction! □

For $m = 1$ the very same technique used in Theorem 3.7 works in any bounded domain Ω if we wish to show that the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ is simple and that the corresponding eigenfunction is of one sign. On the other hand, the L^2-norm remains constant if we replace $u^+ - u^-$ with $u^+ + u^-$. So, for this problem, the decomposition into dual cones works directly, whereas the usual decomposition into positive and negative parts does not prove simplicity of the first eigenvalue nor fixed sign of the first eigenfunction without a further regularity argument.

Theorem 3.7 applies to any bounded domain $\Omega \neq B$ with a positive Green function also for $m \geq 2$. Note that the positivity preserving property (positivity of the Green function) implies fixed sign of the first eigenfunction to (3.1) but the converse implication does not hold in general. One can then wonder whether a positive first eigenfunction can be obtained also for domains which fail to have the positivity preserving property, see Definition 3.1. The answer is delicate and negative in general.

Let us quickly outline what is known for sign-changing first eigenfunctions of

$$\begin{cases} \Delta^2 u = \lambda u\,,\ u \not\equiv 0 & \text{in } \Omega, \\ u = |Du| = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Basically, only this biharmonic eigenvalue problem on bounded domains has been considered so far. Concerning (3.8) it is proven in [212] that for an appropriately defined family of perturbations starting from the ball the positivity preserving property fails to hold strictly before the first eigenfunction loses its fixed sign. So, the converse implication on ellipses as mentioned above is not true, see also Remark 6.4. It does not seem to be rigorously proven yet that the sign of the first eigenfunction changes on ellipses with a large ratio but there exists numerical evidence.

The first example of a sign-changing first eigenfunction is due to Coffman [107] and deals with squares.

Theorem 3.8. *For $\Omega = (0,1)^2$ problem (3.8) has a sign-changing first eigenfunction.*

Independently of previous results in [111], Kozlov-Kondratiev-Maz'ya [252] proved that domains in any space dimension whose boundaries contain suitable cones also have a sign-changing first eigenfunction for (3.8). Their results cover a class of elliptic operators of order $2m$ under Dirichlet boundary conditions. Their proof is based on a result which ensures the absence of zeros of infinite order at the vertex of a cone, for nontrivial nonnegative local solutions of the inequality $Au \leq 0$, where A is an elliptic differential operator with real coefficients. Moreover, they constructed a sequence of smooth convex domains that exhaust the cone and since the corresponding first eigenfunctions are proven to converge to the sign-changing first eigenfunction in the cone the same holds eventually for the approximating domains.

A main assumption that often appears is the convexity of Ω. From Theorem 3.8 and the numerical evidence on eccentric ellipses it is clear that this assumption will not be sufficient to ensure positivity of the first eigenfunction. Ellipses suggest that, possibly, a suitable upper bound for the ratio between the radii of the largest inscribed ball in Ω and the largest filling balls of Ω might yield a sufficient condition for a positive eigenfunction. We recall that B is a filling ball for Ω if Ω is the union of translated B. Clearly, for any bounded domain this ratio is always larger than or equal to 1. An interesting family of domains in this sense are elongated disks, the so-called stadiums, where the radius of the largest inscribed ball equals the radius of the largest filling ball. Numerical approximations of the first eigenfunction on such a domain always resulted in functions apparently of fixed sign.

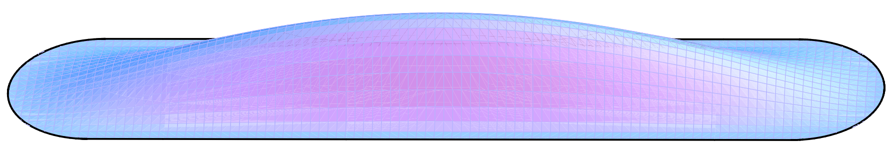

Fig. 3.2 Stadium-like domains seem to have a positive first eigenfunction in the Dirichlet biharmonic case.

Domains which are far from being convex are domains with holes. The standard examples are the annuli

$$A_\varepsilon = \{(x,y) \in \mathbb{R}^2;\, \varepsilon^2 < x^2 + y^2 < 1\} \qquad \text{with } 0 < \varepsilon < 1.$$

For these domains, Coffman-Duffin-Shaffer [109, 110, 155] proved the following somehow surprising statement.

Theorem 3.9. *Let $\Omega = A_\varepsilon$ for some $\varepsilon \in (0,1)$ and consider problem (3.8). There exists $\varepsilon_0 > 0$ such that the following holds.*

1. *If $\varepsilon < \varepsilon_0$, then the first eigenvalue has multiplicity two. There exist two independent eigenfunctions for this first eigenvalue with diametral nodal lines.*
2. *If $\varepsilon = \varepsilon_0$, then the first eigenvalue has multiplicity three. There exists a positive eigenfunction for this eigenvalue and there are two independent eigenfunctions with diametral nodal lines.*

3. *If $\varepsilon > \varepsilon_0$, then the first eigenvalue has multiplicity one and the corresponding eigenfunction is of fixed sign.*

It is not surprising that a large hole yields a positive eigenfunction since the domain becomes somehow close to an infinite strip with periodic boundary conditions w.r.t. the unbounded direction, where the first eigenfunction in the appropriate functional space depends only on one variable and is positive.

Even more, numerical experiments indicate that there exist starshaped domains, where the first eigenfunction is anti-symmetric with respect to a nodal line and hence sign-changing. See [76] and Figure 3.3 which shows the first and second eigenfunction for such a domain.

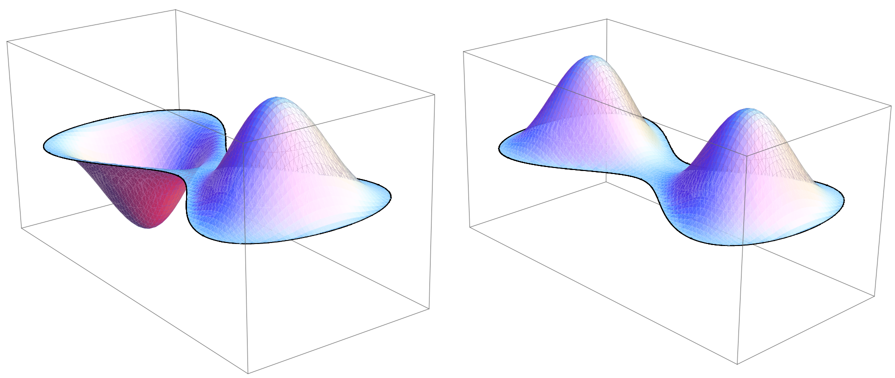

Fig. 3.3 On the left the first eigenfunction for the clamped biharmonic eigenvalue problem on an 8-shaped domain which is sign-changing. On the right the second eigenfunction which is (almost) positive.

3.1.4 Symmetrisation and Talenti's comparison principle

Let $\Lambda_{1,1}(\Omega)$ denote the first Dirichlet eigenvalue for $-\Delta$ in a bounded domain $\Omega \subset \mathbb{R}^n$, see (3.2) with $m = 1$. The celebrated Faber-Krahn [162, 253, 254] inequality states that if one considers the map $\Omega \mapsto \Lambda_{1,1}(\Omega)$ in dependence of domains Ω having all the same measure e_n as the unit ball B, its minimum is achieved precisely for $\Omega = B$ and, moreover, balls of radius 1 are the only minimisers. The crucial tool to prove this statement is *symmetrisation*. We recall here some basic facts about this method.

In 1836, Jacob Steiner noticed that symmetrisation with respect to planes leaves the measure of bounded sets invariant and decreases the measure of their boundary. This is the basic idea for a rigorous proof of the isoperimetric problem. In other words, if Ω^* denotes the ball centered at the origin and having the same measure as a bounded domain Ω, we have $|\Omega^*| = |\Omega|$ and $|\partial\Omega^*| \le |\partial\Omega|$ with strict inequality if Ω is not a ball. The same principle may be applied to functions.

Definition 3.10. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $u \in C_c^\infty(\Omega)$. The *spherical rearrangement* of u is the unique nonnegative measurable function u^* defined in Ω^* such that its level sets $\{x \in \Omega^*;\, u^*(x) > t\}$ are concentric balls with the same measure as the level sets $\{x \in \Omega;\, |u(x)| > t\}$ of $|u|$.

By density arguments we may define the spherical rearrangement of any function in $L^p(\Omega)$ for $p \in [1,\infty)$. We summarise here the basic properties of spherical rearrangements in a statement which makes clear how the symmetrisation method can be applied to obtain the Faber-Krahn result.

Theorem 3.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.*

1. *If $u \in L^p(\Omega)$ for some $p \in [1,\infty)$ then $u^* \in L^p(\Omega^*)$ and $\|u^*\|_{L^p(\Omega^*)} = \|u\|_{L^p(\Omega)}$.*
2. *If $u \in W_0^{1,p}(\Omega)$ for some $p \in [1,\infty)$ then $u^* \in W_0^{1,p}(\Omega^*)$ and $\|\nabla u^*\|_{L^p(\Omega^*)} \le \|\nabla u\|_{L^p(\Omega)}$.*
3. *If $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ for some $p \in (1,\infty)$ with $p' = \frac{p}{p-1}$ its conjugate, then $\|u^*v^*\|_{L^1(\Omega^*)} \ge \|uv\|_{L^1(\Omega)}$.*

Theorem 3.11 has several important applications, for example in the proof of first order Sobolev inequalities and of their sharpness. However, it is unsuitable for higher order derivatives since u^* may not be twice weakly differentiable even if u is very smooth. In their monograph, Pólya-Szegö [343, Section F.5] claim that they can extend the Faber-Krahn result to the Dirichlet biharmonic operator among domains having a first eigenfunction of fixed sign. Not only this assumption does not cover all domains, see Section 3.1.3 above, but also their argument is not correct. They deal with the Laplacian of a symmetrised smooth function and implicitly claim that it belongs to L^2, which is false in general. Incidentally, we point out that this mistake is responsible for the wrong proof in [373], see Section 1.3.3 for the details. This shows that standard symmetrisation methods are not available for higher order problems.

As a possible way out, Cianchi [98] considers larger classes than the Sobolev space, such as the space of functions whose second order distributional derivatives are measures with finite total variation. Alternatively, one can prove an inequality comparing the rearrangement invariant norm of the Hessian matrix of u and a weighted norm in the representation space of $(u^*)'$, see [99]. Unfortunately, none of these tricks works when trying to extend the Faber-Krahn result to the first Dirichlet eigenvalue of the biharmonic operator.

However, as we shall see, in some significant situations a comparison result by Talenti [391] turns out to be extremely useful. For our convenience, we state here an iterated version of this principle which will be used at several different places in the present book.

Theorem 3.12. *Let $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$ be a C^m-smooth bounded domain such that $|\Omega| = |B| = e_n$ and let $H^m_\vartheta(\Omega)$ be the space defined in (2.35), namely*

$$H^m_\vartheta(\Omega) := \left\{ v \in H^m(\Omega);\ \Delta^j v = 0 \text{ on } \partial\Omega \text{ for } j < \frac{m}{2} \right\}.$$

Let $m = 2k$ be an even number, let $f \in L^2(\Omega)$ and let $u \in H^m_\vartheta(\Omega)$ be the unique strong solution to

$$\begin{cases} (-\Delta)^k u = f & \text{in } \Omega, \\ \Delta^j u = 0 & \text{on } \partial\Omega, \end{cases} \qquad j = 0, \dots, k-1. \tag{3.9}$$

Let $f^* \in L^2(B)$ and $u^* \in H^1_0(B)$ denote respectively the spherical rearrangements of f and u (see Definition 3.10) and let $v \in H^m_\vartheta(B)$ be the unique strong solution to

$$\begin{cases} (-\Delta)^k v = f^* & \text{in } B, \\ \Delta^j v = 0 & \text{on } \partial B, \end{cases} \qquad j = 0, \dots, k-1. \tag{3.10}$$

Then $v \geq u^*$ a.e. in B.

Proof. When $k = 1$, Theorem 3.12 is precisely [391, Theorem 1]. For $k \geq 2$ we proceed by finite induction. We may rewrite (3.9) and (3.10) as the following systems:

$$\begin{cases} -\Delta u_1 = f & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta u_i = u_{i-1} & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \qquad i = 2, \dots, k; \tag{3.11}$$

$$\begin{cases} -\Delta v_1 = f^* & \text{in } B, \\ v_1 = 0 & \text{on } \partial B, \end{cases} \qquad \begin{cases} -\Delta v_i = v_{i-1} & \text{in } B, \\ v_i = 0 & \text{on } \partial B, \end{cases} \qquad i = 2, \dots, k. \tag{3.12}$$

Note that $u_k = u$ and $v_k = v$. By Talenti's principle [391, Theorem 1] applied for $i = 1$, we know that $v_1 \geq u_1^*$ a.e. in B. Assume that the inequality $v_i \geq u_i^*$ a.e. in B has been proved for some $i = 1, \dots, k-1$. By (3.11) and (3.12) we then infer

$$\begin{cases} -\Delta u_{i+1} = u_i & \text{in } \Omega \\ u_{i+1} = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta v_{i+1} = v_i \geq u_i^* & \text{in } B \\ v_{i+1} = 0 & \text{on } \partial B. \end{cases}$$

By combining the maximum principle for $-\Delta$ in B with a further application of Talenti's principle, we obtain $v_{i+1} \geq u_{i+1}^*$ a.e. in B. This finite induction shows that $v_k \geq u_k^*$ and proves the statement. □

3.1.5 The Rayleigh conjecture for the clamped plate

We consider here the domain functional given by the first Dirichlet eigenvalue for the biharmonic operator

$$\Omega \mapsto \Lambda_{2,1}(\Omega) \ = \ \min_{H^2_0(\Omega)\setminus\{0\}} \frac{\|u\|^2_{H^2_0}}{\|u\|^2_{L^2}}. \tag{3.13}$$

In 1894, Lord Rayleigh [350, p. 382] conjectured that, among planar domains Ω of given area, the disk minimises $\Lambda_{2,1}(\Omega)$. If Ω^* denotes the symmetrised of Ω, namely the ball having the same measure as Ω, Rayleigh's conjecture reads

$$\Lambda_{2,1}(\Omega^*) \leq \Lambda_{2,1}(\Omega). \tag{3.14}$$

After many attempts, see Section 1.3.1, this conjecture was proved one century later by Nadirashvili [315] and immediately extended by Ashbaugh-Benguria [22] to the case of 3-dimensional domains. More precisely, we have

Theorem 3.13. *In dimensions $n = 2$ or $n = 3$ the ball is the unique minimiser of the first eigenvalue of the clamped plate problem (3.13) among bounded domains of given measure. Hence, (3.14) holds whenever $n = 2$ or $n = 3$ with equality only if Ω is a ball.*

Proof. Thanks to the homogeneity of the map $\Omega \mapsto \Lambda_{2,1}(\Omega)$, we may restrict our attention to the case where $|\Omega| = |B| = e_n$. For such a domain, let u denote a first nontrivial eigenfunction so that

$$\frac{\|u\|^2_{H^2_0}}{\|u\|^2_{L^2}} \ = \ \Lambda_{2,1}(\Omega).$$

By a bootstrap argument, elliptic regularity theory (see Theorem 2.20) ensures that $u \in C^\infty(\Omega)$. Moreover, the unique continuation principle [336, 345] ensures that u cannot be harmonic (in particular, constant) on a subset of positive measure. In view of Section 3.1.3, both the positive and the negative part of u may be nontrivial so that it makes sense to define

$$\Omega_+ = \{x \in \Omega;\, u(x) > 0\}\ , \qquad \Omega_- = \{x \in \Omega;\, u(x) < 0\}.$$

Let $B_a = \Omega_+^*$ and $B_b = \Omega_-^*$ be their spherical symmetrisation, namely the two balls centered at the origin and such that $|\Omega_\pm^*| = |\Omega_\pm|$. Let a and b denote the radii of the balls $B_a = \Omega_+^*$ and $B_b = \Omega_-^*$, then

$$a^n + b^n = 1. \tag{3.15}$$

Let $(\Delta u)_\pm$ denote the positive and negative parts of Δu in Ω; again we point out that they may both be nontrivial. For $s \in [0, e_n]$ let $\sigma(s) := (s/e_n)^{1/n}$ and define the two functions $f, g : [0, e_n] \to \mathbb{R}$ by

$$\begin{aligned} g(s) &:= \big((\Delta u)_+\big)^*(\sigma(s)) - \big((\Delta u)_-\big)^*(\sigma(e_n - s)), \\ f(s) &:= -g(e_n - s). \end{aligned}$$

Note that at most one of $\big((\Delta u)_+\big)^*(\sigma(s))$ and $\big((\Delta u)_-\big)^*(\sigma(e_n - s))$ can be different from 0 for any s and that

$$\big((\Delta u)_+\big)^*(\sigma(s)) \cdot \big((\Delta u)_-\big)^*(\sigma(e_n - s)) \equiv 0. \tag{3.16}$$

The function g sums the contribution of $\big((\Delta u)_+\big)^*$ starting from the center of B and the contribution of $\big((\Delta u)_-\big)^*$ starting from ∂B. The function f switches these two contributions.

With the change of variable $r=\sigma(s)$, by Theorem 3.11 and the divergence theorem, we obtain

$$\begin{aligned}
\int_0^{e_n} g(s)ds &= \int_0^{e_n}\Big(\big((\Delta u)_+\big)^*(\sigma(s)) - \big((\Delta u)_-\big)^*(\sigma(e_n-s))\Big)ds \\
&= \int_0^{e_n}\Big(\big((\Delta u)_+\big)^*(\sigma(s)) - \big((\Delta u)_-\big)^*(\sigma(s))\Big)ds \\
&= ne_n\int_0^1 r^{n-1}\Big(\big((\Delta u)_+\big)^*(r) - \big((\Delta u)_-\big)^*(r)\Big)dr \\
&= \int_B \big((\Delta u)_+\big)^* dx - \int_B \big((\Delta u)_-\big)^* dx = \int_\Omega (\Delta u)_+\, dx - \int_\Omega (\Delta u)_-\, dx \\
&= \int_\Omega \Delta u\, dx = \int_{\partial\Omega} u_\nu\, d\omega = 0.
\end{aligned}$$

A similar computation holds for f so that

$$\int_0^{e_n} g(s)ds = \int_0^{e_n} f(s)ds = 0. \tag{3.17}$$

Let now $v\in H^2\cap H_0^1(B_a)$ and $w\in H^2\cap H_0^1(B_b)$ be the solutions of the problems

$$\begin{cases} -\Delta v = f(e_n|x|^n) & \text{in } B_a,\\ v=0 & \text{on } \partial B_a,\end{cases} \qquad \begin{cases} -\Delta w = g(e_n|x|^n) & \text{in } B_b,\\ w=0 & \text{on } \partial B_b.\end{cases}$$

Therefore, from the definition of f and from (3.15) we infer

$$\Delta v(a)+\Delta w(b) = -f(e_na^n)-g(e_nb^n)=0.$$

Moreover, by (3.17) and the definition of f we get

$$\begin{aligned}
0 &= \int_0^{e_na^n} f(s)ds + \int_{e_na^n}^{e_n} f(s)ds = \int_0^{e_na^n} f(s)ds - \int_0^{e_nb^n} g(s)ds \\
&= ne_n\int_0^a r^{n-1}f(e_nr^n)dr - ne_n\int_0^b r^{n-1}g(e_nr^n)dr = -\int_{B_a}\Delta v\,dx + \int_{B_b}\Delta w\,dx
\end{aligned}$$

so that

$$\int_{B_a}\Delta v\,dx = \int_{B_b}\Delta w\,dx. \tag{3.18}$$

In turn, employing the divergence theorem, (3.18) yields

$$a^{n-1}v'(a) = b^{n-1}w'(b). \tag{3.19}$$

In view of (3.16), we remark that

$$\int_{B_b} |\Delta w|^2 \, dx = \int_0^{|\Omega_b|} g^2(s) \, ds$$
$$= \int_0^{|\Omega_b|} \left(\left| \left((\Delta u)_+ \right)^* (\sigma(s)) \right|^2 + \left| \left((\Delta u)_- \right)^* (\sigma(e_n - s)) \right|^2 \right) ds.$$

Similarly, we have

$$\int_{B_a} |\Delta v|^2 \, dx = \int_0^{|\Omega_a|} \left(\left| \left((\Delta u)_- \right)^* (\sigma(s)) \right|^2 + \left| \left((\Delta u)_+ \right)^* (\sigma(e_n - s)) \right|^2 \right) ds.$$

By adding the last two equations and by Theorem 3.11 we obtain

$$\int_\Omega |\Delta u|^2 \, dx = \int_{B_a} |\Delta v|^2 \, dx + \int_{B_b} |\Delta w|^2 \, dx. \tag{3.20}$$

From Talenti [392, (2.7)] we know that

$$u_+^* \le v \quad \text{in } B_a \,, \qquad u_-^* \le w \quad \text{in } B_b,$$

so that, by Theorem 3.11,

$$\int_\Omega u^2 \, dx \le \int_{B_a} v^2 \, dx + \int_{B_b} w^2 \, dx,$$

with strict inequality if $\Omega \neq B$. With this inequality and (3.20) we obtain

$$\Lambda_{2,1}(\Omega) = \frac{\int_\Omega |\Delta u|^2 \, dx}{\int_\Omega u^2 \, dx} \ge \frac{\int_{B_a} |\Delta v|^2 \, dx + \int_{B_b} |\Delta w|^2 \, dx}{\int_{B_a} v^2 \, dx + \int_{B_b} w^2 \, dx} \tag{3.21}$$

with strict inequality if $\Omega \neq B$. As pointed out by Talenti [392], if $u > 0$ in Ω, then $\Omega_+^* = B_a = B$ and $\Omega_-^* = B_b = \emptyset$ so that (3.21) proves Rayleigh conjecture for domains with first eigenfunction of one sign. Indeed, in this case we have $b = 0$. Therefore, $v_\nu = 0$ on ∂B in view of (3.19).

We define

$$\mu = \mu_{a,b} = \min_{v,w} \frac{\int_{B_a} |\Delta v|^2 \, dx + \int_{B_b} |\Delta w|^2 \, dx}{\int_{B_a} v^2 \, dx + \int_{B_b} w^2 \, dx} \tag{3.22}$$

where the minimum is taken over all radially symmetric functions $v \in H^2 \cap H_0^1(B_a)$ and $w \in H^2 \cap H_0^1(B_b)$ satisfying (3.19). Using standard tools of the calculus of variations, it is shown in [22, Appendix 2] that the minimum in (3.22) is attained by a couple of functions satisfying $\Delta^2 v = \mu v$ in B_a and $\Delta^2 w = \mu w$ in B_b, $v(a) = w(b) = 0$, $a^{n-1} v'(a) = b^{n-1} w'(b)$, $\Delta v(a) + \Delta w(b) = 0$; moreover, as shown in formula (3.12) in [315], the functions v and w may be chosen positive and radially decreasing.

By combining (3.21) and (3.22) we obtain $\Lambda_{2,1}(\Omega) \ge \mu_{a,b}$ and, since a and b are unknown,

$$\Lambda_{2,1}(\Omega) \ge \min_{a,b} \mu_{a,b} \tag{3.23}$$

where the minimum is now taken among all couples $(a,b) \in [0,1]^2$ satisfying (3.15). At this point a delicate and technical analysis of fine properties of Bessel functions is needed. A crucial inequality, which only holds for $n = 2,3$, allows to show that $\min_{a,b} \mu_{a,b} = \mu_{1,0} = \Lambda_{2,1}(B)$. This proves the statement when combined with (3.23), see [22, Section 4] for the details. Indeed, recall that if $\Omega \neq B$ then (3.21) is strict.$\square$

We conclude this section by emphasising that a couple of interesting generalisations of Rayleigh's original conjecture are still missing. First, it remains to prove (3.14) in any space dimension $n \geq 2$ and not only for $n = 2,3$. Second, one might wonder whether one could prove that $\Lambda_{m,1}(\Omega^*) \leq \Lambda_{m,1}(\Omega)$ for any $m \geq 2$ and not only for $m = 2$.

3.2 Buckling load of a clamped plate

Similar questions as for the first eigenvalue of the clamped plate (3.2) arise when considering the buckling load of a clamped plate which may be characterised as follows

$$\mu_1(\Omega) \;=\; \inf_{H_0^2(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_{L^2}^2}{\|\nabla u\|_{L^2}^2}. \tag{3.24}$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded planar domain. Minimisers u to (3.24) solve

$$\begin{cases} \Delta^2 u = -\mu_1 \Delta u & \text{in } \Omega, \\ u = u_\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

This is the Dirichlet version of the Steklov problem (1.22) considered in Section 1.3.2 which describes the linearised von Kármán equations for an elastic plate. For later use, let us mention that an inequality (which holds true in any space dimension) due to Payne [333] states that for any bounded domain $\Omega \subset \mathbb{R}^2$

$$\mu_1(\Omega) \geq \Lambda_{1,2}(\Omega) \qquad \text{with equality if and only if } \Omega \text{ is a disk,} \tag{3.26}$$

where $\Lambda_{1,2}(\Omega)$ denotes the second Dirichlet eigenvalue for the Laplacian in Ω.

Similarly to (3.14), Pólya-Szegö [343, Note F] conjectured that the disk minimises the buckling load among domains of given measure.

Conjecture 3.14 (Pólya-Szegö). For any bounded domain $\Omega \subset \mathbb{R}^2$

$$\mu_1(\Omega^*) \leq \mu_1(\Omega),$$

where Ω^* denotes the symmetrised of Ω.

A complete proof of this conjecture is not known at the moment. However, we show here two interesting results which give some support to its validity. Consider the following special class of (not necessarily bounded) domains having the same measure as the unit disk:

$$\mathbb{B} = \{\Omega \subset \mathbb{R}^2; \Omega \text{ open, connected, simply connected, } |\Omega| = \pi\}.$$

The first result, due to Ashbaugh-Bucur [23], states that an optimal domain exists among domains in the class $\mathbb{B}$.

Theorem 3.15. *There exists $\Omega_o \in \mathbb{B}$ such that $\mu_1(\Omega_o) \le \mu_1(\Omega)$ for any other domain $\Omega \in \mathbb{B}$.*

Proof. Note first that minimising μ_1 in the wider class

$$\mathbb{B}_0 = \{\Omega \subset \mathbb{R}^2; \Omega \text{ open, simply connected, } |\Omega| = \pi\}$$

is equivalent to minimising μ_1 in $\mathbb{B}$, where we understand that "simply connected" means that "each connected component is simply connected". Indeed, if we find a minimiser in $\mathbb{B}_0$, then it is necessarily connected since otherwise, scaling one of its connected components and noticing that $\Omega \mapsto \mu_1(\Omega)$ is homogeneous of degree -2, would contradict minimality.

So, consider a minimising sequence $(\Omega_m) \subset \mathbb{B}_0$ with (u_m) being the corresponding sequence of normalised eigenfunctions, that is, $\int_{\Omega_m} |\nabla u_m|^2 = 1$. Extending u_m by 0 in $\mathbb{R}^2 \setminus \Omega_m$ we may view (u_m) as a bounded sequence in $H^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^2} |\Delta u_m|^2 \, dx \le C_1\,, \quad \int_{\mathbb{R}^2} |\nabla u_m|^2 \, dx = 1\,, \quad \int_{\mathbb{R}^2} |u_m|^2 \, dx \le C_2, \tag{3.27}$$

for suitable $C_1, C_2 > 0$, the L^2-bound following from Poincaré's inequality in $H_0^1(\Omega_m)$ and the fact that $|\Omega_m| = \pi$ for all m. In particular, (3.27) shows that

$$\inf_{\Omega \in \mathbb{B}_0} \mu_1(\Omega) > 0. \tag{3.28}$$

Indeed, we have

$$1 = \int_{\mathbb{R}^2} |\nabla u_m|^2 \, dx = -\int_{\mathbb{R}^2} u_m \Delta u_m \, dx \le C_2^{1/2} \left(\int_{\mathbb{R}^2} |\Delta u_m|^2 \, dx \right)^{1/2},$$

which proves that $\int_{\mathbb{R}^2} |\Delta u_m|^2$ is also bounded away from 0. By (3.27), we may also apply the concentration-compactness principle [276] and deduce that, up to a subsequence, three cases may occur.

1. Vanishing.
$$\lim_{m \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_R(y)} |\nabla u_m|^2 \, dx = 0 \qquad \text{for all } R > 0.$$

2. Dichotomy. There exists $\alpha \in (0,1)$ such that for all $\varepsilon > 0$ there exist two bounded sequences $(u_m^{(1)})$, $(u_m^{(2)}) \subset H^2(\mathbb{R}^2)$ such that
$$\lim_{m \to \infty} \operatorname{dist}\Big(\operatorname{support}(u_m^{(1)}), \operatorname{support}(u_m^{(2)})\Big) = +\infty,$$

$$\lim_{m\to\infty}\int_{\mathbb{R}^2}|\nabla u_m^{(1)}|^2\,dx\to\alpha,\qquad \lim_{m\to\infty}\int_{\mathbb{R}^2}|\nabla u_m^{(2)}|^2\,dx\to 1-\alpha, \tag{3.29}$$

$$\lim_{m\to\infty}\int_{\mathbb{R}^2}\left||\nabla u_m|^2-|\nabla u_m^{(1)}|^2-|\nabla u_m^{(2)}|^2\right|dx\le\varepsilon, \tag{3.30}$$

$$\lim_{m\to\infty}\int_{\mathbb{R}^2}\left(|\Delta u_m|^2-|\Delta u_m^{(1)}|^2-|\Delta u_m^{(2)}|^2\right)dx\ge 0. \tag{3.31}$$

3. Compactness. There exists a sequence $(y_m)\subset\mathbb{R}^2$ such that for all $\varepsilon>0$ there exists $R>0$ and

$$\int_{B_R(y_m)}|\nabla u_m|^2\,dx\ge 1-\varepsilon\qquad\text{for all } m.$$

We first show that vanishing cannot occur. By contradiction, assume that vanishing occurs. Up to a permutation of x_1 and x_2 and up to a subsequence, by (3.27) we have

$$\int_{\mathbb{R}^2}\left(\frac{\partial u_m}{\partial x_1}\right)^2 dx\ge\frac{1}{2}.$$

Moreover, since two integrations by parts yield

$$\int_{\mathbb{R}^2}\frac{\partial^2 u_m}{\partial x_1^2}\frac{\partial^2 u_m}{\partial x_2^2}\,dx=\int_{\mathbb{R}^2}\left(\frac{\partial^2 u_m}{\partial x_1\partial x_2}\right)^2 dx\ge 0,$$

we remark that

$$\int_{\mathbb{R}^2}|\Delta u_m|^2\,dx\ge\int_{\mathbb{R}^2}\left|\nabla\frac{\partial u_m}{\partial x_1}\right|^2 dx.$$

Therefore, we infer that

$$\frac{\|\Delta u_m\|_{L^2}^2}{\|\nabla u_m\|_{L^2}^2}\ \ge\ \frac{1}{2}\,\frac{\|\nabla\frac{\partial u_m}{\partial x_1}\|_{L^2}^2}{\|\frac{\partial u_m}{\partial x_1}\|_{L^2}^2}. \tag{3.32}$$

By assumption any translation of $\frac{\partial u_m}{\partial x_1}$ converges weakly to 0 in $L^2(\mathbb{R}^2)$. Moreover, $\frac{\partial u_m}{\partial x_1}\in H_0^1(\Omega_m)$ and $|\Omega_m|=\pi$. Hence, we may apply [81, Lemma 3.3] to get that, up to a subsequence, $\|\nabla\frac{\partial u_m}{\partial x_1}\|_{L^2}\to\infty$. Since the left hand side of (3.32) is supposed to converge to $\inf_{\Omega\in\mathbb{B}_0}\ \mu_1(\Omega)$, we get a contradiction.

Next, we show that dichotomy cannot occur. By contradiction, assume that dichotomy occurs and fix $\varepsilon>0$. Then the sequences $(u_m^{(1)})$ and $(u_m^{(2)})$ can be chosen as follows, see [276]. Let B_2 denote the ball of radius 2 centered at the origin and let $\varphi\in C_c^\infty(B_2,[0,1])$ be such that $\varphi\equiv 1$ in B (the unit ball). Then for suitable sequences $(R_m),(\rho_m)\to\infty$, we put

$$u_m^{(1)}(x):=\varphi\left(\frac{x}{R_m}\right)u_m(x)\,,\qquad u_m^{(2)}(x):=\left(1-\varphi\left(\frac{x}{\rho_mR_m}\right)\right)u_m(x).$$

Note that support$(u_m^{(1)}) \subset (\overline{\Omega_m} \cap B_{2R_m})$ whereas support$(u_m^{(2)}) \subset (\overline{\Omega_m} \setminus B_{\rho_m R_m})$. By elementary calculus we know that $\frac{x_1+x_2}{y_1+y_2} \ge \min\{\frac{x_1}{y_1}, \frac{x_2}{y_2}\}$ for all $x_1, x_2, y_1, y_2 > 0$. Hence, by (3.30) and (3.31), up to a switch between $u_m^{(1)}$ and $u_m^{(2)}$, we have

$$\inf_{\Omega\in\mathbb{B}_0} \mu_1(\Omega) = \lim_{m\to\infty} \frac{\|\Delta u_m\|^2_{L^2(\Omega_m)}}{\|\nabla u_m\|^2_{L^2(\Omega_m)}} \ge \limsup_{m\to\infty} \frac{\|\Delta u_m^{(1)}\|^2_{L^2(\Omega_m\cap B_{2R_m})}}{\varepsilon + \|\nabla u_m^{(1)}\|^2_{L^2(\Omega_m\cap B_{2R_m})}}. \tag{3.33}$$

Up to a further subsequence, the above "limsup" becomes a limit.

We now claim that there exists $\delta > 0$ such that for m large enough

$$|\Omega_m \setminus B_{\rho_m R_m}| \ge \delta. \tag{3.34}$$

Indeed, if (3.34) were not true, up to a subsequence we would have $\lim_{m\to\infty} |\Omega_m \setminus B_{\rho_m R_m}| = 0$ implying that $\Lambda_{1,1}(\Omega_m \setminus B_{\rho_m R_m}) \to \infty$. In view of (3.26), this would imply $\mu_1(\Omega_m \setminus B_{\rho_m R_m}) \to \infty$. In turn, since (3.29) states that $\|\nabla u_m^{(2)}\|_{L^2(\mathbb{R}^2)}$ is bounded away from zero, this implies that $\|\Delta u_m^{(2)}\|_{L^2(\mathbb{R}^2)} \to \infty$, contradicting (3.27) and (3.31). A similar argument also shows that $\delta > 0$ in (3.34) may be chosen to be independent of ε.

By (3.34) we know that there exists $\gamma \in (0,1)$, independent of ε, such that

$$\limsup_{m\to\infty} |\Omega_m \cap B_{2R_m}| = \gamma\pi \le \pi - \delta.$$

Up to a further subsequence, also the above "limsup" becomes a limit. Combined with (3.28), (3.33), and homogeneity of μ_1, this yields

$$\begin{aligned}
\inf_{\Omega\in\mathbb{B}_0} \mu_1(\Omega) &\ge \lim_{m\to\infty} \frac{\|\Delta u_m^{(1)}\|^2_{L^2(\Omega_m\cap B_{2R_m})}}{\varepsilon + \|\nabla u_m^{(1)}\|^2_{L^2(\Omega_m\cap B_{2R_m})}} \\
&\ge \lim_{m\to\infty} \frac{\|\Delta u_m^{(1)}\|^2_{L^2(\Omega_m\cap B_{2R_m})}}{\|\nabla u_m^{(1)}\|^2_{L^2(\Omega_m\cap B_{2R_m})}} \, \frac{\|\nabla u_m^{(1)}\|^2_{L^2(\Omega_m\cap B_{2R_m})}}{\varepsilon + \|\nabla u_m^{(1)}\|^2_{L^2(\Omega_m\cap B_{2R_m})}} \\
&= \frac{\alpha}{\varepsilon+\alpha} \lim_{m\to\infty} \mu_1(\Omega_m \cap B_{2R_m}) \\
&= \frac{\alpha}{\varepsilon+\alpha} \lim_{m\to\infty} \left(\frac{\pi}{|\Omega_m \cap B_{2R_m}|} \right)^2 \mu_1\left(\pi \frac{\Omega_m \cap B_{2R_m}}{|\Omega_m \cap B_{2R_m}|} \right) \\
&= \frac{\alpha}{\varepsilon+\alpha} \frac{1}{\gamma^2} \lim_{m\to\infty} \mu_1\left(\pi \frac{\Omega_m \cap B_{2R_m}}{|\Omega_m \cap B_{2R_m}|} \right) \\
&\ge \frac{\alpha}{\varepsilon+\alpha} \frac{1}{\gamma^2} \inf_{\Omega\in\mathbb{B}_0} \mu_1(\Omega)
\end{aligned}$$

since $\pi \frac{\Omega_m \cap B_{2R_m}}{|\Omega_m \cap B_{2R_m}|} \in \mathbb{B}_0$. As $\gamma < 1$, by arbitrariness of ε we get a contradiction which rules out dichotomy.

Since we excluded both vanishing and dichotomy, compactness necessarily occurs. Then by arbitrariness of ε, we infer that there exist $(y_m) \subset \mathbb{R}^2$ and $u \in H^2(\mathbb{R}^2)$ such that

$$u_m(\,.+y_m) \rightharpoonup u \quad \text{in } H^2(\mathbb{R}^2) \qquad \text{and} \qquad \|\nabla u\|_{L^2(\mathbb{R}^2)} = 1.$$

By combining $u_m \rightharpoonup u$ in $H^1(\mathbb{R}^2)$ and the conservation of norms, we deduce that $u_m \to u$ in the norm topology of $H^1(\mathbb{R}^2)$. In turn, by Poincaré's inequality applied in domains of uniformly bounded measure, this yields $u_m \to u$ in $L^2(\mathbb{R}^2)$. Therefore, it follows that (Ω_m) converges in the Hausdorff topology to some simply connected domain $\widehat{\Omega} \subset \mathbb{R}^2$ such that $\widehat{\Omega} \supset \text{support}(u)$ and

$$|\widehat{\Omega}| \leq \pi. \tag{3.35}$$

From weak convergence in H^2 and strong convergence in H^1, we get

$$\mu_1(\widehat{\Omega}) \leq \frac{\|\Delta u\|^2_{L^2(\widehat{\Omega})}}{\|\nabla u\|^2_{L^2(\widehat{\Omega})}} \leq \liminf_{m\to\infty} \frac{\|\Delta u_m\|^2_{L^2(\Omega_m)}}{\|\nabla u_m\|^2_{L^2(\Omega_m)}} = \inf_{\Omega \in \mathbb{B}_0} \mu_1(\Omega).$$

By (3.35) and homogeneity of μ_1 we infer that all the above inequalities are in fact equalities. So, the minimiser for μ_1 is found. □

As pointed out in Section 1.3.2, the next step would be to show that the minimiser Ω_o found in Theorem 3.15 has some regularity properties. But already for second order equations this is a very difficult task, see [228]. However, *assuming* smoothness of the boundary, one can show (see [415]) that the optimal domain is indeed the disk.

Theorem 3.16. *If the minimiser Ω_o found in Theorem 3.15 has $C^{2,\gamma}$ boundary, then it is a disk.*

Proof. Let Ω_o be the $C^{2,\gamma}$ minimiser found in Theorem 3.15 and let ϕ denote the corresponding first eigenfunction, namely a solution to (3.25) when $\Omega = \Omega_o$. By performing the shape derivative [228] of the map $\Omega \mapsto \mu_1(\Omega)$ and using the optimality of Ω_o one finds that

$$\Delta\phi \text{ exists and is constant on } \partial\Omega_o. \tag{3.36}$$

We point out that this first step is precisely the part of the proof where smoothness of $\partial\Omega_o$ is needed. Moreover, the connectedness of the boundary $\partial\Omega_o$ is here crucial in order to deduce (3.36).

Since $\phi = 0$ on $\partial\Omega_o$, (3.36) also implies that $\Delta\phi + \mu_1\phi$ is constant on $\partial\Omega_o$. In turn, since $\phi \mapsto \Delta\phi + \mu_1\phi$ is harmonic in Ω_o in view of (3.25), this implies

$$\Delta\phi + \mu_1\phi \quad \text{is constant in } \Omega_o. \tag{3.37}$$

The function ϕ has a critical point in Ω_o which we may assume to be the origin so that $\nabla\phi(0) = 0$.

Next, for $(x,y) \in \Omega_o$ define $w(x,y) := x\phi_y(x,y) - y\phi_x(x,y)$. In polar coordinates (r,θ) this can be written as $w = \phi_\theta$. Therefore, if $w \equiv 0$, then ϕ does not depend on θ so that Ω_o is a disk and we are done. So, assume by contradiction that $w \not\equiv 0$. Since $\phi \in H_0^2(\Omega_o)$, we have $w \in H_0^1(\Omega_o)$ and from (3.37) we deduce that $-\Delta w = \mu_1 w$ in Ω_o. Hence, μ_1 is a Dirichlet eigenvalue for $-\Delta$ in Ω_o and it is the first Dirichlet eigenvalue in each of the nodal zones of w.

Note that $w_x = \phi_y + x\phi_{xy} - y\phi_{xx}$ so that $w_x(0) = 0$, recalling that $\nabla\phi(0) = 0$. Similarly, $w_y(0) = 0$. Hence, both w and ∇w vanish at the origin. This means that the origin 0 is a nodal point of w and a point where a nodal line intersects itself transversally. But then, for topological reasons, this nodal line divides Ω_o into at least three nodal domains and at least one has a measure not exceeding $|\Omega_o|/3$. This would imply the following chain of inequalities

$$\begin{aligned}\mu_1(\Omega_o) &= \Lambda_{1,1}(\text{subdomain of measure } \le |\Omega_o|/3) \ge \Lambda_{1,1}(\text{ball of measure } |\Omega_o|/3)\\ &= 3\Lambda_{1,1}(\Omega_o^*) > \Lambda_{1,2}(\Omega_o^*) = \mu_1(\Omega_o^*),\end{aligned}$$

which contradicts the minimality of Ω_o. In this chain of inequalities we have used one after the other the monotonicity of $\Lambda_{1,1}$ with respect to domain inclusions, the Faber-Krahn inequality [162, 253, 254], a scaling argument, an inequality from [335] and (3.26). $\square$

3.3 Steklov eigenvalues

Let $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$, let $a \in \mathbb{R}$ and consider the boundary eigenvalue problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = \Delta u - a u_\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.38}$$

We are interested in studying the eigenvalues of (3.38), namely those values of a for which the problem admits nontrivial solutions, the corresponding eigenfunctions. By a solution of (3.38) we mean a function $u \in H^2 \cap H_0^1(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v \, dx = a \int_{\partial\Omega} u_\nu v_\nu \, d\omega \qquad \text{for all } v \in H^2 \cap H_0^1(\Omega). \tag{3.39}$$

By taking $v = u$ in (3.39), it is clear that all the eigenvalues of (3.38) are strictly positive.

3.3.1 The Steklov spectrum

The least positive eigenvalue of (3.38) may be characterised variationally as

$$\delta_1 = \delta_1(\Omega) := \min\left\{ \frac{\|\Delta u\|^2_{L^2(\Omega)}}{\|u_\nu\|^2_{L^2(\partial\Omega)}};\ u \in [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega) \right\}. \tag{3.40}$$

We first prove the existence of a function $u \in [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega)$ which achieves equality in (3.40), provided the domain Ω is smooth (C^2) or satisfies a geometric condition which is fulfilled if Ω has no "reentrant corners" (for instance, if Ω is convex). More precisely, we consider domains satisfying a uniform outer ball condition according to Definition 2.30. Then the following existence result for a minimiser of $\delta_1(\Omega)$ holds.

Theorem 3.17. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary and satisfying a uniform outer ball condition. Then the minimum in (3.40) is achieved and, up to a multiplicative constant, the minimiser $\overline{u}$ for (3.40) is unique, superharmonic in Ω (in particular, $\overline{u} > 0$ in Ω and $\overline{u}_\nu < 0$ on $\partial\Omega$) and it solves (3.38) when $a = \delta_1$. Furthermore, $\overline{u} \in C^\infty(\Omega)$ and, up to the boundary, $\overline{u}$ is as smooth as the boundary permits.*

Proof. By Theorem 2.31 we know that $u \mapsto \|\Delta u\|_{L^2}$ is a norm in $H^2(\Omega)$. Let (u_m) be a minimising sequence for $\delta_1(\Omega)$ with $\|\Delta u_m\|_{L^2} = 1$ so that (u_m) is bounded in $H^2(\Omega)$. Up to a subsequence, we may assume that there exists $u \in H^2 \cap H^1_0(\Omega)$ such that $u_m \rightharpoonup u$ in $H^2(\Omega)$. Moreover, since Ω is Lipschitzian and satisfies a uniform outer ball condition, by [321, Chapter 2, Theorem 6.2] we infer that the map

$$H^2 \cap H^1_0(\Omega) \ni u \mapsto \nabla u|_{\partial\Omega} \in \left(L^2(\partial\Omega)\right)^n$$

is well-defined and compact. Hence, we deduce that $(u_m)_\nu \to u_\nu$ in $L^2(\partial\Omega)$ and that $\delta_1(\Omega) > 0$.

Furthermore, since (u_m) is a minimising sequence, $\|\Delta u_m\|_{L^2} = 1$ holds, and $\|(u_m)_\nu\|_{L^2(\partial\Omega)}$ is bounded from below, u_ν is not identically zero on $\partial\Omega$ and

$$\|u_\nu\|^{-2}_{L^2(\partial\Omega)} = \lim_{m\to\infty} \|(u_m)_\nu\|^{-2}_{L^2(\partial\Omega)} = \delta_1(\Omega).$$

Moreover, by weak lower semicontinuity of the norm, we also have

$$\|\Delta u\|^2_{L^2} \le \liminf_{m\to\infty} \|\Delta u_m\|^2_{L^2} = 1$$

and hence $u \in [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega)$ satisfies

$$\frac{\|\Delta u\|^2_{L^2(\Omega)}}{\|u_\nu\|^2_{L^2(\partial\Omega)}} \le \delta_1(\Omega).$$

This proves that u is a minimiser for $\delta_1(\Omega)$.

For all $u \in [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega)$ put

$$I(u) := \frac{\|\Delta u\|^2_{L^2(\Omega)}}{\|u_\nu\|^2_{L^2(\partial\Omega)}}.$$

To show that, up to their sign, the minimisers for (3.40) are superharmonic, we observe that for all $u \in [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega)$ there exists $w \in [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega)$ such that $-\Delta w \geq 0$ in Ω and $I(w) \leq I(u)$. Indeed, for a given u, let w be the unique solution of

$$\begin{cases} -\Delta w = |\Delta u| & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

so that w is superharmonic. Moreover, both $w \pm u$ are superharmonic in Ω and vanish on $\partial\Omega$. This proves that

$$|u| \leq w \quad \text{in } \Omega\,, \qquad |u_\nu| \leq |w_\nu| \quad \text{on } \partial\Omega.$$

In turn, these inequalities (and $-\Delta w = |\Delta u|$) prove that $I(w) \leq I(u)$. We emphasise that this inequality is strict if Δu changes sign.

Any minimiser $\overline{u}$ for (3.40) solves the Euler equation (3.38) and is a smooth function in view of elliptic theory, see the explanation just after (2.22). In order to conclude the proof we still have to show that the minimiser $\overline{u}$ is unique. By contradiction, let $v \in H^2 \cap H^1_0(\Omega)$ be another positive minimiser and for every $c \in \mathbb{R}$, define $v_c := v + c\overline{u}$. Exploiting the fact that both v and $\overline{u}$ solve (3.38) when $a = \delta_1$, we see that also v_c is a minimiser. But unless v is a multiple of $\overline{u}$, there exists some c such that v_c changes sign in Ω. This leads to a contradiction and completes the proof. □

We are now interested in the description of the spectrum of (3.38). To this end, we restrict our attention to smooth domains. As in (2.10), the Hilbert space $H^2 \cap H^1_0(\Omega)$ is endowed with the scalar product

$$(u,v) \mapsto \int_\Omega \Delta u \Delta v\, dx. \tag{3.41}$$

Consider the space

$$Z = \left\{ v \in C^\infty(\overline{\Omega}) : \ \Delta^2 u = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega \right\} \tag{3.42}$$

and let V denote the completion of Z with respect to the scalar product in (3.41). Then we prove

Theorem 3.18. *Assume that $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded domain with C^2-boundary. Then problem (3.38) admits infinitely many (countable) eigenvalues. The only eigenfunction of one sign is the one corresponding to the first eigenvalue. The set of eigenfunctions forms a complete orthonormal system in V.*

Proof. Let Z be as in (3.42), define on Z the scalar product given by

$$(u,v)_W := \int_{\partial\Omega} u_\nu v_\nu \, d\omega \qquad \text{for all } u,v \in Z$$

and let W denote the completion of Z with respect to this scalar product. We first claim that the (Hilbert) space V is compactly embedded into the (Hilbert) space W. Indeed, by definition of δ_1 we have

$$\|u\|_W = \|u_\nu\|_{L^2(\partial\Omega)} \le \delta_1^{-1/2} \|\Delta u\|_{L^2(\Omega)} = \delta_1^{-1/2} \|u\|_V \qquad \text{for all } u \in Z. \quad (3.43)$$

Hence any Cauchy sequence in Z with respect to the norm of V is a Cauchy sequence with respect to the norm of W. Since V is the completion of Z with respect to (3.41), it follows that $V \subset W$. The continuity of this inclusion can be obtained by density from (3.43). In order to prove that this embedding is compact, let $u_m \rightharpoonup u$ in V, so that also $u_m \rightharpoonup u$ in $H^2 \cap H^1_0(\Omega)$. Then by the compact trace embedding $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$ we obtain $u_m \to u$ in W. This proves the claim.

Let $I_1 : V \to W$ denote the embedding $V \subset W$ and $I_2 : W \to V'$ the linear continuous operator defined by

$$\langle I_2 u, v\rangle = (u,v)_W \qquad \text{for all } u \in W \text{ and } v \in V.$$

Moreover, let $L : V \to V'$ be the linear operator given by

$$\langle Lu, v\rangle = \int_\Omega \Delta u \Delta v \, dx \qquad \text{for all } u,v \in V.$$

Then by the Lax-Milgram theorem, L is an isomorphism and in view of the compact embedding $V \subset W$, the linear operator $K = L^{-1} I_2 I_1 : V \to V$ is compact. Since for $n \ge 2$, V is an infinite dimensional Hilbert space and K is a compact self-adjoint operator with strictly positive eigenvalues, V admits an orthonormal basis of eigenfunctions of K and the set of the eigenvalues of K can be ordered in a strictly decreasing sequence (μ_i) which converges to zero. Therefore problem (3.39) admits an infinite set of eigenvalues given by $\delta_i = \frac{1}{\mu_i}$ and the eigenfunctions of (3.39) coincide with the eigenfunctions of K.

To complete the proof we need to show that if δ_k is an eigenvalue of (3.38) corresponding to a positive eigenfunction ϕ_k then necessarily $\delta_k = \delta_1$. So, take $\phi_k > 0$ in Ω and $\phi_k = 0$ on $\partial\Omega$; then $(\phi_k)_\nu \le 0$ on $\partial\Omega$ and, in turn, $\Delta\phi_k = \delta_k (\phi_k)_\nu \le 0$ on $\partial\Omega$. Therefore, by $\Delta^2 \phi_k = 0$ in Ω and the weak comparison principle, we infer $\Delta\phi_k \le 0$ in Ω. Moreover, since $\phi_k > 0$ in Ω and $\phi_k = 0$ on $\partial\Omega$, the Hopf boundary lemma implies that $(\phi_k)_\nu < 0$ on $\partial\Omega$. Let ϕ_1 be a positive eigenfunction corresponding to the first eigenvalue δ_1, see Theorem 3.17. Then ϕ_1 satisfies $(\phi_1)_\nu < 0$ on $\partial\Omega$ and hence from

$$\delta_k \int_{\partial\Omega} (\phi_k)_\nu (\phi_1)_\nu \, d\omega = \int_\Omega \Delta\phi_k \Delta\phi_1 \, dx = \delta_1 \int_{\partial\Omega} (\phi_k)_\nu (\phi_1)_\nu \, d\omega > 0$$

we obtain $\delta_k = \delta_1$. This completes the proof of Theorem 3.18. $\square$

The vector space V also has a different interesting characterisation.

Theorem 3.19. *Assume that $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded domain with C^2-boundary. Then the space $H^2 \cap H^1_0(\Omega)$ admits the following orthogonal decomposition with respect to the scalar product (3.41)*

$$H^2 \cap H^1_0(\Omega) = V \oplus H^2_0(\Omega).$$

Moreover, if $v \in H^2 \cap H^1_0(\Omega)$ and if $v = v_1 + v_2$ is the corresponding orthogonal decomposition, then $v_1 \in V$ and $v_2 \in H^2_0(\Omega)$ are weak solutions of

$$\begin{cases} \Delta^2 v_1 = 0 & \text{in } \Omega, \\ v_1 = 0 & \text{on } \partial\Omega, \\ (v_1)_\nu = v_\nu & \text{on } \partial\Omega, \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta^2 v_2 = \Delta^2 v & \text{in } \Omega, \\ v_2 = 0 & \text{on } \partial\Omega, \\ (v_2)_\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.44}$$

Proof. We start by proving that $Z^\perp = H^2_0(\Omega)$. Let $v \in Z$ and $w \in H^2 \cap H^1_0(\Omega)$. After two integrations by parts we obtain

$$\int_\Omega \Delta v \Delta w \, dx = \int_\Omega \Delta^2 v \, w \, dx + \int_{\partial\Omega} (w_\nu \Delta v - w(\Delta v)_\nu) \, d\omega = \int_{\partial\Omega} w_\nu \Delta v \, d\omega$$

for all $v \in Z$ and $w \in H^2 \cap H^1_0(\Omega)$. This proves that $w_\nu = 0$ on $\partial\Omega$ if and only if $w \in Z^\perp$ and hence $V^\perp = Z^\perp = H^2_0(\Omega)$.

Let $v \in H^2 \cap H^1_0(\Omega)$ and consider the first Dirichlet problem in (3.44), that is

$$\begin{cases} \Delta^2 v_1 = 0 & \text{in } \Omega, \\ v_1 = 0 & \text{on } \partial\Omega, \\ (v_1)_\nu = v_\nu & \text{on } \partial\Omega. \end{cases} \tag{3.45}$$

Since $v_\nu \in H^{1/2}(\partial\Omega)$, by Lax-Milgram's theorem and [275, Ch. 1, Théorème 8.3], we deduce that (3.45) admits a unique solution $v_1 \in H^2 \cap H^1_0(\Omega)$ such that

$$\|\Delta v_1\|_{L^2(\Omega)} \leq C \|v_\nu\|_{H^{1/2}(\partial\Omega)}.$$

This proves that $v_1 \in V$. Let $v_2 = v - v_1$, then $(v_2)_\nu = 0$ on $\partial\Omega$ and, in turn, $v_2 \in H^2_0(\Omega)$. Moreover, by (3.45) we infer

$$\int_\Omega \Delta v_2 \Delta w \, dx = \int_\Omega \Delta v \Delta w \, dx - \int_\Omega \Delta v_1 \Delta w \, dx = \int_\Omega \Delta v \Delta w \, dx \quad \text{for all } w \in H^2_0(\Omega)$$

which proves that v_2 is a weak solution of the second problem in (3.44). $\square$

When $\Omega = B$ (the unit ball in $\mathbb{R}^n$, $n \geq 2$) all the eigenvalues of (3.38) can be determined explicitly. To this end, consider the spaces of harmonic homogeneous polynomials

$$\mathscr{P}_k := \{P \in C^\infty(\mathbb{R}^n);\ \Delta P = 0 \text{ in } \mathbb{R}^n,\ P \text{ is a homogeneous polynomial of degree } k-1\}.$$

Also, let μ_k be the dimension of $\mathscr{P}_k$. By [17, p. 450] we know that

$$\mu_k = \frac{(2k+n-4)\,(k+n-4)!}{(k-1)!\,(n-2)!}.$$

In particular, we have

$$\mathscr{P}_1 = \operatorname{span}\{1\}\ , \qquad \mu_1 = 1,$$

$$\mathscr{P}_2 = \operatorname{span}\{x_i;\ i=1,\dots,n\}\ , \qquad \mu_2 = n,$$

$$\mathscr{P}_3 = \operatorname{span}\{x_i x_j; x_1^2 - x_h^2;\ i,j=1,\dots,n,\ i\neq j,\ h=2,\dots,n\}\ , \quad \mu_3 = \frac{n^2+n-2}{2}.$$

Then we prove

Theorem 3.20. *If $n \geq 2$ and $\Omega = B$, then for all $k = 1,2,3,\dots$*

1. *the eigenvalues of (3.38) are $\delta_k = n + 2(k-1)$;*
2. *the multiplicity of δ_k equals μ_k;*
3. *for all $\psi_k \in \mathscr{P}_k$, the function $\phi_k(x) := (1-|x|^2)\psi_k(x)$ is an eigenfunction corresponding to δ_k.*

Proof. Let $u \in C^\infty(\overline{B})$ be an eigenfunction of (3.38) so that $u = 0$ on ∂B. Therefore, we can write

$$u(x) = (1-|x|^2)\phi(x) \qquad (x \in B) \tag{3.46}$$

for some $\phi \in C^\infty(\overline{B})$. We have $u_{x_i} = -2x_i\phi + (1-|x|^2)\phi_{x_i}$, and on ∂B,

$$u_\nu = x\cdot\nabla u = -2\phi. \tag{3.47}$$

Moreover,

$$\Delta u = -2n\phi - 4x\cdot\nabla\phi + (1-|x|^2)\Delta\phi. \tag{3.48}$$

Hence,

$$\Delta u = -2n\phi - 4\phi_\nu \qquad \text{on } \partial B. \tag{3.49}$$

From (3.48) we get for $i = 1,\dots,n$,

$$(\Delta u)_{x_i} = -(2n+4)\phi_{x_i} - 4\sum_{j=1}^n x_j\phi_{x_j x_i} - 2x_i\Delta\phi + (1-|x|^2)\Delta\phi_{x_i},$$

and therefore

$$(\Delta u)_{x_i x_i} = -2(n+4)\phi_{x_i x_i} - 4x\cdot\nabla(\phi_{x_i x_i}) - 2\Delta\phi - 4x_i(\Delta\phi)_{x_i} + (1-|x|^2)\Delta\phi_{x_i x_i}.$$

Summing with respect to i and recalling that u is biharmonic in B, we obtain

$$\begin{aligned} 0 = \Delta^2 u &= -2(n+4)\Delta\phi - 4x\cdot\nabla\Delta\phi - 2n\Delta\phi - 4x\cdot\nabla\Delta\phi + (1-|x|^2)\Delta^2\phi \\ &= (1-|x|^2)\Delta^2\phi - 8x\cdot\nabla\Delta\phi - 4(n+2)\Delta\phi. \end{aligned} \tag{3.50}$$

Writing (3.50) as an equation in $w = \Delta\phi$, we get

$$(1-|x|^2)\Delta w - 8x\cdot\nabla w - 4(n+2)w = 0 \qquad \text{in } B,$$

so that

$$\begin{aligned} 0 &= -(1-|x|^2)^4\Delta w + 8(1-|x|^2)^3 x\cdot\nabla w + 4(n+2)(1-|x|^2)^3 w \\ &= -\mathrm{div}\Big((1-|x|^2)^4\nabla w\Big) + 4(n+2)(1-|x|^2)^3 w. \end{aligned} \tag{3.51}$$

Multiplying the right hand side of (3.51) by w and integrating by parts over B, we obtain

$$\int_B (1-|x|^2)^4|\nabla w|^2\,dx + 4(n+2)\int_B (1-|x|^2)^3 w^2\,dx = \int_{\partial B}(1-|x|^2)^4 w w_\nu\,d\omega = 0.$$

Hence $\Delta\phi = w = 0$ in B. Now from (3.38), (3.47) and (3.49) we get

$$\phi_\nu = \frac{a-n}{2}\phi \qquad \text{on } \partial B.$$

Therefore, the number a is an eigenvalue of (3.38) with corresponding eigenfunction u if and only if ϕ defined by (3.46) is an eigenfunction of the boundary eigenvalue problem

$$\begin{cases} \Delta\phi = 0 & \text{in } B, \\ \phi_\nu = \gamma\phi & \text{on } \partial B, \end{cases} \tag{3.52}$$

where

$$\gamma = \frac{a-n}{2}. \tag{3.53}$$

We are so led to study the eigenvalues of the second order Steklov problem (3.52). Let us quickly explain how to obtain them. In radial and angular coordinates (r,θ), the equation in (3.52) reads

$$\frac{\partial^2\phi}{\partial r^2} + \frac{n-1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\Delta_\theta\phi = 0,$$

where Δ_θ denotes the Laplace-Beltrami operator on ∂B. From [47, p. 160] we know that $-\Delta_\theta$ admits a sequence of eigenvalues (λ_k) having multiplicity μ_k equal to the number of independent harmonic homogeneous polynomials of degree $k-1$. Moreover, $\lambda_k = (k-1)(n+k-3)$.

Let us write e_k^j $(j = 1,\dots,\mu_k)$ for the independent normalised eigenfunctions corresponding to λ_k. Then one seeks functions $\phi = \phi(r,\theta)$ of the kind

$$\phi(r,\theta) = \sum_{k=1}^{\infty}\sum_{j=1}^{\mu_k}\phi_k^j(r)e_k^j(\theta).$$

Hence, by differentiating the series, we obtain

$$\Delta\phi(r,\theta)=\sum_{k=1}^{\infty}\sum_{j=1}^{\mu_k}\left(\frac{d^2}{dr^2}\phi_k^j(r)+\frac{n-1}{r}\frac{d}{dr}\phi_k^j(r)-\frac{\lambda_k}{r^2}\phi_k^j(r)\right)e_k^j(\theta)=0.$$

Therefore, we are led to solve the equations

$$\frac{d^2}{dr^2}\phi_k^j(r)+\frac{n-1}{r}\frac{d}{dr}\phi_k^j(r)-\frac{\lambda_k}{r^2}\phi_k^j(r)=0 \qquad k=1,2\dots \quad j=1,\dots,\mu_k. \tag{3.54}$$

With the change of variables $r=e^t$ ($t\le 0$), equation (3.54) becomes a linear constant coefficients equation. It has two linearly independent solutions, but one is singular. Hence, up to multiples, the only regular solution of (3.54) is given by $\phi_k^j(r)=r^{k-1}$ because

$$\frac{2-n+\sqrt{(n-2)^2+4\lambda_k}}{2}=k-1.$$

Since the boundary condition in (3.52) reads $\frac{d}{dr}\phi_k^j(1)=\gamma\phi_k^j(1)$ we immediately infer that $\gamma=\bar{k}-1$ for some $\bar{k}$. In turn, (3.53) tells us that

$$\delta_{\bar{k}}=n+2(\bar{k}-1).$$

The proof of Theorem 3.20 is so complete. □

Remark 3.21. Theorems 3.18 and 3.20 become false if $n=1$ since the problem

$$u^{iv}=0 \text{ in } (-1,1)\ , \quad u(\pm 1)=u''(-1)+au'(-1)=u''(1)-au'(1)=0, \tag{3.55}$$

admits only *two* eigenvalues, $\delta_1=1$ and $\delta_2=3$, each one of multiplicity 1. The reason of this striking difference is that the "boundary space" of (3.55) has precisely dimension 2, one for each endpoint of the interval $(-1,1)$. This result is consistent with Theorem 3.20 since $\mu_1=\mu_2=1$ and $\mu_3=0$ whenever $n=1$.

By combining Theorems 3.18 and 3.20 we obtain

Corollary 3.22. *Assume that $n\ge 2$ and that $\Omega=B$. Assume moreover that for all $k\in\mathbb{N}^+$ the set $\{\psi_k^j : j=1,\dots,\mu_k\}$ is a basis of $\mathscr{P}_k$ chosen in such a way that the corresponding functions ϕ_k^j are orthonormal with respect to the scalar product (3.41). Then for any $u\in V$ there exists a sequence $(\alpha_k^j)\subset\ell^2$ ($k\in\mathbb{N}^+$; $j=1,\dots,\mu_k$) such that*

$$u(x)=(1-|x|^2)\sum_{k=1}^{\infty}\sum_{j=1}^{\mu_k}\alpha_k^j\psi_k^j(x) \qquad \textit{for a.e. } x\in B.$$

3.3.2 Minimisation of the first eigenvalue

In this section we take advantage of Theorem 3.17 and we study several aspects of the first Steklov eigenvalue δ_1.

We first give an alternative characterisation of $\delta_1(\Omega)$. Let

$$C_H^2\left(\overline{\Omega}\right) := \left\{v \in C^2\left(\overline{\Omega}\right);\ \Delta v = 0 \text{ in } \Omega\right\}$$

and consider the norm defined by $\|v\|_H := \|v\|_{L^2(\partial\Omega)}$ for all $v \in C_H^2\left(\overline{\Omega}\right)$. Then define

$$\mathbf{H} := \text{the completion of } C_H^2\left(\overline{\Omega}\right) \text{ with respect to the norm } \|\cdot\|_H .$$

Since Ω is assumed to have a Lipschitz boundary, we infer by [238] that $\mathbf{H} \subset H^{1/2}(\Omega) \subset L^2(\Omega)$. Therefore, the quantity

$$\sigma_1(\Omega) := \inf_{h \in \mathbf{H}\setminus\{0\}} \frac{\|h\|^2_{L^2(\partial\Omega)}}{\|h\|^2_{L^2(\Omega)}}$$

is well-defined. Our purpose is now to relate σ_1 with δ_1, see (3.40). To this end, we make use of a suitable version of Fichera's principle of duality [170]. However, in its original version, this principle requires smoothness of the boundary $\partial\Omega$. Since we aim to deal with most general domains, we need to drop this assumption. We consider Lipschitz domains satisfying a uniform outer ball condition, see Definition 2.30. Then regularity results by Jerison-Kenig [237, 238] enable us to prove the following result.

Theorem 3.23. *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, then $\sigma_1(\Omega)$ admits a minimiser $h \in \mathbf{H}\setminus\{0\}$. If Ω also satisfies a uniform outer ball condition then the minimiser is positive, unique up to a constant multiplier and $\sigma_1(\Omega) = \delta_1(\Omega)$.*

Proof. In the first part of this proof, we just assume that Ω is a domain with Lipschitz boundary. Let $(h_m) \subset \mathbf{H}\setminus\{0\}$ be a minimising sequence for $\sigma_1(\Omega)$ with $\|h_m\|_H = \|h_m\|_{L^2(\partial\Omega)} = 1$. Up to a subsequence, we may assume that there exists $h \in \mathbf{H}$ such that $h_m \rightharpoonup h$ in $\mathbf{H}$. By regularity estimates [237, 238], we infer that there exists a constant $C > 0$ such that

$$\|h\|_{H^{1/2}(\Omega)} \le C\|h\|_{L^2(\partial\Omega)} \qquad \text{for all } h \in \mathbf{H}$$

so that $\sigma_1(\Omega) > 0$ and the sequence (h_m) is bounded in $H^{1/2}(\Omega)$, $h_m \rightharpoonup h$ in $H^{1/2}(\Omega)$ up to a subsequence and, by compact embedding, we also have $h_m \to h$ in $L^2(\Omega)$. Therefore, since (h_m) is a minimising sequence, $\|h_m\|_{L^2(\partial\Omega)} = 1$ and $\|h_m\|_{L^2(\Omega)}$ is bounded it follows that $h \in \mathbf{H}\setminus\{0\}$ and

$$\|h\|^{-2}_{L^2(\Omega)} = \lim_{m\to\infty} \|h_m\|^{-2}_{L^2(\Omega)} = \sigma_1(\Omega).$$

Moreover, by weak lower semicontinuity of $\|.\|_H$ we also have

$$\|h\|^2_{L^2(\partial\Omega)} = \|h\|^2_H \le \liminf_{m\to\infty} \|h_m\|^2_H = 1$$

and hence $h \in \mathbf{H} \setminus \{0\}$ satisfies

$$\frac{\|h\|^2_{L^2(\partial\Omega)}}{\|h\|^2_{L^2(\Omega)}} \le \sigma_1(\Omega).$$

This proves that h is a minimiser for $\sigma_1(\Omega)$.

In the rest of the proof, we assume furthermore that Ω satisfies a uniform outer ball condition. Under this condition, we have the existence of a minimiser for $\delta_1(\Omega)$ by Theorem 3.17. We say that σ is a *harmonic boundary eigenvalue* if there exists $g \in \mathbf{H}$ such that

$$\sigma \int_\Omega g v \, dx = \int_{\partial\Omega} g v \, d\omega \qquad \text{for all } v \in \mathbf{H}.$$

Clearly, σ_1 is the least harmonic boundary eigenvalue. We prove that $\sigma_1 = \delta_1$ by showing two inequalities.

Proof of $\sigma_1 \ge \delta_1$. Let h be a minimiser for σ_1, then

$$\sigma_1 \int_\Omega h v \, dx = \int_{\partial\Omega} h v \, d\omega \qquad \text{for all } v \in \mathbf{H}. \tag{3.56}$$

Let $u \in [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega)$ be the unique solution to

$$\begin{cases} \Delta u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Integrating by parts we have

$$\int_\Omega h v \, dx = \int_\Omega v \Delta u \, dx = \int_{\partial\Omega} v u_\nu \, d\omega \quad \text{for all } v \in \mathbf{H} \cap C^2(\overline{\Omega}).$$

By a density argument, the latter follows for all $v \in \mathbf{H}$. Inserting this into (3.56) gives

$$\sigma_1 \int_{\partial\Omega} v u_\nu \, d\omega = \int_{\partial\Omega} v \Delta u \, d\omega \qquad \text{for all } v \in \mathbf{H}.$$

This yields $\Delta u = \sigma_1 u_\nu$ on $\partial\Omega$. Therefore,

$$\sigma_1 = \frac{\|h\|^2_{L^2(\partial\Omega)}}{\|h\|^2_{L^2(\Omega)}} = \frac{\|\Delta u\|^2_{L^2(\partial\Omega)}}{\|\Delta u\|^2_{L^2(\Omega)}} = \sigma_1^2 \frac{\|u_\nu\|^2_{L^2(\partial\Omega)}}{\|\Delta u\|^2_{L^2(\Omega)}}.$$

In turn, this implies that

$$\sigma_1 = \frac{\|\Delta u\|^2_{L^2(\Omega)}}{\|u_\nu\|^2_{L^2(\partial\Omega)}} \ge \min \left\{ \frac{\|\Delta v\|^2_{L^2(\Omega)}}{\|v_\nu\|^2_{L^2(\partial\Omega)}} ;\ v \in [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega) \right\} = \delta_1.$$

Proof of $\sigma_1 \le \delta_1$. Let u be a minimiser for δ_1 in (3.40), then $\Delta u = \delta_1 u_\nu$ on $\partial\Omega$ so that $\Delta u \in H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$ and

$$\int_{\partial\Omega} v\Delta u\, d\omega = \delta_1 \int_{\partial\Omega} v u_\nu\, d\omega \qquad \text{for all } v \in \mathbf{H}. \tag{3.57}$$

Let $h := \Delta u$ so that $h \in L^2(\Omega) \cap L^2(\partial\Omega)$. Moreover, $\Delta h = \Delta^2 u = 0$ in a distributional sense and hence $h \in \mathbf{H}$. Two integrations by parts and a density argument yield

$$\int_\Omega hv\, dx = \int_{\partial\Omega} v u_\nu\, d\omega \qquad \text{for all } v \in \mathbf{H}.$$

Replacing this into (3.57) gives

$$\int_{\partial\Omega} hv\, d\omega = \delta_1 \int_\Omega hv\, dx \qquad \text{for all } v \in \mathbf{H}.$$

This proves that h is an eigenfunction with corresponding harmonic boundary eigenvalue δ_1. Since σ_1 is the least harmonic boundary eigenvalue, we obtain $\delta_1 \geq \sigma_1$.

Then $\sigma_1 = \delta_1$ and there is a one-to-one correspondence between minimisers of $\sigma_1(\Omega)$ and $\delta_1(\Omega)$. Hence, uniqueness of a minimiser for $\sigma_1(\Omega)$ up to a constant multiplier follows from Theorem 3.17. □

We now show that an optimal shape for δ_1 under volume or perimeter constraint does not exist in any space dimension $n \geq 2$.

Theorem 3.24. *Let $D_\varepsilon = \{x \in \mathbb{R}^2;\ \varepsilon < |x| < 1\}$ and let $\Omega_\varepsilon \subset \mathbb{R}^n$ ($n \geq 2$) be such that*

$$\Omega_\varepsilon = D_\varepsilon \times (0,1)^{n-2}\ ;$$

in particular, if $n = 2$ we have $\Omega_\varepsilon = D_\varepsilon$. Then

$$\lim_{\varepsilon \searrow 0} \delta_1(\Omega_\varepsilon) = 0.$$

Proof. We assume first that $n = 2$. For any $\varepsilon \in (0,1)$ let $w_\varepsilon \in H^2 \cap H^1_0(D_\varepsilon)$ be defined by

$$w_\varepsilon(x) = \frac{1-|x|^2}{4} - \frac{1-\varepsilon^2}{4\log\varepsilon} \log|x| \qquad \text{for all } x \in D_\varepsilon. \tag{3.58}$$

Then we have

$$\Delta w_\varepsilon = -1 \qquad \text{in } \Omega_\varepsilon$$

and

$$|\nabla w_\varepsilon(x)|^2 = \left(\frac{|x|}{2} + \frac{1-\varepsilon^2}{4\log\varepsilon} \frac{1}{|x|} \right)^2 \qquad \text{for all } x \in \overline{\Omega}_\varepsilon$$

so that

$$\int_{\Omega_\varepsilon} |\Delta w_\varepsilon|^2\, dx = \pi\left(1 - \varepsilon^2\right)$$

and

$$\int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu^2 \, d\omega = 2\pi\left(\frac{1}{2}+\frac{1-\varepsilon^2}{4\log\varepsilon}\right)^2 + 2\pi\varepsilon\left(\frac{\varepsilon}{2}+\frac{1-\varepsilon^2}{4\varepsilon\log\varepsilon}\right)^2 \tag{3.59}$$
$$= \frac{\pi}{8}\frac{1}{\varepsilon\log^2\varepsilon} + o\left(\frac{1}{\varepsilon\log^2\varepsilon}\right) \to +\infty \qquad \text{as } \varepsilon \searrow 0.$$

It follows immediately that

$$\lim_{\varepsilon\searrow 0} \delta_1(\Omega_\varepsilon) \le \lim_{\varepsilon\searrow 0} \frac{\displaystyle\int_{\Omega_\varepsilon} |\Delta w_\varepsilon|^2 \, dx}{\displaystyle\int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu^2 \, d\omega} = 0.$$

This completes the proof of the theorem for $n = 2$.

We now consider the case $n \ge 3$. Let

$$u_\varepsilon(x) = \left(\prod_{i=3}^{n} x_i(1-x_i)\right) w_\varepsilon(x_1,x_2) \qquad \text{for all } x \in \Omega_\varepsilon$$

where w_ε is as in (3.58); note that u_ε vanishes on $\partial\Omega_\varepsilon$ and $u_\varepsilon \in H^2 \cap H_0^1(\Omega_\varepsilon)$. Then we have

$$\Delta u_\varepsilon = -\prod_{i=3}^{n} x_i(1-x_i) - 2w_\varepsilon(x_1,x_2) \sum_{j=3}^{n} \prod_{\substack{i=3\\ i\neq j}}^{n} x_i(1-x_i)$$

(with the convention that $\prod_{i\in\emptyset}\beta_i = 1$) and

$$\int_{\Omega_\varepsilon} |\Delta u_\varepsilon|^2 \, dx \le 2\int_{\Omega_\varepsilon} \prod_{i=3}^{n} x_i^2(1-x_i)^2 \, dx + 8\int_{\Omega_\varepsilon} w_\varepsilon^2(x_1,x_2) \sum_{j=3}^{n} \prod_{\substack{i=3\\ i\neq j}}^{n} x_i^2(1-x_i)^2 \; dx.$$

Hence, since $|w_\varepsilon(x)| < \frac{1}{2}$ for all $x \in D_\varepsilon$, there exists $C > 0$ such that

$$\int_{\Omega_\varepsilon} |\Delta u_\varepsilon|^2 \, dx \le C \qquad \text{for all } \varepsilon \in (0,1). \tag{3.60}$$

On the other hand, we have

$$|\nabla u_\varepsilon|^2 = \prod_{i=3}^{n} x_i^2(1-x_i)^2 \left(\left(\frac{\partial w_\varepsilon}{\partial x_1}\right)^2 + \left(\frac{\partial w_\varepsilon}{\partial x_2}\right)^2\right)$$
$$+ \sum_{j=3}^{n}\left((1-2x_j)^2 w_\varepsilon^2(x_1,x_2) \prod_{\substack{i=3\\ i\neq j}}^{n} x_i^2(1-x_i)^2\right)$$

and since w_ε vanishes on ∂D_ε we obtain

$$\int_{\partial\Omega_\varepsilon} (u_\varepsilon)_\nu^2 \, d\omega = \int_{\partial\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, d\omega \geq \int_{\partial D_\varepsilon \times (0,1)^{n-2}} |\nabla u_\varepsilon|^2 \, d\omega$$
$$\geq \int_{\partial D_\varepsilon} (w_\varepsilon)_\nu^2 \, d\omega \cdot \prod_{i=3}^n \int_0^1 x_i^2 \, (1-x_i)^2 \, dx_i \to +\infty$$

as $\varepsilon \searrow 0$ in view of (3.59). Therefore, by (3.60) we obtain

$$\lim_{\varepsilon\searrow 0} \delta_1(\Omega_\varepsilon) \leq \lim_{\varepsilon\searrow 0} \frac{\int_{\Omega_\varepsilon} |\Delta u_\varepsilon|^2 \, dx}{\int_{\partial\Omega_\varepsilon} (u_\varepsilon)_\nu^2 \, d\omega} = 0$$

which proves the theorem also when $n \geq 3$. □

Theorem 3.24 has several important consequences. First, it shows that $\delta_1(\Omega)$ has no optimal shape under the constraint that Ω is contained in a fixed ball.

Corollary 3.25. *Let $B_R = \{x \in \mathbb{R}^n;\ |x| < R\}$. Then for any $R > 0$*

$$\inf_{\Omega \subset B_R} \delta_1(\Omega) = 0$$

where the infimum is taken over all domains $\Omega \subset B_R$ such that $\partial\Omega \in C^\infty$ if $n = 2$ and $\partial\Omega$ is Lipschitzian if $n \geq 3$.

A second consequence of Theorem 3.24 is that it disproves the conjecture by Kuttler [258] which states that the disk has the smallest δ_1 among all planar regions having the same perimeter. Let us also mention that, although the ball has no isoperimetric property, it is a stationary domain for the map $\Omega \mapsto \delta_1(\Omega)$ in the class of C^4 domains under smooth perturbations which preserve measure, see [80] for the details.

Theorem 3.24 also shows that the map $\Omega \mapsto \delta_1(\Omega)$ is *not* monotonically decreasing with respect to domain inclusion.

Finally, Theorem 3.24 raises several natural questions. Why do we consider an annulus in the plane and the region between two cylinders in space dimensions $n \geq 3$? What happens if we consider an annulus in any space dimension? The quite surprising answer is given in

Theorem 3.26. *Let $n \geq 3$ and let $\Omega^\varepsilon = \{x \in \mathbb{R}^n;\ \varepsilon < |x| < 1\}$.*

1. *If $n = 3$ then*
$$\lim_{\varepsilon\searrow 0} \delta_1(\Omega^\varepsilon) = 2.$$
2. *If $n \geq 4$ then*
$$\lim_{\varepsilon\searrow 0} \delta_1(\Omega^\varepsilon) = n.$$

For the proof of Theorem 3.26 we refer to [80]. Theorems 3.24 and 3.26 highlight a striking difference between dimension $n = 2$, dimension $n = 3$ and dimensions $n \geq 4$. Since the set Ω^ε is smooth, by Theorem 3.23 it follows that $\delta_1(\Omega^\varepsilon) = \sigma_1(\Omega^\varepsilon)$.

Moreover, since the proof of Theorem 3.26 in [80] uses radial harmonic functions $h = h(r)$ $(r = |x|)$, we may rewrite the ratio defining $\sigma_1(\Omega^\varepsilon)$ as

$$\frac{\displaystyle\int_{\partial\Omega^\varepsilon} h^2 d\omega}{\displaystyle\int_{\Omega^\varepsilon} h^2 dx} = \frac{h(1)^2 + \varepsilon^{n-1} h(\varepsilon)^2}{\displaystyle\int_\varepsilon^1 h(r)^2 r^{n-1} dr}.$$

In this setting, we can treat the space dimension n as a real number. Then we have

Theorem 3.27. *Let $\varepsilon \in (0,1)$, let $K_\varepsilon = \{h \in C^2([\varepsilon,1]); h''(r) + \frac{n-1}{r} h'(r) = 0,\ r \in [\varepsilon,1]\}$ and, for all $n \in [1,\infty)$, let*

$$\gamma_\varepsilon(n) = \inf_{h \in K_\varepsilon \setminus \{0\}} \frac{h(1)^2 + \varepsilon^{n-1} h(\varepsilon)^2}{\displaystyle\int_\varepsilon^1 h(r)^2 r^{n-1} dr}.$$

Then

$$\lim_{\varepsilon \to 0} \gamma_\varepsilon(n) = \begin{cases} 2 & \text{if } n = 1, \\ 0 & \text{if } 1 < n < 3, \\ 2 & \text{if } n = 3, \\ n & \text{if } n > 3. \end{cases}$$

Theorem 3.27 is proved in [80] and shows that dimensions $n = 1$ and $n = 3$ are "discontinuous" dimensions for the behaviour of γ_ε. The reason of this discontinuity is not clear to us.

Finally, we point out that Steklov boundary conditions, producing a boundary integral in the denominator of the Rayleigh quotient, require a *strong* geometric convergence (namely a very fine topology) in order to preserve the perimeter. However, contrary to the Babuška paradox (see Section 1.4.2), we notice that we do have stability of the first eigenvalue on the sequence of regular polygons converging to the disk.

Theorem 3.28. *Let $n = 2$ and let (P_k) be a sequence of regular polygons with k edges circumscribed to the unit disk D centered at the origin. Then*

$$\lim_{k \to \infty} \delta_1(P_k) = \delta_1(D) = 2.$$

The proof of Theorem 3.28 is lengthy and delicate. This is why we refer again to [80].

3.4 Bibliographical notes

An interesting survey on spectral properties of higher-order elliptic operators is also provided by Davies [129].

For the original version of the Kreĭn-Rutman theorem, which generalises Jentzsch's [236] theorem, we refer to [257, Theorem 6.2 and 6.3]. Theorem 3.3 is taken from the appendix of [55] and it follows by combining the variant of the Kreĭn-Rutman result in [359, Theorem 6.6, p. 337] with a result by de Pagter [136].

Theorem 3.4 is due to Moreau [311]. A first application of this decomposition is given in the paper by Miersemann [301] for the positivity in a buckling eigenvalue problem. Proposition 3.6 is the generalisation of [19, Lemma 16] from $m = 2$ to the case $m \geq 2$.

Theorem 3.7 is a straightforward consequence of Kreĭn-Rutman's theorem and Lemma 2.27 but the elementary proof suggested here is taken from [181]. The rest of Section 3.1.3 is taken from the survey paper by Sweers [385].

Concerning Theorem 3.8, numerical results in 1972 and 1980 already predicted that the first eigenfunction on a square changes sign, see [34, 220]. Subsequently, in 1982 Coffman [107] gave an analytic proof of Theorem 3.8. More recently, in 1996, the numerical results on the square have been revisited by Wieners [412] who proved that the sign-changing of the numerically approximated first eigenfunction is rigorous, that is, the sign changing effect is too large to be explained by numerical errors.

Theorem 3.9 is due to Coffman-Duffin-Shaffer. The eigenvalue problem for domains with holes was first studied by Duffin-Shaffer [155]. Subsequently, with Coffman [110] they could show that for the annuli with a small hole, the first eigenfunction changes sign. They used an explicit formula and explicit values of the Bessel functions involved and obtained even a critical number for the ratio of the inner and outer radius. The proof has been simplified in [109].

For further results on sign-changing first eigenfunctions to (3.8), for numerical experiments, and for conjectures on simple domains (such as ellipses, elongated disks, dumb-bells, and limaçons) we refer again to [385].

For some first properties of spherical rearrangements, we refer to [343]. A complete proof of Theorem 3.11 can be found in [10] while its essential Item 2 goes back to Sperner [378] and Talenti [390]. Kawohl [243] discusses the question whether equality in Item 2 of Theorem 3.11 implies symmetry; he shows that the answer is affirmative for analytic functions while it is negative in general. A more general condition ensuring symmetry was subsequently obtained by Brothers-Ziemer [74], see also [100] and references therein for further results on this topic. Theorem 3.12 is an iteration of Talenti's principle [391].

For a fairly complete story of Rayleigh's conjecture [350], we refer to Section 1.3.1. Although it was Nadirashvili [315] who proved first the Rayleigh conjecture in dimension $n = 2$, the proof of Theorem 3.13 follows closely the one by Ashbaugh-Benguria [22] which is more general since it also holds for $n = 3$. It uses some results by Talenti [392].

Theorem 3.15 is due to Ashbaugh-Bucur [23]. Minimisation of the buckling load can be also performed in different classes of domains. For instance, one could argue in the class of convex domains like in [244, Proposition 4.5]. For further classes of domains, such as open sets, quasi-open sets or multiply connected sets, we refer again to [23]. On the occasion of an Oberwolfach meeting in 1995, Willms gave a talk with the proof of Theorem 3.16 according to joint work with Weinberger [415] but the proof was never written by them. With their permission, Kawohl [244, Proposition 4.4] wrote down the proof of the talk by Willms and this is where we have taken it, see also [23]. For more results on buckling eigenvalues, mainly under Dirichlet boundary conditions, we refer to [48, 49, 172, 228, 245, 301, 376] and references therein.

Elliptic problems with parameters in the boundary conditions are called *Steklov problems* from their first appearance in [379]. For the biharmonic operator, these conditions were first considered by Kuttler-Sigillito [260] and Payne [334] who studied the isoperimetric properties of the first eigenvalue δ_1, see also subsequent work by Smith [373, 374] and Kuttler [258, 259]. We also refer to the monograph by Kuttler-Sigillito [261] for some numerical experiments and for a survey of results known at that time. Finally, we refer to Section 1.3.3 for the complete story about the minimisation of δ_1.

Theorem 3.17 is taken from [80] although it was already known in the smooth case $\partial\Omega \in C^2$, see [42]. The characterisation of the first Steklov eigenvalue in the ball and Remark 3.21 are taken from [42]. Subsequently, the whole spectrum of the biharmonic Steklov problem was studied by Ferrero-Gazzola-Weth [165] from where Theorems 3.18, 3.19, and 3.20 are taken. Theorem 3.23 is a generalisation to nonsmooth domains of a particular application of Fichera's principle of duality [170]; in this final form it is proved in [80], see also [165, 170] for previous work in the case $\partial\Omega \in C^2$. All the other statements in Section 3.3.2 are taken from Bucur-Ferrero-Gazzola [80].

Chapter 4
Kernel estimates

In Chapters 5 and 6 we discuss positivity and almost positivity for higher order boundary value problems. The goal of the present chapter is to provide the required estimates, which are also interesting in themselves. In order to avoid a too technical exposition, in many cases the discussion is restricted to fourth order problems. However, whenever it does not require too many additional distinctions, the general case of $2m$-th order operators is also covered.

4.1 Consequences of Boggio's formula

Throughout this chapter we will exploit the following notations.

Notation 4.1 Let $f,g \geq 0$ be functions defined on the same set D.

- We write $f \preceq g$ if there exists $c>0$ such that $f(x) \leq c g(x)$ for all $x \in D$.
- We write $f \simeq g$ if both $f \preceq g$ and $g \preceq f$.

Notation 4.2 For a smooth bounded domain Ω, we define the distance function to the boundary

$$d(x) := \operatorname{dist}(x, \partial\Omega) = \min_{y \in \partial\Omega} |x-y|\,, \quad x \in \Omega. \tag{4.1}$$

Many estimates will be for coordinates inside the unit ball B in $\mathbb{R}^n$ and for this special domain the following expression will be used repeatedly.

Notation 4.3 For $x,y \in \overline{B}$ we write

$$[XY] := \sqrt{|x|^2\,|y|^2 - 2x\cdot y + 1} = \left| |x|y - \frac{x}{|x|} \right| = \left| |y|x - \frac{y}{|y|} \right|. \tag{4.2}$$

As (4.2) shows, $[XY]$ is the distance from $|y|x$ to the projection of y on the unit sphere, which is larger than $|x-y|$. Indeed

$$[XY]^2 - |x-y|^2 = (1-|x|^2)(1-|y|^2) > 0 \text{ for } x,y \in B. \tag{4.3}$$

Since $1-|x| = d(x)$ for $x \in \overline{B}$ it even shows that

$$|x-y|^2 + d(x)d(y) \le [XY]^2 \le |x-y|^2 + 4d(x)d(y). \tag{4.4}$$

We focus on the polyharmonic analogue of the clamped plate boundary value problem

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ D^\alpha u|_{\partial\Omega} = 0 & \text{for } |\alpha| \le m-1. \end{cases} \tag{4.5}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, f a datum in a suitable functional space and u denotes the unknown solution.

In bounded smooth domains, a unique Green function $G_{(-\Delta)^m,\Omega}$ for problem (4.5) exists and the representation formula

$$u(x) = \int_\Omega G_{(-\Delta)^m,\Omega}(x,y) f(y)\,dy, \quad x \in \Omega, \tag{4.6}$$

holds true, see (2.64). Having a positivity preserving property in Ω is equivalent to $G_{(-\Delta)^m,\Omega} \ge 0$. *Almost positivity* will mean that the negative part of $G_{(-\Delta)^m,\Omega}$ is small in a sense to be specified when compared with its positive part. The main goal of Chapters 5 and 6 is to identify domains and also further differential operators enjoying almost positivity or even a positivity preserving property. To this end, we provide in the present chapter fine estimates for the Green function and the other kernels involved in the solution of higher order boundary value problems.

With $[XY]$ as in (4.2), the Green function from Lemma 2.27 by Boggio for the Dirichlet problem (4.5) with $\Omega = B$, the unit ball, is given by

$$G_{m,n}(x,y) = k_{m,n}\,|x-y|^{2m-n} \int_1^{[XY]/|x-y|} (v^2-1)^{m-1} v^{1-n}\,dv\,. \tag{4.7}$$

In Section 4.2 we give the following characterisation of $G_{m,n}$, which will be much more convenient than Boggio's original formula in discussing positivity issues:

$$G_{m,n}(x,y) \simeq \begin{cases} |x-y|^{2m-n} \min\left\{1, \dfrac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\} & \text{if } n > 2m, \\ \log\left(1 + \dfrac{d(x)^m d(y)^m}{|x-y|^{2m}}\right) & \text{if } n = 2m, \\ d(x)^{m-\frac{n}{2}} d(y)^{m-\frac{n}{2}} \min\left\{1, \dfrac{d(x)^{\frac{n}{2}} d(y)^{\frac{n}{2}}}{|x-y|^n}\right\} & \text{if } n < 2m. \end{cases}$$

A more detailed discussion of the boundary terms will be given below. We further deduce related estimates for the derivatives $|D_x^\alpha G_{m,n}(x,y)|$. All these are used to prove so-called 3-G-type theorems in Section 4.2.2, which will help us to develop a perturbation theory of positivity.

It is an obvious question whether in general domains $\Omega \subset \mathbb{R}^n$, where one does not have positivity preserving, estimates for $|G_{(-\Delta)^m,\Omega}|$ as above are available. This question is addressed in Section 4.5. In order to avoid too many technicalities, we confine ourselves here to the biharmonic case. The following estimate is proven in any bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^{4,\gamma}$.

$$\left|G_{\Delta^2,\Omega}(x,y)\right| \preceq \begin{cases} |x-y|^{4-n} \min\left\{1, \dfrac{d(x)^2 d(y)^2}{|x-y|^4}\right\} & \text{if } n>4, \\ \log\left(1+\dfrac{d(x)^2 d(y)^2}{|x-y|^4}\right) & \text{if } n=4, \\ d(x)^{2-\frac{n}{2}} d(y)^{2-\frac{n}{2}} \min\left\{1, \dfrac{d(x)^{\frac{n}{2}} d(y)^{\frac{n}{2}}}{|x-y|^n}\right\} & \text{if } n<4. \end{cases} \tag{4.8}$$

These estimates, also being quite interesting in themselves, will prove to be basic for the positivity and almost positivity results in Chapter 6.

Finally, kernel estimates and 3-G-type results are collected in Section 4.3 to prepare the discussion of positivity in the Steklov problem which will be given in Section 5.4.

4.2 Kernel estimates in the ball

4.2.1 Direct Green function estimates

Let $G_{m,n} : \overline{B} \times \overline{B} \to \mathbb{R} \cup \{\infty\}$ denote the Green function for $(-\Delta)^m$ under homogeneous Dirichlet boundary conditions, see (4.7), and let

$$\mathscr{G}_{m,n} : L^p(B) \to W^{2m,p} \cap W_0^{m,p}(B), \quad (\mathscr{G}_{m,n} f)(x) = \int_B G_{m,n}(x,y) f(y)\, dy, \tag{4.9}$$

be the corresponding Green operator. In order to base a perturbation theory of positivity on this formula, we first condense the key information on the behaviour of $G_{m,n}$ and its derivatives in more convenient expressions, which also allow for a more direct interpretation of its behaviour, see Theorems 4.6 and 4.7 below.

The first lemma characterises the crucial distinction between the cases "x and y are closer to the boundary ∂B than to each other" and vice versa.

Lemma 4.4. *Let $x, y \in \overline{B}$. If $|x-y| \geq \frac{1}{2}[XY]$, then*

$$d(x)d(y) \leq 3\,|x-y|^2, \tag{4.10}$$

$$\max\{d(x), d(y)\} \leq 3\,|x-y|. \tag{4.11}$$

If $|x-y| \leq \frac{1}{2}[XY]$, *then*

$$\begin{aligned} \tfrac{3}{4}|x-y|^2 &\leq \tfrac{3}{16}\,[XY]^2 \;\leq\; d(x)d(y), && (4.12)\\ \tfrac{1}{4}d(x) &\leq d(y) \leq 4d(x), && (4.13)\\ |x-y| &\leq 3\min\{d(x),d(y)\}, && (4.14)\\ [XY] &\leq 5\min\{d(x),d(y)\}. && (4.15) \end{aligned}$$

Moreover, for all $x,y \in \overline{B}$ *we have*

$$d(x) \leq [XY]\,, \qquad d(y) \leq [XY]\,, \tag{4.16}$$

$$[XY] \simeq d(x) + d(y) + |x-y|\,. \tag{4.17}$$

Proof. Let $|x-y| \geq \frac{1}{2}[XY]$. Then one has

$$d(x)d(y) \leq \left(1-|x|^2\right)\left(1-|y|^2\right) = [XY]^2 - |x-y|^2 \leq 3\,|x-y|^2\,,$$

hence (4.10). The estimate (4.11) follows from

$$\begin{gathered} d(x)^2 \leq d(x)\;\left(d(y)+|x-y|\right) \leq 3\,|x-y|^2 + |x-y|\,d(x) \leq 4\,|x-y|^2 + \tfrac{1}{4}d(x)^2 \\ \Rightarrow\;\; d(x)^2 \leq \tfrac{16}{3}\,|x-y|^2\,, \end{gathered}$$

and a corresponding estimate for y.

Now, let $|x-y| \leq \frac{1}{2}[XY]$. Then it follows

$$d(x)d(y) \geq \tfrac{1}{4}\left(1-|x|^2\right)\left(1-|y|^2\right) = \tfrac{1}{4}\left([XY]^2 - |x-y|^2\right) \geq \tfrac{3}{16}[XY]^2 \geq \tfrac{3}{4}\,|x-y|^2\,,$$

hence (4.12). Inequalities (4.13) can be deduced from

$$\begin{gathered} d(y) \leq d(x) + |x-y| \leq d(x) + \left(\tfrac{4}{3}d(x)d(y)\right)^{1/2} \leq \left(1+\tfrac{2}{3}\right)d(x) + \tfrac{1}{2}d(y) \\ \Rightarrow\; d(y) \leq \tfrac{10}{3}d(x), \end{gathered}$$

and the analogous computation with x and y interchanged; (4.14) and (4.15) are now obvious.

Finally, for all $x,y \in \overline{B}$ we have

$$[XY]^2 = \left|\,|x|\,y - \frac{x}{|x|}\right|^2 \geq 1 - 2\,|x|\,|y| + |x|^2\,|y|^2 = (1-|x|\,|y|)^2 \geq \begin{cases} (1-|x|)^2 = d(x)^2 \\ (1-|y|)^2 = d(y)^2 \end{cases}$$

thereby proving (4.16). For (4.17), formulae (4.3) and (4.16) show "$\succeq$". On the other hand, $[XY]^2 - |x-y|^2 = \left(1-|x|^2\right)\left(1-|y|^2\right) \leq 4d(x)d(y) \leq 2d(x)^2 + 2d(y)^2$ showing also "$\preceq$". □

In the ball, the following lemma is a direct consequence of the preceding one. However, since the result is needed also in general domains we prove it in this framework.

Lemma 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $p, q \geq 0$ be fixed. For $(x,y) \in \overline{\Omega}^2$ we have:*

$$\min\left\{1, \frac{d(y)}{|x-y|}\right\} \simeq \min\left\{1, \frac{d(y)}{d(x)}, \frac{d(y)}{|x-y|}\right\}, \tag{4.18}$$

$$\min\left\{1, \frac{d(x)\,d(y)}{|x-y|^2}\right\} \simeq \min\left\{\frac{d(y)}{d(x)}, \frac{d(x)}{d(y)}, \frac{d(x)\,d(y)}{|x-y|^2}\right\}, \tag{4.19}$$

$$\min\left\{1, \frac{d(x)^p\,d(y)^q}{|x-y|^{p+q}}\right\} \simeq \min\left\{1, \frac{d(x)^p}{|x-y|^p}, \frac{d(y)^q}{|x-y|^q}, \frac{d(x)^p\,d(y)^q}{|x-y|^{p+q}}\right\}, \tag{4.20}$$

$$\min\left\{1, \frac{d(x)^p\,d(y)^q}{|x-y|^{p+q}}\right\} \simeq \min\left\{1, \frac{d(x)}{|x-y|}\right\}^p \min\left\{1, \frac{d(y)}{|x-y|}\right\}^q, \tag{4.21}$$

and assuming moreover that $p+q>0$, we also have

$$\log\left(1+\frac{d(x)^p\,d(y)^q}{|x-y|^{p+q}}\right) \simeq \log\left(2+\frac{d(y)}{|x-y|}\right)\min\left\{1, \frac{d(x)^p\,d(y)^q}{|x-y|^{p+q}}\right\}. \tag{4.22}$$

Proof. *Case $d(x) \geq 2|x-y|$ or $d(y) \geq 2|x-y|$.*
If $d(x) \geq 2|x-y|$ we also have

$$d(y) \geq d(x) - |x-y| \geq d(x) - \tfrac{1}{2}d(x) = \tfrac{1}{2}d(x) \geq |x-y|\,,$$
$$d(y) \leq d(x) + |x-y| \leq \tfrac{3}{2}d(x).$$

If, on the other hand, $d(y) \geq 2|x-y|$ one concludes similarly that

$$|x-y| \leq \tfrac{1}{2}d(y) \leq d(x) \leq \tfrac{3}{2}d(y).$$

Hence, in what follows we may use that

$$|x-y| \leq d(x) \text{ and } |x-y| \leq d(x) \text{ and } d(x) \simeq d(y). \tag{4.23}$$

This shows that in (4.18) - (4.21) we have that the left hand sides as well as the right hand sides all satisfy $\simeq 1$. As for (4.22), we have in this case thanks to $p+q>0$

$$\begin{aligned}\log\left(1+\frac{d(x)^p d(y)^q}{|x-y|^{p+q}}\right) &\simeq \log\left(\frac{d(x)^p d(y)^q}{|x-y|^{p+q}}\right) \simeq (p+q)\log\left(\frac{d(y)}{|x-y|}\right)\\ &\simeq \log\left(2+\frac{d(y)}{|x-y|}\right) \simeq \log\left(2+\frac{d(y)}{|x-y|}\right)\min\left\{1, \frac{d(x)^p d(y)^q}{|x-y|^{p+q}}\right\}.\end{aligned}$$

Case $d(x) < 2|x-y|$ and $d(y) < 2|x-y|$.
As for (4.18), inequality "$\succeq$" is obvious, while "$\preceq$" follows from

$$\min\left\{1, \frac{d(y)}{|x-y|}\right\} \simeq \frac{d(y)}{|x-y|} \leq 2\frac{d(y)}{d(x)}.$$

For (4.19) one uses $\min\{t, \frac{1}{t}\} \leq 1$ (for all $t > 0$) to prove "$\succeq$". For "$\preceq$"one may observe that

$$\frac{1}{|x-y|^2} \leq \frac{4}{d(x)^2} \text{ and } \frac{1}{|x-y|^2} \leq \frac{4}{d(y)^2}.$$

In the case considered claims (4.20) and (4.21) are obvious. Finally, through

$$\begin{aligned}\log\left(1+\frac{d(x)^p d(y)^q}{|x-y|^{p+q}}\right) &\simeq \frac{d(x)^p d(y)^q}{|x-y|^{p+q}} \simeq \min\left\{1, \frac{d(x)^p d(y)^q}{|x-y|^{p+q}}\right\} \\ &\simeq \log\left(2+\frac{d(y)}{|x-y|}\right)\min\left\{1, \frac{d(x)^p d(y)^q}{|x-y|^{p+q}}\right\}\end{aligned}$$

we find (4.22). □

We are now ready to establish the basic Green function estimates. In what follows the estimates of $G_{m,n}$ *from below* will be crucial.

Theorem 4.6. *(Two-sided estimates of the Green function) In $\overline{B} \times \overline{B}$ we have*

$$G_{m,n}(x,y) \simeq \begin{cases} |x-y|^{2m-n} \min\left\{1, \dfrac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\} & \text{if } n > 2m; \\ \log\left(1+\dfrac{d(x)^m d(y)^m}{|x-y|^{2m}}\right) & \text{if } n = 2m; \\ d(x)^{m-\frac{n}{2}} d(y)^{m-\frac{n}{2}} \min\left\{1, \dfrac{d(x)^{\frac{n}{2}} d(y)^{\frac{n}{2}}}{|x-y|^n}\right\} & \text{if } n < 2m. \end{cases} \tag{4.24}$$

Proof. According to Lemma 4.4 it is essential to distinguish the two cases "$|x-y| \geq \frac{1}{2}[XY]$" and "$|x-y| \leq \frac{1}{2}[XY]$".
1st case: $|x-y| \leq \frac{1}{2}[XY]$. Here (4.10) applies and we have to show

$$G_{m,n}(x,y) \simeq \begin{cases} |x-y|^{2m-n} & \text{if } n > 2m, \\ \log\left(1+\dfrac{d(x)^m d(y)^m}{|x-y|^{2m}}\right) & \text{if } n = 2m, \\ d(x)^{m-\frac{n}{2}} d(y)^{m-\frac{n}{2}} & \text{if } n < 2m. \end{cases} \tag{4.25}$$

It is not too hard to see that

$$a \in [2,\infty) \;\Rightarrow\; \int_1^a (v^2-1)^{m-1} v^{1-n}\, dv \simeq \int_1^a v^{2m-n-1}\, dv$$

holds true. According to our assumption we may conclude in this case from formula (4.7) for the Green function

$$
\begin{aligned}
G_{m,n}(x,y) &\simeq |x-y|^{2m-n} \int_1^{[XY]/|x-y|} (v^2-1)^{m-1} v^{1-n}\, dv \\
&\simeq |x-y|^{2m-n} \int_1^{[XY]/|x-y|} v^{2m-n-1}\, dv \\
&\simeq \begin{cases} |x-y|^{2m-n} & \text{if } n > 2m, \\ \log\left(\dfrac{[XY]}{|x-y|}\right) & \text{if } n = 2m, \\ [XY]^{2m-n} - |x-y|^{2m-n} \simeq [XY]^{2m-n} & \text{if } n < 2m. \end{cases}
\end{aligned} \tag{4.26}
$$

If $n > 2m$, statement (4.25) is already proved. In order to proceed also in small dimensions $n \leq 2m$ we combine (4.15) and (4.16). We obtain in this case

$$[XY] \simeq d(x) \simeq d(y).$$

Hence, (4.25) is now obvious also for $n < 2m$. If $n = 2m$, we observe further that

$$a \in [2,\infty) \ \Rightarrow\ \log a \simeq \log(1+a^m). \tag{4.27}$$

The discussion of the case $|x-y| \leq \frac{1}{2}[XY]$ is now complete.

2nd case: $|x-y| \geq \frac{1}{2}[XY]$.
In this case we have $\frac{d(x)}{|x-y|} \leq 3$, $\frac{d(y)}{|x-y|} \leq 3$. Hence, independently of whether $n > 2m$, $n = 2m$ or $n < 2m$, we have to show

$$G_{m,n}(x,y) \simeq |x-y|^{-n} d(x)^m d(y)^m. \tag{4.28}$$

When using formula (4.7) for $G_{m,n}$, we note that the upper integration bound $[XY]/|x-y|$ is in $[1,2]$. On this interval one has $v^{-n} \simeq 1$ and may conclude

$$
\begin{aligned}
G_{m,n}(x,y) &\simeq |x-y|^{2m-n} \int_1^{[XY]/|x-y|} (v^2-1)^{m-1}\, v\, dv \\
&\simeq |x-y|^{2m-n} \left(\frac{[XY]^2}{|x-y|^2} - 1\right)^m = |x-y|^{-n} \left([XY]^2 - |x-y|^2\right)^m \\
&= |x-y|^{-n} \left((1-|x|^2)(1-|y|^2)\right)^m \simeq |x-y|^{-n} d(x)^m d(y)^m.
\end{aligned}
$$

The proof of (4.28), and hence of Theorem 4.6, is complete. □

In the spirit of Theorem 4.6 we also have estimates for the derivatives.

Theorem 4.7. *(Estimates of the derivatives of the Green function)*
Let $\alpha \in \mathbb{N}^n$ be a multiindex. Then in $\overline{B} \times \overline{B}$ we have

$$|D_x^\alpha G_{m,n}(x,y)| \preceq (*)$$

with $(*)$ as follows:

1. *if $|\alpha| \geq 2m-n$ and n odd, or if $|\alpha| > 2m-n$ and n even*

$$
(*) = \begin{cases} |x-y|^{2m-n-|\alpha|} \min\left\{1, \dfrac{d(x)^{m-|\alpha|} d(y)^m}{|x-y|^{2m-|\alpha|}}\right\} & \textit{for } |\alpha| < m, \\[2ex] |x-y|^{2m-n-|\alpha|} \min\left\{1, \dfrac{d(y)^m}{|x-y|^m}\right\} & \textit{for } |\alpha| \geq m; \end{cases}
$$

2. if $|\alpha| = 2m-n$ and n even

$$
(*) = \begin{cases} \log\left(2 + \dfrac{d(y)}{|x-y|}\right) \min\left\{1, \dfrac{d(x)^{m-|\alpha|} d(y)^m}{|x-y|^{2m-|\alpha|}}\right\} & \textit{for } |\alpha| < m, \\[2ex] \log\left(2 + \dfrac{d(y)}{|x-y|}\right) \min\left\{1, \dfrac{d(y)^m}{|x-y|^m}\right\} & \textit{for } |\alpha| \geq m; \end{cases}
$$

3. if $|\alpha| \leq 2m-n$ and n odd, or if $|\alpha| < 2m-n$ and n even

$$
(*) = \begin{cases} d(x)^{m-\frac{n}{2}-|\alpha|} d(y)^{m-\frac{n}{2}} \min\left\{1, \dfrac{d(x)^{\frac{n}{2}} d(y)^{\frac{n}{2}}}{|x-y|^n}\right\} & \textit{for } |\alpha| < m - \frac{n}{2}, \\[2ex] d(y)^{2m-n-|\alpha|} \min\left\{1, \dfrac{d(x)^{m-|\alpha|} d(y)^{n-m+|\alpha|}}{|x-y|^n}\right\} & \textit{for } m - \frac{n}{2} \leq |\alpha| < m, \\[2ex] d(y)^{2m-n-|\alpha|} \min\left\{1, \dfrac{d(y)^{n-m+|\alpha|}}{|x-y|^{n-m+|\alpha|}}\right\} & \textit{for } |\alpha| \geq m. \end{cases}
$$

Proof. 1. We claim that on $\{(x,y) \in \overline{B} \times \overline{B} : \ |x-y| \geq \frac{1}{2}[XY]\}$ it holds true that

$$
|D_x^\alpha G_{m,n}(x,y)| \preceq |x-y|^{2m-n-|\alpha|} \left(\frac{d(x)}{|x-y|}\right)^{\max\{m-|\alpha|,0\}} \left(\frac{d(y)}{|x-y|}\right)^m. \tag{4.29}
$$

To this end we use the transformation $s = 1 - \frac{1}{v^2}$ in formula (4.7) in order to show the boundary behaviour of the Green function more clearly. We have

$$
G_{m,n}(x,y) = \frac{k_{m,n}}{2} |x-y|^{2m-n} f_{m,n}(A_{x,y}), \tag{4.30}
$$

where

$$
f_{m,n}(t) := \int_0^t s^{m-1}(1-s)^{\frac{n}{2}-m-1}\, ds, \tag{4.31}
$$

$$
A_{x,y} := \frac{[XY]^2 - |x-y|^2}{[XY]^2} = \frac{(1-|x|^2)(1-|y|^2)}{[XY]^2} \simeq \frac{d(x)d(y)}{[XY]^2}. \tag{4.32}
$$

According to the assumption we have

$$A_{x,y} \leq \frac{3}{4}. \tag{4.33}$$

Here, i.e for $t \in [0, \frac{3}{4}]$, we know

$$\left| f_{m,n}^{(j)}(t) \right| \preceq t^{\max\{m-j,0\}}. \tag{4.34}$$

Since $d(x) \leq [XY]$, by (4.16), for every multiindex $\beta \in \mathbb{N}^n$ one has

$$\left| D_x^\beta A_{x,y} \right| \preceq d(y)[XY]^{-1-|\beta|}. \tag{4.35}$$

Application of a general product and chain rule yields

$$|D_x^\alpha G_{m,n}(x,y)| \preceq \sum_{\beta \leq \alpha} \left| D_x^{\alpha-\beta} |x-y|^{2m-n} \right| \cdot \left| D_x^\beta f_{m,n}(A_{x,y}) \right|$$

$$\begin{aligned}
&\preceq |x-y|^{2m-n-|\alpha|} \cdot \left| f_{m,n}(A_{x,y}) \right| \\
&\quad + \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0}} |x-y|^{2m-n-|\alpha|+|\beta|} \cdot \sum_{j=1}^{|\beta|} \left\{ \left| f_{m,n}^{(j)}(A_{x,y}) \right| \cdot \sum_{\substack{\sum_{i=1}^{j} \beta^{(i)} = \beta \\ |\beta^{(i)}| \geq 1}} \prod_{i=1}^{j} \left| D_x^{\beta^{(i)}} A_{x,y} \right| \right\} \\
&\preceq |x-y|^{2m-n-|\alpha|} \frac{d(x)^m d(y)^m}{[XY]^{2m}} \\
&\quad + \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0}} |x-y|^{2m-n-|\alpha|+|\beta|} \cdot \sum_{j=1}^{|\beta|} \left\{ \left(\frac{d(x)d(y)}{[XY]^2} \right)^{\max\{m-j,0\}} \cdot \frac{d(y)^j}{[XY]^{j+|\beta|}} \right\} \\
&\qquad \text{by (4.32), (4.34), (4.35)} \\
&\preceq \sum_{\beta \leq \alpha} |x-y|^{2m-n-|\alpha|} \left(\frac{d(x)}{[XY]} \right)^{\max\{m-|\beta|,0\}} \left(\frac{d(y)}{[XY]} \right)^m \left(\frac{|x-y|}{[XY]} \right)^{|\beta|} \\
&\qquad \text{by (4.16)} \\
&\preceq |x-y|^{2m-n-|\alpha|} \left(\frac{d(x)}{[XY]} \right)^{\max\{m-|\alpha|,0\}} \left(\frac{d(y)}{[XY]} \right)^m \\
&\qquad \text{by (4.3) and (4.16).}
\end{aligned}$$

Thanks to inequality (4.3) the estimate (4.29) follows.
2. We claim that in $\left\{ (x,y) \in \overline{B} \times \overline{B} : \ |x-y| \leq \frac{1}{2}[XY] \right\}$ one has

$$|D_x^\alpha G_{m,n}(x,y)| \preceq \begin{cases} |x-y|^{2m-n-|\alpha|} & \text{if } |\alpha| > 2m-n, \\ \log\left(\dfrac{[XY]}{|x-y|}\right) & \text{if } |\alpha| = 2m-n \text{ and } n \text{ even}, \\ 1 & \text{if } |\alpha| = 2m-n \text{ and } n \text{ odd}, \\ [XY]^{2m-n-|\alpha|} & \text{if } |\alpha| < 2m-n. \end{cases} \tag{4.36}$$

In contrast with the proof of (4.29) here we do not have to discuss the behaviour of the Green function close to the boundary but "close to the singularity $x = y$". For this reason it is suitable to expand formula (4.7) first and then to carry out the integration explicitly. The integrand contains a term like $\frac{1}{v}$ if and only if n even and $n \le 2m$. It follows for suitable numbers $c_j = c_j(m,n) \in \mathbb{R}$, $j = 0, \dots, m$:

$$G_{m,n}(x,y) = \begin{cases} c_m|x-y|^{2m-n} + \displaystyle\sum_{j=0}^{m-1} c_j [XY]^{2m-n-2j}|x-y|^{2j} \\ \qquad\qquad \text{if } n > 2m \text{ or } n \text{ odd}, \\ c_m|x-y|^{2m-n}\log\dfrac{[XY]}{|x-y|} + \displaystyle\sum_{j=0}^{m-1} c_j [XY]^{2m-n-2j}|x-y|^{2j} \\ \qquad\qquad \text{if } n \le 2m \text{ and } n \text{ even}. \end{cases} \tag{4.37}$$

When differentiating we take into account that $|x-y|^{2j}$ is a polynomial of degree $2j$, whose derivatives of order $> 2j$ vanish identically. Moreover, taking advantage of $|x-y| \le [XY]$, $|D_x^\alpha [XY]^k| \preceq [XY]^{k-|\alpha|}$ and $|D_x^\alpha |x-y|^k| \preceq |x-y|^{k-|\alpha|}$:

$$|D_x^\alpha G_{m,n}(x,y)| \preceq \begin{cases} |x-y|^{2m-n-|\alpha|} + [XY]^{2m-n-|\alpha|} \\ \qquad\qquad \text{if } n > 2m-|\alpha| \text{ or } n \text{ odd}, \\ |x-y|^{2m-n-|\alpha|}\left(1+\log\dfrac{[XY]}{|x-y|}\right) + [XY]^{2m-n-|\alpha|} \\ \qquad\qquad \text{if } n \le 2m-|\alpha| \text{ and } n \text{ even}. \end{cases} \tag{4.38}$$

This already proves (4.36) except in the case where n is even and $n < 2m-|\alpha|$. In this case, we use $a \in [1,\infty) \Rightarrow 0 \le \log a \le a$ and conclude from (4.38):

$$|D_x^\alpha G_{m,n}(x,y)| \preceq |x-y|^{2m-n-|\alpha|-1}[XY] + [XY]^{2m-n-|\alpha|} \preceq [XY]^{2m-n-|\alpha|}.$$

Therefore, (4.36) holds in any case.
3. We conclude the proof of the theorem by using (4.29) and (4.36). Let $x, y \in \overline{B}$ be arbitrary. According to Lemma 4.4 two cases have to be distinguished.
1st case: $|x-y| \le \frac{1}{2}[XY]$.
Here $d(x) \simeq d(y)$, and using Lemma 4.4 we obtain for $p, q \ge 0$:

$$\min\left\{1, \left(\frac{d(x)}{|x-y|}\right)^p \left(\frac{d(y)}{|x-y|}\right)^q\right\} \simeq 1.$$

We have to show that

$$|D_x^\alpha G_{m,n}(x,y)| \preceq \begin{cases} |x-y|^{2m-n-|\alpha|} & \text{if } |\alpha| > 2m-n, \\ \log\left(2+\dfrac{d(y)}{|x-y|}\right) & \text{if } |\alpha| = 2m-n \text{ and } n \text{ even}, \\ 1 & \text{if } |\alpha| = 2m-n \text{ and } n \text{ odd}, \\ d(y)^{2m-n-|\alpha|} & \text{if } |\alpha| < 2m-n. \end{cases}$$

This estimate follows from (4.36), since $d(x) \simeq d(y) \simeq [XY]$ according to (4.15) and (4.16). For the logarithmic term one should observe further (4.27). Making use of $[XY]/|x-y| \geq 2$ and $[XY] \leq 5d(y)$ we obtain

$$\log\frac{[XY]}{|x-y|} \preceq \log\left(1+\frac{1}{5}\frac{[XY]}{|x-y|}\right) \leq \log\left(2+\frac{d(y)}{|x-y|}\right).$$

2nd case: $|x-y| \geq \frac{1}{2}[XY]$.
According to Lemma 4.4 we have for $p,q \geq 0$:

$$\log\left(2+\frac{d(y)}{|x-y|}\right) \simeq 1,$$

$$\min\left\{1, \left(\frac{d(x)}{|x-y|}\right)^p \left(\frac{d(y)}{|x-y|}\right)^q\right\} \simeq \left(\frac{d(x)}{|x-y|}\right)^p \left(\frac{d(y)}{|x-y|}\right)^q.$$

The estimates for $(*)$ as in the statement follow immediately from (4.29). □

The Green function for the Laplacian $(m=1, n>2)$ satisfies the estimates above in arbitrary bounded $C^{2,\gamma}$-smooth domains, see e.g. [411]. This result is proved with the help of general maximum principles and Harnack's inequality. For higher order equations we proceed just in the opposite way, namely, we deduce the above estimates from Boggio's explicit formula and, in turn, use them to prove some comparison principles.

In general the following estimate is weaker than Item 3 of Theorem 4.7 but still appropriate and more convenient for our purposes.

Corollary 4.8. *For $|\alpha| \leq 2m-n$ and n odd, or, $|\alpha| < 2m-n$ and n even we have*

$$|D_x^\alpha G_{m,n}(x,y)| \preceq d(x)^{m-\frac{n}{2}-|\alpha|}\, d(y)^{m-\frac{n}{2}} \, \min\left\{1, \frac{d(x)^{\frac{n}{2}}\, d(y)^{\frac{n}{2}}}{|x-y|^n}\right\}.$$

4.2.2 A 3-G-type theorem

In Chapter 5 we develop a perturbation theory of positivity for Boggio's prototype situation of the polyharmonic operator in the ball. This will be achieved by means of Neumann series and estimates of iterated Green operators. The latter are consequences of the following 3-G-type result, which provides an estimate for a term of three Green functions.

Theorem 4.9 (3-G-theorem). *Let $\alpha \in \mathbb{N}^n$ be a multiindex. Then on $B \times B \times B$ we have*

$$\frac{G_{m,n}(x,z)\left|D_z^\alpha G_{m,n}(z,y)\right|}{G_{m,n}(x,y)} \preceq$$
$$\preceq \begin{cases} |x-z|^{2m-n-|\alpha|} + |y-z|^{2m-n-|\alpha|} & \text{if } |\alpha| > 2m-n, \\ \log\left(\dfrac{3}{|x-z|}\right) + \log\left(\dfrac{3}{|y-z|}\right) & \text{if } |\alpha| = 2m-n \text{ and } n \text{ even}, \\ 1 & \text{if } |\alpha| = 2m-n \text{ and } n \text{ odd}, \\ 1 & \text{if } |\alpha| < 2m-n. \end{cases} \tag{4.39}$$

The proof is crucially based on the Green function estimates in Theorems 4.6 and 4.7 and a number of technical inequalities and equivalencies which we are going to prove first.

Lemma 4.10. *For $s,t > 0$ it holds that*

$$\frac{\log(1+t)}{\log(1+s)} \le 1 + \frac{t}{s}. \tag{4.40}$$

Proof. For $s > 0$ and $\alpha \ge 1$ concavity of the logarithm yields

$$\log(1+s) = \log\left(\frac{1}{\alpha}(1+\alpha s) + (1-\frac{1}{\alpha}\cdot 1)\right) \ge \frac{1}{\alpha}\log(1+\alpha s),$$

i.e. $\log(1+\alpha s)/\log(1+s) \le \alpha$. For $0 < \alpha \le 1$ it is obvious that $\log(1+\alpha s) \le \log(1+s)$. Combining these estimates we have for $s, \alpha > 0$

$$\frac{\log(1+\alpha s)}{\log(1+s)} \le 1 + \alpha.$$

The claim (4.40) follows by taking $\alpha = \frac{t}{s}$. □

Boggio's formula is the reason that we can prove the 3-G-theorem 4.9 only in balls. The following lemmas, however, hold true in any bounded domain.

Lemma 4.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that $p,q,r \ge 0$, $r \le p+q$. Further let $s \in \mathbb{R}$ be such that $\frac{r}{2} - p \le s \le q - \frac{r}{2}$. Then, on $\overline{\Omega} \times \overline{\Omega}$, we have*

$$\min\left\{1,\left(\frac{d(x)}{|x-y|}\right)^p\left(\frac{d(y)}{|x-y|}\right)^q\right\}\preceq\left(\frac{d(y)}{d(x)}\right)^s\min\left\{1,\frac{d(x)d(y)}{|x-y|^2}\right\}^{\frac{r}{2}}. \tag{4.41}$$

Proof. We make the same distinction as in the proof of Lemma 4.5.
Case $d(x)\geq 2|x-y|$ or $d(y)\geq 2|x-y|$. According to (4.23) we have that

$$|x-y|\leq d(x) \text{ and } |x-y|\leq d(x) \text{ and } d(x)\simeq d(y)$$

which directly yields (4.41).
Case $d(x)<2|x-y|$ and $d(y)<2|x-y|$. Under this assumption we obtain

$$\begin{aligned}\min\left\{1,\left(\frac{d(x)}{|x-y|}\right)^p\left(\frac{d(y)}{|x-y|}\right)^q\right\}&\simeq\left(\frac{d(x)}{|x-y|}\right)^p\left(\frac{d(y)}{|x-y|}\right)^q\\ &=\left(\frac{d(y)}{d(x)}\right)^s\left(\frac{d(x)d(y)}{|x-y|^2}\right)^{\frac{r}{2}}\left(\frac{d(x)}{|x-y|}\right)^{p+s-\frac{r}{2}}\left(\frac{d(y)}{|x-y|}\right)^{q-s-\frac{r}{2}}\end{aligned}$$

and, since $p+s-\frac{r}{2}$ and $q-s-\frac{r}{2}$ are nonnegative, the estimate in (4.41). □

Lemma 4.12. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain. On $\Omega\times\Omega\times\Omega$, it holds true that*

$$Q(x,y,z):=\frac{\min\left\{1,\frac{d(x)d(z)}{|x-z|^2}\right\}\min\left\{1,\frac{d(z)d(y)}{|z-y|^2}\right\}}{\min\left\{1,\frac{d(x)d(y)}{|x-y|^2}\right\}}\preceq 1, \tag{4.42}$$

$$R(x,y,z):=\frac{\min\left\{1,\frac{d(x)d(z)}{|x-z|^2}\right\}\min\left\{1,\frac{d(y)}{|z-y|}\right\}}{\min\left\{1,\frac{d(x)d(y)}{|x-y|^2}\right\}}\preceq 1+\frac{|y-z|}{|x-z|}, \tag{4.43}$$

$$S(x,y,z):=\frac{\log\left(1+\frac{d(x)d(z)}{|x-z|^2}\right)\min\left\{1,\frac{d(y)}{|z-y|}\right\}}{\log\left(1+\frac{d(x)d(y)}{|x-y|^2}\right)}\preceq 1+\frac{|y-z|}{|x-z|}, \tag{4.44}$$

$$T(x,y,z):=\frac{|x-y|}{|x-z|\,|z-y|}\leq\frac{1}{|x-z|}+\frac{1}{|z-y|}. \tag{4.45}$$

Proof. Estimate (4.45) is an immediate consequence of the triangle inequality. To prove the remaining estimates we distinguish several cases as in Lemmas 4.5 and 4.11.
Case $d(x)\geq 2|x-y|$ or $d(y)\geq 2|x-y|$. Again, we refer to (4.23):

$$|x-y|\leq d(x) \text{ and } |x-y|\leq d(x) \text{ and } d(x)\simeq d(y).$$

This shows that the denominators of Q and R are bounded from below. Estimating the numerators by 1 from above proves (4.42) and (4.43). In order to estimate S, we make also use of (4.22) and Lemma 4.10.

$$
\begin{aligned}
S(x,y,z) &\preceq \frac{\log\left(2+\frac{d(x)}{|x-z|}\right)}{\log\left(2+\frac{d(x)}{|x-y|}\right)}\cdot 1 \preceq 1+\frac{1+\frac{d(x)}{|x-z|}}{1+\frac{d(x)}{|x-y|}}\\
&\le 2+\frac{\frac{d(x)}{|x-z|}}{1+\frac{d(x)}{|x-y|}} \le 2+\frac{|x-y|}{|x-z|} \le 3+\frac{|y-z|}{|x-z|}.
\end{aligned}
$$

Case $d(x)<2|x-y|$ and $d(y)<2|x-y|$. Under this assumption we have

$$
\min\left\{1,\frac{d(x)d(y)}{|x-y|^2}\right\} \simeq \log\left(1+\frac{d(x)d(y)}{|x-y|^2}\right) \simeq \frac{d(x)d(y)}{|x-y|^2}.
$$

A further distinction with respect to z seems inevitable.

Assume first that $|x-z|\ge \frac{1}{2}|x-y|$. Then (4.18), (4.19), and $\log(1+x)\le x$ yield

$$
\left.\begin{array}{l} Q(x,y,z)\\ R(x,y,z)\\ S(x,y,z)\end{array}\right\} \preceq \frac{|x-y|^2}{d(x)d(y)}\cdot\frac{d(x)d(z)}{|x-z|^2}\cdot\frac{d(y)}{d(z)} \preceq 1.
$$

Assume now that $|x-z|<\frac{1}{2}|x-y|$. Then $|y-z|\ge |y-x|-|x-z|\ge \frac{1}{2}|x-y|$. We obtain by applying Lemma 4.5

$$
\begin{aligned}
Q(x,y,z) &\preceq \frac{|x-y|^2}{d(x)d(y)}\cdot\frac{d(x)}{d(z)}\cdot\frac{d(z)d(y)}{|y-z|^2} \preceq 1,\\
R(x,y,z) &\preceq \frac{|x-y|^2}{d(x)d(y)}\cdot\frac{d(x)}{|x-z|}\cdot\frac{d(y)}{|y-z|} \preceq \frac{|x-y|}{|x-z|} \preceq 1+\frac{|y-z|}{|x-z|},\\
S(x,y,z) &\preceq \frac{|x-y|^2}{d(x)d(y)}\cdot\log\left(2+\frac{d(x)}{|x-z|}\right)\cdot\min\left\{1,\frac{d(x)}{|x-z|}\right\}\cdot\frac{d(y)}{|y-z|}\\
&\preceq \frac{|x-y|}{|x-z|} \preceq 1+\frac{|y-z|}{|x-z|}.
\end{aligned}
$$

$\square$

Proof of the 3-G-theorem 4.9. According to Theorems 4.6 and 4.7 several cases have to be distinguished.

The case: $n>2m$.

$$
\begin{aligned}
&\frac{G_{m,n}(x,z)\left|D_z^\alpha G_{m,n}(z,y)\right|}{G_{m,n}(x,y)}\\
&\preceq \frac{|x-y|^{n-2m}\min\left\{1,\frac{d(x)^m d(z)^m}{|x-z|^{2m}}\right\}\min\left\{1,\left(\frac{d(z)}{|z-y|}\right)^{\max\{m-|\alpha|,0\}}\left(\frac{d(y)}{|z-y|}\right)^m\right\}}{|x-z|^{n-2m}|z-y|^{n+|\alpha|-2m}\min\left\{1,\frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\}}\\
&\preceq \frac{1}{|y-z|^{|\alpha|}}\left(T(x,y,z)\right)^{n-2m}\left(Q(x,y,z)\right)^{\max\{m-|\alpha|,0\}}\left(R(x,y,z)\right)^{\min\{|\alpha|,m\}} =: (\star_0)
\end{aligned}
$$

thanks to (4.21). We continue by Lemma 4.12 to find

$$
\begin{aligned}
(\star_0) &\preceq \frac{1}{|y-z|^{|\alpha|}}\left(\frac{1}{|x-z|}+\frac{1}{|y-z|}\right)^{n-2m}\left(1+\frac{|y-z|}{|x-z|}\right)^{\min\{|\alpha|,m\}} \\
&\preceq |x-z|^{2m-n}|y-z|^{-|\alpha|}+|y-z|^{2m-n-|\alpha|} \\
&\quad + |x-z|^{2m-n-\min\{|\alpha|,m\}}|y-z|^{-|\alpha|+\min\{|\alpha|,m\}} \\
&\quad + |x-z|^{-\min\{|\alpha|,m\}}|y-z|^{2m-n-|\alpha|+\min\{|\alpha|,m\}} \\
&\preceq |x-z|^{2m-n-|\alpha|}+|y-z|^{2m-n-|\alpha|}.
\end{aligned}
$$

The case: $n=2m$ and $\alpha=0$.

$$
\begin{aligned}
&\frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} \\
&\preceq \frac{\log\left(2+\frac{d(x)}{|x-z|}\right)\log\left(2+\frac{d(y)}{|y-z|}\right)\min\left\{1,\frac{d(x)^m d(z)^m}{|x-z|^{2m}}\right\}\min\left\{1,\frac{d(z)^m d(y)^m}{|z-y|^{2m}}\right\}}{\max\left\{\log\left(2+\frac{d(x)}{|x-y|}\right),\log\left(2+\frac{d(y)}{|x-y|}\right)\right\}\min\left\{1,\frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by virtue of (4.22)} \\
&\preceq \frac{\log\left(2+\frac{d(x)}{|x-z|}\right)\log\left(2+\frac{d(y)}{|y-z|}\right)}{\max\left\{\log\left(2+\frac{d(x)}{|x-y|}\right),\log\left(2+\frac{d(y)}{|x-y|}\right)\right\}}\,(Q(x,y,z))^m =: (\star_1).
\end{aligned}
$$

If $|x-z|\geq\frac{1}{2}|x-y|$, then $\log\left(2+\frac{d(x)}{|x-z|}\right)\preceq\log\left(2+\frac{d(x)}{|x-y|}\right)$. If, on the other hand, $|x-z|<\frac{1}{2}|x-y|$, then the reverse inequality $|y-z|\geq|x-y|-|x-z|\geq\frac{1}{2}|x-y|$ follows and hence $\log\left(2+\frac{d(y)}{|y-z|}\right)\preceq\log\left(2+\frac{d(y)}{|x-y|}\right)$. Combining this estimate with Lemma 4.12, (4.42) yields

$$
(\star_1)\preceq\log\left(2+\frac{d(x)}{|x-z|}\right)+\log\left(2+\frac{d(y)}{|y-z|}\right)\preceq\log\left(\frac{3}{|x-z|}\right)+\log\left(\frac{3}{|y-z|}\right).
$$

The case: $n=2m$ and $|\alpha|>0$.

$$
\begin{aligned}
&\frac{G_{m,n}(x,z)\left|D_z^\alpha G_{m,n}(z,y)\right|}{G_{m,n}(x,y)} \\
&\preceq \frac{\log\left(1+\frac{d(x)d(z)}{|x-z|^2}\right)\min\left\{1,\frac{d(x)^{m-1}d(z)^{m-1}}{|x-z|^{2m-2}}\right\}\min\left\{1,\frac{d(z)^{\max\{m-|\alpha|,0\}}d(y)^m}{|z-y|^{m+\max\{m-|\alpha|,0\}}}\right\}}{\log\left(1+\frac{d(x)d(y)}{|x-y|^2}\right)|y-z|^{|\alpha|}\min\left\{1,\frac{d(x)^{m-1}d(y)^{m-1}}{|x-y|^{2m-2}}\right\}} \\
&\preceq |y-z|^{-|\alpha|}S(x,y,z)\,(Q(x,y,z))^{\max\{m-|\alpha|,0\}}\,(R(x,y,z))^{\min\{|\alpha|,m\}-1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by (4.21)}
\end{aligned}
$$

$$\preceq |y-z|^{-|\alpha|}\left(1+\frac{|y-z|}{|x-z|}\right)^{\min\{|\alpha|,m\}} \qquad \text{by Lemma 4.12}$$
$$\preceq |x-z|^{-|\alpha|}+|y-z|^{-|\alpha|}.$$

The case: $n<2m$ and $|\alpha|<2m-n$,
or: $n<2m$ and $|\alpha|\le 2m-n$ and n odd.
Here we use Corollary 4.8. Together with Theorem 4.6 we obtain

$$\frac{G_{m,n}(x,z)\left|D_z^\alpha G_{m,n}(z,y)\right|}{G_{m,n}(x,y)}$$
$$\preceq \frac{d(x)^{m-\frac{n}{2}}d(z)^{2m-n-|\alpha|}d(y)^{m-\frac{n}{2}}\min\left\{1,\frac{d(x)^{\frac{n}{2}}d(z)^{\frac{n}{2}}}{|x-z|^n}\right\}\min\left\{1,\frac{d(z)^{\frac{n}{2}}d(y)^{\frac{n}{2}}}{|z-y|^n}\right\}}{d(x)^{m-\frac{n}{2}}d(y)^{m-\frac{n}{2}}\min\left\{1,\frac{d(x)^{\frac{n}{2}}d(y)^{\frac{n}{2}}}{|x-y|^n}\right\}}$$
$$\preceq d(z)^{2m-n-|\alpha|}\left(Q(x,y,z)\right)^{\frac{n}{2}}\preceq 1 \qquad \text{due to (4.42).}$$

The case: $n<2m$ and $|\alpha|=2m-n$ and n even.
We employ Lemma 4.11 with $p=\max\{m-|\alpha|,0\}$, $q=m$, $s=m-\frac{n}{2}$ and $r=n$. In the present case, due to $|\alpha|=2m-n$, one has: $p+q=\max\{n-m,0\}+m=\max\{n,m\}\ge n=r$; $q-\frac{r}{2}=m-\frac{n}{2}=s=\frac{n}{2}-(n-m)\ge\frac{r}{2}-p$.

$$\frac{G_{m,n}(x,z)\left|D_z^\alpha G_{m,n}(z,y)\right|}{G_{m,n}(x,y)}$$
$$\preceq \frac{d(x)^{m-\frac{n}{2}}d(z)^{m-\frac{n}{2}}\min\left\{1,\frac{d(x)^{\frac{n}{2}}d(z)^{\frac{n}{2}}}{|x-z|^n}\right\}}{d(x)^{m-\frac{n}{2}}d(y)^{m-\frac{n}{2}}\min\left\{1,\frac{d(x)^{\frac{n}{2}}d(y)^{\frac{n}{2}}}{|x-y|^n}\right\}}$$
$$\times\ \log\left(2+\frac{d(y)}{|z-y|}\right)\min\left\{1,\left(\frac{d(z)}{|z-y|}\right)^{\max\{m-|\alpha|,0\}}\left(\frac{d(y)}{|z-y|}\right)^{m}\right\}$$
$$\preceq \frac{\log\left(2+\frac{d(y)}{|z-y|}\right)d(z)^{m-\frac{n}{2}}\min\left\{1,\frac{d(x)^{\frac{n}{2}}d(z)^{\frac{n}{2}}}{|x-z|^n}\right\}\left(\frac{d(y)}{d(z)}\right)^{m-\frac{n}{2}}\min\left\{1,\frac{d(z)^{\frac{n}{2}}d(y)^{\frac{n}{2}}}{|z-y|^n}\right\}}{d(y)^{m-\frac{n}{2}}\min\left\{1,\frac{d(x)^{\frac{n}{2}}d(y)^{\frac{n}{2}}}{|x-y|^n}\right\}}$$
$$\preceq \log\left(2+\frac{d(y)}{|z-y|}\right)\left(Q(x,y,z)\right)^{\frac{n}{2}}\preceq \log\left(\frac{3}{|y-z|}\right) \qquad \text{by virtue of (4.42).}$$

The case: $n < 2m$ *and* $|\alpha| > 2m - n$.

$$\begin{aligned}
\frac{G_{m,n}(x,z)\left|D_z^\alpha G_{m,n}(z,y)\right|}{G_{m,n}(x,y)} &\preceq \frac{d(x)^{m-\frac{n}{2}}d(z)^{m-\frac{n}{2}}\min\left\{1,\frac{d(x)^{\frac{n}{2}}d(z)^{\frac{n}{2}}}{|x-z|^n}\right\}}{d(x)^{m-\frac{n}{2}}d(y)^{m-\frac{n}{2}}\min\left\{1,\frac{d(x)^{\frac{n}{2}}d(y)^{\frac{n}{2}}}{|x-y|^n}\right\}} \\
&\quad\times |z-y|^{2m-n-|\alpha|}\min\left\{1,\left(\frac{d(z)}{|z-y|}\right)^{\max\{m-|\alpha|,0\}}\left(\frac{d(y)}{|z-y|}\right)^m\right\} \\
&= |y-z|^{2m-n-|\alpha|}\frac{d(z)^{m-\frac{n}{2}}\min\left\{1,\frac{d(x)^{\frac{n}{2}}d(z)^{\frac{n}{2}}}{|x-z|^n}\right\}}{d(y)^{m-\frac{n}{2}}\min\left\{1,\frac{d(x)^{\frac{n}{2}}d(y)^{\frac{n}{2}}}{|x-y|^n}\right\}} \\
&\quad\times \min\left\{1,\left(\frac{d(z)}{|z-y|}\right)^{\max\{m-|\alpha|,0\}}\left(\frac{d(y)}{|z-y|}\right)^m\right\} =: (\star_2).
\end{aligned}$$

In order to proceed, we have to distinguish further cases.

In addition, we assume first that $|\alpha| \le 2m - \frac{n}{2}$.
We apply (4.21) of Lemma 4.5 to the "dangerous" term in $(\star_2)$. Here one has to observe that $|\alpha|+n-2m > 0$ as well as $3m-n-|\alpha| \ge 3m-n-2m+\frac{n}{2} = m-\frac{n}{2} > 0$. In a second step we make use of Lemma 4.11 with $p = \max\{m-|\alpha|,0\} \ge 0$, $q = 3m-n-|\alpha| \ge 0$, $r = 4m-n-2|\alpha| \ge 0$ and $s = m-\frac{n}{2}$. Obviously $p+q-r = \max\{|\alpha|-m,0\} \ge 0$, $q-\frac{r}{2} = s$, $\frac{r}{2}-p = m-\frac{n}{2}-\max\{|\alpha|-m,0\} \le s$.

$$\begin{aligned}
&\min\left\{1,\left(\frac{d(z)}{|z-y|}\right)^{\max\{m-|\alpha|,0\}}\left(\frac{d(y)}{|z-y|}\right)^m\right\} \\
&\simeq \min\left\{1,\frac{d(y)}{|y-z|}\right\}^{|\alpha|+n-2m}\min\left\{1,\left(\frac{d(z)}{|z-y|}\right)^{\max\{m-|\alpha|,0\}}\left(\frac{d(y)}{|z-y|}\right)^{3m-n-|\alpha|}\right\} \\
&\preceq \min\left\{1,\frac{d(y)}{|y-z|}\right\}^{|\alpha|+n-2m}\left(\frac{d(y)}{d(z)}\right)^{m-\frac{n}{2}}\min\left\{1,\frac{d(z)d(y)}{|z-y|^2}\right\}^{2m-\frac{n}{2}-|\alpha|}.
\end{aligned}$$

With the aid of this estimate and of Lemma 4.12 we obtain further

$$\begin{aligned}
(\star_2) &\preceq |y-z|^{2m-n-|\alpha|}\left(Q(x,y,z)\right)^{2m-\frac{n}{2}-|\alpha|}\left(R(x,y,z)\right)^{|\alpha|+n-2m} \\
&\preceq |y-z|^{2m-n-|\alpha|}\left(1+\frac{|y-z|}{|x-z|}\right)^{|\alpha|+n-2m} \preceq |y-z|^{2m-n-|\alpha|}+|x-z|^{2m-n-|\alpha|}.
\end{aligned}$$

Now we assume that additionally $|\alpha| > 2m - \frac{n}{2}$ *holds true.*
Here one has to deal with the "dangerous" term in $(\star_2)$ in a different manner. Obviously, one has that $|\alpha| > m + \left(m-\frac{n}{2}\right) > m$. We apply repeatedly Lemma 4.5, observing that $\frac{n}{2} < m$.

$$\min\left\{1,\left(\frac{d(z)}{|z-y|}\right)^{\max\{m-|\alpha|,0\}}\left(\frac{d(y)}{|z-y|}\right)^{m}\right\}\simeq\min\left\{1,\left(\frac{d(y)}{|z-y|}\right)^{m}\right\}$$
$$\simeq\min\left\{1,\frac{d(y)}{|y-z|}\right\}^{\frac{n}{2}}\min\left\{1,\frac{d(y)}{|z-y|}\right\}^{m-\frac{n}{2}}\preceq\min\left\{1,\frac{d(y)}{|y-z|}\right\}^{\frac{n}{2}}\left(\frac{d(y)}{d(z)}\right)^{m-\frac{n}{2}}.$$

By means of this estimate and of Lemma 4.12 we further conclude that

$$\begin{aligned}(\star_2)&\preceq|y-z|^{2m-n-|\alpha|}\,(R(x,y,z))^{\frac{n}{2}}\preceq|y-z|^{2m-n-|\alpha|}\left(1+\frac{|y-z|}{|x-z|}\right)^{\frac{n}{2}}\\&\preceq|y-z|^{2m-n-|\alpha|}+|y-z|^{2m-\frac{n}{2}-|\alpha|}\,|x-z|^{-\frac{n}{2}}\\&\preceq|y-z|^{2m-n-|\alpha|}+|x-z|^{2m-n-|\alpha|}.\end{aligned}$$

To apply Young's inequality in the last step, one has to exploit the assumption $2m-\frac{n}{2}-|\alpha|<0$ of this case. □

4.3 Estimates for the Steklov problem

In the previous section we considered the higher order operator $(-\Delta)^m$ under Dirichlet boundary conditions starting from the explicit formula of Boggio and hence we necessarily had to restrict ourselves to the ball as domain. Under different boundary conditions the boundary value problem may be rewritten as a second order system. The present section prepares for such a situation so that general bounded smooth domains are allowed. So we consider the second order Green operator $\mathscr{G}$ and the Poisson operator $\mathscr{K}$ on a general domain Ω, that is, $w=\mathscr{G}f+\mathscr{K}g$ formally solves

$$\begin{cases}-\Delta w=f & \text{in } \Omega,\\ w=g & \text{on } \partial\Omega.\end{cases}$$

For bounded C^2-domains the operators $\mathscr{G}$ and $\mathscr{K}$ can be represented by integral kernels G and K, namely

$$(\mathscr{G}f)(x)=\int_\Omega G(x,y)f(y)\,dy\quad\text{and}\quad(\mathscr{K}g)(x)=\int_{\partial\Omega}K(x,y)g(y)\,d\omega_y.\tag{4.46}$$

Moreover, it holds that

$$K(x,y)=\frac{-\partial}{\partial\nu_y}G(x,y)\qquad\text{for all } (x,y)\in\Omega\times\partial\Omega.\tag{4.47}$$

According to (4.6), the Green function G in (4.46) should be written as $G_{-\Delta,\Omega}$. However, since this function is frequently used in this section, we drop the subscripts for a simpler notation.

In this section we prove some estimates for the kernels G and K in general bounded domains Ω such that $\partial\Omega \in C^2$. These estimates will be intensively used in Section 5.4 in order to prove positivity properties for the biharmonic Steklov problem.

Based on several estimates due to Zhao [420, 421] (see also [118, 384]), Grunau-Sweers [213] were able to show:

Proposition 4.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^2$. Then the following uniform estimates hold for $(x,y) \in \Omega \times \Omega$:*

$$\text{for } n > 4: \int_\Omega G(x,z)G(z,y)\,dz \simeq |x-y|^{4-n} \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}, \tag{4.48}$$

$$\text{for } n = 4: \int_\Omega G(x,z)G(z,y)\,dz \simeq \log\left(1 + \frac{d(x)d(y)}{|x-y|^2}\right), \tag{4.49}$$

$$\text{for } n = 3: \int_\Omega G(x,z)G(z,y)\,dz \simeq \sqrt{d(x)d(y)}\min\left\{1, \frac{\sqrt{d(x)d(y)}}{|x-y|}\right\}, \tag{4.50}$$

$$\text{for } n = 2: \int_\Omega G(x,z)G(z,y)\,dz \simeq d(x)d(y)\log\left(2 + \frac{1}{|x-y|^2 + d(x)d(y)}\right). \tag{4.51}$$

We will exploit these estimates combined with the following "geometric" result:

Lemma 4.14. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with $\partial\Omega \in C^2$. For $x \in \Omega$ let $x^* \in \partial\Omega$ be any point such that $d(x) = |x - x^*|$.*

- *Then there exists $r_\Omega > 0$ such that for $x \in \Omega$ with $d(x) \leq r_\Omega$ there is a unique $x^* \in \partial\Omega$.*
- *Then the following uniform estimates hold:*

$$\text{for } (x,y) \in \Omega \times \Omega: \quad |x-y| \preceq d(x) + d(y) + |x^* - y^*|, \tag{4.52}$$

$$\text{for } (x,y) \in \Omega \times \Omega: \quad \frac{d(x)}{d(x) + d(y) + |x^* - y^*|} \preceq \min\left\{1, \frac{d(x)}{|x-y|}\right\}, \tag{4.53}$$

$$\text{for } (x,z) \in \Omega \times \partial\Omega: \quad |x-z| \simeq d(x) + |x^* - z|. \tag{4.54}$$

And for $(x,y,z) \in \Omega \times \Omega \times \partial\Omega$:

$$\text{if } d(y) \leq d(x) \text{ and } |x^* - y^*| \leq d(x) + d(y), \text{ then } |x-z| \simeq d(x) + |y^* - z|. \tag{4.55}$$

Proof. Since $\partial\Omega \in C^2$, there exists $r_1 > 0$ such that Ω can be filled with balls of radius r_1. Set $r_\Omega = \frac{1}{2}r_1$. For $x \in \Omega$ with $d(x) \leq r_\Omega$ there is a unique $x^* \in \partial\Omega$.

Estimate (4.52) is just the triangle inequality. Estimate (4.54) follows from the three inequalities

$$|x-z| \leq |x-x^*| + |x^*-z| = d(x) + |x^*-z|,$$
$$d(x) \leq |x-z| \quad \text{and} \quad |x^*-z| \leq |x^*-x| + |x-z| \leq 2|x-z|.$$

In order to prove (4.55), we first remark that under the assumptions made we have $d(x) \geq \frac{1}{2}|x^* - y^*|$. This yields the two inequalities

$$\begin{aligned} d(x) + |x^* - z| &\leq d(x) + |x^* - y^*| + |y^* - z| \leq 3d(x) + |y^* - z| \leq 3(d(x) + |y^* - z|), \\ d(x) + |y^* - z| &\leq d(x) + |x^* - y^*| + |x^* - z| \leq 3d(x) + |x^* - z| \leq 3(d(x) + |x^* - z|). \end{aligned}$$

In turn these inequalities read as $d(x) + |x^* - z| \simeq d(x) + |y^* - z|$. This, combined with (4.54), proves (4.55).

To prove (4.53), we distinguish two cases. If $|x - y| \leq \frac{1}{2}\max(d(x), d(y))$, then $\frac{1}{2}d(x) \leq d(y) \leq 2d(x)$ and $|x - y| \preceq d(x) \simeq d(y)$. It follows that

$$\frac{d(x)}{d(x) + d(y) + |x^* - y^*|} \preceq 1 \simeq \min\left\{1, \frac{d(x)}{|x - y|}\right\}$$

and a similar estimate with x and y interchanged. If $|x - y| \geq \frac{1}{2}\max\left(d(x), d(y)\right)$, we use (4.52) to find that

$$\frac{d(x)}{d(x) + d(y) + |x^* - y^*|} \preceq \frac{d(x)}{|x - y|} \simeq \min\left\{1, \frac{d(x)}{|x - y|}\right\}$$

and a similar estimate with x and y interchanged. □

We are now ready to prove the estimates which are needed for the study of the Steklov problem.

Lemma 4.15. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with $\partial\Omega \in C^2$. Then the following uniform estimates hold for $(x,z) \in \Omega \times \partial\Omega$:*

$$\int_\Omega G(x,\xi)K(\xi,z)d\xi \simeq \begin{cases} d(x)\ |x - z|^{2-n} & \text{for } n \geq 3, \\ d(x)\ \log\left(2 + \frac{1}{|x-z|^2}\right) & \text{for } n = 2. \end{cases}$$

Proof. Let

$$H(x,z) := \int_\Omega G(x,\xi)G(\xi,z)d\xi \qquad \text{for all } (x,z) \in \Omega \times \partial\Omega.$$

In view of (4.47), and since $H(x,z) = 0$ for $z \in \partial\Omega$, we have

$$\int_\Omega G(x,\xi)K(\xi,z)d\xi = \frac{-\partial}{\partial\nu_z}H(x,z) = \lim_{t \to 0}\frac{H(x,z - t\nu_z)}{t}. \tag{4.56}$$

Note also that if r_Ω is as in Lemma 4.14, then $d(z - t\nu_z) = t$ for all $z \in \partial\Omega$ and $t \leq r_\Omega$. Hence, by (4.48) we obtain for $n > 4$

$$\lim_{t \to 0}\frac{H(x,z - t\nu_z)}{t} \simeq \lim_{t \to 0}\frac{|x - z + t\nu_z|^{4-n}\ \min\left\{1, \frac{td(x)}{|x-z+t\nu_z|^2}\right\}}{t} = d(x)\ |x - z|^{2-n}.$$

For $n = 4$ we use (4.49) to obtain

$$\lim_{t\to 0} \frac{H(x, z-t\nu_z)}{t} \simeq \lim_{t\to 0} \frac{\log\left(1+\frac{td(x)}{|x-z+t\nu_z|^2}\right)}{t} \simeq d(x)\ |x-z|^{-2}.$$

For $n = 3$ we use (4.50) to obtain

$$\lim_{t\to 0} \frac{H(x, z-t\nu_z)}{t} \simeq \lim_{t\to 0} \frac{\sqrt{td(x)}\ \min\left\{1, \frac{\sqrt{td(x)}}{|x-z+t\nu_z|}\right\}}{t} = d(x)\ |x-z|^{-1}.$$

And finally for $n = 2$ we use (4.51) to obtain

$$\lim_{t\to 0} \frac{H(x, z-t\nu_z)}{t} \simeq \lim_{t\to 0} \frac{td(x)\log\left(2+\frac{1}{|x-z+t\nu_z|^2+td(x)}\right)}{t} = d(x)\ \log\left(2+\frac{1}{|x-z|^2}\right).$$

By (4.56) the statement is so proved for any $n \geq 2$. $\square$

Lemma 4.16. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with $\partial\Omega \in C^2$. Then the following uniform estimates hold for $(x,y) \in \Omega \times \Omega$:*

$$\begin{aligned} &\int_\Omega \int_{\partial\Omega} \int_\Omega G(x,\xi)K(\xi,z)\frac{-\partial}{\partial\nu_z}G(z,w)G(w,y)\,d\xi\, d\omega_z dw \\ &\quad\preceq \begin{cases} d(x)d(y)\ \left(d(x)+d(y)+|x^*-y^*|\right)^{2-n} & \text{for } n \geq 3, \\ d(x)d(y)\ \log\left(2+\frac{1}{d(x)+d(y)+|x^*-y^*|)}\right) & \text{for } n = 2, \end{cases} \end{aligned} \tag{4.57}$$

respectively for $(x,y) \in \Omega \times \partial\Omega$:

$$\begin{aligned} &\int_\Omega \int_{\partial\Omega} \int_\Omega G(x,\xi)K(\xi,z)\frac{-\partial}{\partial\nu_z}G(z,w)K(w,y)\,d\xi\, d\omega_z dw \\ &\quad\preceq \begin{cases} d(x)\ |x-y|^{2-n} & \text{for } n \geq 3, \\ d(x)\ \log\left(2+\frac{1}{|x-y|)}\right) & \text{for } n = 2. \end{cases} \end{aligned} \tag{4.58}$$

Proof. Setting

$$R(x,y) := \int_\Omega \int_{\partial\Omega} \int_\Omega G(x,\xi)K(\xi,z)\frac{-\partial}{\partial\nu_z}G(z,w)G(w,y)\,d\xi\, d\omega_z dw,$$

and using (4.47) and the estimates from Lemma 4.15, the following holds:

$$\begin{aligned} R(x,y) &\preceq d(x)d(y)\int_{\partial\Omega} |x-z|^{2-n}\,|z-y|^{2-n}\,d\omega_z && \text{for } n \geq 3, \\ R(x,y) &\preceq d(x)d(y)\int_{\partial\Omega} \log\left(2+\frac{1}{|x-z|^2}\right)\log\left(2+\frac{1}{|y-z|^2}\right)d\omega_z && \text{for } n = 2. \end{aligned}$$

Let r_Ω be as in Lemma 4.14. We distinguish three cases, according to the positions of $x, y \in \Omega$.

Case 1: $\max(d(x), d(y)) \geq r_\Omega$.
By symmetry we may assume that $d(y) \geq r_\Omega$ and find for $n \geq 3$ that

$$\int_{\partial\Omega} |x-z|^{2-n} |z-y|^{2-n} \, d\omega_z \preceq \int_{\partial\Omega} |x-z|^{2-n} \, d\omega_z \preceq \int_0^1 \frac{r^{n-2}}{(d(x)+r)^{n-2}} dr \preceq 1,$$

and for $n = 2$

$$\int_{\partial\Omega} \log\left(2 + \tfrac{1}{|x-z|^2}\right) \log\left(2 + \tfrac{1}{|z-y|^2}\right) d\omega_z \preceq \int_0^1 \log\left(2 + \tfrac{1}{(d(x)+r)^2}\right) dr \preceq 1,$$

which imply (4.57) since $d(y) \geq r_\Omega$.

Case 2: $\max(d(x), d(y)) < r_\Omega$ *and* $|x^* - y^*| \geq d(x) + d(y)$.
In this case, in view of Lemma 4.14, we have that (4.54) holds for both x and y. So, for $n \geq 3$ we have

$$\int_{\partial\Omega} |x-z|^{2-n} |z-y|^{2-n} \, d\omega_z \preceq \int_{\partial\Omega} \frac{1}{(d(x)+|x^*-z|)^{n-2}} \frac{1}{(d(y)+|y^*-z|)^{n-2}} d\omega_z.$$

We split this integral as $I_x + I_y$ where I_x is over $\partial\Omega_x = \{z \in \partial\Omega; |x^*-z| \leq |y^*-z|\}$ and I_y over $\partial\Omega_y = \partial\Omega \backslash \partial\Omega_x$. Over $\partial\Omega_x$ we have

$$|x^*-z| + |x^*-y^*| \leq |x^*-z| + |x^*-z| + |y^*-z| \leq 3|y^*-z|.$$

Hence we find

$$\begin{aligned} I_x &\preceq \int_{\partial\Omega_x} \frac{1}{(d(x)+|x^*-z|)^{n-2}} \frac{1}{(d(y)+|x^*-z|+|x^*-y^*|)^{n-2}} d\omega_z \\ &\preceq \frac{1}{|x^*-y^*|^{n-2}} \int_0^1 \frac{r^{n-2}}{(d(x)+r)^{n-2}} dr \preceq |x^*-y^*|^{2-n} \\ &\preceq (d(x)+d(y)+|x^*-y^*|)^{2-n} \end{aligned}$$

where we used $|x^*-y^*| \geq d(x)+d(y)$ in the last estimate.
Similarly, for $n = 2$ we find

$$\begin{aligned} I_x &\preceq \int_{\partial\Omega_x} \log\left(2 + \frac{1}{d(x)+|x^*-z|}\right) \log\left(2 + \frac{1}{d(y)+|x^*-z|+|x^*-y^*|}\right) d\omega_z \\ &\preceq \log\left(2 + \frac{1}{d(y)+|x^*-y^*|}\right) \int_0^1 \log\left(2 + \frac{1}{d(x)+r}\right) dr \\ &\preceq \log\left(2 + \frac{1}{d(y)+|x^*-y^*|}\right) \preceq \log\left(2 + \frac{1}{d(x)+d(y)+|x^*-y^*|}\right). \end{aligned}$$

Analogous estimates hold for I_y. All together these estimates prove (4.57) in Case 2.

Case 3: $\max(d(x),d(y)) < r_\Omega$ *and* $|x^*-y^*| \le d(x)+d(y)$.
By symmetry, we may assume that $d(y) \le d(x)$. Then we may use both (4.54) and (4.55). So, for $n \ge 3$ we find

$$\begin{aligned}
&\int_{\partial\Omega} |x-z|^{2-n}\,|z-y|^{2-n}\,d\omega_z \\
&\preceq \int_{\partial\Omega} \frac{1}{(d(x)+|y^*-z|)^{n-2}}\,\frac{1}{(d(y)+|y^*-z|)^{n-2}}\,d\omega_z \\
&\preceq \int_0^1 \frac{r^{n-2}}{(d(x)+r)^{n-2}}\,\frac{1}{(d(y)+r)^{n-2}}\,dr \\
&\preceq \frac{1}{d(x)^{n-2}} \preceq (d(x)+d(y)+|x^*-y^*|)^{2-n},
\end{aligned}$$

and for $n=2$

$$\begin{aligned}
&\int_{\partial\Omega} \log\left(2+\frac{1}{|x-z|}\right)\log\left(2+\frac{1}{|y-z|}\right)d\omega_z \\
&\preceq \int_{\partial\Omega} \log\left(2+\frac{1}{d(x)+|y^*-z|}\right)\log\left(2+\frac{1}{d(y)+|y^*-z|}\right)d\omega_z \\
&\preceq \int_0^1 \log\left(2+\frac{1}{d(x)+r}\right)\log\left(2+\frac{1}{d(y)+r}\right)dr \\
&\preceq \log\left(2+\frac{1}{d(x)}\right) \preceq \log\left(2+\frac{1}{d(x)+d(y)+|x^*-y^*|}\right).
\end{aligned}$$

This proves (4.57) in Case 3.

For the estimates in (4.58) one divides the estimates in (4.57) by $d(y)$, takes the limit for $d(y) \to 0$, and uses (4.54), namely that $d(x)+|x^*-y| \simeq |x-y|$ for $y \in \partial\Omega$. $\square$

4.4 General properties of the Green functions

In this section we collect some smoothness properties of biharmonic functions and derive some preliminary pointwise estimates. That is, we first give a more precise statement concerning the smoothness of the Green functions simultaneously with respect to *both* variables. Next we will show some pointwise estimates for the Green function that follow almost directly from its construction through the fundamental solution.

4.4.1 Regularity of the biharmonic Green function

For brevity we here write $G = G_{\Delta^2,\Omega}$ for the Green function in the domain Ω, see (4.6).

Proposition 4.17. *Let Ω be a bounded $C^{4,\gamma}$-smooth domain. Let G be the Green function for the biharmonic Dirichlet problem. Then*

$$G \in C^{4,\gamma}\left(\overline{\Omega} \times \overline{\Omega} \setminus \left\{(x,x):\ x \in \overline{\Omega}\right\}\right).$$

Proof. Suppose $\alpha \in \mathbb{N}^n$ with $i = |\alpha| \le 3$ and let $p \in (n, n+1)$. In particular it holds that $4 - \frac{n}{p} > i$. Let $\varphi \in C_c^\infty(\Omega)$ and consider $\psi \in C^{4,\gamma}(\overline{\Omega})$ such that $\Delta^2\psi = \varphi$ in Ω and $\psi = \psi_\nu = 0$ on $\partial\Omega$. It follows from Theorem 2.20 and Sobolev's embedding theorem 2.6 that

$$\|\psi\|_{C^{i,\mu}(\overline{\Omega})} \le C\|\psi\|_{W^{4,p}(\Omega)} \le C\|\varphi\|_{L^p(\Omega)}$$

for all $\mu \in (0,1)$ with $i + \mu \le 4 - \frac{n}{p}$. Since $\psi(x) = \int_\Omega G(x,y)\varphi(y)\,dy$, we get that

$$\left|\int_\Omega \left(D_x^\alpha G(x,y) - D_x^\alpha G(x',y)\right)\varphi(y)\,dy\right| \le C_2\|\varphi\|_{L^p(\Omega)}\left|x - x'\right|^\mu \text{ for all } x,x' \in \Omega.$$

By duality, we then obtain $y \mapsto D_x^\alpha G(x,y) \in L^q(\Omega)$ for all $q \in (\frac{n+1}{n}, \frac{n}{n-1})$ and moreover, for all $\mu \le 4 - i - n + \frac{n}{q}$ with $\mu \in (0,1)$, that

$$\|D_x^\alpha G(x,\,.\,) - D_x^\alpha G(x',\,.\,)\|_q \le C(q)\left|x - x'\right|^\mu \text{ for all } x,x' \in \Omega.$$

Since the functions $y \mapsto G(x,y)$ are biharmonic in $\Omega \setminus \{x\}$, so is $y \mapsto D_x^\alpha G(x,y)$. Fix x and consider $y \mapsto D_x^\alpha G(x,y)$. Since $D_x^\alpha G(x,\,.\,) = \frac{\partial}{\partial\nu} D_x^\alpha G(x,\,.\,) = 0$ on $\partial\Omega$, regularity theory, as one may find in Theorem 2.19, gives that $D_x^\alpha G(x,\,.\,) \in C^{4,\gamma}(\overline{\Omega} \setminus \{x\})$. Moreover, for all $\delta > 0$ there exists $C(\delta) > 0$ such that

$$\|D_x^\alpha G(x,\,.\,) - D_x^\alpha G(x',\,.\,)\|_{C^{4,\gamma}(\overline{\Omega}\setminus(B_\delta(x)\cup B_\delta(x')))} \le C(\delta)\left|x - x'\right|^\mu \text{ for all } x,x' \in \Omega.$$

This is valid for $|\alpha| \le 3$. Using the symmetry of the Green function, we have a similar result for $|\alpha| = 4$ with respect to the $C^{3,\gamma}(\overline{\Omega} \setminus (B_\delta(x) \cup B_\delta(x')))$-norm. So, all derivatives of order 4 are covered and we find that $G \in C^{4,\gamma}(\overline{\Omega} \times \overline{\Omega} \setminus \{(x,x) : x \in \overline{\Omega}\})$. $\square$

4.4.2 Preliminary estimates for the Green function

We start with a relatively straightforward application of the Schauder theory to the construction of Green's functions.

Lemma 4.18. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{4,\gamma}$-smooth domain and let $d(\,.\,)$ be as in (4.1). Then for the biharmonic Green function $G_{\Delta^2,\Omega}$ the following estimates hold true:*

$$\left|G_{\Delta^2,\Omega}(x,y)\right| \le C(\Omega)\cdot \begin{cases} |x-y|^{4-n} + \max\{d(x),d(y)\}^{4-n} & \text{if } n>4,\\ \log\left(1+|x-y|^{-1}+\max\{d(x),d(y)\}^{-1}\right) & \text{if } n=4,\\ 1 & \text{if } n=2,3. \end{cases} \tag{4.59}$$

For $n=2,3,4$ also the following gradient estimates hold true:

$$\left|\nabla_x G_{\Delta^2,\Omega}(x,y)\right| \le C(\Omega)\cdot \begin{cases} |x-y|^{-1} + \max\{d(x),d(y)\}^{-1} & \text{if } n=4,\\ 1 & \text{if } n=2,3. \end{cases} \tag{4.60}$$

By symmetry (4.60) also holds for $\left|\nabla_y G_{\Delta^2,\Omega}(x,y)\right|$. The dependence of the constants $C(\Omega)$ on Ω is explicit via the $C^{4,\gamma}$-properties of $\partial\Omega$.

Proof. For brevity we write $G(x,y)=G_{\Delta^2,\Omega}(x,y)$. We recall a fundamental solution for Δ^2 on $\mathbb{R}^n$:

$$F_n(x) = \begin{cases} c_n|x|^{4-n} & \text{if } n\notin\{2,4\},\\ -2c_4\log|x| & \text{if } n=4,\\ 2c_2|x|^2\log|x| & \text{if } n=2, \end{cases} \tag{4.61}$$

where c_n is defined through $e_n=|B|$ and

$$c_n = \begin{cases} \dfrac{1}{2(n-4)(n-2)ne_n} & \text{if } n\notin\{2,4\},\\[2ex] \dfrac{1}{8ne_n} & \text{if } n\in\{2,4\}. \end{cases}$$

The Green function is given by $G(x,y)=F_n(|x-y|)+h(x,y)$, where $h(x,.)$ is a solution of the following Dirichlet problem:

$$\begin{cases} \Delta_y^2 h(x,y)=0 & \text{in } \Omega,\\ h(x,y) = -F_n(|x-y|) & \text{for } y\in\partial\Omega,\\ \frac{\partial}{\partial\nu_y}h(x,y) = -\frac{\partial}{\partial\nu_y}F_n(|x-y|) & \text{for } y\in\partial\Omega. \end{cases} \tag{4.62}$$

We first discuss extensively the generic case $n>4$. At the end we comment on the changes and additional arguments which have to be made for $n\in\{2,3,4\}$.

Case $n>4$. In (4.62), the $C^{1,\gamma}$-norm of the datum for $h(x,.)|_{\partial\Omega}$ and the $C^{0,\gamma}$-norm of the datum for $\frac{\partial}{\partial\nu_y}h(x,.)|_{\partial\Omega}$ are bounded by $C(\partial\Omega)d(x)^{3-n-\gamma}$. The dependence of the constant $C(\partial\Omega)$ on $\partial\Omega$ is constructive and explicit via its curvature properties and their derivatives. According to $C^{1,\gamma}$-estimates for boundary value problems in variational form like (4.62) we see with the help of Theorem 2.19 that

$$\|h(x,.)\|_{C^{1,\gamma}(\overline{\Omega})} \leq C(\partial\Omega)d(x)^{3-n-\gamma}. \tag{4.63}$$

One should observe that the differential operators are uniformly coercive, so that no $h(x,.)$-term needs to appear on the right-hand-side of (4.63).

As long as $d(y) \leq d(x)$, (4.63) shows $h(x,y) \leq C(\partial\Omega)d(x)^{4-n}$ and hence that

$$|G(x,y)| \leq C(\partial\Omega)\left(|x-y|^{4-n} + d(x)^{4-n}\right). \tag{4.64}$$

For $d(y) > d(x)$, we conclude from (4.64) by exploiting the symmetry of the Green function:

$$|G(x,y)| = |G(y,x)| \leq C(\partial\Omega)\left(|x-y|^{4-n} + d(y)^{4-n}\right). \tag{4.65}$$

Combining (4.64) and (4.65) yields (4.59) for $n > 4$.

Case $n = 4$. As above we find that

$$\|h(x,.)\|_{C^{1,\gamma}(\overline{\Omega})} \leq C(\partial\Omega)d(x)^{-1-\gamma}. \tag{4.66}$$

As long as $d(y) \leq d(x)$, (4.66) shows that

$$\left|\nabla_y G(x,y)\right| \leq C(\partial\Omega)\left(|x-y|^{-1} + d(x)^{-1}\right). \tag{4.67}$$

In order to exploit the symmetry of $G(x,y)$ we need a similar estimate also for $|\nabla_x G(x,y)|$. To this end one has to differentiate (4.62) with respect to x being considered here as a parameter and obtains as before that for $d(y) \leq d(x)$

$$|\nabla_x G(x,y)| \leq C(\partial\Omega)\left(|x-y|^{-1} + d(x)^{-1}\right). \tag{4.68}$$

By symmetry $G(x,y) = G(y,x)$, and (4.68) shows that for $d(x) \leq d(y)$ one has

$$\left|\nabla_y G(x,y)\right| \leq C(\partial\Omega)\left(|x-y|^{-1} + d(y)^{-1}\right) \tag{4.69}$$

while (4.67) yields

$$|\nabla_x G(x,y)| \leq C(\partial\Omega)\left(|x-y|^{-1} + d(y)^{-1}\right). \tag{4.70}$$

Combining (4.67)-(4.70) proves (4.60) and hence (4.59) in the case $n = 4$.

Case $n = 3$. As in the previous cases one comes up with

$$\|h(x,.)\|_{C^{1,\gamma}(\overline{\Omega})} \leq C(\partial\Omega)d(x)^{-\gamma}.$$

Proceeding as for $n = 4$ yields (4.60) and hence (4.59) also in the case $n = 3$.

Case $n = 2$. Here, one directly finds that

$$\|h(x,.)\|_{C^{1,\gamma}(\overline{\Omega})} \leq C(\partial\Omega)$$

and the claims (4.59), (4.60) immediately follow. $\square$

4.5 Uniform Green functions estimates in $C^{4,\gamma}$-families of domains

Later on we will need convergence properties of Green functions defined on a converging family of domains. For the sake of simplicity we restrict ourselves also in this section to biharmonic operators. Moreover, in order to avoid too many technicalities, we restrict ourselves to special families of bounded domains that may be parametrised with the help of global coordinate charts over the closure of a fixed bounded smooth domain.

To be more precise: we will consider the family of the biharmonic Green functions $G_k = G_{\Delta^2,\Omega_k}$ and $G = G_{\Delta^2,\Omega}$ in Ω_k and Ω respectively, where $(\Omega_k)_{k\in\mathbb{N}}$ is a family of domains converging to a bounded domain $\Omega \subset \mathbb{R}^n$ in the following sense.

Definition 4.19. We say that the sequence $(\Omega_k)_{k\in\mathbb{N}}$ is a $C^{4,\gamma}$-perturbation of the bounded $C^{4,\gamma}$-smooth domain Ω, if there exists a neighbourhood U of $\overline{\Omega}$ and for each $k \in \mathbb{N}$ a $C^{4,\gamma}$-diffeomorphism $\Psi_k : U \to \Psi_k(U)$ with $\Psi_k(\overline{\Omega}) = \overline{\Omega_k}$ such that one has

$$\lim_{k\to\infty} \|Id - \Psi_k\|_{C^{4,\gamma}(\overline{U})} = 0.$$

The remaining section is divided in a part without and a part with boundary terms and we finish with some results on the convergence of these Green functions.

4.5.1 *Uniform global estimates without boundary terms*

As for the diffeomorphisms Ψ_k we refer to Definition 4.19.

Theorem 4.20. *Assume that $(\Omega_k)_{k\in\mathbb{N}}$ is a $C^{4,\gamma}$-perturbation of the bounded $C^{4,\gamma}$-smooth domain $\Omega \subset \mathbb{R}^n$ and let $G_k = G_{\Delta^2,\Omega_k}$ be the biharmonic Green function in Ω_k under Dirichlet boundary conditions. Then there exists a constant $C = C((\Omega_k)_{k\in\mathbb{N}})$, which is independent of k, such that for all $k \in \mathbb{N}$ and $\alpha,\beta \in \mathbb{N}^n$ with $|\alpha|+|\beta| \le 4$:*

- *If $|\alpha|+|\beta|+n > 4$:*

$$\left|D_x^\alpha D_y^\beta G_k(x,y)\right| \le C\ |x-y|^{4-n-|\alpha|-|\beta|} \text{ for all } x,y \in \Omega_k. \tag{4.71}$$

- *If $|\alpha|+|\beta|+n = 4$ and n is even*

$$\left|D_x^\alpha D_y^\beta G_k(x,y)\right| \le C\ \log\left(1+|x-y|^{-1}\right) \text{ for all } x,y \in \Omega_k. \tag{4.72}$$

- *If $|\alpha|+|\beta|+n = 4$ and n is odd, or if $|\alpha|+|\beta|+n < 4$*

$$\left|D_x^\alpha D_y^\beta G_k(x,y)\right| \le C \text{ for all } x,y \in \Omega_k. \tag{4.73}$$

This kind of estimates was given by Krasovskiĭ [255, 256] in a very general framework. Here we provide an independent proof, which is simpler but nevertheless still quite involved. We shall proceed in several steps, where the proof of Proposition 4.22 is the most important part.

Lemma 4.21. *Assume that* $n \geq 4$*. Let* $(\Omega_k)_{k\in\mathbb{N}}$ *be a* $C^{4,\gamma}$*-perturbation of the bounded* $C^{4,\gamma}$*-smooth domain* Ω*. Let* G_k *denote the Green functions for* Δ^2 *in* Ω_k *under Dirichlet boundary conditions and* $d(\,.\,)$ *the distance function to the boundary* $\partial\Omega_k$*. For any* $q \in \left(\frac{n}{n-3}, \frac{n}{n-4}\right)$ *there exists* $C(q) > 0$ *such that for all* k *and all* $x \in \Omega_k$ *we have*

$$\|G_k(x,\,.\,)\|_{L^q(\Omega_k)} \leq C(q)\; d(x)^{4-n+\frac{n}{q}}. \tag{4.74}$$

The constant $C(q)$ *can be chosen uniformly for the family* $(\Omega_k)_{k\in\mathbb{N}}$*.*

Proof. We proceed with the help of a duality argument. Let $\psi \in C_c^\infty(\Omega_k)$ and let $\varphi \in C^{4,\gamma}(\overline{\Omega_k})$ be a solution of

$$\begin{cases} \Delta^2\varphi = \psi & \text{in } \Omega_k, \\ \varphi = \varphi_\nu = 0 & \text{on } \partial\Omega_k. \end{cases}$$

Let $q \in \left(\frac{n}{n-3}, \frac{n}{n-4}\right)$ and let $q' = \frac{q}{q-1}$ be the dual exponent, so that in particular $\frac{n}{4} < q' < \frac{n}{3}$. It follows from Theorem 2.20 (in particular, Corollary 2.21) that there exists $C_3 > 0$ independent of φ, ψ and k such that

$$\|\varphi\|_{W^{4,q'}(\Omega_k)} \leq C_3 \|\psi\|_{L^{q'}(\Omega_k)}.$$

The embedding $W^{4,q'}(\Omega_k) \subset C^{0,\mu}(\overline{\Omega_k})$ (see Theorem 2.6) with $\mu = 4 - \frac{n}{q'} = 4 - n + \frac{n}{q}$ being continuous uniformly in k shows that there exists $C_4 > 0$ independent of φ and k such that $\|\varphi\|_{C^{0,\mu}(\overline{\Omega_k})} \leq C_4 \|\varphi\|_{W^{4,q'}(\Omega_k)}$. Let $x \in \Omega_k$ and $x' \in \partial\Omega_k$. We then get that

$$|\varphi(x)| = \left|\varphi(x) - \varphi(x')\right| \leq \|\varphi\|_{C^{0,\mu}(\overline{\Omega_k})} \left|x - x'\right|^\mu \leq C_3 C_4 \|\psi\|_{L^{q'}(\Omega_k)} \left|x - x'\right|^\mu.$$

Moreover, it follows from Green's representation formula that

$$\varphi(x) = \int_{\Omega_k} G_k(x,y)\psi(y)\,dy \text{ for all } x \in \Omega_k.$$

Therefore, taking the infimum with respect to $x' \in \partial\Omega_k$, we have that

$$\left|\int_{\Omega_k} G_k(x,y)\psi(y)\,dy\right| \leq C_3 C_4 \|\psi\|_{L^{q'}(\Omega_k)}\, d(x)^\mu$$

for all $\psi \in C_c^\infty(\Omega_k)$. Inequality (4.74) then follows. □

4.5.1.1 Zero and first derivative estimates

Proposition 4.22. *Let $(\Omega_k)_{k\in\mathbb{N}}$ be a $C^{4,\gamma}$-perturbation of the bounded $C^{4,\gamma}$-smooth domain Ω. Let G_k be as in Lemma 4.21. Then there exists a constant $C_1 > 0$ such that for all k and all $x, y \in \Omega_k$ with $x \neq y$ one has that*

$$|G_k(x,y)| \le C_1 \cdot \begin{cases} |x-y|^{4-n} & \text{if } n > 4, \\ \log\left(1+|x-y|^{-1}\right) & \text{if } n = 4, \\ 1 & \text{if } n = 2,3. \end{cases} \tag{4.75}$$

Moreover, for $n = 2,3,4$ and for all $k \in \mathbb{N}$ and $x, y \in \Omega_k$ with $x \neq y$

$$|\nabla_x G_k(x,y)| \le C_1 \cdot \begin{cases} |x-y|^{-1} & \text{if } n = 4, \\ 1 & \text{if } n = 2,3. \end{cases} \tag{4.76}$$

By symmetry the last estimate also holds for $\left|\nabla_y G_k(x,y)\right|$.

Proof. If $n = 2,3$, the statement of Lemma 4.18 is already strong enough and nothing remains to be proved.

We start with the case $n > 4$. We use an argument by contradiction and assume that there exist two sequences $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ with $x_k, y_k \in \Omega_{\ell_k}$ for a suitable sequence $(\ell_k) \subset \mathbb{N}$ such that $x_k \neq y_k$ for all $k \in \mathbb{N}$ and such that

$$\lim_{k\to+\infty} |x_k - y_k|^{n-4} \left|G_{\ell_k}(x_k, y_k)\right| = +\infty. \tag{4.77}$$

It is enough to consider $\ell_k = k$; other situations may be reduced to this by relabeling or are even more special. After possibly passing to a subsequence, it follows from (4.59) that there exists $x_\infty \in \partial\Omega$ such that

$$\lim_{k\to+\infty} x_k = x_\infty \quad \text{and} \quad \lim_{k\to+\infty} \frac{d(x_k)}{|x_k - y_k|} = 0. \tag{4.78}$$

We remark that the constant in (4.59) can be chosen uniformly for the family $(\Omega_k)_{k\in\mathbb{N}}$.

Next we claim that if (4.77) holds, then

$$\lim_{k\to+\infty} |x_k - y_k| = 0. \tag{4.79}$$

Assume by contradiction that $|x_k - y_k|$ does not converge to 0. After extracting a subsequence we may then assume that there exists $\delta > 0$ such that for all k we have $x_k \in B_\delta(x_\infty)$ and $y_k \in \Omega_k \setminus \overline{B_{3\delta}(x_\infty)}$. We consider q as in Lemma 4.21. In particular we know that $\|G_k(x,\,.\,)\|_{L^q(\Omega_k)} \le C$ uniformly in k. By applying local elliptic estimates (see Theorem 2.20) combined with Sobolev embeddings in $\Omega_k \setminus \overline{B_{2\delta}(x_\infty)}$ we find that

$$\|G_k(x_k,\,.\,)\|_{L^\infty\left(\Omega_k\setminus\overline{B_{3\delta}(x_\infty)}\right)} \le C(q,\delta)$$

uniformly in k. In particular, we would have

$$|G_k(x_k,y_k)| \le C(q,\delta) \quad \text{and} \quad |x_k-y_k|^{n-4}\,|G_k(x_k,y_k)| \le C(q,\delta)$$

independent of k. This contradicts the hypothesis (4.77) and proves the claim in (4.79).

Let $\Phi : U \to \mathbb{R}^n$, $\Phi(0) = x_\infty$ be a fixed coordinate chart for $\overline{\Omega}$ around ∞. We put $\Phi_k := \Psi_k \circ \Phi$ and have that

$$\Phi_k(U\cap\{x_1<0\}) = \Phi_k(U)\cap\Omega_k \;\text{ and }\; \Phi_k(U\cap\{x_1=0\}) = \Phi_k(U)\cap\partial\Omega_k.$$

Let $x_k = \Phi_k(x'_k)$ and $y_k = \Phi_k(y'_k)$. Therefore, (4.78) rewrites as

$$\lim_{k\to+\infty} x'_k = 0 \;\text{ and }\; \lim_{k\to+\infty} \frac{x'_{k,1}}{|x'_k-y'_k|} = 0. \tag{4.80}$$

We define for R and k large enough

$$\tilde{G}_k(z) = |x'_k-y'_k|^{n-4}\, G_k\left(\Phi_k(x'_k), \Phi_k\left(x'_k + |x'_k-y'_k|\,(z-\rho_k\mathbf{e}_1)\right)\right)$$

in $B_R(0)\cap\{x_1<0\}$, where $\rho_k := \frac{x'_{k,1}}{|x'_k-y'_k|}$ and $\mathbf{e}_1$ is the first unit vector. The biharmonic equation $\Delta^2 G_k(x,\,.\,) = 0$, complemented with Dirichlet boundary conditions, is rewritten as

$$\Delta^2_{g_k}\tilde{G}_k = 0 \text{ in } (B_R(0)\cap\{z_1<0\})\setminus\{\rho_k\mathbf{e}_1\}, \qquad \tilde{G}_k = \partial_1\tilde{G}_k = 0 \text{ on } \{z_1=0\}.$$

Here, $g_k(z) = \Phi_k^*(\mathscr{E})(x'_k + |x'_k-y'_k|(z-\rho_k\mathbf{e}_1))$, $\mathscr{E} = (\delta_{ij})$ the Euclidean metric, and Δ_{g_k} denotes the Laplace-Beltrami operator with respect to this scaled and translated pull back of the Euclidean metric under Φ_k. Then for some $q \in \left(\frac{n}{n-3}, \frac{n}{n-4}\right)$ and $\tau > 0$ being chosen suitably small, it follows from elliptic estimates (see Theorem 2.20) and Sobolev embeddings that there exists $C(R,\tau,q) > 0$ such that

$$|\tilde{G}_k(z)| \le C(R,q,\tau)\,\|\tilde{G}_k\|_{L^q(B_R(0)\setminus B_\tau(0))} \tag{4.81}$$

for all $z \in B_{R/2}(0)\setminus B_{2\tau}(0)$, $z_1 \le 0$. In order to estimate the L^q-norm on the right-hand side we use (4.74) and obtain that

$$\begin{aligned}\int_{B_R(0)\cap\{\zeta_1<0\}} |\tilde{G}_k(\zeta)|^q\,d\zeta &\le C\,|x'_k-y'_k|^{q(n-4)-n}\int_{\Omega_k}|G_k(x_k,y)|^q\,dy\\ &\le C\,|x'_k-y'_k|^{q(n-4)-n}\,d(x_k)^{(4-n)q+n} \le C\left(\frac{d(x_k)}{|x'_k-y'_k|}\right)^{n-q(n-4)}.\end{aligned}$$

Therefore, with (4.78), we get that $\lim_{k\to+\infty}\|\tilde{G}_k\|_{L^q(B_R(0)\setminus B_\tau(0))} = 0$, and (4.81) yields

$$\lim_{k\to+\infty}\tilde{G}_k = 0 \text{ in } C^0((B_{R/2}(0)\setminus B_{2\tau}(0))\cap\{z_1\le 0\}).$$

In particular, since $\lim_{k\to+\infty}\rho_k = 0$, we have that

$$\lim_{k\to+\infty} \tilde{G}_k\left(\frac{y_k'-x_k'}{\left|x_k'-y_k'\right|}+\rho_k\mathbf{e}_1\right)=0.$$

This limit rewrites as

$$\lim_{k\to+\infty} |x_k-y_k|^{n-4}\,|G_k(x_k,y_k)|=0,$$

contradicting (4.77). This completes the proof of Proposition 4.22 for the case $n>4$.

Now let us consider the case $n=4$. Here it is enough to prove (4.76) for ∇_y, exploiting the symmetry of the Green function. We argue by contradiction and, as in the proof for $n>4$, we may assume that there exist two sequences $(x_k)_{k\in\mathbb{N}},(y_k)_{k\in\mathbb{N}}$ with $x_k,y_k\in\Omega_k$ such that $x_k\neq y_k$ and

$$\lim_{k\to+\infty} |x_k-y_k|\,\left|\nabla_y G_k(x_k,y_k)\right|=+\infty. \tag{4.82}$$

After possibly passing to a subsequence it follows from (4.60) that there exists $x_\infty\in\partial\Omega$ such that

$$\lim_{k\to+\infty} x_k=x_\infty \text{ and } \lim_{k\to+\infty}\frac{d(x_k)}{|x_k-y_k|}=0. \tag{4.83}$$

Lemma 4.21 may be applied with some $q>4$. The analogue of (4.79) is proved in exactly the same way as above. Like above we now put for R and k large enough

$$\tilde{G}_k(z)=G_k\left(\Phi_k(x_k'),\Phi_k(x_k'+\left|x_k'-y_k'\right|(z-\rho_k\mathbf{e}_1))\right)$$

in $B_R(0)\cap\{z_1<0\}$, where $x_k=\Phi_k(x_k')$, $y_k=\Phi_k(y_k')$, $\rho_k:=\frac{x_{k,1}'}{|x_k'-y_k'|}$. As above we find for $\tau>0$ small enough that there exists $C(R,\tau,q)>0$ such that

$$\left|\nabla\tilde{G}_k(z)\right|\le C(R,q,\tau)\left\|\tilde{G}_k\right\|_{L^q(B_R(0)\setminus B_\tau(0))}$$

for all $z\in B_{R/2}(0)\setminus B_{2\tau}(0)$, $z_1\le 0$. Using (4.74) we obtain that

$$\begin{aligned}\int_{B_R(0)\cap\{\zeta_1<0\}}|\tilde{G}_k(\zeta)|^q\,d\zeta &\le C\left|x_k'-y_k'\right|^{-4}\int_{\Omega_k}|G_k(x_k,y)|^q\,dy\\ &\le C\left(\frac{d(x_k)}{\left|x_k'-y_k'\right|}\right)^4.\end{aligned}$$

In the same way as in the generic case $n>4$ this yields first that

$$\lim_{k\to+\infty}\nabla\tilde{G}_k=0 \text{ in } C^0((B_{R/2}(0)\setminus B_{2\tau}(0))\cap\{z_1\le 0\})$$

and back in the original coordinates

$$\lim_{k\to\infty}|x_k-y_k|\,\left|\nabla_y G_k(x_k,y_k)\right|=0.$$

So, we achieve a contradiction also if $n = 4$. This proves (4.76). By integrating (4.76) we get (4.75). The proof of Proposition 4.22 is complete. □

4.5.1.2 First and higher derivatives for $n \geq 3$

Proposition 4.23. *Suppose that $n \geq 3$ and let $(\Omega_k)_{k\in\mathbb{N}}$ be a $C^{4,\gamma}$-perturbation of the bounded $C^{4,\gamma}$-smooth domain Ω. Let G_k be as in Lemma 4.21. Then there exists a constant $C > 0$ such that for all k, all $\alpha, \beta \in \mathbb{N}^n$ with $1 \leq |\alpha| + |\beta| < 4$, and all $x, y \in \Omega_k$ with $x \neq y$ one has that*

$$\left| D_x^\alpha D_y^\beta G_k(x,y) \right| \leq C \cdot |x-y|^{4-n-|\alpha|-|\beta|}. \tag{4.84}$$

Proof. For $|\alpha| + |\beta| = 1$ with $n = 3, 4$ the result is found in (4.76).

To obtain estimates for (higher) derivatives we will use the following local estimate, see (2.19) for biharmonic functions. This local estimate is fundamental. Moreover, it also holds near the boundary part where homogeneous Dirichlet boundary conditions are satisfied. For any two concentric balls $B_R \subset B_{2R}$ and $|\alpha| \leq 4$ we have

$$\|D^\alpha v\|_{L^\infty(B_R \cap \Omega_k)} \leq \frac{C}{R^{|\alpha|}} \|v\|_{L^\infty(B_{2R} \cap \Omega_k)}. \tag{4.85}$$

The constant is uniform in k and R. The behaviour with respect to (small) R is obtained by means of scaling.

Case $n > 4$. Keeping $x \in \Omega_k$ fixed, for any $y \in \Omega_k \setminus \{x\}$ we choose $R = |x-y|/4$ and apply (4.85) and (4.75) of Proposition 4.22 in $B_R(y) \subset B_{2R}(y)$ to $G_k(x, .)$. This proves (4.84) for $|\alpha| = 0$. By symmetry the same estimate holds for $|\alpha| > 0$ and $|\beta| = 0$. Since also $D_x^\alpha G_k(x, .)$ is biharmonic with homogeneous Dirichlet boundary conditions we may repeat the argument to find estimates for mixed derivatives.

Case $n = 3, 4$. The result follows from a similar argument as above but now starting with the first order estimate in (4.76). □

4.5.1.3 Second and higher derivatives for $n = 2$

Lemma 4.24. *Let $n = 2$ and $\delta > 0$. Then there exists a constant $C = C(\delta, (\Omega_k)_{k\in\mathbb{N}})$ such that for $\alpha, \beta \in \mathbb{N}^2$ with $|a| + |\beta| = 2$*

$$x, y \in \Omega_k, \quad \max\{d(x), d(y)\} \geq \delta \Rightarrow \left| D_x^\alpha D_y^\beta G_k(x,y) \right| \leq C \log\left(1 + |x-y|^{-1}\right).$$

Proof. The Green function can be written as $G(x,y) = F_{n,2}(|x-y|) + h(x,y)$ with $h(x, .)$ the solution of (4.62). For $d(x) > \delta$ one finds as a direct consequence of Schauder estimates that $\|h(x, .)\|_{C^m(\overline{\Omega})} < C(\delta, m)$ for any $m \in \mathbb{N}$ and uniformly for all x with $d(x) > \delta$. Hence, for $|\beta| = 2$ one obtains

$$\left|D_y^\beta G(x,y)\right| \le C_1 \left|D_y^\beta F_{n,2}(x,y)\right| + C(\delta),$$

which shows the estimate in Lemma 4.24 for $\alpha = 0$. For $|\beta| < 2$ and hence $|\alpha| > 0$ one considers the function $D_x^\alpha h(x,.)$ and proceeds similar as before. So one has found the estimates in Lemma 4.24 for $d(x) > \delta$. Since the Green function is symmetric one may interchange the role of x and y and a similar result holds when $d(y) > \delta$. $\square$

Proposition 4.25. *Let $n = 2$. There exists a constant $C = C((\Omega_k)_{k\in\mathbb{N}})$ such that for $\alpha, \beta \in \mathbb{N}^2$ with $|a| + |\beta| \ge 3$*

$$\left|D_x^\alpha D_y^\beta G_k(x,y)\right| \le C\,|x-y|^{2-|\alpha|-|\beta|}.$$

The proposition requires a somehow technical proof which will be performed in several steps. However, combining the lemma and the proposition obviously gives a proof of the remaining cases of (4.72)-(4.73) and the proof of Theorem 4.20 will then be complete for $n = 2$.

As a starting point we prove an L^q-estimate for second derivatives of the Green functions.

Lemma 4.26. *Let $n = 2$. For any $q > 2$, there exists a constant $C = C(q, (\Omega_k)_{k\in\mathbb{N}})$ such that*

$$\left\|\nabla_y^2 G_k(x,.)\right\|_{L^q(\Omega_k)} \le C\, d(x)^{2/q}; \tag{4.86}$$

$$\left\|\nabla_x \nabla_y G_k(x,.)\right\|_{L^q(\Omega_k)} \le C\, d(x)^{2/q}. \tag{4.87}$$

Proof. We argue along the lines of the proof of Lemma 4.21 to which we refer for more detailed arguments. We prove first (4.86). For $\psi \in L^{q'}(\Omega_k)$, $q' = \frac{q}{q-1} \in (1,2)$ let $\varphi \in W^{2,q'}(\Omega_k)$ be the solution of

$$\begin{cases} \Delta^2\varphi = \nabla^2\psi & \text{in } \Omega_k, \\ \varphi = \varphi_\nu = 0 & \text{on } \partial\Omega_k. \end{cases}$$

For biharmonic equations in integral form $L^{q'}$-estimates (see Theorem 2.22) yield

$$\|\varphi\|_{W^{2,q'}} \le C\,\|\psi\|_{L^{q'}}.$$

Since $q' \in (1,2)$ we have that $2 - 2/q' \in (0,1)$ and employing also Sobolev's embedding theorem gives

$$|\varphi(x)| \le C\,\|\psi\|_{L^{q'}}\, d(x)^{2-2/q'}. \tag{4.88}$$

We observe the following representation formula, homogeneous Dirichlet boundary data of the Green functions and integrate by parts:

$$\varphi(x) = \int_{\Omega_k} G_k(x,y)\nabla_y^2 \psi(y)\,dy = \int_{\Omega_k} \nabla_y^2 G_k(x,y)\psi(y)\,dy.$$

Together with (4.88) and $2-2/q' = 2/q$ this shows (4.86).

In order to prove (4.87) we solve

$$\begin{cases} \Delta^2 \varphi = \nabla\psi & \text{in } \Omega_k, \\ \varphi = \varphi_\nu = 0 & \text{on } \partial\Omega_k. \end{cases}$$

and get

$$\|\varphi\|_{W^{3,q'}} \le C\,\|\psi\|_{L^{q'}}.$$

We proceed similarly as above and find

$$|\nabla\varphi(x)| \le C\,\|\psi\|_{L^{q'}}\, d(x)^{2-2/q'}$$

as well as

$$\nabla\varphi(x) = -\int_{\Omega_k} \nabla_x \nabla_y G_k(x,y)\psi(y)\,dy$$

and so, finally, (4.87). □

Proof of Proposition 4.25. We first prove the statement for $D_y^\beta G_k(x,y)$ with $|\beta| = 3$. We assume by contradiction that, after suitably relabeling, there exist sequences (x_k), (y_k) with $x_k, y_k \in \Omega_k$ and $x_k \neq y_k$, such that

$$\lim_{k\to\infty} |x_k - y_k|\, D_y^\beta G_k(x_k,y_k) = \infty. \tag{4.89}$$

As in Proposition 4.22, local elliptic estimates show that

$$\lim_{k\to\infty} |x_k - y_k| = 0.$$

Hence, we may assume that there exists $x_\infty \in \overline{\Omega}$ with $x_\infty = \lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k$. This shows that local elliptic estimates around x_k and y_k may be rescaled and hold with uniform constants.

First case: $d(x_k) < 2|x_k - y_k|$. Here we work in $B_{4|x_k-y_k|}(x_k) \setminus B_{|x_k-y_k|/2}(x_k)$, which certainly hit the boundaries $\partial\Omega_k$ where we have homogeneous Dirichlet boundary data for $G_k(x_k, .)$. These allow to apply local rescaled elliptic estimates and a localised Poincaré inequality to show that

$$\begin{aligned} |D_y^\beta G_k(x_k,y_k)| &\le C|x_k - y_k|^{-3-2/q} \|G_k(x_k,.)\|_{L^q(\Omega\cap(B_{4|x_k-y_k|}(x_k)\setminus B_{|x_k-y_k|/2}(x_k)))} \\ &\le C|x_k - y_k|^{-1-2/q} \|\nabla_y^2 G_k(x_k,.)\|_{L^q(\Omega\cap(B_{4|x_k-y_k|}(x_k)\setminus B_{|x_k-y_k|/2}(x_k)))} \\ &\le C|x_k - y_k|^{-1-2/q} d(x_k)^{2/q} \le C|x_k - y_k|^{-1}, \end{aligned}$$

where $q > 2$ is some arbitrarily chosen number. This inequality contradicts the assumption (4.89).

Second case: $d(x_k) \geq 2|x_k - y_k|$. We change our point of view and consider now y_k as parameter and the boundary value problem for the regular part of $\nabla_y^3 G_k(\,.\,,y_k)$. Arguing as in Lemma 4.18 and integrating local Schauder estimates yields

$$|D_y^\beta G_k(x_k,y_k)| \leq C\left(\frac{1}{|x_k-y_k|} + \frac{d(x_k)^{1+\gamma}}{d(y_k)^{2+\gamma}}\right). \tag{4.90}$$

By assumption we have $d(x_k) \geq 2|x_k - y_k|$, which implies that

$$\begin{aligned} d(x_k) \leq |x_k-y_k| + d(y_k) \leq \frac{1}{2}d(x_k) + d(y_k),\\ \Rightarrow \quad d(x_k) \leq 2d(y_k). \end{aligned}$$

Inserting this into (4.90) gives

$$|D_y^\beta G_k(x_k,y_k)| \leq C\left(\frac{1}{|x_k-y_k|} + \frac{1}{d(x_k)}\right) \leq C\,\frac{1}{|x_k-y_k|},$$

again a contradiction to the assumption (4.89).

We comment now on how to prove the statement for $\nabla_x\nabla_y^2 G_k(x,\,.\,)$. The remaining cases then follow by exploiting the symmetry of the Green functions. We assume by contradiction that – after a suitable relabeling – there exist sequences (x_k), (y_k), with $x_k, y_k \in \Omega_k$ and $x_k \neq y_k$, such that

$$\lim_{k\to\infty} |x_k - y_k| \cdot \nabla_x\nabla_y^2 G_k(x_k,y_k) = \infty. \tag{4.91}$$

As for (4.79), local elliptic estimates show that

$$\lim_{k\to\infty} |x_k - y_k| = 0.$$

Hence, we may assume that there exists $x_\infty \in \overline{\Omega}$ with $x_\infty = \lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k$. This shows that local elliptic estimates around x_k and y_k may be rescaled and hold with uniform constants.

First case: $d(x_k) < 2|x_k - y_k|$. As above we work in $B_{4|x_k-y_k|}(x_k) \setminus B_{|x_k-y_k|/2}(x_k)$ and find that

$$\begin{aligned} &|\nabla_x\nabla_y^2 G_k(x_k,y_k)| \leq C|x_k-y_k|^{-2-2/q}\|\nabla_x G_k(x_k,\,.\,)\|_{L^q(\Omega\cap(B_{4|x_k-y_k|}(x_k)\setminus B_{|x_k-y_k|/2}(x_k)))}\\ &\leq C|x_k-y_k|^{-1-2/q}\|\nabla_x\nabla_y G_k(x_k,\,.\,)\|_{L^q(\Omega\cap(B_{4|x_k-y_k|}(x_k)\setminus B_{|x_k-y_k|/2}(x_k)))}\\ &\leq C|x_k-y_k|^{-1-2/q} d(x_k)^{2/q} \leq C|x_k-y_k|^{-1}, \end{aligned}$$

where $q > 2$ is some arbitrarily chosen number. This inequality contradicts the assumption (4.91).

Second case: $d(x_k) \geq 2|x_k - y_k|$. Again we change our point of view and consider now y_k as parameter and the boundary value problem for the regular part of $\nabla_y^2 G_k(\,.\,,y_k)$. Arguing as in Lemma 4.18, integrating local Schauder estimates yield

$$|\nabla_x \nabla_y^2 G_k(x_k, y_k)| \leq C \left(\frac{1}{|x_k - y_k|} + \frac{d(x_k)^\gamma}{d(y_k)^{1+\gamma}} \right). \tag{4.92}$$

As above we may insert $d(x_k) \leq 2d(y_k)$ into (4.92) and obtain

$$|\nabla_x \nabla_y^2 G_k(x_k, y_k)| \leq C \left(\frac{1}{|x_k - y_k|} + \frac{1}{d(x_k)} \right) \leq C \frac{1}{|x_k - y_k|},$$

again a contradiction to the assumption (4.91).

Once the estimates for $|\alpha| + |\beta| = 3$ have been derived, we may proceed as in the proof of Proposition 4.23 to obtain the estimates for $|\alpha| + |\beta| = 4$. $\square$

Proposition 4.27. *Let $n = 2$. There exists a constant $C = C((\Omega_k)_{k\in\mathbb{N}})$ such that for $\alpha, \beta \in \mathbb{N}^2$ with $|a| + |\beta| = 2$*

$$\left| D_x^\alpha D_y^\beta G_k(x,y) \right| \leq C \log \left(1 + |x-y|^{-1} \right).$$

Proof. For x or y away from the boundary the result is found in Lemma 4.24. For $\delta > 0$ small enough take $y_0 \in \Omega_k$ with $d(y_0) > 2\delta$ and assume both $d(x) < \delta$ and $d(y) < \delta$. Let $r_0 > 0$ be small enough such that $\Omega_k \setminus B_{2r_0}(z)$ is still connected for each $z \in \Omega_k$. Using the estimate from Proposition 4.23 and integrating along a path γ from y_0 to y that avoids $B_r(x)$ with $r = \min\{r_0, |x-y|\}$, one finds

$$\begin{aligned}
\left| D_x^\alpha D_y^\beta G_k(x,y) \right| &\leq \left| D_x^\alpha D_y^\beta G_k(x,y_0) \right| + \int_\gamma \left| D_x^\alpha \nabla_y D_y^\beta G_k(x,\gamma(s)) \right| d\gamma(s) \\
&\leq C \left(1 + \int_{|x-y|}^M r^{-1} dr \right) \leq C' \log \left(1 + |x-y|^{-1} \right),
\end{aligned}$$

which shows the claim. $\square$

4.5.1.4 The proof of the uniform estimates

Proof of Theorem 4.20. For $n \geq 3$ and $|\alpha| + |\beta| + n > 4$ the estimate in (4.71) follows from Propositions 4.22 and 4.23. For $n = 2$ the estimate (4.71) follows from Proposition 4.25. The estimate in (4.72) is stated in Proposition 4.22 for $n = 4$ and in Proposition 4.27 for $n = 2$. The estimate in (4.73) is contained in Proposition 4.22. $\square$

4.5.2 *Uniform global estimates including boundary terms*

Theorem 4.28. *We assume that $(\Omega_k)_{k\in\mathbb{N}}$ is a $C^{4,\gamma}$-perturbation of the bounded $C^{4,\gamma}$-smooth domain $\Omega \subset \mathbb{R}^n$. Let $G_k = G_{\Delta^2,\Omega_k}$ denote the biharmonic Green function in Ω_k under Dirichlet boundary conditions. Then there exists a constant $C = C((\Omega_k)_{k\in\mathbb{N}})$, independent of k, such that for all $k \in \mathbb{N}$ it holds that*

$$\left| D_x^\alpha D_y^\beta G_k(x,y) \right| \le C\,(*), \tag{4.93}$$

where $(*)$ is as in Table 4.1. In this table the following abbreviations are used:

$$d(x) = d(x, \partial\Omega_k) \ \ \textit{and} \ \ d(y) = d(y, \partial\Omega_k),$$
$$W_x = \min\left\{1, \frac{d(x)}{|x-y|}\right\} \ \ \textit{and} \ \ W_y = \min\left\{1, \frac{d(y)}{|x-y|}\right\}.$$

Table 4.1 Expressions to be inserted into $(*)$ of (4.93).

$\vert\alpha\vert = \vert\beta\vert = 2:$	$\vert x-y\vert^{-n}$
$\vert\alpha\vert = 1$ and $\vert\beta\vert = 2:$	$\vert x-y\vert^{1-n}\,W_x$
$\vert\alpha\vert = 2$ and $\vert\beta\vert = 1:$	$\vert x-y\vert^{1-n}\,W_y$
$\vert\alpha\vert = 0$ and $\vert\beta\vert = 2:$	$\vert x-y\vert^{2-n}\,W_x^2$ for $n \ge 3$ $\log\left(1+\frac{d(x)^2}{\vert x-y\vert^2}\right)$ for $n=2$
$\vert\alpha\vert = \vert\beta\vert = 1:$	$\vert x-y\vert^{2-n}\,W_xW_y$ for $n \ge 3$ $\log\left(1+\frac{d(x)d(y)}{\vert x-y\vert^2}\right)$ for $n=2$
$\vert\alpha\vert = 2$ and $\vert\beta\vert = 0:$	$\vert x-y\vert^{2-n}\,W_y^2$ for $n \ge 3$ $\log\left(1+\frac{d(y)^2}{\vert x-y\vert^2}\right)$ for $n=2$
$\vert\alpha\vert = 0$ and $\vert\beta\vert = 1:$	$\vert x-y\vert^{3-n}\,W_x^2W_y$ for $n \ge 4$ $W_x^2W_y$ for $n=3$ $d(x)\,W_xW_y$ for $n=2$
$\vert\alpha\vert = 1$ and $\vert\beta\vert = 0:$	$\vert x-y\vert^{3-n}\,W_xW_y^2$ for $n \ge 4$ $W_xW_y^2$ for $n=3$ $d(y)\,W_xW_y$ for $n=2$
$\vert\alpha\vert = \vert\beta\vert = 0:$	$\vert x-y\vert^{4-n}\,W_x^2W_y^2$ for $n \ge 5$ $\log\left(1+\frac{d(x)^2d(y)^2}{\vert x-y\vert^4}\right)$ for $n=4$ $d(x)^{1/2}d(y)^{1/2}\,W_x^{3/2}W_y^{3/2}$ for $n=3$ $d(x)\,d(y)\,W_xW_y$ for $n=2$

Proof. The ingredients of this proof are the estimates in Theorem 4.20, the construction of appropriate curves connecting x with a boundary point x^* that avoid singular points, and integral estimates along these curves. The $C^{4,\gamma}$-smoothness only comes in through the constant that appear in Theorem 4.20. So we may suppress the dependence on k in the present proof.

Claim 1: Let $x, y \in \Omega$. There exists a piecewise smooth curve Γ_x connecting x with the boundary $\partial\Omega$ such that $d(\Gamma_x, y) \ge \frac{1}{2}|x-y|$ and if we parametrise Γ_x by arclength, it holds that:

$$\tfrac{2}{3}s \leq |\Gamma_x(s) - x| \leq s, \tag{4.94}$$

$$|\Gamma_x(s) - y| \geq \tfrac{1}{8}|x-y| + \tfrac{1}{8}|\Gamma_x(s) - x|. \tag{4.95}$$

Let x^* be such that $d(x) = |x - x^*|$. If the interval $[x,x^*]$ does not intersect $B_{\frac{1}{2}|x-y|}(y)$, then we take $\Gamma_x = [x,x^*]$. If the set $[x,x^*] \cap B_{\frac{1}{2}|x-y|}(y)$ is nonempty while $B_{\frac{1}{2}|x-y|}(y) \cap \partial\Omega$ is empty, one modifies Γ_x by replacing $[x,x^*] \cap B_{\frac{1}{2}|x-y|}(y)$ by a shortest path on $\partial B_{\frac{1}{2}|x-y|}(y)$ that connects the two points of $[x,x^*] \cap \partial B_{\frac{1}{2}|x-y|}(y)$. If both $[x,x^*] \cap B_{\frac{1}{2}|x-y|}(y)$ and $B_{\frac{1}{2}|x-y|}(y) \cap \partial\Omega$ are nonempty, the part of $[x,x^*] \cap B_{\frac{1}{2}|x-y|}(y)$ is replaced by the shortest path on $\partial B_{\frac{1}{2}|x-y|}(y)$ that connects with the boundary, see Figure 4.1.

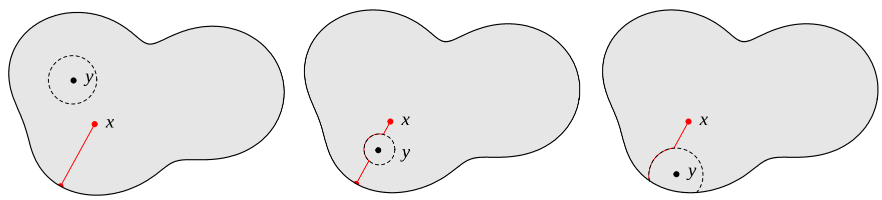

Fig. 4.1 Curves connecting x with the boundary by a path of length less than $\frac{3}{2}d(x)$ that avoid the singularity in y by staying outside of $B_{\frac{1}{2}|x-y|}(y)$.

Geometric arguments show that $s \leq \frac{1}{3}(\pi+1)|\Gamma_x(s) - x|$. Using $\frac{1}{3}(\pi+1) < \frac{3}{2}$, (4.94) follows. Moreover, writing $z = \Gamma_x(s)$, we have $|z-y| \geq \frac{1}{2}|x-y|$. So, if $|z-x| \leq 2|x-y|$, then $|z-x| \leq 4|z-y|$. If $|z-x| \geq 2|x-y|$, then $|z-y| \geq |z-x| - |x-y| \geq \frac{1}{2}|z-x|$. Combining we obtain (4.95).

Claim 2: Let $k \geq 2$ and $\nu_1, \nu_2 \geq 0$. If $H(x,y) = 0$ for all $x \in \partial\Omega$ and $y \in \Omega$ and if for some $C \in \mathbb{R}^+$

$$|\nabla_x H(x,y)| \leq C\ |x-y|^{-k} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2} \text{for } x,y \in \Omega,$$

then there is $\tilde{C} \in \mathbb{R}^+$ such that

$$|H(x,y)| \leq \tilde{C}\ |x-y|^{1-k} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1+1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2} \text{for } x,y \in \Omega.$$

Let $s \mapsto x(s)$ parametrise Γ_x as above by arclength connecting $x^* \in \partial\Omega$ with x. Then

$$H(x,y) = H(x^*,y) + \int_{\Gamma_x} \nabla_x H(x(s),y) \cdot \tau(s)\, ds \tag{4.96}$$

and using Lemma 4.5

$$
\begin{aligned}
|H(x,y)| &\le \int_{\Gamma_x} |\nabla_x H(x(s),y)|\, ds \\
&\le \int_{\Gamma_x} C\,|x(s)-y|^{-k} \min\left\{1, \frac{d(x(s))^{\nu_1} d(y)^{\nu_2}}{|x(s)-y|^{\nu_1+\nu_2}}\right\} ds.
\end{aligned}
$$

It follows from (4.95) that

$$
\begin{aligned}
|H(x,y)| &\le c_1 \int_0^{\frac{3}{2}d(x)} (|x-y|+s)^{-k} \min\left\{1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{(|x-y|+s)^{\nu_1+\nu_2}}\right\} ds \\
&= c_1\,|x-y|^{1-k} \int_0^{\frac{3}{2}\frac{d(x)}{|x-y|}} (1+t)^{-k} \min\left\{1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{(1+t)^{\nu_1+\nu_2}\,|x-y|^{\nu_1+\nu_2}}\right\} dt.
\end{aligned}
$$

We distinguish two cases. If $d(x) \le |x-y|$, then

$$
\begin{aligned}
|H(x,y)| &\le c_1\,|x-y|^{1-k} \int_0^{\frac{3}{2}\frac{d(x)}{|x-y|}} \min\left\{1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}}\right\} dt \\
&\le c_2\,|x-y|^{1-k} \min\left\{1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}}\right\} \frac{d(x)}{|x-y|} \\
&\le c_3\,|x-y|^{1-k} \min\left\{1, \frac{d(x)^{\nu_1+1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2+1}}\right\}. \qquad (4.97)
\end{aligned}
$$

If $d(x) \ge |x-y|$, then

$$
\begin{aligned}
|H(x,y)| &\le c_1\,|x-y|^{1-k} \int_0^{\frac{3}{2}\frac{d(x)}{|x-y|}} (1+t)^{-k} dt \cdot \min\left\{1, \frac{d(y)^{\nu_2}}{|x-y|^{\nu_2}}\right\} \\
&\le c_2\,|x-y|^{1-k} \min\left\{1, \frac{d(y)^{\nu_2}}{|x-y|^{\nu_2}}\right\} \\
&\le c_3\,|x-y|^{1-k} \min\left\{1, \frac{d(x)^{\nu_1+1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2+1}}\right\}.
\end{aligned}
$$

Claim 3: Let $\nu_1, \nu_2 \ge 0$. If $H(x,y) = 0$ for $x \in \partial\Omega$ and for some $C \in \mathbb{R}^+$

$$
|\nabla_x H(x,y)| \le C\ |x-y|^{-1} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2} \text{for } x,y \in \Omega,
$$

then there is $\tilde{C} \in \mathbb{R}^+$ such that for all $x,y \in \Omega$

$$
|H(x,y)| \le \tilde{C}\ \log\left(2 + \frac{d(x)}{|x-y|}\right) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1+1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2}.
$$

The steps of Claim 2 remain valid until

$$|H(x,y)| \le c_1 \int_0^{\frac{3}{2}\frac{d(x)}{|x-y|}} (1+t)^{-1} \min\left\{1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{(1+t)^{\nu_1+\nu_2} |x-y|^{\nu_1+\nu_2}}\right\} dt,$$

and inclusive (4.97). For $d(x) \le |x-y|$ the claim follows. If $d(x) \ge |x-y|$, then

$$\begin{aligned}
&\int_0^{\frac{3}{2}\frac{d(x)}{|x-y|}} (1+t)^{-1} \min\left\{1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{(1+t)^{\nu_1+\nu_2} |x-y|^{\nu_1+\nu_2}}\right\} dt \\
&\qquad \le \log\left(2+\frac{d(x)}{|x-y|}\right) \min\left\{1, \frac{d(x)^{\nu_1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2}}\right\} \\
&\qquad \le \log\left(2+\frac{d(x)}{|x-y|}\right) \min\left\{1, \frac{d(x)^{\nu_1+1} d(y)^{\nu_2}}{|x-y|^{\nu_1+\nu_2+1}}\right\}.
\end{aligned}$$

Claim 4: Let $k \ge 2$ and $\nu_1, \nu_2, \alpha_1, \alpha_2 \ge 0$. If $H(x,y) = 0$ for $x \in \partial\Omega$ and for some $C \in \mathbb{R}^+$

$$|\nabla_x H(x,y)| \le C\, d(x)^{\alpha_1} d(y)^{\alpha_2} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2} \text{ for } x,y \in \Omega,$$

then there is $\tilde{C} \in \mathbb{R}^+$ such that

$$|H(x,y)| \le \tilde{C}\, d(x)^{\alpha_1+1} d(y)^{\alpha_2} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2} \text{ for } x,y \in \Omega.$$

This is a direct consequence of (4.96), (4.94) and (4.95).

Claim 5: Let $k \ge 2$ and $\nu_1, \nu_2, \alpha_1, \alpha_2 \ge 0$. If $H(x,y) = 0$ for $x \in \partial\Omega$ and if there exists $C \in \mathbb{R}^+$ such that

$$|\nabla_x H(x,y)| \le C\, \log\left(2+\frac{d(x)d(y)}{|x-y|^2}\right) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2}$$

for $x,y \in \Omega$, then there is $\tilde{C} \in \mathbb{R}^+$ such that

$$|H(x,y)| \le \tilde{C}\, d(x) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2} \quad \text{for } x,y \in \Omega.$$

We first observe that

$$\begin{aligned}
&\log\left(2+\frac{d(x)d(y)}{|x-y|^2}\right) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2} \\
&\simeq \log\left(2+\frac{d(x)}{|x-y|}\right) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{\nu_1} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{\nu_2}
\end{aligned}$$

If $\frac{d(x)}{|x-y|} \leq 1$, then $\log\left(2+\frac{d(x)}{|x-y|}\right)$ is bounded and the result is again a direct consequence of (4.96), (4.94) and (4.95). If $\frac{d(x)}{|x-y|} \geq 1$, then for $z \in \Gamma_x$

$$\frac{d(z)}{|x-z|} \leq \frac{8d(z)}{|x-y|+s}$$

and one finds

$$|\nabla_x H(x(s),y)| \leq c_1 \log\left(2+\frac{d(x)}{|x-y|+s}\right) \cdot \min\left\{1, \frac{d(y)^{\nu_2}}{|x-y|^{\nu_2}}\right\}.$$

Hence

$$\begin{aligned}|H(x,y)| &\leq c_1 \int_0^{\frac{3}{2}d(x)} \log\left(2+\frac{d(x)}{|x-y|+s}\right) ds \cdot \min\left\{1, \frac{d(y)^{\nu_2}}{|x-y|^{\nu_2}}\right\} \\ &\leq c_2 d(x) \min\left\{1, \frac{d(y)^{\nu_2}}{|x-y|^{\nu_2}}\right\}.\end{aligned}$$

In order to complete the proof of Theorem 4.28 one starts from the estimates of Theorem 4.20. We find, using the Claims 2 to 5 and working our way down, the estimates as in Table 4.1 except for $n=3$ with $|\alpha|+|\beta| \leq 1$. Suppose $\alpha = 0$ and $|\beta| = 1$. Then

$$\left|\nabla_x D_y^\beta G(x,y)\right| \leq C\,|x-y|^{-1} \min\left\{1, \frac{d(x)}{|x-y|}\right\} \min\left\{1, \frac{d(y)}{|x-y|}\right\}$$

implies

$$\left|D_y^\beta G(x,y)\right| \leq C \log\left(1+\frac{d(x)^2 d(y)}{|x-y|^3}\right).$$

Together with (4.76) we obtain

$$\left|D_y^\beta G(x,y)\right| \leq C \min\left\{1, \frac{d(x)}{|x-y|}\right\}^2 \min\left\{1, \frac{d(y)}{|x-y|}\right\}.$$

For the zeroth order in case $n=3$ one finds through

$$\left|\nabla_y G(x,y)\right| \leq C \min\left\{1, \frac{d(x)}{|x-y|}\right\}^2 \min\left\{1, \frac{d(y)}{|x-y|}\right\}$$

that

$$|G(x,y)| \leq C\,d(y) \min\left\{1, \frac{d(x)}{|x-y|}\right\}^2 \min\left\{1, \frac{d(y)}{|x-y|}\right\}$$

and through the similar estimate for $|\nabla_x G(x,y)|$ that

$$|G(x,y)| \le C M(x,y) \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}$$

with

$$M(x,y) = \min\left\{d(y)\min\left\{1, \frac{d(x)}{|x-y|}\right\}, d(x)\min\left\{1, \frac{d(y)}{|x-y|}\right\}\right\}.$$

Since $M(x,y) \le \sqrt{d(x)d(y)}\min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}^{3/2}$ the proof is complete. □

In a similar way one may derive estimates for the Poisson kernels. Consider

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u|_{\partial\Omega} = \psi, & -\frac{\partial u}{\partial \nu}|_{\partial\Omega} = \varphi. \end{cases} \tag{4.98}$$

If $G = G_{\Delta^2,\Omega}$ is the Green function for this boundary value problem, then the solution of (4.98) is written as

$$u(x) = \int_\Omega G(x,y)\, f(y)\, dy + \int_{\partial\Omega} K(x,y)\, \psi(y)\, d\omega_y + \int_{\partial\Omega} L(x,y)\, \varphi(y)\, d\omega_y,$$

with $K, L : \overline{\Omega} \times \partial\Omega \to \mathbb{R}$ defined by

$$K(x,y) = \frac{\partial}{\partial \nu_y}\Delta_y G(x,y),$$
$$L(x,y) = \Delta_y G(x,y).$$

Theorem 4.29. *Let $(\Omega_k)_{k\in\mathbb{N}}$ be as in Theorem 4.28 and let K_{Ω_k} and L_{Ω_k} be the corresponding Poisson kernels. Then there exists $C = C\left((\Omega_k)_{k\in\mathbb{N}}\right)$ such that for all $(x,y) \in \Omega \times \partial\Omega$:*

$$\left|K_{\Omega_k}(x,y)\right| \le C\frac{d(x)^2}{|x-y|^{n+1}} \text{ and } \left|L_{\Omega_k}(x,y)\right| \le C\frac{d(x)^2}{|x-y|^n}.$$

For $n = 2$ one obtains $\left|L_{\Omega_k}(x,y)\right| \le C$.

4.5.3 *Convergence of the Green function in domain approximations*

Proposition 4.30. *Let $x_k \in \Omega_k$ and assume that $\lim_{k\to\infty} x_k = x_\infty \in \Omega$. Then we have*

$$G_k(x_k,.) \to G(x_\infty,.) \text{ in } C^4_{loc}(\Omega \setminus \{x_\infty\}),$$
$$G_k(x_k,.) \to G(x_\infty,.) \text{ in } L^1(\mathbb{R}^n),$$
$$G_k(x_k,.)\circ\Psi_k \to G(x_\infty,.) \text{ in } C^4_{loc}(\overline{\Omega} \setminus \{x_\infty\}).$$

If $n = 3$ we have in addition that

$$G_k(\,.\,,\,.\,) \to G(\,.\,,\,.\,) \text{ in } C^0_{loc}(\Omega \times \Omega).$$

Proof. According to Theorem 4.20 we know that

$$|G_k(x,y)| \le C \begin{cases} |x-y|^{4-n} & \text{if } n > 4, \\ \log\left(1+|x-y|^{-1}\right) & \text{if } n = 4, \\ 1 & \text{if } n = 3; \end{cases} \tag{4.99}$$

uniformly in k. This shows that in particular

$$\|G_k(x,\,.\,)\|_{L^1(\Omega_k)} \le C \text{ uniformly in } x \text{ and } k.$$

Moreover, since $x_k \to x_\infty$, we may assume that all x_k are in a small neighbourhood around x_∞. Let $\Omega_0 \subset\subset \Omega$ be arbitrary; local Schauder estimates (see Theorem 2.19) show that $(G_k(x_k,\,.\,))_{k\in\mathbb{N}}$ is locally bounded in $C^4_{loc}\left(\overline{\Omega_0}\setminus\{x_\infty\}\right)$. Hence, after selecting a suitable subsequence we see that for each such $\Omega_0 \subset\subset \Omega$ one has $G_k(x_k,\,.\,) \to \varphi$ in $C^4_{loc}\left(\overline{\Omega_0}\setminus\{x_\infty\}\right)$ and $G_k(x_k,\,.\,)\circ\Psi_k \to \varphi$ in $C^4_{loc}(\overline{\Omega}\setminus\{x_\infty\})$ with a suitable $\varphi \in C^{4,\gamma}(\overline{\Omega}\setminus\{x_\infty\})$. Thanks to this compactness and the fact that in any case the limit is the uniquely determined Green function, we have convergence on the whole sequence towards $G(x_\infty,\,.\,)$.

Finally, since we have pointwise convergence, (4.99) allows for applying Vitali's convergence theorem to show that

$$G_k(x_k,\,.\,) \to G(x_\infty,\,.\,) \text{ in } L^1(\mathbb{R}^n).$$

The statement concerning $C^0_{loc}(\Omega\times\Omega)$-convergence in $n = 3$ is a consequence of $|\nabla G_k(\,.\,,\,.\,)| \le C$, see (4.60). □

In order to have enough smoothness to conclude also for the last case in Theorem 6.30 we also need a convergence result simultaneous in both variables.

Proposition 4.31. *We have that*

$$G_k(\,.\,,\,.\,)\circ(\Psi_k\times\Psi_k) \to G(\,.\,,\,.\,) \text{ in } C^4_{loc}\left(\overline{\Omega}\times\overline{\Omega}\setminus\{(x,x)\,;x\in\overline{\Omega}\}\right).$$

Proof. We combine the ideas of the proofs of Propositions 4.30 and 4.17. One should observe that Theorem 4.20 guarantees uniform L^1-bounds for G_k as in the proof of Proposition 4.30. □

4.6 Weighted estimates for the Dirichlet problem

As a side result the estimates in the previous section for the homogeneous biharmonic Dirichlet problem allow weighted L^p-L^q estimates for elliptic boundary value problems under homogeneous boundary conditions with $d\,(\,.\,)^\theta$ as weight function.

Here we will restrict ourselves to the biharmonic case. A more general version of this theorem is found in [118].

If u and f are such that

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = u_\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.100}$$

then we have

Theorem 4.32. *Let Ω be a bounded $C^{4,\gamma}$-smooth domain and let $u \in C^4\left(\overline{\Omega}\right)$, $f \in C\left(\overline{\Omega}\right)$ be as in (4.100). Then the following hold:*

1. *For $n < 4$ there exists $C = C(\Omega)$ such that for all $\theta \in [0,1]$:*

$$\left\| d(\,.\,)^{\theta n - 2} u \right\|_{L^\infty(\Omega)} \le C \left\| d(\,.\,)^{2-(1-\theta)n} f \right\|_{L^1(\Omega)}. \tag{4.101}$$

2. *For all $n \ge 2$ if $p, q \in [1,\infty]$ are such that $0 \le \frac{1}{p} - \frac{1}{q} < \alpha \le \min\left\{1, \frac{4}{n}\right\}$, then there exists $C = C(\Omega, \alpha)$ such that for all $\theta \in [0,1]$:*

$$\left\| d(\,.\,)^{\theta n\alpha - 2} u \right\|_{L^q(\Omega)} \le C \left\| d(\,.\,)^{2-(1-\theta)n\alpha} f \right\|_{L^p(\Omega)}. \tag{4.102}$$

Remark 4.33. Notice that the shift in the exponent of $d(\,.\,)$ in (4.102) is $4 - n\alpha$ with $\alpha > 0$. If $p = q$ this shift can be arbitrarily close to 4 but will not reach 4.

Before proving this theorem we recall an estimate involving the Riesz potential

$$\left(K_\gamma * f\right)(x) := \int_\Omega |x-y|^{-\gamma} f(y) dy.$$

We prove a classical convolution estimate which can e.g. be found in [231, Corollary 4.5.2].

Lemma 4.34. *Let $\Omega \subset \mathbb{R}^n$ be bounded, $\gamma \in (0,n)$ and $1 \le p, q \le \infty$. If $\frac{\gamma}{n} < \frac{1}{r} := \min\left\{1, 1 + \frac{1}{q} - \frac{1}{p}\right\}$, then there exists $C = C(\mathrm{diam}(\Omega), n - \gamma r) \in \mathbb{R}^+$ such that for all $f \in L^p(\Omega)$:*

$$\left\| K_\gamma * f \right\|_{L^q(\Omega)} \le C \, \|f\|_{L^p(\Omega)}. \tag{4.103}$$

Proof. We let $p' \in [1,\infty]$ denote the conjugate of $p \in [1,\infty]$: $\frac{1}{p} + \frac{1}{p'} = 1$ etc. Set

$$C_{\Omega,s} = \max_{x \in \Omega} \left\| |x - .\,|^{-s} \right\|_{L^1(\Omega)}$$

and notice that $C_{\Omega,s}$ is bounded for $s < n$.

If $q = 1$, then $r = 1$ and a change in the order of integration gives

$$\begin{aligned} \left\| K_\gamma * f \right\|_{L^1(\Omega)} &\le \int_\Omega \left(\int_\Omega |x-y|^{-\gamma} dx \right) |f(y)|\, dy \\ &\le C_{\Omega,\gamma r} \|f\|_{L^1(\Omega)} \le C \, \|f\|_{L^p(\Omega)}. \end{aligned} \tag{4.104}$$

If $p \geq q$, then $r = 1$ and

$$\begin{aligned}\left\|K_\gamma * f\right\|^q_{L^q(\Omega)} &= \int_\Omega \left|\int_\Omega |x-y|^{-\gamma} f(y)dy\right|^q dx \\ &\leq \int_\Omega \left(\int_\Omega |x-y|^{-\gamma} dy\right)^{\frac{q}{q'}} \left(\int_\Omega |x-y|^{-\gamma} |f(y)|^q dx\right) dy \\ &\leq C^{q/q'}_{\Omega,\gamma r} \left\|K_\gamma * |f|^q\right\|_{L^1(\Omega)}\end{aligned}$$

and one continues with (4.104).

One finds for $1 < p < q < \infty$, since $\frac{1}{r'} + \frac{1}{p'} + \frac{1}{q} = 1$, that

$$\begin{aligned}&\left|\left(K_\gamma * f\right)(x)\right| \leq \\ &\leq \left(\int_\Omega |f(y)|^p dy\right)^{\frac{1}{r'}} \left(\int_\Omega |x-y|^{-r\gamma} dy\right)^{\frac{1}{p'}} \left(\int_\Omega |x-y|^{-r\gamma} |f(y)|^p dy\right)^{\frac{1}{q}} \qquad (4.105)\end{aligned}$$

and, by changing the order of integration and using $pq/r' + p = q$, also

$$\begin{aligned}\left\|K_\gamma * f\right\|^q_{L^q(\Omega)} &\leq \|f\|^{pq/r'}_{L^p(\Omega)} C^{q/p'}_{\Omega,\gamma r} \int_\Omega \left(\int_\Omega |x-y|^{-r\gamma} dx\right) |f(y)|^p dy \\ &\leq C^{1+q/p'}_{\Omega,\gamma r} \|f\|^q_{L^p(\Omega)}. \qquad (4.106)\end{aligned}$$

For $q = \infty$ and has $r = p'$ and the proof reduces to

$$\left|\left(K_\gamma * f\right)(x)\right| \leq \left\||x-\,.\,|^{-\gamma}\right\|_{L^{p'}(\Omega)} \|f\|_{L^p(\Omega)} \leq C^{1/r}_{\Omega,\gamma r} \|f\|_{L^p(\Omega)}.$$

For $p = 1$ one replaces (4.105) by

$$\left|\left(K_\gamma * f\right)(x)\right| \leq \left(\int_\Omega |f(y)|\, dy\right)^{\frac{1}{q'}} \left(\int_\Omega |x-y|^{-q\gamma} |f(y)|\, dy\right)^{\frac{1}{q}}$$

and continues similar as in (4.106) by changing the order of integration. □

Proof of Theorem 4.32. Again we use the notation (4.1). The estimate that we will use repeatedly is a consequence of Lemma 4.5:

$$\min\left\{1, \frac{d(x)\,d(y)}{|x-y|^2}\right\} \leq C\left(\frac{d(x)\,d(y)}{|x-y|^2}\right)^{1-s} \left(\frac{d(y)}{d(x)}\right)^{s(2\theta-1)} \text{ for all } s,\theta \in [0,1]. \qquad (4.107)$$

To prove Item 1 in Theorem 4.32 we apply this estimate for $s = 1$ to find for $n < 4$

$$|G(x,y)| \leq C d(x)^{2-\frac{1}{2}n} d(y)^{2-\frac{1}{2}n} \left(\frac{d(y)}{d(x)}\right)^{\frac{1}{2}n(2\theta-1)} = C d(x)^{2-n\theta} d(y)^{2-n(1-\theta)}$$

and a straightforward integration shows (4.101).

For Item 2, we need Lemma 4.34. If $n > 4$ we use from Theorem 4.28 the estimate for the Green function itself, and (4.107) for $s = \frac{1}{4}n\alpha$, which is allowed since $0 \leq \frac{1}{4}n\alpha \leq 1$,

$$\begin{aligned}
|G(x,y)| &\leq C\,|x-y|^{4-n} \min\left\{1, \frac{d(x)\,d(y)}{|x-y|^2}\right\}^2 \\
&\leq C\,|x-y|^{4-n} \left(\frac{d(x)\,d(y)}{|x-y|^2}\right)^{2\left(1-\frac{1}{4}n\alpha\right)} \left(\frac{d(y)}{d(x)}\right)^{\frac{1}{2}n\alpha(2\theta-1)} \\
&= C\,|x-y|^{n(\alpha-1)}\,d(x)^{2-\theta n\alpha}\,d(y)^{2-(1-\theta)n\alpha}.
\end{aligned}$$

For $n = 4$ it holds that

$$|G(x,y)| \leq C \log\left(1 + \frac{d(x)^2\,d(y)^2}{|x-y|^2}\right) \leq C\,|x-y|^{-\varepsilon} \min\left\{1, \frac{d(x)\,d(y)}{|x-y|^2}\right\}^2$$

and we may continue as before if we choose ε small enough such that $\alpha - \frac{1}{4}\varepsilon > \frac{1}{p} - \frac{1}{q}$.

If $n = 3$, one has for $\alpha, \theta \in [0,1]$

$$\begin{aligned}
|G(x,y)| &\leq C d(x)^{\frac{1}{2}}\,d(y)^{\frac{1}{2}} \min\left\{1, \frac{d(x)\,d(y)}{|x-y|^2}\right\}^{\frac{3}{2}} \\
&\leq C d(x)^{\frac{1}{2}}\,d(y)^{\frac{1}{2}} \left(\frac{d(x)\,d(y)}{|x-y|^2}\right)^{\frac{3}{2}(1-\alpha)} \left(\frac{d(y)}{d(x)}\right)^{\frac{3}{2}\alpha(2\theta-1)} \\
&= C\,|x-y|^{-3(1-\alpha)}\,d(x)^{2-3\alpha\theta}\,d(y)^{2-3\alpha(1-\theta)}.
\end{aligned}$$

Similarly, if $n = 2$ one finds for $\alpha, \theta \in [0,1]$

$$|G(x,y)| \leq C\,|x-y|^{-2(1-\alpha)}\,d(x)^{2-2\alpha\theta}\,d(y)^{2-2\alpha(1-\theta)}.$$

We have found for all $n \neq 4$ (and with a minor change for $n = 4$) that

$$\left|d(x)^{\theta n\alpha-2}\,u(x)\right| \leq C\left|\int_\Omega |x-y|^{-n(1-\alpha)}\,d(y)^{2-(1-\theta)n\alpha}\,f(y)dy\right|.$$

By Lemma 4.34 we have

$$\left\|d(\,.\,)^{\theta n\alpha-2}\,u\right\|_{L^q(\Omega)} \leq C\left\|d(\,.\,)^{2-(1-\theta)n\alpha} f\right\|_{L^p(\Omega)}$$

whenever $\alpha > \frac{1}{p} - \frac{1}{q}$ and with $\alpha \leq \frac{1}{4}n$ for $n \geq 4$ and $\alpha \leq 1$ for $n < 4$. $\square$

4.7 Bibliographical notes

Characterisations of Green's functions like in Section 4.2.1 are well-known by now in the context of second order problems on arbitrary smooth domains. Estimates from above go back to works of Grüter-Widman [215, 411]. The two-sided sharp estimates for the Green function of the Dirichlet-Laplacian are due to Zhao in [419, 420, 421]. Hueber-Sieveking [233] and Cranston-Fabes-Zhao [114] proved bounds for the Green function for general second order operators based on Harnack inequalities. For the importance of so-called 3-G-theorems in the potential theory of Schrödinger operators and the link with stochastic processes we refer to [97]. A first place where the optimal two-sided estimates are listed is [384].

For the higher order problems considered here the situation is quite different, namely, no general maximum principles and in particular no general Harnack inequalities are available. In the polyharmonic situation the starting point is Boggio's explicit formula for the Green's function in balls [63] from 1905, see (2.65). His formula led to optimal two-sided estimates for the polyharmonic Green function in case of a ball and inspired the estimates for the absolute value of the Green function in general domains. The subsequent estimates and 3-G-theorems were developed by Grunau-Sweers [210]. For further classical material on polyharmonic operators we refer to the book of Nicolesco [323].

As for the biharmonic Steklov boundary value problem in Section 4.3, we follow Gazzola-Sweers [191]. Proposition 4.13 is taken from [213, Lemmas 3.1 and 3.2] and is based on previous estimates by Zhao [420, 421], see also [118, 384]. Some of the results in Section 4.3 can be obtained under the assumption that $\partial\Omega \in C^{1,1}$, see [191].

Estimates of Green's functions for general higher order elliptic operators are due to Krasovskiĭ [255, 256]. However, due the general situation considered there, high regularity was imposed on the boundary. Since we restrict ourselves to biharmonic Green's functions and for the reader's convenience, we give a more elementary derivation of such estimates which only need to refer to Agmon-Douglis-Nirenberg [5], i.e. to Section 2.5 of the present book. The actual estimates are based on Dall'Acqua-Sweers and Grunau-Robert [118, 207]. For generalisations of Green function estimates to nonsmooth domains see also Mayboroda-Maz'ya [286].

Chapter 5
Positivity and lower order perturbations

As already mentioned in Section 1.2, in general one does not have positivity preserving for higher order Dirichlet problems. Nevertheless, in Chapter 6 we shall identify some families of domains where the biharmonic — or more generally the polyharmonic — Dirichlet problem enjoys a positivity preserving property. Moreover, there we shall prove "almost positivity" for the biharmonic Dirichlet problem in any bounded smooth domain $\Omega \subset \mathbb{R}^n$.

As an intermediate step, taking advantage of the kernel estimates proved in Chapter 4, we study lower order perturbations of the prototype $((-\Delta)^m, B \subset \mathbb{R}^n)$ where B is again the unit ball. In Theorem 5.1 we prove positivity for Dirichlet problems

$$\begin{cases} Lu := (-\Delta)^m u + \sum\limits_{|\beta| \leq 2m-1} a_\beta(x) D^\beta u = f & \text{in } B, \\ D^\alpha u|_{\partial B} = 0 & \text{for } |\alpha| \leq m-1, \end{cases} \tag{5.1}$$

with "small" coefficients a_β. Its proof is based on Green's function estimates, estimates for iterated Green's functions via the 3-G-theorem 4.9, and Neumann series. With the help of Riemann's theorem on conformal mappings and a reduction to normal form, this result will be used to prove the more general Theorem 6.3 where this approach permits to consider also highest-order perturbations in two dimensions.

If we remove in (5.1) the smallness assumptions on the coefficients a_β, we are still able to prove a *local* maximum principle for differential inequalities, which is true also in arbitrary domains Ω, see Theorem 5.19.

In the same spirit we study in Section 5.4 positivity preserving for the Steklov boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - a u_\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

with data $a \in C^0(\partial\Omega)$ and suitable $f \geq 0$. It turns out that when a is below the corresponding positive first Steklov eigenvalue (see (3.40)) and above a negative critical parameter, one has positivity preserving, i.e. $f \geq 0 \Rightarrow u \geq 0$. This critical parameter may also be $-\infty$. This issue is somehow related to positivity in the corresponding

Dirichlet problem. As an application we will see that in a *convex* planar domain the hinged plate described by (1.10) satisfies the positivity preserving property, namely upwards pushing yields upwards bending.

It is another interesting question to ask which is the role of nontrivial Dirichlet boundary data with regard to the positivity of the solution. We look first at the inhomogeneous problem for the clamped plate equation:

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = \psi, & -\dfrac{\partial u}{\partial \nu}|_{\partial\Omega} = \varphi. \end{cases} \tag{5.2}$$

One could think that, at least in the unit ball B, nonnegative data $\psi \geq 0$, $\varphi \geq 0$ yield a nonnegative solution $u \geq 0$. Actually, for B this is true with respect to φ in any dimension and with respect to ψ if $n \leq 4$. But for $n \geq 5$, the corresponding integral kernel changes sign! This issue will be discussed in detail in Section 5.2. A perturbation theory of positivity, allowing also for highest order perturbations, can be developed also with respect to φ in the special case $\psi = 0$. This can be generalised to equations of arbitrary order, see Theorems 5.6 and 5.7 and Remark 6.8. With respect to ψ we can prove only a rather restricted perturbation result, see Theorem 5.15.

5.1 A positivity result for Dirichlet problems in the ball

In order to avoid unnecessarily strong assumptions on the coefficients, a reasonable framework for the positivity results is L^p-theory. For existence and regularity we refer to Chapter 2. The next statement should be compared with Corollary 5.5 below for the necessity of the smallness assumptions on the coefficients.

Theorem 5.1. *There exists $\varepsilon_0 = \varepsilon_0(m,n) > 0$ such that if the $a_\beta \in C^0(\overline{B})$ satisfy the smallness condition $\|a_\beta\|_{C^0(\overline{B})} \leq \varepsilon_0$, $|\beta| \leq 2m-1$, then for every $f \in L^p(B)$, $1 < p < \infty$, there exists a solution $u \in W^{2m,p} \cap W_0^{m,p}(B)$ of the Dirichlet problem (5.1). Moreover, if $f \gneqq 0$ then the solution is strictly positive,*

$$u > 0 \text{ in } B.$$

To explain the strategy of the proof, we rewrite the boundary value problem (5.1) as

$$\begin{cases} ((-\Delta)^m + \mathscr{A})\, u = f & \text{in } B, \\ D^\alpha u|_{\partial B} = 0 & \text{for } |\alpha| \leq m-1, \end{cases} \tag{5.3}$$

where we put

$$\mathscr{A} u := \sum_{|\beta| \leq 2m-1} a_\beta(\,.\,) D^\beta u(\,.\,), \qquad a_\beta \in C^0(\overline{B}).$$

We recall from Section 4.2.1 the definition of the Green operator $\mathscr{G}_{m,n}$ for the boundary value problem (5.3) with $\mathscr{A} = 0$. In order to prove Theorem 5.1, we write the solution of (5.3) in the form

$$u = (\mathscr{I} + \mathscr{G}_{m,n}\mathscr{A})^{-1}\mathscr{G}_{m,n}f,$$

where $\mathscr{I}$ is the identity operator, and we shall estimate

$$(\mathscr{I} + \mathscr{G}_{m,n}\mathscr{A})^{-1}\mathscr{G}_{m,n} \geq \frac{1}{C}\mathscr{G}_{m,n}.$$

For this purpose we introduce the following notation.

Definition 5.2. For two operators $\mathscr{S}, \mathscr{T} : L^p(B) \to L^p(B)$ we write

$$\mathscr{S} \geq \mathscr{T},$$

if for all $f \in L^p(B)$ satisfying $f \geq 0$, one has that $\mathscr{S}f \geq \mathscr{T}f$.

Lemma 5.3. *Let $1 < p < \infty$. Then $\mathscr{G}_{m,n}\mathscr{A} : W^{2m,p} \cap W_0^{m,p}(B) \to W^{2m,p} \cap W_0^{m,p}(B)$ is a bounded linear operator. Furthermore, there exists $\varepsilon_1 = \varepsilon_1(m,n) > 0$ such that the following holds true.*

Assume that $\|a_\beta\|_{C^0(\overline{B})} \leq \varepsilon_1$ for all $|\beta| \leq 2m-1$. Then $\mathscr{I} + \mathscr{G}_{m,n}\mathscr{A} : W^{2m,p} \cap W_0^{m,p}(B) \to W^{2m,p} \cap W_0^{m,p}(B)$ is boundedly invertible. For each $f \in L^p(B)$ the boundary value problem (5.3) has precisely one solution. This solution is given by

$$u = (\mathscr{I} + \mathscr{G}_{m,n}\mathscr{A})^{-1}\mathscr{G}_{m,n}f. \tag{5.4}$$

The Green operator for the boundary value problem (5.3)

$$(\mathscr{I} + \mathscr{G}_{m,n}\mathscr{A})^{-1}\mathscr{G}_{m,n} : L^p(B) \to W^{2m,p} \cap W_0^{m,p}(B) \subset L^p(B)$$

is compact.

Proof. This is an immediate consequence of Corollary 2.21: $\mathscr{G}_{m,n} : L^p(B) \to W^{2m,p} \cap W_0^{m,p}(B)$ is a bounded linear operator. □

The Green operator for (5.3) is studied with the help of a Neumann series. Theorem 5.1 then follows from the next result.

Theorem 5.4. *Assume that $1 < p < \infty$. There exists $\varepsilon_0 = \varepsilon_0(m,n) > 0$ such that if $\|a_\beta\|_{C^0(\overline{B})} \leq \varepsilon_0$ for all $|\beta| \leq 2m-1$, then the Green operator*

$$\mathscr{G}_{m,n,\mathscr{A}} := (\mathscr{I} + \mathscr{G}_{m,n}\mathscr{A})^{-1}\mathscr{G}_{m,n} : L^p(B) \to W^{2m,p} \cap W_0^{m,p}(B)$$

for the boundary value problem (5.3) exists. The corresponding Green function $G_{m,n,\mathscr{A}} : \overline{B} \times \overline{B} \to \mathbb{R} \cup \{\infty\}$ is defined according to:

$$\left(\mathscr{G}_{m,n,\mathscr{A}}f\right)(x) = \int_B G_{m,n,\mathscr{A}}(x,y)f(y)\,dy$$

and satisfies the following estimate with a constant $C = C(m,n) > 0$:

$$\frac{1}{C}\mathscr{G}_{m,n} \le \mathscr{G}_{m,n,\mathscr{A}} \le C\mathscr{G}_{m,n}. \tag{5.5}$$

On $\overline{B}\times\overline{B}$, this reads:

$$\frac{1}{C}G_{m,n}(x,y) \le G_{m,n,\mathscr{A}}(x,y) \le C\,G_{m,n}(x,y). \tag{5.6}$$

Proof. Let $\varepsilon := \max_{|\beta|\le 2m-1}\|a_\beta\|_{C^0(\overline{B})}$. If $\varepsilon \le \varepsilon_1$, then according to Lemma 5.3, $\mathscr{G}_{m,n,\mathscr{A}}$ exists and enjoys the properties listed there. If we choose moreover ε small enough such that $\|\mathscr{G}_{m,n}\mathscr{A}\| < 1$ in the sense of bounded linear operators in $W^{2m,p}\cap W_0^{m,p}(B)$, we may use a Neumann series to see that for all $f\in L^p(B)$

$$\mathscr{G}_{m,n,\mathscr{A}}f = (\mathscr{I}+\mathscr{G}_{m,n}\mathscr{A})^{-1}\mathscr{G}_{m,n}f = \sum_{i=0}^{\infty}(-1)^i\,(\mathscr{G}_{m,n}\mathscr{A})^i\,\mathscr{G}_{m,n}f.$$

Here, with the aid of the Fubini-Tonelli theorem and analogously to [197, Lemma 4.1], we conclude:

$$\begin{aligned}
\mathscr{G}^{(i)}f &:= (-1)^i\,(\mathscr{G}_{m,n}\mathscr{A})^i\,\mathscr{G}_{m,n}f\\
&= (-1)^i\int_B G_{m,n}(\,.\,,z_1)\mathscr{A}_{z_1}\int_B G_{m,n}(z_1,z_2)\mathscr{A}_{z_2}\ldots\\
&\qquad\ldots\mathscr{A}_{z_i}\int_B G_{m,n}(z_i,y)f(y)\,dydz_i\ldots dz_1\\
&= \int_B\Big\{(-1)^i\int_B\cdots\int_B G_{m,n}(\,.\,,z_1)\,(\mathscr{A}_{z_1}G_{m,n}(z_1,z_2))\ldots\\
&\qquad\ldots(\mathscr{A}_{z_i}G_{m,n}(z_i,y))\,d(z_1,\ldots,z_i)\Big\}f(y)\,dy\\
&=: \int_B G^{(i)}(\,.\,,y)f(y)\,dy.
\end{aligned}$$

We use the following version of the 3-G-theorem 4.9:

$$\int_B\frac{G_{m,n}(x,z)\,|\mathscr{A}_zG_{m,n}(z,y)|}{G_{m,n}(x,y)}\,dz \le \varepsilon M < \infty$$

where $M = M(m,n) > 0$ is independent of ε and obtain:

$$\begin{aligned}
&\left|G^{(i)}(x,y)\right| =\\
&= \left|\int_B\cdots\int_B\frac{G_{m,n}(x,z_1)\,(\mathscr{A}_{z_1}G_{m,n}(z_1,z_2))}{G_{m,n}(x,z_2)}\;\frac{G_{m,n}(x,z_2)\,(\mathscr{A}_{z_2}G_{m,n}(z_2,z_3))}{G_{m,n}(x,z_3)}\cdots\right.\\
&\qquad\left.\cdots\frac{G_{m,n}(x,z_i)\,(\mathscr{A}_{z_i}G_{m,n}(z_i,y))}{G_{m,n}(x,y)}G_{m,n}(x,y)\,d(z_1,\ldots,z_i)\right| \le
\end{aligned}$$

$$
\begin{aligned}
&\leq G_{m,n}(x,y) \prod_{j=1}^{i} \sup_{\xi,\eta\in B} \int_B \frac{G_{m,n}(\xi,z_j)\left|\mathscr{A}_{z_j} G_{m,n}(z_j,\eta)\right|}{G_{m,n}(\xi,\eta)}\, dz_j \\
&\leq (\varepsilon M)^i\, G_{m,n}(x,y). \qquad (5.7)
\end{aligned}
$$

For $\varepsilon M < 1$, thanks to $\sum_{i=0}^{\infty}(\varepsilon M)^i = (1-\varepsilon M)^{-1} < \infty$, we have absolute local uniform convergence in $x \neq y$ of

$$
G_{m,n,\mathscr{A}}(x,y) := \sum_{i=0}^{\infty} G^{(i)}(x,y). \qquad (5.8)
$$

It follows that

$$
\left|G_{m,n,\mathscr{A}}(x,y)\right| \leq \frac{1}{1-\varepsilon M} G_{m,n}(x,y). \qquad (5.9)
$$

On the other hand, Lebesgue's theorem yields

$$
\begin{aligned}
\left(\mathscr{G}_{m,n,\mathscr{A}} f\right)(x) &= \sum_{i=0}^{\infty} \left(\mathscr{G}^{(i)} f\right)(x) = \sum_{i=0}^{\infty} \int_B G^{(i)}(x,y) f(y)\, dy \\
&= \int_B \left(\sum_{i=0}^{\infty} G^{(i)}(x,y) f(y)\right) dy = \int_B G_{m,n,\mathscr{A}}(x,y) f(y)\, dy.
\end{aligned}
$$

Finally, thanks to (5.7) we have

$$
\begin{aligned}
G_{m,n,\mathscr{A}}(x,y) &= G_{m,n}(x,y) + \sum_{i=1}^{\infty} G^{(i)}(x,y) \\
&\geq G_{m,n}(x,y) - \left(\sum_{i=1}^{\infty} (\varepsilon M)^i\right) G_{m,n}(x,y) = \frac{1-2\varepsilon M}{1-\varepsilon M} G_{m,n}(x,y).
\end{aligned}
$$

Choosing $\varepsilon_0 \leq 1/(4M)$ yields the crucial part of the estimate (5.6) from below for the Green function of the perturbed boundary value problem (5.3), provided $\varepsilon \in [0, \varepsilon_0]$. $\square$

If we confine ourselves to perturbations of order zero, we may show the necessity of the smallness conditions in Theorem 5.1. Consider the problem

$$
\begin{cases}
(-\Delta)^m u + a(x) u = f \text{ in } \Omega, \\
D^\alpha u|_{\partial\Omega} = 0 \qquad \text{for } |\alpha| \leq m-1,
\end{cases} \qquad (5.10)
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2m,\gamma}$-smooth domain. For coefficients $a \in C^0(\overline{\Omega})$, where we have uniqueness and hence existence in (5.10), let $\mathscr{G}_{m,\Omega,a}$ be the corresponding Green operator. In case of a constant coefficient a, $\mathscr{G}_{m,\Omega,a}$ is the resolvent operator. Recalling the meaning of the symbols

$$
\phi > 0, \quad \phi \ngeq 0, \quad \phi \gneqq 0,
$$

in the Notations-Section, the positivity properties of Dirichlet problem (5.10) may be summarised as follows.

Corollary 5.5. *Let $m > 1$ and let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain such that the (Dirichlet-) Green function for $(-\Delta)^m$ is positive in $\Omega \times \Omega$. Let $\Lambda_{m,1}$ denote the first Dirichlet eigenvalue of $(-\Delta)^m$ in Ω.*

Then there exists a critical number $a_c \in [0,\infty)$ such that for $a \in C^0(\overline{\Omega})$ we have:

1. *If $a > a_c$ on $\overline{\Omega}$, then $\mathscr{G}_{m,\Omega,a}$ does not preserve positivity:*

$$\text{there exists } f \gneqq 0: \quad \mathscr{G}_{m,\Omega,a} f \not\geq 0. \tag{5.11}$$

On the other hand we have:

$$\text{for all } f \gneqq 0: \ \mathscr{G}_{m,\Omega,a} f \not\leq 0, \tag{5.12}$$

$$\text{there exists } f \gneqq 0: \ \mathscr{G}_{m,\Omega,a} f \geq 0. \tag{5.13}$$

2. *If $-\Lambda_{m,1} < a \leq a_c$ (or $-\Lambda_{m,1} < a < a_c$, respectively), then $\mathscr{G}_{m,\Omega,a}$ is positivity preserving (or strongly positivity preserving, respectively), that is,*

$$\text{for all } f \gneqq 0: \quad \mathscr{G}_{m,\Omega,a} f \geq 0 \ \ (\text{or } \mathscr{G}_{m,\Omega,a} f > 0 \text{ in } \Omega, \text{ respectively}). \tag{5.14}$$

3. *If $a = -\Lambda_{m,1}$ and $f \gneqq 0$, then (5.10) has no solution.*
4. *If $a < -\Lambda_{m,1}$, then (5.10) kills positivity, that is, if $f \gneqq 0$ and u is a solution to (5.10), then $u \not\geq 0$ in Ω.*

For a proof, the strategy of which is related to – but simpler than – the arguments in Section 5.4, we refer to [210, Section 6] and [212, Lemma 1].

Case 1 does not occur in second order equations. This different behaviour may be responsible for the difficulties in classical solvability of semilinear boundary value problems of higher order, see e.g. [209, 395, 404, 405]. If $m > 1$, we have that for a large enough the resolvent is always sign changing, see e.g. Coffman-Grover [111]. As was noted e.g. by Bernis [51], this is equivalent to instantaneous change of sign for the corresponding parabolic heat kernel, see also related contributions to local eventual positivity by Ferrero-Gazzola-Grunau [164, 183, 184].

5.2 The role of positive boundary data

This section is devoted to the role of nonhomogeneous boundary data with regard to the sign of the solution. As already mentioned in the introduction this problem is rather subtle. In general we cannot expect that fixed sign of any particular Dirichlet datum leads to fixed sign of the solution. It seems that a perturbation theory of positivity (analogous to that above with regard to the right-hand side) exists in general only for the Dirichlet datum of highest order. As a dual result we obtain also a Hopf-type boundary lemma in the ball for perturbed polyharmonic Dirichlet problems.

With respect to lower order data, only much more restricted results can be achieved, see Theorem 5.15 below.

5.2.1 The highest order Dirichlet datum

Here, we consider the following boundary value problem:

$$\begin{cases} ((-\Delta)^m + \mathscr{A})\, u = f & \text{in } B, \\ D^\alpha u|_{\partial B} = 0 & \text{for } |\alpha| \le m-2, \\ \begin{cases} -\dfrac{\partial}{\partial \nu} \Delta^{(m/2)-1} u|_{\partial B} = \varphi & \text{if } m \text{ is even,} \\ \Delta^{(m-1)/2} u|_{\partial B} = \varphi & \text{if } m \text{ is odd.} \end{cases} \end{cases} \tag{5.15}$$

Here $f \in C^0(\overline{B})$, $\varphi \in C^0(\partial B)$ and

$$\mathscr{A} = \sum_{|\beta| \le 2m-1} a_\beta(\,.\,) D^\beta, \qquad a_\beta \in C^{|\beta|}(\overline{B}), \tag{5.16}$$

is a sufficiently small lower order perturbation. For existence of solutions $u \in W^{2m,p}_{\mathrm{loc}}(B) \cap C^{m-1}(\overline{B})$, $p > 1$, we refer to the local L^p-theory in Theorem 2.20 and the lines following it and to the Agmon-Miranda maximum estimates of Theorem 2.25. The latter already require the strong regularity assumptions on the coefficients a_β. These have to be imposed whenever the adjoint operator $(-\Delta)^m + \mathscr{A}^*$ is involved.

Theorem 5.6. *There exists $\varepsilon_0 = \varepsilon_0(m,n) > 0$ such that the following holds true.*

If for all $|\beta| \le 2m-1$ the smallness condition $\|a_\beta\|_{C^{|\beta|}(\overline{B})} \le \varepsilon_0$ is fulfilled, then for every $f \in C^0(\overline{B})$ and $\varphi \in C^0(\partial B)$ there exists a solution $u \in W^{2m,p}_{\mathrm{loc}}(B) \cap C^{m-1}(\overline{B})$, $1 < p < \infty$, to the Dirichlet problem (5.15). Moreover, $f \ge 0$ and $\varphi \ge 0$, with $f \not\equiv 0$ or $\varphi \not\equiv 0$, implies that $u > 0$.

If $m = 1$, we recover a special form of the strong maximum principle for second order elliptic equations. The next result, in some sense dual to the previous one, may be viewed as a higher order analogue to the Hopf boundary lemma.

Theorem 5.7. *Assume $a_\beta \in C^0(\overline{B})$, $|\beta| \le 2m-1$. There exists $\varepsilon_0 = \varepsilon_0(m,n) > 0$ such that the following holds:*

If $\|a_\beta\|_{C^0(\overline{B})} \le \varepsilon_0$, $|\beta| \le 2m-1$, then for every $f \in C^0(\overline{B})$ the Dirichlet problem (5.1) has a solution $u \in W^{2m,p}(B) \cap C^{2m-1}(\overline{B})$, $p > 1$ arbitrary. Moreover $0 \not\equiv f \ge 0$ implies $u > 0$ in B and for every $x \in \partial B$

$$\begin{cases} \Delta^{(m/2)}u(x) > 0 & \text{if } m \text{ even,} \\ -\dfrac{\partial}{\partial \nu}\Delta^{(m-1)/2}u(x) > 0 & \text{if } m \text{ odd.} \end{cases} \tag{5.17}$$

The common key point in the proof of both theorems is the observation that the corresponding Green function $G_{m,n,\mathscr{A}}$ vanishes (in both variables) on ∂B precisely of order m, see Theorem 4.7 and estimate (5.6). In the proof of Theorem 5.6 we observe further that, for $x \in B$, $y \in \partial B$, the Poisson kernel for φ is given by

$$\begin{cases} \Delta_y^{m/2} G_{m,n,\mathscr{A}}(x,y) & \text{if } m \text{ even,} \\ \left(-\dfrac{\partial}{\partial \nu_y}\Delta_y^{(m-1)/2}\right) G_{m,n,\mathscr{A}}(x,y) & \text{if } m \text{ odd.} \end{cases}$$

In order to prove the theorems we need a precise characterisation of the growth properties near ∂B of the Green function $G_{m,n,\mathscr{A}}$ for the boundary value problem (5.15). These estimates were proved in a more general setting but under more restrictive assumptions on the coefficients by Krasovskiĭ [255, 256], see also Theorem 4.20. In the present special situation we provide an elementary proof which combines Theorems 4.6 and 5.4. Moreover, we need to verify suitable smoothness.

Lemma 5.8. *We assume that $a_\beta \in C^0(\overline{B})$. Then there exists $\varepsilon_1 = \varepsilon_1(m,n) > 0$ such that the following holds true.*

If $\|a_\beta\|_{C^0(\overline{B})} \le \varepsilon_1$ for all $|\beta| \le 2m-1$, then the Green function $G_{m,n,\mathscr{A}}(\,.\,,\,.\,)$ for the boundary value problem (5.15) exists. For each $y \in B$, $G_{m,n,\mathscr{A}}(\,.\,,y) \in C^{2m-1}(\overline{B}\setminus\{y\})$. Furthermore, there exist constants $C = C(m,n)$ such that for $|\beta| \le 2m-1$, $x,y \in \overline{B}$

$$\begin{cases} G_{m,n,\mathscr{A}}(\,.\,,y) \in C^{|\beta|}(\overline{B}) & \text{if } 0 \le |\beta| < 2m-n, \\ \left|D_x^\beta G_{m,n,\mathscr{A}}(x,y)\right| \le C & \text{if } 0 \le |\beta| < 2m-n, \\ \left|D_x^\beta G_{m,n,\mathscr{A}}(x,y)\right| \le C\log\left(\dfrac{3}{|x-y|}\right) & \text{if } |\beta| = 2m-n \text{ and } n \text{ even,} \\ \left|D_x^\beta G_{m,n,\mathscr{A}}(x,y)\right| \le C & \text{if } |\beta| = 2m-n \text{ and } n \text{ odd,} \\ \left|D_x^\beta G_{m,n,\mathscr{A}}(x,y)\right| \le C\,|x-y|^{2m-n-|\beta|} & \text{if } 2m-n < |\beta| < 2m. \end{cases} \tag{5.18}$$

Moreover, one has $D_x^\beta G_{m,n,\mathscr{A}} \in C^0\left(\overline{B}\times\overline{B}\setminus\{(x,y) : x = y\}\right)$.

Proof. We come back to Theorem 5.4, making use of the notations and formulae in its proof. The following holds true, provided ε_1 is chosen sufficiently small.

The Green function $G_{m,n,\mathscr{A}}$ exists, and one has

$$G_{m,n,\mathscr{A}}(x,y) = \sum_{i=0}^{\infty} G^{(i)}(x,y),$$

where

$$
\begin{aligned}
G^{(0)}(x,y) &= G_{m,n}(x,y),\\
G^{(i)}(x,y) &= (-1)^i \int_B \cdots \int_B G_{m,n}(x,z_1)\,(\mathscr{A}_{z_1} G_{m,n}(z_1,z_2))\\
&\qquad\qquad \times \ldots\ (\mathscr{A}_{z_i} G_{m,n}(z_i,y))\, d(z_1,\ldots,z_i).
\end{aligned}
$$

In particular, we have $G^{(i)}(\,.\,,y) \in C^{2m-1}(\overline{B}\setminus\{y\})$, $G^{(i)}(\,.\,,y) \in C^{|\beta|}(\overline{B})$ for $0 \le |\beta| < 2m-n$. If $|\beta| \le 2m-1$ and $i \ge 1$, one has with constants $C_j = C_j(m,n)$ which are independent of i

$$
\begin{aligned}
&\left| D_x^\beta G^{(i)}(x,y) \right|\\
&\le \int_B \cdots \int_B \left| D_x^\beta G_{m,n}(x,z_1) \right| |\mathscr{A}_{z_1} G_{m,n}(z_1,z_2)| \ \ldots\ |\mathscr{A}_{z_i} G_{m,n}(z_i,y)|\, d(z_1,\ldots,z_i)\\
&\le \varepsilon_1^i\, C_1^{i+1} \int_B \cdots \int_B \Gamma\,(|x-z_1|)\ |z_1-z_2|^{1-n}\ \ldots\ |z_i-y|^{1-n} d(z_1,\ldots,z_i).
\end{aligned}
$$

Here, in view of Theorem 4.7, we define

$$
\Gamma(\rho) := \begin{cases}
1 & \text{if } 0 \le |\beta| < 2m-n,\\
\log\left(\dfrac{3}{\rho}\right) & \text{if } |\beta| = 2m-n \text{ and } n \text{ even},\\
1 & \text{if } |\beta| = 2m-n \text{ and } n \text{ odd},\\
\rho^{2m-n-|\beta|} & \text{if } |\beta| > 2m-n.
\end{cases}
$$

Applying repeatedly $\int_B |\xi - z|^{1-n} |z-\eta|^{1-n}\, dz \le C_2 |\xi-\eta|^{1-n}$, we conclude:

$$
\begin{aligned}
\left| D_x^\beta G^{(i)}(x,y) \right| &\le \varepsilon_1^i\, C_1^{i+1}\, C_2^{i-1} \int_B \Gamma\,(|x-z_1|)\, |z_1-y|^{1-n}\, dz_1\\
&\le \varepsilon_1^i\, C_1^{i+1}\, C_2^{i-1} \begin{cases}
C_2 & \text{if } |\beta| \le 2m-n,\\
C_2 |x-y|^{2m-n-|\beta|} & \text{if } 2m-n < |\beta| < 2m.
\end{cases}
\end{aligned}
$$

For sufficiently small $\varepsilon_1 > 0$ we achieve absolute uniform convergence of the series $\sum_{i=0}^\infty D_x^\beta G^{(i)}(\,.\,,y)$ in $\overline{B}$ if $|\beta| \le 2m-n$, and in $\overline{B}\setminus B_\delta(y)$ otherwise, where $\delta > 0$ is arbitrary. Taking the properties of $G^{(0)} = G_{m,n}$ into account we obtain the estimates for $D_x^\beta G_{m,n,\mathscr{A}}$ as well as the stated smoothness. $\square$

Lemma 5.9. *We assume that $a_\beta \in C^{|\beta|}(\overline{B})$. Then there exists $\varepsilon_2 = \varepsilon_2(m,n) > 0$ such that the following holds true.*

If $\|a_\beta\|_{C^{|\beta|}(\overline{B})} \le \varepsilon_2$ for all $|\beta| \le 2m-1$, the Green function $G_{m,n,\mathscr{A}}(\,.\,,\,.\,)$ for the boundary value problem (5.15) exists. Moreover, for each $x \in B$ we have $G_{m,n,\mathscr{A}}(x,\,.\,) \in C^{2m-1}(\overline{B}\setminus\{x\})$. Furthermore, for $|\beta| \le 2m-1$ one has with constants $C = C(m,n)$ being independent of x,y:

$$\begin{cases} G_{m,n,\mathscr{A}}(x,\,.\,) \in C^{|\beta|}(\overline{B}) & \textit{if } 0 \le |\beta| < 2m-n, \\ \left| D_y^\beta G_{m,n,\mathscr{A}}(x,y) \right| \le C & \textit{if } 0 \le |\beta| < 2m-n, \\ \left| D_y^\beta G_{m,n,\mathscr{A}}(x,y) \right| \le C \log\left(\dfrac{3}{|x-y|}\right) & \textit{if } |\beta| = 2m-n \textit{ and } n \textit{ even}, \\ \left| D_y^\beta G_{m,n,\mathscr{A}}(x,y) \right| \le C & \textit{if } |\beta| = 2m-n \textit{ and } n \textit{ odd}, \\ \left| D_y^\beta G_{m,n,\mathscr{A}}(x,y) \right| \le C\,|x-y|^{2m-n-|\beta|} & \textit{if } 2m-n < |\beta| < 2m. \end{cases} \tag{5.19}$$

Moreover, $D_y^\beta G_{m,n,\mathscr{A}}$ *is continuous outside the diagonal of* $\overline{B} \times \overline{B}$*.*

Proof. Thanks to the strong differentiability assumptions on the coefficients a_β we may consider the adjoint boundary value problem

$$\begin{cases} (-\Delta)^m u + \mathscr{A}^* u = f & \text{in } B, \\ D^\alpha u|_{\partial B} = 0 & \text{for } |\alpha| \le m-1, \end{cases}$$

where $(\mathscr{A}^* u)(x) = \sum_{|\beta| \le 2m-1} (-1)^{|\beta|} D^\beta \left(a_\beta(x) u(x) \right)$. If ε_2 is small enough, the corresponding Green function $G_{m,n,\mathscr{A}^*}$ exists and satisfies $G_{m,n,\mathscr{A}}(x,y) = G_{m,n,\mathscr{A}^*}(y,x)$. This observation allows us to apply Lemma 5.8 and the claim follows. □

Proof of Theorem 5.6. Let $\varepsilon_0 > 0$ be sufficiently small so that Theorem 5.4 and Lemma 5.9 are applicable.

The required smoothness of the Green function $G_{m,n,\mathscr{A}}$ has just been proved in Lemma 5.9. For solutions of the boundary value problem (5.15) we have the following representation formula:

$$u(x) = \begin{cases} \displaystyle\int_B G_{m,n,\mathscr{A}}(x,y) f(y)\,dy + \int_{\partial B} \Delta_y^{m/2} G_{m,n,\mathscr{A}}(x,y) \varphi(y)\,d\omega(y) \\ \hfill \text{if } m \text{ even}, \\ \displaystyle\int_B G_{m,n,\mathscr{A}}(x,y) f(y) dy + \int_{\partial B} \left(-\frac{\partial}{\partial \nu_y} \Delta_y^{(m-1)/2} \right) G_{m,n,\mathscr{A}}(x,y) \varphi(y) d\omega(y) \\ \hfill \text{if } m \text{ odd}. \end{cases}$$

We keep arbitrary $x \in B$ fixed and consider y “close” to ∂B. Then an application of Theorems 5.4 and 4.6 yields

$$G_{m,n,\mathscr{A}}(x,y) \succeq G_{m,n}(x,y) \succeq |x-y|^{-n} d(x)^m d(y)^m \succeq d(y)^m.$$

It follows for each $x \in B$ that

$$\begin{cases} \Delta_y^{m/2} G_{m,n,\mathscr{A}}(x,\,.\,)|_{\partial B} > 0 & \text{for even } m, \\ -\dfrac{\partial}{\partial \nu_y} \Delta_y^{(m-1)/2} G_{m,n,\mathscr{A}}(x,\,.\,)|_{\partial B} > 0 & \text{for odd } m. \end{cases}$$

Together with the positivity of $G_{m,n,\mathscr{A}}$, the claim of Theorem 5.6 is now obvious. $\square$

Proof of Theorem 5.7. This proof is "dual" to the previous one. Let $\varepsilon_0 > 0$ be sufficiently small. Differentiating the representation formula

$$u(x) = \int_B G_{m,n,\mathscr{A}}(x,y) f(y)\, dy$$

gives for $x \in \partial B$:

$$\begin{cases} \Delta^{m/2} u(x) = \int_B \left(\Delta_x^{m/2} G_{m,n,\mathscr{A}}(x,y) \right) f(y)\, dy & m \text{ even}, \\ -\dfrac{\partial}{\partial \nu} \Delta^{(m-1)/2} u(x) = \int_B \left(-\dfrac{\partial}{\partial \nu_x} \Delta_x^{(m-1)/2} G_{m,n,\mathscr{A}}(x,y) \right) f(y)\, dy & m \text{ odd}. \end{cases}$$

Keeping an arbitrary $y \in B$ fixed, we see that for $\tilde{x}$ "close" to ∂B

$$G_{m,n,\mathscr{A}}(\tilde{x},y) \succeq G_{m,n}(\tilde{x},y) \succeq |\tilde{x} - y|^{-n} d(\tilde{x})^m d(y)^m \succeq d(\tilde{x})^m$$

and consequently for $x \in \partial B$, $y \in B$

$$\begin{cases} \Delta_x^{m/2} G_{m,n,\mathscr{A}}(x,y) > 0 & \text{for even } m, \\ -\dfrac{\partial}{\partial \nu_x} \Delta_x^{(m-1)/2} G_{m,n,\mathscr{A}}(x,y) > 0 & \text{for odd } m. \end{cases}$$

Now it is immediate that $\Delta^{m/2} u(x) > 0$ or $-\frac{\partial}{\partial \nu} \Delta^{(m-1)/2} u(x) > 0$, according to whether m is even or odd. $\square$

5.2.2 *Also nonzero lower order boundary terms*

Now we turn to investigating further conditions on $\varphi \geq 0$ and $\psi \geq 0$ such that the solution u of the Dirichlet problem

$$\begin{cases} \Delta^2 u = 0 \quad \text{in } B, \\ u|_{\partial B} = \psi, \quad \left(-\dfrac{\partial u}{\partial \nu} \right) |_{\partial B} = \varphi, \end{cases} \tag{5.20}$$

is positive, i.e. $u \geq 0$. We recall that we have the following explicit formula for the solution u of (5.20), see [323, p.34]:

$$u(x) = \int_{\partial B} K_{2,n}(x,y) \psi(y)\, d\omega(y) + \int_{\partial B} L_{2,n}(x,y) \varphi(y)\, d\omega(y), \quad x \in B, \tag{5.21}$$

where

$$K_{2,n}(x,y) = \frac{1}{2ne_n}\frac{(1-|x|^2)^2}{|x-y|^{n+2}}\left(2+(n-4)x\cdot y-(n-2)|x|^2\right), \tag{5.22}$$

$$L_{2,n}(x,y) = \frac{1}{2ne_n}\frac{(1-|x|^2)^2}{|x-y|^{n}}, \tag{5.23}$$

with $x \in B$, $y \in \partial B$, and $ne_n = |\partial B|$. Evidently $L_{2,n} > 0$ for any n, while $K_{2,n} > 0$ only for $n \le 4$ and $K_{2,n}$ changes sign for $n \ge 5$.

We will show that the Dirichlet problem (5.20) may be reformulated in such a way that we have a positivity result with respect to both boundary data in any dimension. Moreover for $n \le 3$ and in particular for $n = 2$ the above mentioned result may be sharpened so that if $\psi(x_0) > 0$ for some $x_0 \in \partial B$. Also negative values for φ near x_0 are admissible.

Finally, we switch to polyharmonic Dirichlet problems of arbitrary order $2m$. We will admit some "small" lower order perturbations of the differential operator. Positivity with respect to the Dirichlet data of order $(m-1)$ and $(m-2)$ will be shown in any dimension n, provided the other boundary data are prescribed homogeneously and the positivity assumption is posed in a suitable way.

5.2.2.1 The appropriate positivity assumption for the clamped plate equation

In order to find the adequate positivity assumption on the boundary data in the Dirichlet problem (5.20), one may observe that adding a suitable multiple of $L_{2,n}$ to $K_{2,n}$ yields a positive kernel.

Lemma 5.10. *Let $s \in \mathbb{R}$, $s \ge \frac{1}{2}(n-4)$. Then for*

$$\hat{K}_{2,n,s}(x,y) := K_{2,n}(x,y) + sL_{2,n}(x,y), \quad x \in B, \quad y \in \partial B, \tag{5.24}$$

we have

$$\hat{K}_{2,n,s}(x,y) > 0.$$

Proof. We observe that for $x \in B$, $y \in \partial B$ (i.e. $|y| = 1$) we have

$$\begin{aligned} K_{2,n}(x,y) &= \frac{1}{2ne_n}\frac{(1-|x|^2)^2}{|x-y|^{n+2}}\left(\frac{n}{2}(1-|x|^2)-\frac{1}{2}(n-4)|x-y|^2\right) \\ &= \frac{1}{4e_n}\frac{(1-|x|^2)^3}{|x-y|^{n+2}} - \frac{1}{2}(n-4)L_{2,n}(x,y), \quad ne_n = |\partial B|. \qquad \square \end{aligned}$$

Proposition 5.11. *Let $\varphi \in C^0(\partial B)$, $\psi \in C^1(\partial B)$ and $s \ge \frac{1}{2}(n-4)$. If we assume that*

$$\psi(x) \ge 0 \quad \textit{and} \quad \varphi(x) \ge s\psi(x) \quad \textit{for } x \in \partial B,$$

then the uniquely determined solution $u \in C^4(B) \cap C^1(\overline{B})$ of the Dirichlet problem (5.20) is positive:

$$u \ge 0 \quad \textit{in } B.$$

Proof. From (5.21) and (5.24) we obtain:

$$u(x) = \int_{\partial B} K_{2,n}(x,y)\,\psi(y)\,d\omega(y) + \int_{\partial B} L_{2,n}(x,y)\varphi(y)\,d\omega(y)$$
$$= \int_{\partial B} \hat{K}_{2,n,s}(x,y)\,\psi(y)\,d\omega(y) + \int_{\partial B} L_{2,n}(x,y)\Big(\varphi(y) - s\,\psi(y)\Big)\,d\omega(y). \qquad \square$$

One may observe that for $n = 1,2,3$ also negative values for s are admissible. On $B_R(0)$ the condition on s is $s \geq \frac{1}{2R}(n-4)$.

We are interested in whether this positivity result remains under perturbations of the prototype problem (5.20). Since in higher order Dirichlet problems quite similar phenomena can be observed, we develop the perturbation theory for the biharmonic Dirichlet problem (5.20) as a special case of the perturbation theory for the polyharmonic Dirichlet problem (5.25) below.

5.2.2.2 Higher order equations. Perturbations

In this section we assume $m \geq 2$.

First we consider the polyharmonic prototype Dirichlet problem:

$$\begin{cases} (-\Delta)^m u = 0 & \text{in } B, \\ \left(-\dfrac{\partial}{\partial\nu}\right)^j u = 0 & \text{on } \partial B \quad \text{for} \quad j = 0,\dots,m-3, \\ \left(-\dfrac{\partial}{\partial\nu}\right)^{m-2} u = \psi & \text{on } \partial B, \\ \left(-\dfrac{\partial}{\partial\nu}\right)^{m-1} u = \varphi & \text{on } \partial B. \end{cases} \tag{5.25}$$

No uniform positivity result can be expected with respect to the boundary data of order $0,\dots,m-3$, as we will explain below in Example 5.14. So, these data are prescribed homogeneously. Such behaviour is in contrast with the radially symmetric case $u = u(|x|)$, where $(-\Delta)^m u \geq 0$ in B, $(-1)^j u^{(j)}(1) \geq 0$ $(j = 0,\dots,m-1)$ implies that $u \geq 0$ in B, see Soranzo [375, Proposition 1, Remark 9].

After some elementary calculations we find from Boggio's formula (2.65) (see also [158]) that for $\varphi \in C^0(\partial B)$, $\psi \in C^1(\partial B)$ the solution $u \in C^{2m}(B) \cap C^{m-1}(\overline{B})$ to the Dirichlet problem (5.25) is given by

$$u(x) = \int_{\partial B} K_{m,n}(x,y)\psi(y)\,d\omega(y) + \int_{\partial B} L_{m,n}(x,y)\varphi(y)\,d\omega(y), \quad x \in B. \tag{5.26}$$

Here, the Poisson kernels are defined by

$$K_{m,n}(x,y) = \frac{1}{2^m(m-2)!ne_n}\frac{(1-|x|^2)^m}{|x-y|^{n+2}}\left(n(1-|x|^2)-(n-2-m)|x-y|^2\right), \quad (5.27)$$

$$L_{m,n}(x,y) = \frac{1}{2^{m-1}(m-1)!ne_n}\frac{(1-|x|^2)^m}{|x-y|^n}, \quad (5.28)$$

with $x \in B$, $y \in \partial B$. The following result generalises Lemma 5.10.

Lemma 5.12. *Let $s \in \mathbb{R}$ satisfy $s \geq \frac{1}{2}(n-2-m)(m-1)$. Then for*

$$\hat{K}_{m,n,s}(x,y) := K_{m,n}(x,y) + sL_{m,n}(x,y), \quad x \in B, \quad y \in \partial B, \quad (5.29)$$

we have

$$\hat{K}_{m,n,s}(x,y) > 0.$$

Proof.

$$\begin{aligned} \hat{K}_{m,n,s}(x,y) &= \frac{1}{2^m\,(m-2)!\,ne_n}\frac{(1-|x|^2)^m}{|x-y|^{n+2}} \\ &\quad \times \left(n(1-|x|^2)+\left(\frac{2s}{m-1}-(n-2-m)\right)|x-y|^2\right). \end{aligned} \quad (5.30)$$

□

Proposition 5.13. *Let $\varphi \in C^0(\partial B)$, $\psi \in C^1(\partial B)$ and $s \geq \frac{1}{2}(n-2-m)(m-1)$. If*

$$\psi(x) \geq 0 \quad \textit{and} \quad \varphi(x) \geq s\psi(x) \quad \textit{for } x \in \partial B,$$

then the uniquely determined solution $u \in C^{2m}(B) \cap C^{m-1}(\overline{B})$ of the Dirichlet problem (5.25) is positive:

$$u \geq 0 \quad \textit{in } B.$$

Example 5.14. In the triharmonic Dirichlet problem

$$\begin{cases} (-\Delta)^3 u = 0 & \text{in } B, \\ u = \chi & \text{on } \partial B, \\ \left(-\frac{\partial}{\partial\nu}\right)u = \psi & \text{on } \partial B, \\ \left(\frac{\partial}{\partial\nu}\right)^2 u = \varphi & \text{on } \partial B, \end{cases}$$

the solution is given by

$$\begin{aligned} u(x) &= \int_{\partial B} H_{3,n}(x,y)\chi(y)\,d\omega(y) + \int_{\partial B} K_{3,n}(x,y)\psi(y)\,d\omega(y) \\ &\quad + \int_{\partial B} L_{3,n}(x,y)\varphi(y)\,d\omega(y), \qquad x \in B. \end{aligned}$$

The kernels $K_{3,n}$ and $L_{3,n}$ are defined above and

$$H_{3,n}(x,y) = \frac{1}{16ne_n}\frac{(1-|x|^2)^3}{|x-y|^{n+4}}\Big(n(n+2)(1-|x|^2)^2+(n-4)(n-8)|x-y|^4 \\ -2n(n-7)(1-|x|^2)|x-y|^2-4n|x-y|^2\Big)$$

with $x\in B$, $y\in\partial B$. For any n, $x\to y$, x "very close" to the boundary, $H_{3,n}$ takes on also negative values. By adding multiples of $L_{3,n}$ and $K_{3,n}$, only the terms $|x-y|^4$ and $(1-|x|^2)|x-y|^2$ in the curved brackets could be effected. In any case the most dangerous term $-4n|x-y|^2$ remains.

Theorem 5.15. *Let $s>\frac{1}{2}(n-2-m)(m-1)$. Then there exists $\varepsilon_0=\varepsilon_0(m,n,s)>0$ such that the following holds.*

If $\|a_\beta\|_{C^{|\beta|}(\overline{B})}\le\varepsilon_0$ for $|\beta|\le 2m-2$, then for every $\varphi\in C^0(\partial B)$ and $\psi\in C^1(\partial B)$ with

$$\left.\begin{array}{l}\psi\ge 0\\ \varphi\ge s\psi\end{array}\right\}\ \text{on } \partial B,\ \ \psi\not\equiv 0 \text{ or } \varphi\not\equiv 0,$$

the Dirichlet problem

$$\begin{cases}(-\Delta)^m u+\sum\limits_{|\beta|\le 2m-2}a_\beta(x)D^\beta u=0 & \text{in } B,\\ \left(-\dfrac{\partial}{\partial\nu}\right)^j u=0 & \text{on } \partial B \text{ for } j=0,\dots,m-3,\\ \left(-\dfrac{\partial}{\partial\nu}\right)^{m-2} u=\psi & \text{on } \partial B,\\ \left(-\dfrac{\partial}{\partial\nu}\right)^{m-1} u=\varphi & \text{on } \partial B,\end{cases} \tag{5.31}$$

has a solution $u\in W^{2m,p}_{\mathrm{loc}}(B)\cap C^{m-1}(\overline{B})$ ($p>1$ arbitrary) which is strictly positive: $u>0$ in B.

In order to prove this result we first need to describe the essential properties of the integral kernels $\hat{K}_{m,n,s}$ and $L_{m,n}$.

Lemma 5.16. *1. Let $s\ge\frac{1}{2}(n-2-m)(m-1)$. On $B\times\partial B$ (i.e. for $x\in B$, $y\in\partial B$) we have*

$$\hat{K}_{m,n,s}(x,y)\begin{cases}\preceq |x-y|^{-n-1}d(x)^m,\\ \succeq |x-y|^{-n-2}d(x)^{m+1},\end{cases} \tag{5.32}$$

$$L_{m,n}(x,y)\simeq |x-y|^{-n}d(x)^m. \tag{5.33}$$

2. If we assume additionally that $s>\frac{1}{2}(n-2-m)(m-1)$, then we have on $B\times\partial B$:

$$\hat{K}_{m,n,s}(x,y) \begin{cases} \preceq |x-y|^{-n-1} d(x)^m, \\ \succeq |x-y|^{-n} d(x)^m. \end{cases} \tag{5.34}$$

Proof. The claim follows from $1-|x|^2 \simeq d(x)$, $d(x) \leq |x-y|$ and the expression in (5.30). $\square$

Remark 5.17. 1. The estimation constants in (5.34) depend strongly on s.

2. If $s = \frac{1}{2}(n-2-m)(m-1)$ then we have $\hat{K}_{m,n,s}(x,y) \simeq |x-y|^{-n-2} d(x)^{m+1}$, i.e. for $x \to \partial B \setminus \{y\}$ we have a zero of order $(m+1)$. We would have expected, and actually need in order to prove perturbation results, a zero of order m. Consequently in what follows we have to assume $s > \frac{1}{2}(n-2-m)(m-1)$. The estimate (5.34) is more appropriate. But as $\hat{K}_{m,n,s}(x,y) \not\simeq |x-y|^{-n-1} d(x)^m$ our perturbation result Theorem 5.15 below is less general than the corresponding results in Theorems 5.1, 6.3 and 6.29. In particular, domain perturbations are not considered.

For our purposes the following "3-G-type" estimates are essential. We recall that $G_{m,n} = G_{(-\Delta)^m,B}$ denotes the Dirichlet Green function for $(-\Delta)^m$ in the unit ball $B \subset \mathbb{R}^n$.

Lemma 5.18. *Let $s > \frac{1}{2}(n-2-m)(m-1)$, $\beta \in \mathbb{N}^n$. Then on $B \times \partial B \times B$ (i.e. for $x \in B$, $y \in \partial B$, $z \in B$) we have the following:*

$$\frac{\left|D_z^\beta G_{m,n}(x,z)\right| \hat{K}_{m,n,s}(z,y)}{\hat{K}_{m,n,s}(x,y)} \preceq \begin{cases} 1 & \text{if } |\beta| < 2m-n, \\ |x-z|^{2m-1-n-|\beta|} + |y-z|^{2m-1-n-|\beta|} & \text{if } |\beta| \geq 2m-n; \end{cases} \tag{5.35}$$

$$\frac{\left|D_z^\beta G_{m,n}(x,z)\right| L_{m,n}(z,y)}{L_{m,n}(x,y)} \preceq \begin{cases} 1 & \text{if } |\beta| < 2m-n, \\ 1 & \text{if } |\beta| = 2m-n \text{ and } n \text{ odd}, \\ \log\left(\frac{3}{|x-z|}\right) & \text{if } |\beta| = 2m-n \text{ and } n \text{ even}, \\ |x-z|^{2m-n-|\beta|} + |y-z|^{2m-n-|\beta|} & \text{if } |\beta| > 2m-n. \end{cases} \tag{5.36}$$

The proof is quite similar to that of Theorem 4.9 and is based on the Green's functions estimates of Theorems 4.6 and 4.7, Corollary 4.8 and the boundary kernel estimates of Lemma 5.16. For this reason we skip the proof here and refer to [211, Lemma 3.4].

Proof of Theorem 5.15. For existence and regularity we refer to Theorem 2.25. First, we assume additionally that $\psi \in C^{m+2,\gamma}(\partial B)$, $\varphi \in C^{m+1,\gamma}(\partial B)$. We write $\hat{\varphi}_s = \varphi -$

$s\psi$ and let $p>1$ be arbitrary. The operator

$$\mathscr{L}_{m,n}\hat{\varphi}_s(x) := \int_{\partial B} L_{m,n}(x,y)\hat{\varphi}_s(y)\,d\omega(y)$$

satisfies $\mathscr{L}_{m,n}:\ C^{m+1,\gamma}(\partial B)\to C^{2m,\gamma}(\overline{B})\subset W^{2m,p}(B)$, the operator

$$\hat{\mathscr{K}}_{m,n,s}\psi(x) := \int_{\partial B} \hat{K}_{m,n,s}(x,y)\psi(y)\,d\omega(y)$$

satisfies $\hat{\mathscr{K}}_{m,n,s}:\ C^{m+2,\gamma}(\partial B)\to C^{2m,\gamma}(\overline{B})\subset W^{2m,p}(B)$, while the Green operator

$$\mathscr{G}_{m,n}f(x) := \int_B G_{m,n}(x,y)f(y)\,dy$$

satisfies $\mathscr{G}_{m,n}:\ L^p(B)\to W^{2m,p}\cap W_0^{m,p}(B)$, see Theorem 2.19 and Corollary 2.21. We write $\mathscr{A} := \sum_{|\beta|\le 2m-2} a_\beta(\,.\,)D^\beta$. The solution of (5.31) is given by

$$u = -\mathscr{G}_{m,n}\mathscr{A}\,u + \hat{\mathscr{K}}_{m,n,s}\psi + \mathscr{L}_{m,n}\hat{\varphi}_s \quad\text{or}\quad (\mathscr{I}+\mathscr{G}_{m,n}\mathscr{A})\,u = \hat{\mathscr{K}}_{m,n,s}\psi + \mathscr{L}_{m,n}\hat{\varphi}_s.$$

Here, $\mathscr{I}+\mathscr{G}_{m,n}\mathscr{A}$ is a bounded linear operator in $W^{2m,p}(B)$ which for sufficiently small ε_0 is invertible. Hence

$$\begin{aligned} u &= (\mathscr{I}+\mathscr{G}_{m,n}\mathscr{A})^{-1}\,\hat{\mathscr{K}}_{m,n,s}\psi + (\mathscr{I}+\mathscr{G}_{m,n}\mathscr{A})^{-1}\,\mathscr{L}_{m,n}\hat{\varphi}_s \\ &= \hat{\mathscr{K}}_{m,n,s}\psi + \sum_{i=1}^{\infty}(-\mathscr{G}_{m,n}\mathscr{A})^i\,\hat{\mathscr{K}}_{m,n,s}\psi + \mathscr{L}_{m,n}\hat{\varphi}_s + \sum_{i=1}^{\infty}(-\mathscr{G}_{m,n}\mathscr{A})^i\,\mathscr{L}_{m,n}\hat{\varphi}_s. \end{aligned}$$

We only show how to deal with the first Neumann series containing $\hat{\mathscr{K}}_{m,n,s}$, the second series containing $\mathscr{L}_{m,n}$ can be treated in the same way with some obvious simplifications. For $i\ge 1$ we integrate by parts. As $\mathscr{A}$ is of order $\le 2m-2$ and $\hat{\mathscr{K}}_{m,n,s}\psi$ vanishes on ∂B of order $m-2$ no additional boundary integrals arise. By means of the Fubini-Tonelli theorem we obtain for $x\in B$

$$\begin{aligned} (-\mathscr{G}_{m,n}\mathscr{A})^i\,\hat{\mathscr{K}}_{m,n,s}\psi(x) &= (-1)^i\int_{z_1\in B} G_{m,n}(x,z_1)\ \mathscr{A}_{z_1}\int_{z_2\in B} G_{m,n}(z_1,z_2) \\ &\quad\times\dots\ \mathscr{A}_{z_{i-1}}\int_{z_i\in B} G_{m,n}(z_{i-1},z_i)\ \mathscr{A}_{z_i}\int_{y\in\partial B}\hat{K}_{m,n,s}(z_i,y)\psi(y)\,d\omega(y)dz_i\dots dz_1 \\ &= (-1)^i\int_{z_1\in B}\left(\mathscr{A}^*_{z_1}G_{m,n}(x,z_1)\right)\int_{z_2\in B}\left(\mathscr{A}^*_{z_2}G_{m,n}(z_1,z_2)\right) \\ &\quad\times\dots\int_{z_i\in B}\left(\mathscr{A}^*_{z_i}G_{m,n}(z_{i-1},z_i)\right)\int_{y\in\partial B}\hat{K}_{m,n,s}(z_i,y)\psi(y)d\omega(y)dz_i\dots dz_1 \\ &= (-1)^i\int_B\cdots\int_B\int_{\partial B}\left(\mathscr{A}^*_{z_1}G_{m,n}(x,z_1)\right)\left(\mathscr{A}^*_{z_2}G_{m,n}(z_1,z_2)\right) \\ &\quad\times\dots\ \left(\mathscr{A}^*_{z_i}G_{m,n}(z_{i-1},z_i)\right)\hat{K}_{m,n,s}(z_i,y)\psi(y)d\omega(y)d(z_1,\dots,z_i). \end{aligned}$$

Here $\mathscr{A}^*_{\cdot} = \sum_{|\beta|\le 2m-2}(-1)^{|\beta|}D^\beta(a_\beta\ .\ )$ is the (formally) adjoint operator of the perturbation $\mathscr{A}$. The estimates (5.35) and (5.36) in Lemma 5.18 are integrable with respect to $z\in B$ uniformly in $x\in B$, $y\in\partial B$ if $|\beta|\le 2m-2$. They yield

$$\begin{aligned}
&\left|(-\mathscr{G}_{m,n}\mathscr{A})^i\,\hat{\mathscr{K}}_{m,n,s}\psi(x)\right|\\
&\le \int_{\partial B}\int_B\cdots\int_B \hat{K}_{m,n,s}(x,y)\frac{\left|\mathscr{A}^*_{z_1}G_{m,n}(x,z_1)\right|\hat{K}_{m,n,s}(z_1,y)}{\hat{K}_{m,n,s}(x,y)}\\
&\quad\times\frac{\left|\mathscr{A}^*_{z_2}G_{m,n}(z_1,z_2)\right|\hat{K}_{m,n,s}(z_2,y)}{\hat{K}_{m,n,s}(z_1,y)}\\
&\quad\times\ldots\ \frac{\left|\mathscr{A}^*_{z_i}G_{m,n}(z_{i-1},z_i)\right|\hat{K}_{m,n,s}(z_i,y)}{\hat{K}_{m,n,s}(z_{i-1},y)}\,\psi(y)\,d(z_1,\ldots,z_i)d\omega(y)\\
&\le (C_0\varepsilon_0)^i\int_{\partial B}\hat{K}_{m,n,s}(x,y)\psi(y)\,d\omega(y)=(C_0\varepsilon_0)^i\left(\hat{\mathscr{K}}_{m,n,s}\psi\right)(x).
\end{aligned}$$

Analogously we have:

$$\left|(-\mathscr{G}_{m,n}\mathscr{A})^i\,\mathscr{L}_{m,n}\hat{\varphi}_s(x)\right|\le\left(\hat{C}_0\varepsilon_0\right)^i(\mathscr{L}_{m,n}\hat{\varphi}_s)(x).$$

The constants $C_0=C_0(m,n,s)$, $\hat{C}_0=\hat{C}_0(m,n)$ do not depend on i.

If $\varepsilon_0=\varepsilon_0(m,n,s)>0$ is chosen sufficiently small, we come up with

$$u\ge\frac{1}{C}\hat{\mathscr{K}}_{m,n,s}\psi+\frac{1}{C}\mathscr{L}_{m,n}\hat{\varphi}_s. \tag{5.37}$$

The general case $\varphi\in C^0(\partial B)$, $\psi\in C^1(\partial B)$ follows from (5.37) with the help of an approximation, the maximum modulus estimates of Theorem 2.25 and local L^p-estimates, see Theorem 2.20. □

5.3 Local maximum principles for higher order differential inequalities

The comparison results of Section 5.1 together with the observations of Section 5.2 on the Poisson boundary kernels will yield local maximum principles for differential inequalities, which are valid for a large class of operators. Here lower order perturbations are no longer subject to smallness restrictions.

We consider $C^{2m,\gamma}$-smooth domains $\Omega\subset\mathbb{R}^n$ and differential operators L like

$$Lu:=\left(-\sum_{i,j=1}^n\tilde{a}_{ij}\frac{\partial^2}{\partial x_i\partial x_j}\right)^m u+\sum_{|\beta|\le 2m-1}a_\beta(\,.\,)D^\beta u, \tag{5.38}$$

with constant highest order coefficients $\tilde{a}_{ij}=\tilde{a}_{ji}$ obeying the ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} \tilde{a}_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n.$$

The ellipticity constants are subject to the condition $0 < \lambda \leq \Lambda$. The lower order coefficients are merely assumed to be smooth,

$$a_\beta \in C^{|\beta|,\gamma}(\overline{\Omega}).$$

Under these assumptions, we have:

Theorem 5.19. *Assume that $q \geq 1$, $q > \frac{n}{2m}$ and that $K \subset \Omega$ is a compact subset. Then there exists a constant*

$$C = C\Big(n, m, \lambda, \Lambda, q, \max_{|\beta|\leq 2m-1} \|a_\beta\|_{C^{|\beta|}(\overline{\Omega})}, \operatorname{dist}(K, \partial\Omega)\Big)$$

such that, for every $f \in C^0(\overline{\Omega})$ and every subsolution $u \in C^{2m}(\overline{\Omega})$ of the differential inequality

$$Lu \leq f,$$

the following local maximum estimate holds true:

$$\sup_K u \leq C\left(\|f^+\|_{L^q} + \|u\|_{W^{m-1,1}}\right). \tag{5.39}$$

Proof. With the help of a linear transformation we may achieve $\tilde{a}_{ij} = \delta_{ij}$. So, in what follows we consider the principal part $(-\Delta)^m$.

We want to apply Theorem 5.4 and Lemma 5.9. Let $\varepsilon_0 = \varepsilon_0(m,n) > 0$ be such that both results hold true in the unit ball B for all differential operators $\tilde{L} = (-\Delta)^m + \sum_{|\beta|\leq 2m-1} \tilde{a}_\beta D^\beta$ with $\max_{|\beta|\leq 2m-1} \|\tilde{a}_\beta\|_{C^{|\beta|}(\overline{B})} \leq \varepsilon_0$. For the differential operator $L = (-\Delta)^m + \sum_{|\beta|\leq 2m-1} a_\beta D^\beta$ defined in Ω we want to achieve the required smallness by means of scaling.

Let $x_0 \in K$ be arbitrary, after translation we may assume $x_0 = 0$. We put

$$\begin{aligned} M &:= \max_{|\beta|\leq 2m-1} \|a_\beta\|_{C^{|\beta|}(\overline{\Omega})}, \\ \rho_0 &:= \min\left\{1, \frac{1}{2}\operatorname{dist}(K, \partial\Omega), \frac{\varepsilon_0}{M}\right\}. \end{aligned} \tag{5.40}$$

For $\rho \in (0, \rho_0]$ we introduce the following scaled functions $\overline{B} \to \mathbb{R}$:

$$u_\rho(x) := u(\rho x), \qquad f_\rho(x) := \rho^{2m} f(\rho x), \qquad a_{\beta,\rho}(x) := \rho^{2m-|\beta|} a_\beta(\rho x).$$

For these functions we have on $\overline{B}$ the following differential inequality

$$L_\rho u_\rho(x) := (-\Delta)^m u_\rho(x) + \sum_{|\beta|\leq 2m-1} a_{\beta,\rho}(x) D^\beta u_\rho(x) \leq f_\rho(x). \tag{5.41}$$

Here, thanks to our choice (5.40) of ρ_0, on $\overline{B}$ the coefficients $a_{\beta,\rho}$, $|\beta| \le 2m-1$, are subject of the following smallness condition

$$\begin{aligned}\|a_{\beta,\rho}\|_{C^{|\beta|}(\overline{B})} &= \sum_{|\mu|\le|\beta|} \max_{x\in\overline{B}} \left|D^\mu a_{\beta,\rho}(x)\right| = \sum_{|\mu|\le|\beta|} \max_{x\in\overline{B}} \left(\rho^{2m-|\beta|+|\mu|}\left|\left(D^\mu a_\beta\right)(\rho x)\right|\right)\\ &\le \rho_0 \|a_\beta\|_{C^{|\beta|}(\overline{\Omega})} \le \rho_0 M \le \varepsilon_0.\end{aligned}$$

Let $G_{L_\rho,B}$ be the Green function for L_ρ in B. Theorem 5.4 and Lemma 5.9 show that there exist constants $C = C(m,n,\varepsilon_0(m,n)) = C(m,n)$, independent of $\rho \in (0,\rho_0]$, such that we have:

$$\begin{cases} G_{L_\rho,B}(x,y) > 0 & \text{in } B\times B,\\ G_{L_\rho,B}(x,y) \le C|x-y|^{2m-n} & \text{in } B\times B \text{ if } n>2m,\\ G_{L_\rho,B}(x,y) \le C\log\left(\dfrac{3}{|x-y|}\right) & \text{in } B\times B \text{ if } n=2m,\\ G_{L_\rho,B}(x,y) \le C & \text{in } B\times B \text{ if } n<2m,\\ \left|D_y^\beta G_{L_\rho,B}(0,y)\right| \le C & \text{for } |\beta|\le 2m-1,\ y\in\partial B.\end{cases} \tag{5.42}$$

To estimate $u(0) = u_\rho(0)$ we use the representation formula for u_ρ. Beside the Dirichlet data $D^\beta u_\rho$, $|\beta| \le m-1$ and terms of the kind $D_y^\beta G_{L_\rho,B}(0,y)$, $m \le |\beta| \le 2m-1$, the boundary integrals contain factors $a_{\beta,\rho}$ and their derivatives up to order $\le \max\{0, |\beta|-m-1\}$. Making use of (5.42), we obtain independently of $\rho \in (0,\rho_0]$:

$$\begin{aligned} u(0) = u_\rho(0) &\le \int_B G_{L_\rho,B}(0,y) f_\rho^+(y)\,dy\\ &\quad + C(m,n,M)\sum_{|\beta|\le m-1}\int_{\partial B}\left|D^\beta u_\rho(y)\right|\,d\omega(y)\\ &\le C(m,n,q)\|f_\rho^+\|_{L^q(B)} + C(m,n,M)\sum_{|\beta|\le m-1}\rho^{|\beta|}\int_{\partial B}\left|\left(D^\beta u\right)(\rho y)\right|\,d\omega(y)\\ &\le C(m,n,q)\rho^{2m-(n/q)}\|f^+\|_{L^q(B_\rho)}\\ &\quad + C(m,n,M)\sum_{|\beta|\le m-1}\rho^{|\beta|-n+1}\int_{|y|=\rho}\left|D^\beta u(y)\right|\,d\omega(y). \end{aligned}$$

Integration with respect to $\rho \in [\frac{1}{2}\rho_0, \rho_0]$ yields

$$u(0) \le C\left(\|f^+\|_{L^q(\Omega)} + \|u\|_{W^{m-1,1}(\Omega)}\right)$$

with a constant $C = C(m,n,q,M,\rho_0)$. Here $C = O(\rho_0^{-n})$ for $\rho_0 \searrow 0$. □

Remark 5.20. This local maximum principle may also be applied to nonlinear problems which are not subject to the standard (controllable) growth conditions as in [404, 405], see [209]. For instance, one finds "almost" classical solutions $u \in C^{2m,\gamma}(\Omega) \cap H_0^m(\Omega)$ to the Dirichlet problem for $Lu = e^u$ where L is as in (5.38).

5.4 Steklov boundary conditions

Let Ω be a bounded domain of $\mathbb{R}^n$ ($n \geq 2$) with $\partial\Omega \in C^2$ and consider the boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - a\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.43}$$

where $a \in C^0(\partial\Omega)$, $f \in L^2(\Omega)$ and ν is the outside normal (we will also use $u_\nu = \frac{\partial u}{\partial \nu}$). In this section we study the positivity preserving property for (5.43), namely under which conditions on Ω and on the boundary coefficient a the assumption $f \geq 0$ implies that the solution u exists and is positive. Let us first make precise what is meant by a solution of (5.43).

Definition 5.21. For $f \in L^2(\Omega)$ we say that u is a *weak solution* of (5.43) if $u \in H^2 \cap H_0^1(\Omega)$ and

$$\int_\Omega \Delta u \Delta v \, dx - \int_{\partial\Omega} a \, u_\nu v_\nu \, d\omega = \int_\Omega f v \, dx \qquad \text{for all } v \in H^2 \cap H_0^1(\Omega). \tag{5.44}$$

Note that weak solutions are well-defined for $a \in C^0(\partial\Omega)$. For $u \in H^4(\Omega)$ one may integrate by parts to find indeed that a weak solution of (5.44) satisfies the boundary value problem in (5.43). This means that the second boundary condition in (5.43) is hidden in the choice of the space $H^2 \cap H_0^1(\Omega)$ of admissible testing functions. For regularity results related to problem (5.43) we refer to Corollary 2.23.

In the next section we state the positivity preserving properties for (5.43) and we give the first part of their proof. The second part of their proof is more delicate and requires a Schauder setting and a different notion of solution. This is the reason why it is postponed to Section 5.4.3. In turn, the Schauder setting takes advantage of the positivity properties of the operators involved in the solution of (5.43). These properties are proven in Section 5.4.2 with a strong use of the kernel estimates of Section 4.3.

5.4.1 Positivity preserving

The first statement describes existence, uniqueness and positivity of a weak solution to (5.43). A crucial role is played by a "weighted first eigenvalue". Fix a nontrivial positive weight function $b \in C^0(\partial\Omega)$ and set

$$J_b(u) = \frac{\int_\Omega |\Delta u|^2 \, dx}{\int_{\partial\Omega} b \, u_\nu^2 \, d\omega} \quad \text{for} \quad \int_{\partial\Omega} b \, u_\nu^2 \, d\omega \neq 0 \tag{5.45}$$

and $J_b(u) = \infty$ otherwise. For every $u \in H^2 \cap H_0^1(\Omega)$ the functional in (5.45) is strictly positive, possibly ∞. Since the linear map $H^2(\Omega) \to L^2(\partial\Omega)$ defined by $u \mapsto u_\nu|_{\partial\Omega}$ is compact, there exists a minimiser for the problem

$$\delta_{1,b} = \delta_{1,b}(\Omega) := \inf_{u \in H^2 \cap H_0^1(\Omega)} J_b(u). \tag{5.46}$$

Hence $\delta_{1,b} > 0$ and it may be viewed as a kind of first Steklov eigenvalue with respect to the weight function b and any minimiser as a corresponding eigenfunction. This definition should be compared with (3.40) in Section 3.3.1.

The next statement summarises the positivity preserving results for (5.43), see the Notations-Section for the interpretation of the symbols.

Theorem 5.22. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with $\partial\Omega \in C^2$ and let $0 \lneqq b \in C^0(\partial\Omega)$. Let the eigenvalue $\delta_{1,b}$ be as defined in (5.46). Then there exists a number $\delta_{c,b} := \delta_{c,b}(\Omega) \in [-\infty, 0)$ such that the following holds for any function $a \in C^0(\partial\Omega)$.*

1. *If $a \geq \delta_{1,b} b$ and if $0 \lneqq f \in L^2(\Omega)$, then (5.43) has no nontrivial positive weak solution.*
2. *If $a = \delta_{1,b} b$, then there exists a positive eigenfunction, that is, problem (5.43) with $f = 0$ admits a weak solution $u_{1,b}$ that satisfies $u_{1,b} > 0$ and $-\Delta u_{1,b} > 0$ in Ω, $\frac{\partial}{\partial \nu} u_{1,b} < 0$ on $\partial\Omega$. This eigenfunction $u_{1,b}$ is unique, up to a constant multiplier.*
3. *If $a \lneqq \delta_{1,b} b$, then for any $f \in L^2(\Omega)$ problem (5.43) admits a unique weak solution u.*
   a. *If $\delta_{c,b} b \leq a \lneqq \delta_{1,b} b$ and if $0 \lneqq f \in L^2(\Omega)$, then $u \gneqq 0$.*
   b. *If $\delta_{c,b} b < a \lneqq \delta_{1,b} b$ and if $0 \lneqq f \in L^2(\Omega)$, then for some $c_f > 0$ it holds that $u \geq c_f\, d$ with d as in (4.1). Furthermore, if $a(x_0) < 0$ for some $x_0 \in \partial\Omega$, then $-\Delta u \ngeq 0$ in Ω, whereas if $a \geq 0$, then $0 \lneqq f$ implies $-\Delta u \geq 0$ in Ω.*
   c. *If $a < \delta_{c,b} b$, then there are $0 \lneqq f \in L^2(\Omega)$ such that the corresponding solution u of (5.43)is not positive: $0 \nleq u$.*

Proof. We first prove Item 2, then Item 1 and we end with Item 3.

Proof of Item 2. Let $u_1 := u_{1,b} \in H^2 \cap H_0^1(\Omega)$ be a minimiser for (5.46) and let $\tilde{u}_1$ be the unique solution in $H^2 \cap H_0^1(\Omega)$ of $-\Delta \tilde{u}_1 = |\Delta u_1|$. Then by the maximum principle we infer that $|u_1| \leq \tilde{u}_1$ in Ω and $\left|\frac{\partial}{\partial \nu} u_1\right| \leq \left|\frac{\partial}{\partial \nu} \tilde{u}_1\right|$ on $\partial\Omega$. If Δu_1 changes sign, then these inequalities are strict and imply $J_b(u_1) > J_b(\tilde{u}_1)$. Hence, Δu_1 is of fixed sign, say $-\Delta u_1 \geq 0$, so that the maximum principle implies $\frac{\partial}{\partial \nu} u_1 < 0$ on $\partial\Omega$ and $u_1 \geq cd$ in Ω, where d is as in (4.1). Similarly, if u_1 and u_2 are two minimisers which are not multiples of each other, then there is a linear combination which is a sign changing minimiser and one proceeds as above to find a contradiction. This proves Item 2.

Proof of Item 1. Let us suppose by contradiction that $a \geq \delta_{1,b}$, that $f \gneqq 0$ and that u is a nontrivial positive solution to (5.43). Hence $u_\nu \leq 0$ on $\partial\Omega$. Let $u_{1,b}$ be a minimiser for (5.46) as obtained above. By taking $v = u_{1,b}$ in (5.44) one finds

$$
\begin{aligned}
0 &< \int_\Omega f\, u_{1,b}\, dx = \int_\Omega \Delta u\, \Delta u_{1,b}\, dx - \int_{\partial\Omega} a\, u_\nu \left(u_{1,b}\right)_\nu\, d\omega \\
&\le \int_\Omega \Delta u\, \Delta u_{1,b}\, dx - \int_{\partial\Omega} \delta_{1,b}\, b\, u_\nu \left(u_{1,b}\right)_\nu\, d\omega = 0,
\end{aligned}
$$

a contradiction. The last equality follows by the fact that $u_{1,b}$ minimises (5.46). This proves Item 1.

Proof of Item 3. On the space $H^2 \cap H_0^1(\Omega)$ we define the energy functional

$$
I(u) := \tfrac{1}{2}\int_\Omega |\Delta u|^2\, dx - \tfrac{1}{2}\int_{\partial\Omega} a u_\nu^2\, d\omega - \int_\Omega f u\, dx \qquad u \in H^2 \cap H_0^1(\Omega).
$$

Critical points of I are weak solutions of (5.43) in the sense of Definition 5.21. We will show that for $a \lneqq \delta_{1,b} b$ the functional I has a unique critical point.

If $a < \delta_{1,b} b$, one sets

$$
\varepsilon := \frac{\min\left\{\delta_{1,b} b(x) - a(x);\, x \in \partial\Omega\right\}}{\max\left\{\delta_{1,b} b(x);\, x \in \partial\Omega\right\}} > 0, \tag{5.47}
$$

and finds that $a \le (1-\varepsilon)\delta_{1,b} b$. By the definition of $\delta_{1,b}$ we have for all $u \in H^2 \cap H_0^1(\Omega)$

$$
\begin{aligned}
&\int_\Omega |\Delta u|^2\, dx - \int_{\partial\Omega} a\, u_\nu^2\, d\omega \\
&\ge \varepsilon \int_\Omega |\Delta u|^2\, dx + (1-\varepsilon)\left(\int_\Omega |\Delta u|^2\, dx - \int_{\partial\Omega} \delta_{1,b} b\, u_\nu^2\, d\omega\right) \\
&\ge \varepsilon \int_\Omega |\Delta u|^2\, dx,
\end{aligned} \tag{5.48}
$$

so that the functional I is coercive. Since it is also strictly convex the functional I admits a unique critical point which is its global minimum over $H^2 \cap H_0^1(\Omega)$.

In order to deal with the case that $a^+ \lneqq \delta_{1,b} b$, but $a^+(x) = \delta_{1,b} b(x)$ for some $x \in \partial\Omega$, we set

$$
\tilde{b} := \frac{1}{2}\left(b + \delta_{1,b}^{-1} a^+\right).
$$

Let u_1 be a minimiser of $J_{\tilde{b}}$ and u_2 of J_b. For the definition see (5.45). Then, since $0 \lneqq \tilde{b} \lneqq b$ and $\left(\frac{\partial}{\partial\nu} u_1\right)^2 > 0$ on $\partial\Omega$, we find $\delta_{1,\tilde{b}} = J_{\tilde{b}}(u_1) > J_b(u_1) \ge J_b(u_2) = \delta_{1,b}$. Instead of (5.47) we set

$$
\varepsilon := 1 - \delta_{1,b}/\delta_{1,\tilde{b}} > 0,
$$

find for $x \in \partial\Omega$ that $a \le a^+ = \delta_{1,b}(2\tilde{b} - b) \le \delta_{1,b}\tilde{b} = (1-\varepsilon)\delta_{1,\tilde{b}}\tilde{b}$ and proceed by replacing all b in (5.48) with $\tilde{b}$.

If $a^+ = \delta_{1,b} b$ and $a^- \gneqq 0$, then one may not proceed directly as before. However, instead of the functional in (5.45), one may use

$$J_b^{a^-}(u) = \left(\int_\Omega |\Delta u|^2 \, dx + \int_{\partial\Omega} a^- \, u_\nu^2 \, d\omega \right) \left(\int_{\partial\Omega} b \, u_\nu^2 \, d\omega \right)^{-1}.$$

Then, defining $\delta_{1,b}^{a^-}$ for $J_b^{a^-}$ as in (5.46), this minimum is assumed, say by $u_{1,b}^{a^-}$. Since

$$\delta_{1,b}^{a^-} = J_b^{a^-}(u_{1,b}^{a^-}) \geq J_b(u_{1,b}^{a^-}) \geq J_b(u_{1,b}) = \delta_{1,b},$$

with the last inequality strict if $u_{1,b}^{a^-} \neq c \; u_{1,b}$ and with the first inequality strict if $u_{1,b}^{a^-} = c \; u_{1,b}$ since $(u_{1,b})_\nu^2 > 0$, we find $\delta_{1,b}^{a^-} > \delta_{1,b}$. So,

$$\int_\Omega |\Delta u|^2 \, dx + \int_{\partial\Omega} a^- \, u_\nu^2 \, d\omega \geq \delta_{1,b}^{a^-} \int_{\partial\Omega} b \, u_\nu^2 \, d\omega \quad \text{for all } u \in H^2 \cap H_0^1(\Omega)$$

and by setting

$$\varepsilon := 1 - \delta_{1,b}/\delta_{1,b}^{a^-} > 0$$

we find the result that replaces (5.48). Indeed

$$\begin{aligned}
&\int_\Omega |\Delta u|^2 \, dx - \int_{\partial\Omega} a \, u_\nu^2 \, d\omega = \int_\Omega |\Delta u|^2 \, dx + \int_{\partial\Omega} a^- \, u_\nu^2 \, d\omega - \int_{\partial\Omega} \delta_{1,b} b \, u_\nu^2 \, d\omega \\
&\geq \varepsilon \int_\Omega |\Delta u|^2 \, dx + (1-\varepsilon) \left(\int_\Omega |\Delta u|^2 \, dx + \int_{\partial\Omega} a^- \, u_\nu^2 \, d\omega - \int_{\partial\Omega} \delta_{1,b}^{a^-} b \, u_\nu^2 \, d\omega \right) \\
&\geq \varepsilon \int_\Omega |\Delta u|^2 \, dx.
\end{aligned}$$

Hence, I is coercive and strictly convex and we conclude as for (5.48). The existence and uniqueness is so proved.

Assume now that there exists $x_0 \in \partial\Omega$ such that $a(x_0) < 0$. If the weak solution u were superharmonic, then by Hopf's boundary lemma we would have $u_\nu(x_0) < 0$. Using the second boundary condition in (5.43), we would then obtain $\Delta u(x_0) > 0$, a contradiction.

If $a \geq 0$ and $f \gneqq 0$, let $\tilde{u}$ be the unique solution in $H^2 \cap H_0^1(\Omega)$ of $-\Delta \tilde{u} = |\Delta u|$ in Ω. Since $\tilde{u} > u$ or $\tilde{u} = u$ in Ω, and $|\tilde{u}_\nu| \geq |u_\nu|$ on $\partial\Omega$, one finds for $f \gneqq 0$ that

$$I(\tilde{u}) - I(u) = -\frac{1}{2} \int_{\partial\Omega} a \left(\tilde{u}_\nu^2 - u_\nu^2 \right) \, d\omega - \int_\Omega f \; (\tilde{u} - u) \; dx \leq 0.$$

Equality occurs only when $\tilde{u} = u$. Since I is strictly convex there is at most one critical point which is a minimum. So $u = \tilde{u} > 0$ and $-\Delta u = -\Delta \tilde{u} = |\Delta u| \geq 0$. This completes the proof of existence and uniqueness whenever $a \lneqq \delta_{1,b} b$. The proof of the remaining statements (a), (b), (c) in Item 3 is more lengthy and delicate and we give it in Section 5.4.3, see Theorem 5.37. □

Note that in Theorem 5.22 it may happen that $b(x) = 0$ on some part $\Gamma_1 \subset \partial\Omega$ and $b(x) > 0$ on the remaining part $\Gamma_0 = \partial\Omega \setminus \Gamma_1$. If moreover $\delta_{c,b} = -\infty$, then the limit problem for which the positivity preserving property holds (that is, $a = \delta_{c,b} b$) becomes a mixed Dirichlet-Navier problem with boundary conditions

$$u = 0 \text{ on } \partial\Omega, \qquad u_\nu = 0 \text{ on } \Gamma_0, \qquad \Delta u = 0 \text{ on } \Gamma_1.$$

As a first consequence of Theorem 5.22 we have the positivity preserving property for the hinged plate model in planar convex domains. As we have seen in Section 1.1.2, the physical bounds for the Poisson ratio are given by

$$-1 < \sigma < 1. \tag{5.49}$$

Under this constraint, the following result holds.

Corollary 5.23. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with C^2-boundary and assume (5.49). For any $f \in L^2(\Omega)$ there exists a unique minimiser $u \in H^2 \cap H^1_0(\Omega)$ of the elastic energy functional (1.11) that is, of*

$$J(u) = \int_\Omega \left(\frac{|\Delta u|^2}{2} - f\,u \right) dx - \frac{1-\sigma}{2} \int_{\partial\Omega} \kappa u_\nu^2 \, d\omega,$$

where κ denotes the curvature of $\partial\Omega$. The minimiser u is the unique weak solution to

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\,\kappa u_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $f \gneqq 0$ implies that there exists $c_f > 0$ such that $u(x) \ge c_f\, d(x)$ and u is superharmonic in Ω.

Proof. We first show that the energy functional J coincides with the form given in (1.5). This can be done on a dense subset of smooth functions. Since $u|_{\partial\Omega} = 0$, one has $u_x = u_\nu \nu_1$ and $u_y = u_\nu \nu_2$ and may conclude that

$$\begin{aligned} &2\int_\Omega \left(u_{xy}^2 - u_{xx}u_{yy}\right) dxdy \\ &= \int_{\partial\Omega} \left(u_{xy}u_y\nu_1 + u_{xy}u_x\nu_2 - u_{xx}u_y\nu_2 - u_{yy}u_x\nu_1\right) d\omega \\ &= \int_{\partial\Omega} u_\nu \left(2u_{xy}\nu_1\nu_2 - u_{xx}\nu_2^2 - u_{yy}\nu_1^2\right) d\omega = -\int_{\partial\Omega} \kappa u_\nu^2 \, d\omega, \end{aligned}$$

where in the last step we used (1.8). Hence, existence and uniqueness of a minimiser u follow from Proposition 2.35.

Since $\partial\Omega \in C^2$ and Ω is convex we have $0 \lneqq \kappa \in C^0(\partial\Omega)$. In Proposition 2.35, it is also shown that J is strictly convex so that $(1-\sigma)\kappa \lneqq \delta_{1,\kappa}\kappa$. Hence, if $f \gneqq 0$ it follows first from statement 3.(a) in Theorem 5.22 that $u \gneqq 0$ in Ω and so that $u_\nu|_{\partial\Omega} \le 0$. In view of the boundary value problem solved by u we obtain $-\Delta u \gneqq 0$ in Ω. This superharmonicity finally yields the other properties stated for u. □

More generally, if we take $b = 1$ in Theorem 5.22, we obtain the following

Corollary 5.24. *Let $\Omega \subset \mathbb{R}^n$ ($n \ge 2$) be a bounded domain with $\partial\Omega \in C^2$ and let*

$$\delta_1 := \delta_1(\Omega) := \inf\left\{ \frac{\int_\Omega |\Delta u|^2 \, dx}{\int_{\partial\Omega} u_\nu^2 \, d\omega}; u \in H^2 \cap H^1_0(\Omega) \setminus H^2_0(\Omega) \right\} \in (0,\infty) \tag{5.50}$$

be the first Steklov eigenvalue. Then there exists a number $\delta_c := \delta_c(\Omega) \in [-\infty, 0)$ such that the following holds for any function $a \in C^0(\partial\Omega)$.

1. *If $a \geq \delta_1$ and if $0 \lneq f \in L^2(\Omega)$, then (5.43) has no nontrivial positive weak solution.*
2. *If $a = \delta_1$, then there exists a positive eigenfunction, that is, problem (5.43) admits a nontrivial weak solution u_1 with $u_1 > 0$ in Ω for $f = 0$. Moreover, the function u_1 is, up to multiples, the unique solution of (5.43) with $f = 0$ and $a = \delta_1$.*
3. *If $a \lneq \delta_1$, then for any $f \in L^2(\Omega)$ problem (5.43) admits a unique weak solution u.*

   a. *If $\delta_c \leq a \lneq \delta_1$, then $0 \lneq f \in L^2(\Omega)$ implies $u \gneq 0$ in Ω.*
   b. *If $\delta_c < a \lneq \delta_1$, then $0 \lneq f \in L^2(\Omega)$ implies $u \geq c_f\, d > 0$ in Ω for some $c_f > 0$.*
   c. *If $a < \delta_c$, then there are $0 \lneq f \in L^2(\Omega)$ with $0 \not\leq u$.*

The result described in Corollary 5.24 quite closely resembles the structure for the resolvent of the biharmonic operator under Navier boundary conditions – see McKenna-Walter [297] and Kawohl-Sweers [246] – or for the biharmonic operator under Dirichlet boundary conditions in case the domain is a ball – see Corollary 5.5 where $(-a)$ plays the same role as a here. For all these problems the scheme is as follows.

$\exists f > 0$ with $u \not\geq 0$	$\forall f > 0:\ \exists u$ and $u \geq 0$	$\forall f > 0$ if $\exists u$ then $u \not\geq 0$
δ_c	0	δ_1 $\qquad a \longrightarrow$

Under Dirichlet boundary conditions such that the corresponding Green function is positive, Corollary 5.5 tells us that the set of constant coefficients $a \in \mathbb{R}$ for which $\Delta^2 u \geq au$ implies $u \geq 0$ is an interval $(a_c, \Lambda_{2,1})$ with $a_c \in (-\infty, 0]$. By combining Theorem 5.22 with Lemma 5.35 below, we immediately see that a similar result holds for (5.43).

Theorem 5.25. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with $\partial\Omega \in C^2$ and let $a_i \in C^0(\partial\Omega)$ with $i = 1, 2$. Suppose that $a_1 \leq 0 \leq a_2$ are such that both for $a = a_1$ and $a = a_2$ we have the following: for all $f \in L^2(\Omega)$ there exists a weak solution $u = u_i$ ($i = 1, 2$) for (5.43), and moreover*

$$f \gneq 0 \text{ implies } u \gneq 0. \tag{5.51}$$

Then for any $a \in C^0(\partial\Omega)$ satisfying $a_1 \leq a \leq a_2$ and for each $f \in L^2(\Omega)$, a unique weak solution of (5.43) exists and (5.51) holds true.

However, a crucial difference with the Dirichlet boundary value problem for $\Delta^2 u \geq au$ is that $a_c \in (-\infty, 0]$ while for problem (5.43) it might happen that $\delta_c(\Omega) = -\infty$ although for general domains one cannot expect to have the positivity preserving property for any negative a. This is stated in the next results which show that the limit situation where $\delta_c(\Omega) = -\infty$ is closely related to the positivity preserving property for the biharmonic Dirichlet problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = u_\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.52}$$

To this end, let us recall once more that the positivity preserving property does not hold in general domains $\Omega \subset \mathbb{R}^n$ for (5.52), see Section 1.2. It is clear that (5.43) with $|a| = +\infty$ corresponds to (5.52). However, if $a \to +\infty$ then a crosses the spectrum of $-\Delta$ under Steklov boundary conditions, see Theorem 3.18, whereas the next statement justifies the feeling that (5.52) only corresponds to the limit case $a = -\infty$.

Theorem 5.26. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with $\partial\Omega \in C^2$. If for every $m \in \mathbb{N}^+$ and $0 \lneqq f \in L^2(\Omega)$ the weak solution of (5.43) with $a = -m$ is nontrivial and positive, then for every $0 \lneqq f \in L^2(\Omega)$ the solution $u \in H_0^2(\Omega)$ of (5.52) satisfies $u \gneqq 0$.*

Proof. Let us first recall the two boundary value problems addressed in the statement, namely

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \left(\Delta + m\frac{\partial}{\partial \nu}\right) u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = u_\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.53}$$

For all $m > 0$ let $u_m \in H^2 \cap H_0^1(\Omega)$ be the unique weak solution of the problem on the left in (5.53). Then according to (5.44) we have

$$\int_\Omega \Delta u_m \Delta \phi \, dx + m \int_{\partial\Omega} \frac{\partial u_m}{\partial \nu} \frac{\partial \phi}{\partial \nu} \, d\omega = \int_\Omega f \phi \, dx \ \text{ for all } \phi \in H^2 \cap H_0^1(\Omega) \,. \tag{5.54}$$

Taking $\phi = u_m$ in (5.54) and using Hölder and Poincaré inequalities, gives for all $m > 0$

$$\|\Delta u_m\|_{L^2}^2 \leq \|\Delta u_m\|_{L^2}^2 + m \int_{\partial\Omega} \left| \frac{\partial u_m}{\partial \nu} \right|^2 d\omega = \int_\Omega f u_m \, dx \leq c \|f\|_{L^2} \|\Delta u_m\|_{L^2} \,. \tag{5.55}$$

Inequality (5.55) shows that the sequence (u_m) is bounded in $H^2(\Omega)$ so that, up to a subsequence, we have

$$u_m \rightharpoonup \overline{u} \quad \text{in } H^2(\Omega) \quad \text{as } m \to \infty \tag{5.56}$$

for some $\overline{u} \in H^2 \cap H_0^1(\Omega)$. Once boundedness is established, if we let $m \to \infty$ then (5.55) also tells us that

$$\frac{\partial u_m}{\partial \nu} \to 0 \quad \text{in } L^2(\partial\Omega) \quad \text{as } m \to \infty \,.$$

Therefore, $\overline{u} \in H_0^2(\Omega)$. Now take any function $\phi \in H_0^2(\Omega)$ in (5.54) and let $m \to \infty$. By (5.56) we obtain

$$\int_\Omega \Delta \overline{u} \Delta \phi \, dx = \int_\Omega f \phi \, dx \qquad \text{for all } \phi \in H_0^2(\Omega) .$$

Hence, $\overline{u}$ is the unique solution of the corresponding Dirichlet problem (5.52). Since (5.56) also implies that, up to a subsequence, $u_m(x) \to \overline{u}(x)$ for a.e. $x \in \Omega$, one finds that $\overline{u} \gneqq 0$. □

Theorem 5.26 states that there exists some link between the Steklov and the Dirichlet problems. This link is confirmed by the special case when Ω is the unit ball. In this case, from Theorem 3.20 we know that the first Steklov eigenvalue as defined in (5.50) satisfies $\delta_1 = n$ and the following holds.

Theorem 5.27. *Let $\Omega = B$, the unit ball in $\mathbb{R}^n$ ($n \geq 2$). Then, for all $0 \lneqq f \in L^2(B)$ and all $a \in C^0(\partial B)$ such that $a \lneqq n$, there exists $c > 0$ such that the weak solution u of (5.43) satisfies $u(x) \geq c d(x)$ in B.*

The constant c depends both on f and a, $c = c_{f,a}$. For a fixed $0 \lneqq f \in L^2(B)$ we expect that $c = c_{f,a} \to 0$ as $a \to -\infty$.

Also the proof of Theorem 5.27 requires a Schauder setting and an approximation procedure. For this reason it is postponed to the end of Section 5.4.3.

5.4.2 *Positivity of the operators involved in the Steklov problem*

We consider the second order Green operator $\mathscr{G}$ and the Poisson operator $\mathscr{K}$, that is, $w = \mathscr{G} f + \mathscr{K} g$ formally solves

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = g & \text{on } \partial\Omega. \end{cases}$$

For C^2-domains, the operators $\mathscr{G}$ and $\mathscr{K}$ can be represented by integral kernels G and K, see (4.46) in Section 4.3. Let $(\mathscr{P} w)(x) := -\nu \cdot \nabla w(x) = -w_\nu(x)$ for $x \in \partial\Omega$. In this section we use the kernel estimates obtained in Section 4.3 in order to prove some positivity properties of these operators. First, we fix the appropriate setting so that $\mathscr{G}$, $\mathscr{K}$ and $\mathscr{P}$ are well-defined operators.

Notation 5.28 Let d denote the distance to $\partial\Omega$ as defined in (4.1). Set

$$C_d(\overline{\Omega}) = \left\{ u \in C^0(\overline{\Omega});\, \text{there exists } w \in C^0(\overline{\Omega}) \text{ such that } u = dw \right\}$$

with norm

$$\|u\|_{C_d(\overline{\Omega})} = \sup \left\{ \frac{|u(x)|}{d(x)}; x \in \Omega \right\} .$$

Set also $C_0(\overline{\Omega}) = \left\{ u \in C^0(\overline{\Omega});\, u = 0 \text{ on } \partial\Omega \right\}$ so that $C_d(\overline{\Omega}) \subsetneq C_0(\overline{\Omega})$.

We consider the three above operators in the following setting.

$$\mathscr{G}: C^0(\overline{\Omega}) \to C_d(\overline{\Omega}), \quad \mathscr{K}: C^0(\partial\Omega) \to C^0(\overline{\Omega}), \quad \mathscr{P}: C_d(\overline{\Omega}) \to C^0(\partial\Omega).$$

We also define the embedding

$$\mathscr{I}_d : C_d(\overline{\Omega}) \to C^0(\overline{\Omega}). \tag{5.57}$$

The space $C_d(\overline{\Omega})$ is a Banach lattice, that is, a Banach space with the ordering such that $|u| \le |v|$ implies $\|u\|_{C_d(\overline{\Omega})} \le \|v\|_{C_d(\overline{\Omega})}$, see Definition 3.2 or [13, 309, 359]. The positive cone

$$C_d(\overline{\Omega})^+ = \left\{u \in C_d(\overline{\Omega});\ u(x) \ge 0 \text{ in } \overline{\Omega}\right\}$$

is solid (namely, it has nonempty interior) and reproducing (that is, every $w \in C_d(\overline{\Omega})$ can be written as $w = u - v$ for some $u, v \in C_d(\overline{\Omega})^+$). Similarly, we define $C^0(\partial\Omega)^+$ and $C^0(\overline{\Omega})^+$.

Note that the interiors of the cones in these spaces are as follows:

$$\begin{aligned} C^0(\partial\Omega)^{+,\circ} &= \left\{v \in C^0(\partial\Omega);\ v(x) \ge c \text{ for some } c > 0\right\}, \\ C^0(\overline{\Omega})^{+,\circ} &= \left\{u \in C^0(\overline{\Omega});\ u(x) \ge c \text{ for some } c > 0\right\}, \\ C_d(\overline{\Omega})^{+,\circ} &= \left\{u \in C_d(\overline{\Omega});\ u(x) \ge c\, d(x) \text{ for some } c > 0\right\}. \end{aligned}$$

Definition 5.29. The operator $\mathscr{F} : \mathscr{C}_1 \to \mathscr{C}_2$ is called

- *nonnegative*, $\mathscr{F} \ge 0$, when $g \in \mathscr{C}_1^+ \Rightarrow \mathscr{F} g \in \mathscr{C}_2^+$;
- *strictly positive*, $\mathscr{F} \gneq 0$, when $g \in \mathscr{C}_1^+ \backslash \{0\} \Rightarrow \mathscr{F} g \in \mathscr{C}_2^+ \backslash \{0\}$;
- *strongly positive*, $\mathscr{F} > 0$, when $g \in \mathscr{C}_1^+ \backslash \{0\} \Rightarrow \mathscr{F} g \in \mathscr{C}_2^{+,\circ}$.

If $\mathscr{F} \ge 0$ and $\mathscr{F} \ne 0$, that is, for some $g \in \mathscr{C}_1^+$ we find $\mathscr{F} g \gneq 0$, we call $\mathscr{F}$ positive. Similarly, two operators are ordered through $\ge$ (respectively $\gneq$ or $>$) whenever their difference is nonnegative (respectively strictly or strongly positive).

We now prove a positivity result.

Proposition 5.30. *Suppose that $\partial\Omega \in C^2$ and $a \in C^0(\partial\Omega)$. Let $\mathscr{G}$, $\mathscr{K}$ and $\mathscr{P}$ be defined as above. Then $\mathscr{G}\mathscr{K} a \mathscr{P} : C_d(\overline{\Omega}) \to C_d(\overline{\Omega})$ is a well-defined compact linear operator. If in addition $a \gneq 0$, then $\mathscr{G}\mathscr{K} a \mathscr{P}$ is positive and even such that*

$$u \in C_d(\overline{\Omega})^+ \quad \textit{implies either} \quad \mathscr{G}\mathscr{K} a \mathscr{P} u = 0 \quad \textit{or} \quad \mathscr{G}\mathscr{K} a \mathscr{P} u \in C_d(\overline{\Omega})^{+,\circ}. \tag{5.58}$$

Proof. Take $\gamma \in (0,1)$, $p > n(1-\gamma)^{-1}$ and fix the embeddings $I_1 : C^0(\overline{\Omega}) \to L^p(\Omega)$, $I_2 : W^{2,p}(\Omega) \to C^{1,\gamma}(\overline{\Omega})$ and $I_3 : C^{1,\gamma} \cap C_0(\overline{\Omega}) \to C_d(\overline{\Omega})$. Since $\partial\Omega \in C^2$, for every $p \in (1,\infty)$ there exists a bounded linear operator $\mathscr{G}_p : L^p(\Omega) \to W^{2,p} \cap W_0^{1,p}(\Omega)$ such that $-\Delta \mathscr{G}_p f = f$ for all $f \in L^p(\Omega)$, see Theorem 2.20. If $\mathscr{I}_d$ is as in (5.57), then the Green operator from $C_d(\overline{\Omega})$ to $C_d(\overline{\Omega})$ should formally be denoted $\mathscr{G}\mathscr{I}_d$, where $\mathscr{G} = I_3 I_2 \mathscr{G}_p I_1$. Note that the embedding $I_1 : C^0(\overline{\Omega}) \to L^p(\Omega)$ is bounded and the embedding $I_2 : W^{2,p}(\Omega) \to C^{1,\gamma}(\overline{\Omega})$ is compact, see Theorem 2.6. Since $W^{2,p} \cap W_0^{1,p}(\Omega) \subset C^{1,\gamma} \cap C_0(\overline{\Omega})$ and $I_3 : C^{1,\gamma} \cap C_0(\overline{\Omega}) \to C_d(\overline{\Omega})$ is bounded, $\mathscr{G}$ is not only well-defined but even compact. The strong maximum principle and Hopf's boundary point lemma allow then to conclude that $\mathscr{G} : C^0(\overline{\Omega}) \to C_d(\overline{\Omega})$ is a compact linear operator and it is strongly positive.

Since $\partial\Omega \in C^2$ and Ω is bounded all boundary points are regular. According to Perron's method [197, Theorem 2.14] the Dirichlet boundary value problem is solvable for arbitrary continuous boundary values by

$$(\mathscr{K}\phi)(x) = \sup\{v(x); v \le \phi \text{ on } \partial\Omega \text{ and } v \text{ subharmonic in } \Omega\}.$$

For $\phi \in C^0(\partial\Omega)$ one obtains $\mathscr{K}\phi \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ and by the maximum principle

$$\sup_{x\in\Omega} (\mathscr{K}\phi)(x) = \max_{x\in\partial\Omega} \phi(x) \quad \text{and} \quad \inf_{x\in\Omega} (\mathscr{K}\phi)(x) = \min_{x\in\partial\Omega} \phi(x)$$

implying not only that $\|\mathscr{K}\phi\|_{L^\infty(\Omega)} = \|\phi\|_{L^\infty(\partial\Omega)}$, but also that $\mathscr{K} : C^0(\partial\Omega) \to C^0(\overline{\Omega})$ is a strictly positive bounded linear operator.

Finally, from the fact that every function $u \in C_d(\overline{\Omega})$ can be written as $u = d\,w$ for some $w \in C^0(\overline{\Omega})$ and $\mathscr{P} d\,w = w|_{\partial\Omega}$, we infer that $\mathscr{P} : C_d(\overline{\Omega}) \to C^0(\partial\Omega)$ is a positive bounded linear operator.

From the just proved properties of $\mathscr{G}$, $\mathscr{K}$ and $\mathscr{P}$ we infer compactness and positivity of $\mathscr{G}\mathscr{K}a\mathscr{P}$ when $a \ge 0$ and that $\mathscr{K}a\mathscr{P}u \gneqq 0$ implies that $\mathscr{G}\mathscr{K}a\mathscr{P}u \in C_d(\overline{\Omega})^{+,\circ}$. □

Proposition 5.30 enables us to compare (using the notations of Definition 5.29) some of the operators involved in the Steklov problem.

Proposition 5.31. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^2$ and let $\mathscr{I}_d$ be as in (5.57). Then there exists a constant $M_\Omega > 0$ such that*

$$\mathscr{G}\mathscr{K}\mathscr{P}\mathscr{G}\mathscr{I}_d\mathscr{G} \le M_\Omega \mathscr{G}\mathscr{I}_d\mathscr{G} \ \text{ and } \ \mathscr{G}\mathscr{K}\mathscr{P}\mathscr{G}\mathscr{K} \le M_\Omega \mathscr{G}\mathscr{K}.$$

Proof. We know that the integral kernel which corresponds to $\mathscr{G}\mathscr{K}\mathscr{P}\mathscr{G}\mathscr{I}_d\mathscr{G}$ satisfies the estimates in Lemma 4.16. By Proposition 4.13 we know estimates from below for $\mathscr{G}\mathscr{I}_d\mathscr{G}$. We have to compare these estimates. To this end, we use the following trivial fact

$$\min(1,\alpha)\min(1,\beta) \le \min(1,\alpha\beta) \quad \text{for all } \alpha,\beta \ge 0,$$

combined with (4.53) and (4.52). Considering the different dimensions separately we then have the following. For $n \ge 5$, if $x^* \in \partial\Omega$ is such that $|x-x^*| = d(x)$,

$$(d(x)+d(y)+|x^*-y^*|)^{2-n}\, d(x)d(y) \preceq |x-y|^{4-n} \, \min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right).$$

This, combined with Lemma 4.16 and (4.48), proves the statement for $n \geq 5$.

For $n = 4$ we argue as for $n = 5$ to find

$$(d(x)+d(y)+|x^*-y^*|)^{-2}\, d(x)d(y) \preceq \min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right) \preceq \log\left(1+\frac{d(x)d(y)}{|x-y|^2}\right).$$

This, combined with Lemma 4.16 and (4.49), proves the statement for $n = 4$.

For $n = 3$ we have

$$(d(x)+d(y)+|x^*-y^*|)^{-1}\, d(x)d(y) \preceq$$
$$\preceq \sqrt{d(x)d(y)\min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right)} = \sqrt{d(x)d(y)}\min\left(1, \frac{\sqrt{d(x)d(y)}}{|x-y|}\right).$$

This, combined with Lemma 4.16 and (4.50), proves the statement for $n = 3$.

For $n = 2$, by using (4.52) we find as a variation of (4.53) that

$$\log\left(2+\frac{1}{d(x)+d(y)+|x^*-y^*|}\right) \preceq \log\left(2+\frac{1}{|x-y|^2+d(x)d(y)}\right).$$

This, combined with Lemma 4.16 and (4.51), proves the statement for $n = 2$. □

5.4.3 Relation between Hilbert and Schauder setting

In this section we complete the proof of Theorem 5.22 and we give the proof of Theorem 5.27. For Theorem 5.22, it remains to prove statements (a), (b) and (c) in Item 3, see Theorem 5.37 below. In these situations it is more convenient to set the problem in spaces of continuous functions. This forces us to argue in a Schauder setting and we rewrite (5.43) as an integral equation. Then we proceed by approximation.

As in (5.57), let $\mathscr{I}_d : C_d(\overline{\Omega}) \to C^0(\overline{\Omega})$ denote the embedding operator, then (5.43) is equivalent to

$$u = \mathscr{G}\mathscr{K}a\mathscr{P}u + \mathscr{G}\mathscr{I}_d\mathscr{G}f. \tag{5.59}$$

Definition 5.32. For $f \in C^0(\overline{\Omega})$ we say that u is a $\mathscr{C}$-solution of (5.43) if $u \in C_d(\overline{\Omega})$ satisfies (5.59).

Proposition 5.33. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with $\partial\Omega \in C^2$ and let $a \in C^0(\partial\Omega)$. If $f \in C^0(\overline{\Omega})$, then a $\mathscr{C}$-solution of (5.43) is also a weak solution in the sense of Definition 5.21.*

Proof. If $f \in C^0(\overline{\Omega})$ and $u \in C_d(\overline{\Omega})$ then by (5.59) it follows that $w = \mathscr{K}a\mathscr{P}u + \mathscr{I}_d\mathscr{G}f \in C^0(\overline{\Omega}) \subset L^2(\Omega)$ and hence $u = \mathscr{G}w \in H^2 \cap H_0^1(\Omega)$. Moreover, for such u and for any $v \in H^2 \cap H_0^1(\Omega)$ we have

$$\int_\Omega \Delta u \,\Delta v\, dx = \int_\Omega (\mathscr{K} a\mathscr{P} u + \mathscr{G} f)\ \Delta v\, dx = \int_{\partial\Omega} a u_\nu v_\nu\, d\omega_x + \int_\Omega f\, v\, dx,$$

which is precisely (5.44). □

Next, we note that (possibly by changing its sign) the minimiser $u_{1,b}$ for (5.46) lies in $C_d(\overline{\Omega})^{+,\circ}$.

Lemma 5.34. *Let $\partial\Omega \in C^2$ and suppose that $a \in C^0(\partial\Omega)$ is such that $a \lneqq \delta_{1,b} b$. Then*

$$\mathscr{E}^a_{\mathscr{G}} := (\mathscr{I} - \mathscr{G}\mathscr{K} a\mathscr{P})^{-1} \mathscr{G}\mathscr{I}_d \mathscr{G} : C^0(\overline{\Omega}) \to C_d(\overline{\Omega}),$$
$$\mathscr{E}^a_{\mathscr{K}} := (\mathscr{I} - \mathscr{G}\mathscr{K} a\mathscr{P})^{-1} \mathscr{G}\mathscr{K} : C^0(\partial\Omega) \to C_d(\overline{\Omega}),$$

are well-defined operators. Moreover, the following holds.

- *For $f \in C^0(\overline{\Omega})$ the unique $\mathscr{C}$-solution of problem (5.43) is $u = \mathscr{E}^a_{\mathscr{G}} f$.*
- *The function $u_{1,b}$ defined in Theorem 5.22 (Item 2) is a positive eigenfunction of $\mathscr{E}^a_{\mathscr{K}} \left(\delta_{1,b} b - a\right) \mathscr{P} : C_d(\overline{\Omega}) \to C_d(\overline{\Omega})$ with eigenvalue 1. Any other nonnegative eigenfunction $\tilde{u}$ of $\mathscr{E}^a_{\mathscr{K}} \left(\delta_{1,b} b - a\right) \mathscr{P}$ satisfies $\left(\delta_{1,b} b - a\right) \mathscr{P}\tilde{u} = 0$ on $\partial\Omega$.*

Proof. By Theorem 5.22 (Item 3) one finds for $a \lneqq \delta_{1,b} b$ that $\mu = 1$ is not an eigenvalue of the (compact) operator $\mathscr{G}\mathscr{K} a\mathscr{P}$. Therefore, the operator $(\mathscr{I} - \mathscr{G}\mathscr{K} a\mathscr{P})$ is invertible in $L^2(\Omega)$ and hence in $C_d(\overline{\Omega})$.

- Equation (5.59) reads as $u = (\mathscr{I} - \mathscr{G}\mathscr{K} a\mathscr{P})^{-1} \mathscr{G}\mathscr{I}_d \mathscr{G} f$.
- One directly checks that $u_{1,b}$ is an eigenfunction of $\mathscr{E}^a_{\mathscr{K}} \left(\delta_{1,b} b - a\right) \mathscr{P}$ with $\lambda = 1$ for all $a \lneqq \delta_{1,b} b$. By Theorem 5.22 (Item 2), up to its multiples, it is the unique eigenfunction with $\lambda = 1$. Let $\tilde{u}$ be another nonnegative eigenfunction of $\mathscr{E}^a_{\mathscr{K}} \left(\delta_{1,b} b - a\right) \mathscr{P}$ corresponding to some eigenvalue $\lambda \neq 1$. One finds that $\lambda = 0$ if and only if $(\delta_{1,b} b - a)\mathscr{P}\tilde{u} = 0$. For $\lambda \neq 0$ it holds that

$$\tilde{u} - \mathscr{G}\mathscr{K}\delta_{1,b} b\mathscr{P}\tilde{u} = \left(\lambda^{-1} - 1\right) \mathscr{G}\mathscr{K}(\delta_{1,b} b - a)\mathscr{P}\tilde{u}. \tag{5.60}$$

We have $u_{1,b}, \tilde{u} \in H^2 \cap H^1_0(\Omega)$. This fact allows us to use (5.60) and to find a contradiction in the case that $(\delta_{1,b} b - a)\mathscr{P}\tilde{u} \gneqq 0$. Indeed,

$$\begin{aligned} 0 &= \int_\Omega \Delta u_{1,b}\, \Delta\tilde{u}\, dx - \int_{\partial\Omega} \delta_{1,b} b\, (u_{1,b})_\nu \tilde{u}_\nu\, d\omega \\ &= \int_\Omega \Delta u_{1,b}\, \Delta\left(\tilde{u} - \mathscr{G}\mathscr{K}\delta_{1,b} b\mathscr{P}\tilde{u}\right) dx \\ &= \left(\lambda^{-1} - 1\right) \int_\Omega \Delta u_{1,b}\, \mathscr{G}\mathscr{K}(\delta_{1,b} b - a)\mathscr{P}\tilde{u}\, dx \\ &= \left(1 - \lambda^{-1}\right) \int_\Omega u_{1,b}\, \mathscr{K}(\delta_{1,b} b - a)\mathscr{P}\tilde{u}\, dx, \end{aligned}$$

and this last expression has a sign if $\lambda \neq 1$. □

Lemma 5.35. *Let $\partial\Omega \in C^2$ and suppose that $a \in C^0(\partial\Omega)$ is such that $a \lneqq \delta_{1,b}b$. Let $\mathscr{E}^a_{\mathscr{G}}$ and $\mathscr{E}^a_{\mathscr{K}}$ be as in Lemma 5.34 and suppose that $\mathscr{E}^a_{\mathscr{G}}$ is a positive operator.*

1. *Then $\mathscr{E}^a_{\mathscr{G}}$, $\mathscr{E}^a_{\mathscr{K}}$, $\mathscr{P}\mathscr{E}^a_{\mathscr{G}}$ and $\mathscr{P}\mathscr{E}^a_{\mathscr{K}}$ are strictly positive operators.*
2. *If $\tilde{a} \in C^0(\partial\Omega)$ is such that $a \le \tilde{a} \lneqq \delta_{1,b}b$, then $\mathscr{E}^{\tilde{a}}_{\mathscr{G}} \ge \mathscr{E}^a_{\mathscr{G}}$, $\mathscr{E}^{\tilde{a}}_{\mathscr{K}} \ge \mathscr{E}^a_{\mathscr{K}}$, $\mathscr{P}\mathscr{E}^{\tilde{a}}_{\mathscr{G}} \ge \mathscr{P}\mathscr{E}^a_{\mathscr{G}}$ and $\mathscr{P}\mathscr{E}^{\tilde{a}}_{\mathscr{K}} \ge \mathscr{P}\mathscr{E}^a_{\mathscr{K}}$.*
3. *If $\tilde{a} \in C^0(\partial\Omega)$ is such that $a < \tilde{a} \lneqq \delta_{1,b}b$, then $\mathscr{E}^{\tilde{a}}_{\mathscr{G}} > \mathscr{E}^a_{\mathscr{G}}$, $\mathscr{E}^{\tilde{a}}_{\mathscr{K}} > \mathscr{E}^a_{\mathscr{K}}$, $\mathscr{P}\mathscr{E}^{\tilde{a}}_{\mathscr{G}} > \mathscr{P}\mathscr{E}^a_{\mathscr{G}}$ and $\mathscr{P}\mathscr{E}^{\tilde{a}}_{\mathscr{K}} > \mathscr{P}\mathscr{E}^a_{\mathscr{K}}$.*

Proof. Assume that $0 \lneqq f \in C^0(\overline{\Omega})$ and $0 \lneqq \varphi \in C^0(\partial\Omega)$. Writing $u_a = \mathscr{E}^a_{\mathscr{G}} f$ and $v_a = \mathscr{E}^a_{\mathscr{K}} \varphi$ one gets

$$(\mathscr{I} - \mathscr{G}\mathscr{K}\, a\mathscr{P})\, u_a = \mathscr{G}\mathscr{I}_d\mathscr{G} f \quad \text{and} \quad (\mathscr{I} - \mathscr{G}\mathscr{K}\, a\mathscr{P})\, v_a = \mathscr{G}\mathscr{K}\, \varphi.$$

1. If $u_a = \mathscr{E}^a_{\mathscr{G}} f = 0$ for $f \gneqq 0$, then

$$u_a = \mathscr{G}\mathscr{K}\, a\mathscr{P} u_a + \mathscr{G}\mathscr{I}_d\mathscr{G} f = \mathscr{G}\mathscr{I}_d\mathscr{G} f > 0$$

by the maximum principle, a contradiction. So $\mathscr{E}^a_{\mathscr{G}}$ positive implies that $\mathscr{E}^a_{\mathscr{G}}$ is strictly positive. Since $K(x,y^*) = \lim_{t\searrow 0} G(x, y^* - t\nu)/t$ for $x \in \Omega$, $y^* \in \partial\Omega$ and ν the exterior normal at y^*, we find that positivity of $\mathscr{E}^a_{\mathscr{G}}$ implies that $\mathscr{E}^a_{\mathscr{K}}$ is positive. We even have strict boundary positivity. Indeed, if $\mathscr{P} u_a = 0$ then $u_a = \mathscr{G}\mathscr{I}_d\mathscr{G} f$ and Hopf's boundary point lemma gives $\mathscr{P} u_a > 0$, a contradiction. A similar argument holds for v_a. This proves the first set of claims.
2. Let $a \le \tilde{a} \lneqq \delta_{1,b}b$. We have

$$(\mathscr{I} - \mathscr{G}\mathscr{K}\, a\mathscr{P})\, u_{\tilde{a}} = \mathscr{G}\mathscr{K}\, (\tilde{a} - a)\, \mathscr{P} u_{\tilde{a}} + \mathscr{G}\mathscr{I}_d\mathscr{G} f$$

and, in turn, since $(\mathscr{I} - \mathscr{G}\mathscr{K}\, a\mathscr{P})$ is invertible in view of Lemma 5.34,

$$(\mathscr{I} - \mathscr{E}^a_{\mathscr{K}} (\tilde{a} - a)\mathscr{P})\, u_{\tilde{a}} = u_a. \tag{5.61}$$

For $\|\tilde{a} - a\|_{L^\infty(\partial\Omega)}$ small enough (say $\|\tilde{a} - a\|_{L^\infty(\partial\Omega)} < \varepsilon$) one may invert the operator in (5.61) and find an identity with a convergent series, that is

$$\mathscr{E}^{\tilde{a}}_{\mathscr{G}} = \mathscr{E}^a_{\mathscr{G}} + \sum_{k=1}^{\infty} \left(\mathscr{E}^a_{\mathscr{K}}\, (\tilde{a} - a)\, \mathscr{P}\right)^k \mathscr{E}^a_{\mathscr{G}}. \tag{5.62}$$

Since $\mathscr{E}^a_{\mathscr{K}}\, (\tilde{a} - a)\, \mathscr{P} \ge 0$ holds, one finds that $u_{\tilde{a}} = \mathscr{E}^{\tilde{a}}_{\mathscr{G}} f \ge \mathscr{E}^a_{\mathscr{G}} f = u_a$. The series formula (5.62) holds for $\|\tilde{a} - a\|_{L^\infty(\partial\Omega)} < \varepsilon$. However, if $\|\tilde{a} - a\|_{L^\infty(\partial\Omega)} \ge \varepsilon$ then the above argument can be repeated by considering some intermediate $a := a_0 \lneqq a_1 \lneqq \ldots \lneqq a_k := \tilde{a}$ such that $\|a_{i+1} - a_i\|_{L^\infty(\partial\Omega)} < \varepsilon$ for all i. A similar reasoning applies to $v_{\tilde{a}}$, v_a. This proves the second set of claims.
3. Let us consider the sequence $(\varphi_m) \subset C_d(\overline{\Omega})$, defined by

$$\begin{aligned} \varphi_0 &= \mathscr{E}^a_{\mathscr{G}} f, \\ \varphi_{m+1} &= \mathscr{E}^a_{\mathscr{K}} \left(\delta_{1,b}b - a\right) \mathscr{P} \varphi_m \qquad \text{for } m \ge 0. \end{aligned}$$

Since $\mathscr{E}_{\mathscr{G}}^{a} f \gneqq 0$ we find that $\varphi_m \geq 0$ for all $m \geq 0$. Moreover, since $\mathscr{E}_{\mathscr{K}}^{a}\left(\delta_{1,b}b - a\right)\mathscr{P}$ is compact, two cases may occur;

(i) there exists $m_0 > 0$ such that $\varphi_m \gneqq 0$ for $m < m_0$ and $\varphi_m = 0$ for all $m \geq m_0$;

(ii) $\varphi_m / \|\varphi_m\|_{C_d(\overline{\Omega})} \to \varphi_\infty$ where φ_∞ is a nonnegative eigenfunction (with $\lambda = 1$) of the operator

$$\mathscr{E}_{\mathscr{K}}^{a}\left(\delta_{1,b}b - a\right)\mathscr{P}\varphi_\infty = \lambda\varphi_\infty.$$

If (i) occurs, then $\mathscr{E}_{\mathscr{K}}^{a}(\delta_{1,b}b - a)\mathscr{P}\varphi_{m_0} = 0$ so that by Item 1 we infer $(\delta_{1,b}b - a)\mathscr{P}\varphi_{m_0} = 0$ and hence $\mathscr{P}\varphi_{m_0} = 0$. We find a contradiction since as in the proof of Item 1 it follows that $\varphi_{m_0} = \mathscr{G}\mathscr{I}_d\mathscr{G}\varphi_{m_0-1}$ and $\mathscr{P}\varphi_{m_0} > 0$ holds by Hopf's boundary point lemma.

Therefore, case (ii) occurs. Then φ_∞ is a multiple of the unique positive eigenfunction $u_{1,b}$, see Lemma 5.34. So for m_1 large enough we find that there exists $c_2 > c_1 > 0$ such that

$$c_1 u_{1,b} \leq \frac{\varphi_m}{\|\varphi_m\|_{C_d(\overline{\Omega})}} \leq c_2 u_{1,b} \qquad \text{for all } m \geq m_1.$$

Now set

$$\psi_0 = \mathscr{E}_{\mathscr{G}}^{a} f, \qquad \psi_{m+1} = \mathscr{E}_{\mathscr{K}}^{a}\left(\tilde{a} - a\right)\mathscr{P}\psi_m \quad \text{for } m \geq 0. \tag{5.63}$$

Since for some $\varepsilon > 0$ it holds that

$$\varepsilon\left(\delta_{1,b}b - a\right) \leq \tilde{a} - a \leq \delta_{1,b}b - a,$$

we obtain $\psi_m \geq \varepsilon^m \varphi_m$ for all m and by (5.63)

$$\psi_m \geq \varepsilon^m \varphi_m \geq c_1 \varepsilon^m \|\varphi_m\|_{C_d(\overline{\Omega})} u_{1,b} \qquad \text{for all } m \geq m_1.$$

Then from (5.62) it follows that there exists $c_3 > 0$ such that

$$\mathscr{E}_{\mathscr{G}}^{\tilde{a}} f \geq \mathscr{E}_{\mathscr{G}}^{a} f + c_3 u_{1,b}.$$

In a similar way we proceed with $\mathscr{E}_{\mathscr{K}}$, $\mathscr{P}\mathscr{E}_{\mathscr{G}}$ and $\mathscr{P}\mathscr{E}_{\mathscr{K}}$. □

With the result in Lemma 5.34 it will be sufficient to have positivity preserving for a negative $a \in C^0(\partial\Omega)$ in order to ensure that this property will hold for any sign-changing $\tilde{a}$ with $a \leq \tilde{a} \lneqq \delta_{1,b}b$. So we may restrict ourselves to $a \leq 0$. We now prove a crucial "comparison" statement in the case where $\mathscr{G}\mathscr{K}a\mathscr{P}$ has a small spectral radius.

Lemma 5.36. *Let $\partial\Omega \in C^2$ and assume that $0 \geq a \in C^0(\partial\Omega)$ is such that*

$$r_\sigma(\mathscr{G}\mathscr{K}a\mathscr{P}) < 1.$$

If there exists $M > 0$ such that

$$\mathscr{G}\mathscr{K}\mathscr{P}\mathscr{G}\mathscr{I}_d\mathscr{G} \leq M\,\mathscr{G}\mathscr{I}_d\mathscr{G}, \tag{5.64}$$

and if $\|a\|_{L^\infty(\partial\Omega)} < M^{-1}$*, then* $\mathscr{E}^a_{\mathscr{G}} > 0$.

Proof. Clearly, $a = -a^-$. Since $r_\sigma(\mathscr{G}\mathscr{K}a^-\mathscr{P}) < 1$ the equation (5.59) can be rewritten as a Neumann series

$$u = \left(\mathscr{I} + \mathscr{G}\mathscr{K}a^-\mathscr{P}\right)^{-1}\mathscr{G}\mathscr{I}_d\mathscr{G}f = \sum_{k=0}^{\infty}\left(-\mathscr{G}\mathscr{K}a^-\mathscr{P}\right)^k\mathscr{G}\mathscr{I}_d\mathscr{G}f,$$

which reads

$$u = \left(\sum_{k=0}^{\infty}\left(\mathscr{G}\mathscr{K}a^-\mathscr{P}\right)^{2k}\right)\left(\mathscr{I} - \mathscr{G}\mathscr{K}a^-\mathscr{P}\right)\mathscr{G}\mathscr{I}_d\mathscr{G}f \tag{5.65}$$

after joining the odd and even powers. Next, notice that in view of (5.65) it suffices to show that the operator $(\mathscr{I} - \mathscr{G}\mathscr{K}a^-\mathscr{P})\mathscr{G}\mathscr{I}_d\mathscr{G}$ is strongly positive. This fact is a direct consequence of (5.64) and $\|a^-\|_{L^\infty(\partial\Omega)} \le M^{-1}$. □

By combining the previous statements, we obtain the following result, which completes the proof of Item 3 of Theorem 5.22. The proof uses estimates for the kernels involved and for this reason it seems more suitable to employ a Schauder setting and to approximate.

Theorem 5.37. *There exists* $\delta_{c,b} := \delta_{c,b}(\Omega) \in [-\infty, 0)$ *such that the following holds for a weak solution u of (5.43):*

1. *for* $\delta_{c,b}b \le a \lneqq \delta_{1,b}b$ *it follows that if* $0 \lneqq f \in L^2(\Omega)$*, then* $u \gneqq 0$*;*
2. *for* $\delta_{c,b}b < a \lneqq \delta_{1,b}b$ *it follows that if* $0 \lneqq f \in L^2(\Omega)$*, then* $u \ge c_f\, d$ *for some* $c_f > 0$ *(depending on f), d being the distance function from (4.1);*
3. *for* $a < \delta_{c,b}b$ *there are* $0 \lneqq f \in L^2(\Omega)$ *with u somewhere negative.*

Proof. Let M_Ω be as in Proposition 5.31 and put $\delta := -\left(M_\Omega \max_{x\in\partial B} b(x)\right)^{-1} < 0$. Then by Lemmas 5.34 and 5.36 we infer that

$$\text{if } \delta b \le a \lneqq \delta_{1,b}b \ \text{ and } \ 0 \lneqq f \in C^0(\overline{\Omega}) \quad \text{then} \quad u \gneqq 0 \text{ in } \Omega, \tag{5.66}$$

where u is the unique $\mathscr{C}$-solution of (5.43). Let $\delta_{c,b}$ be the (negative) infimum of all such δ which satisfy (5.66). We have so proved that there exists $\delta_{c,b} := \delta_{c,b}(\Omega) \in [-\infty, 0)$ such that, if $\delta_{c,b}b \le a \lneqq \delta_{1,b}b$ and $0 \lneqq f \in C^0(\overline{\Omega})$, then $u \gneqq 0$, where u is the $\mathscr{C}$-solution of (5.43). Moreover, if $\delta_{c,b}b < a$ and $0 \lneqq f \in C^0(\overline{\Omega})$, then Lemma 5.35 yields the existence of c_f such that $u \ge c_f\, d$. Finally, the above definition of $\delta_{c,b}$ shows that, if $a < \delta_{c,b}b$, then there are $0 \lneqq f \in C^0(\overline{\Omega})$ with u somewhere negative. In view of Proposition 5.33, this proves Item 3.

For Item 1 we use a density argument. Assume that $\delta_{c,b}b \le a \lneqq \delta_{1,b}b$ and $0 \lneqq f \in L^2(\Omega)$. Let $u \in H^2 \cap H^1_0(\Omega)$ be the unique weak solution of (5.43), according to Item 3 in Theorem 5.22. Consider a sequence of functions $(f_k) \subset C^0(\overline{\Omega})$ such that $f_k \gneqq 0$ for all $k \in \mathbb{N}$ and $f_k \to f$ in $L^2(\Omega)$ as $k \to \infty$. Let u_k be the $\mathscr{C}$-solution to

$$\Delta^2 u_k = f_k \text{ in } \Omega, \qquad u_k = \Delta u_k - a\frac{\partial u_k}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Then, by (5.66), $u_k \gneqq 0$ in Ω for all k. Moreover, by Corollary 2.23, the sequence (u_k) is bounded in $H^2(\Omega)$ so that, up to a subsequence, it converges weakly and pointwise to some $u \in H^2 \cap H^1_0(\Omega)$. By Definition 5.21, we know that

$$\int_\Omega \Delta u_k \Delta v\,dx - \int_{\partial\Omega} a\,(u_k)_\nu v_\nu\,d\omega = \int_\Omega f_k v\,dx \qquad \text{for all } v \in H^2 \cap H^1_0(\Omega).$$

Therefore, letting $k \to \infty$, we deduce that u is a weak solution to the original problem and

$$u \gneqq 0 \qquad \text{in } \Omega. \tag{5.67}$$

The proof of Item 2 is more delicate. Assume that $\delta_{c,b} b < a \lneqq \delta_{1,b} b$ and $0 \lneqq f \in L^2(\Omega)$. Let $u \in H^2 \cap H^1_0(\Omega)$ be the unique weak solution to (5.43). Let

$$g(x) := \min\{1, f(x)\}, \qquad x \in \Omega,$$

and let $v \in H^2 \cap H^1_0(\Omega)$ be the unique weak solution to

$$\begin{cases} \Delta^2 v = g & \text{in } \Omega, \\ v = \Delta v - a v_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $g \le f$, we deduce by Lemma 5.35 and a density argument that

$$u \ge v \qquad \text{in } \Omega. \tag{5.68}$$

Moreover, since $g \in L^\infty(\Omega)$, by Corollary 2.23 and Theorem 2.6 we infer that $v \in C^1(\overline{\Omega})$.

Let $\delta_{c,b}$ be as at the beginning of this proof, take a function $a_0 \in C^0(\partial\Omega)$ such that $\delta_{c,b} b < a_0 < a$ (if $\delta_{c,b} > -\infty$ one can also take $a_0 = \delta_{c,b} b$) and consider also the unique weak solution w to

$$\begin{cases} \Delta^2 w = g & \text{in } \Omega, \\ w = \Delta w - a_0 w_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Again, we have $w \in C^1(\overline{\Omega})$. Since $w \ge 0$ in Ω in view of Item 1, we know that $w_\nu \le 0$ on $\partial\Omega$. Moreover, it cannot be that $w_\nu \equiv 0$ since otherwise the boundary condition would imply $-\Delta w = 0$ on $\partial\Omega$ with $-\Delta w$ superharmonic in Ω. This would imply first that $-\Delta w > 0$ in Ω and next, by Hopf's lemma, that $w_\nu < 0$ on $\partial\Omega$, a contradiction. Therefore,

$$\psi := (a_0 - a) w_\nu \gneqq 0, \qquad \psi \in C^0(\partial\Omega).$$

Finally, let $z := v - w$. Then $z \in C^1(\overline{\Omega})$ and z is the unique weak solution to

$$\begin{cases} \Delta^2 z = 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \\ \Delta z - a z_\nu = -\psi & \text{on } \partial\Omega. \end{cases}$$

In fact, by Corollary 2.23 and Theorem 2.6, we have that $z \in C_d(\overline{\Omega})^+$ and $z = \mathscr{E}^a_{\mathscr{K}} \psi$. By Lemma 5.35 we know that $\mathscr{E}^a_{\mathscr{K}} > \mathscr{E}^{a_0}_{\mathscr{K}} \gneq 0$ so that there exists $c > 0$ with

$$z(x) \geq c\, d(x). \tag{5.69}$$

Note that c depends on ψ and therefore also on w. Hence, it depends on f so that $c = c_f$. By combining (5.68) with (5.69) we obtain

$$u(x) \geq v(x) = z(x) + w(x) \geq z(x) \geq c_f\, d(x)$$

and Item 2 follows. □

Proof of Theorem 5.27. We first assume that $f \in C^\infty_c(B)$. In this case, by Corollary 2.23 we know that the weak solution u satisfies $u \in W^{2,p}(B)$ for all $p > 1$. In turn, by Theorem 2.6, this proves that $u \in C^1(\overline{B})$ and hence $\Delta u = a u_\nu \in C^0(\partial B)$. Therefore, Theorems 2.19 and 2.25 yield $u \in C^\infty(B) \cap C^2(\overline{B})$. In particular, by Lemma 5.34, u is a $\mathscr{C}$-solution.

Consider the auxiliary function $\phi \in C^\infty(B) \cap C^0(\overline{B})$ defined by

$$\phi(x) = (|x|^2 - 1)\Delta u(x) - 4x \cdot \nabla u(x) - 2(n-4)u(x) \quad \text{ for } x \in \overline{B}.$$

Since $x = \nu$ and $u = 0$ on ∂B, we have

$$\phi = -4u_\nu \qquad \text{on } \partial B. \tag{5.70}$$

Moreover, for $x \in B$ we have

$$\nabla\phi = (2\Delta u)x + (|x|^2 - 1)\nabla\Delta u + 2(2-n)\nabla u - 4D^2 u \cdot x, \tag{5.71}$$

$$-\Delta\phi = (1 - |x|^2) f(x) \geq 0, \tag{5.72}$$

where $D^2 u$ denotes the Hessian matrix of u. By (5.71) we find

$$\phi_\nu = 2\Delta u + 2(2-n)u_\nu - 4\langle \nu, D^2 u \cdot \nu\rangle \qquad \text{on } \partial B.$$

Since $\langle \nu, D^2 u \cdot \nu\rangle = u_{\nu\nu}$, by recalling that $u = 0$ on ∂B and using the expression of Δu on the boundary, the previous equation reads $\phi_\nu = -2\Delta u + 2n u_\nu$. Finally, taking into account the second boundary condition in (5.43), we obtain

$$\phi_\nu = 2(n-a)u_\nu \qquad \text{on } \partial B. \tag{5.73}$$

So, combining (5.70), (5.72) and (5.73) we find that ϕ satisfies the second order Steklov boundary value problem

$$\begin{cases} -\Delta\phi = (1 - |x|^2) f \geq 0 \text{ in } B, \\ \phi_\nu + \frac{1}{2}(n-a)\phi = 0 \qquad \text{on } \partial B. \end{cases}$$

As $a \gneqq n$, by the maximum principle (for this second order problem!) we infer that $\phi > 0$ in $\overline{B}$ and hence by (5.70) that $u_\nu \le 0$ on ∂B. By (2.65) and Proposition 5.11, we deduce that $u > 0$ in B whenever $0 \lneqq f \in C_c^\infty(B)$.

Assume now that $0 \lneqq f \in L^2(B)$ and let $u \in H^2 \cap H_0^1(B)$ be the unique weak solution to (5.43), according to Definition 5.21. Then the same density argument leading to (5.67) shows that $u \gneqq 0$ in B. Hence, by Corollary 5.24 3.(c), we infer that $\delta_c = -\infty$. In turn the lower bound $u(x) \ge cd(x)$ in B follows from Corollary 5.24, part 3.(b). $\square$

5.5 Bibliographical notes

The lower order perturbation theory of positivity was developed in [210], see also [204]. These results are based on Green function estimates, 3-G-theorems and Neumann series. This strategy was used e.g. by Chung, Cranston, Fabes, Hueber, Sieveking, Zhao [97, 114, 233, 420, 421] in the context of Schrödinger operators and conditioned Brownian motion and e.g. by Mitidieri-Sweers [309, 383, 384] to study positivity in noncooperatively coupled second order systems.

The discussion of the local maximum principle and of the role of boundary data follows [209, 211]. The underlying formulae for the Poisson kernels for the biharmonic Dirichlet problem go back to Lauricella-Volterra [268, 402] and were collected in the book [323] by Nicolesco. For the polyharmonic Poisson kernels we refer to Edenhofer [158, 159]. Estimates as in Lemmas 5.8 and 5.9 were proved in a more general setting but under more restrictive assumptions on the coefficients by Krasovskiĭ [255, 256]. A local maximum principle for fourth order operators was first deduced by Tomi [395].

For the positivity preserving property of the biharmonic Steklov boundary value problem (5.43), we follow Gazzola-Sweers [191], where one can also find a discussion on existence and positivity of the solution to (5.43) when $a - \delta_1$ changes sign on $\partial\Omega$. Moreover, in [191] one can also find a different proof of Theorem 5.27 which is strongly based on the behaviour of the biharmonic Green function for the Dirichlet problem, see (2.65). Corollary 5.23 is due to Parini-Stylianou [331]. We also refer to [42, 44] for some related nonlinear problems and for a first attempt to describe the positivity preserving property for (5.43).

Chapter 6
Dominance of positivity in linear equations

In Section 1.2 we mentioned that although the Green function $G_{\Delta^2,\Omega}$ for the clamped plate boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = |\nabla u| = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

is in general sign changing, it is very hard to display its negative part in numerical simulations or in real world experiments. Moreover, numerical work in nonlinear elliptic fourth order equations suggests that maximum or comparison principles are violated only to a "small extent". Nevertheless, we do not yet have tools at hand to give this feeling a precise form and, in particular, a quantitative form which might prove to be useful also for nonlinear higher order equations.

This chapter may be considered as a first preliminary step in this direction. We study the negative part of the biharmonic Green function $G^-_{\Delta^2,\Omega}$ and show that it is small when compared with its positive part $G^+_{\Delta^2,\Omega}$. For a precise formulation see Theorems 6.15 and 6.24 and the subsequent interpretations. We emphasise that these are not just continuous dependence on data results. Green's functions are *families* of functions with the position of the pole as a parameter and the main problem consists in gaining uniformity with respect to the position of the pole when it approaches the boundary. In proving these results, one has to distinguish between the dimensions $n = 2$ and $n \geq 3$. The second case seems to be much simpler and is carried out in detail. We are confident that the arguments can be extended to fourth order operators where the principal part is a square of a second order operator and which may contain also lower order perturbations. Uniformity with respect to unbounded families of such perturbations can, however, in general not be expected. The proof ($n \geq 3$) heavily relies on uniform Krasovskiĭ-type estimates for biharmonic Green's functions $G_{\Delta^2,\Omega}$ in general domains, which are deduced in Section 4.5. Local positivity results from Section 6.3 are used as an essential first step which, in the particular case $n = 3$, were observed first by Nehari [322]. Although in the two-dimensional case one has holomorphic maps at hand, the result there requires a much more involved proof, which we sketch in Section 6.2.2 and where for details we refer to the literature [117]. This proof is based on the explicit biharmonic Green functions in the “limaçons de Pascal”, on carefully putting together parts of boundaries of several prototype domains and delicate asymptotic estimates.

A second main objective of this chapter is to show that positivity of the biharmonic Green function $G_{\Delta^2,B}$ in the unit ball $B \subset \mathbb{R}^n$ is *not* just a singular event but remains true under small $C^{4,\gamma}$-smooth perturbations Ω of B. For $n \geq 3$ see Theorem 6.29; its proof is quite similar to that of the small negative part result mentioned before. For $n = 2$, see Theorem 6.3. Here we build on the lower order perturbation theory developed in Theorem 5.1 and benefit from holomorphic maps and reduction to normal form. These tools are special for $n = 2$ and allow for considering also any m-th power of a regular second order elliptic operator being close enough to the polyharmonic prototype $(-\Delta)^m$ in domains Ω close enough to the unit disk. Having such a perturbation theory of positivity is remarkable since, again, this is not just a continuous dependence on data result.

6.1 Highest order perturbations in two dimensions

In two dimensions also perturbations of highest order of the polyharmonic prototype may be taken into account. Here we consider

$$\begin{cases} Lu = f & \text{in } \Omega, \\ D^\alpha u|_{\partial\Omega} = 0 & \text{for } |\alpha| \leq m-1, \end{cases} \tag{6.2}$$

with

$$Lu := \left(-\sum_{i,j=1}^{2} \tilde{a}_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \right)^m u + \sum_{|\beta| \leq 2m-1} a_\beta(x) D^\beta u, \tag{6.3}$$

where $\tilde{a}_{ij} = \tilde{a}_{ji} \in C^{2m-1,\gamma}(\overline{\Omega})$, $a_\beta \in C^{0,\gamma}(\overline{\Omega})$. In view of the maximum principle for second order operators we assume throughout the whole chapter that

$$m \geq 2.$$

First we define an appropriate notion of closeness for domains and operators.

Definition 6.1. We assume that $\Omega^\natural$ and Ω are bounded $C^{k,\gamma}$-smooth domains. Let $\varepsilon \geq 0$. We call Ω *ε-close* to $\Omega^\natural$ in $C^{k,\gamma}$-sense, if there exists a $C^{k,\gamma}$-mapping $g : \overline{\Omega^\natural} \to \overline{\Omega}$ such that $g(\overline{\Omega^\natural}) = \overline{\Omega}$ and

$$\|g - Id\|_{C^{k,\gamma}(\overline{\Omega^\natural})} \leq \varepsilon.$$

We remark that if $k \geq 1$, $\Omega^\natural$ is convex and ε is sufficiently small, then $g^{-1} \in C^{k,\gamma}(\overline{\Omega})$ exists and $\|g^{-1} - Id\|_{C^{k,\gamma}(\overline{\Omega})} = O(\varepsilon)$ as $\varepsilon \to 0$.

Definition 6.2. Let $\varepsilon \geq 0$ and assume that L is as in (6.3). We call the operator L *ε-close* to $(-\Delta)^m$ in $C^{k,\gamma}$-sense, if (in the case $k \geq 2m$) additionally $\tilde{a}_{ij} \in C^{k,\gamma}(\overline{\Omega})$ and

$$\|\tilde{a}_{ij} - \delta_{ij}\|_{C^{k,\gamma}(\overline{\Omega})} \leq \varepsilon,$$
$$\|a_\beta\|_{C^0(\overline{\Omega})} \leq \varepsilon \text{ for } |\beta| \leq 2m-1.$$

If $\varepsilon \geq 0$ is small, then L is uniformly elliptic.

The following is the main perturbation result if $n = 2$.

Theorem 6.3. *There exists $\varepsilon_0 = \varepsilon_0(m) > 0$ such that we have for $0 \leq \varepsilon \leq \varepsilon_0$:*

If the bounded $C^{2m,\gamma}$-smooth domain $\Omega \subset \mathbb{R}^2$ is ε-close to the unit disk B in $C^{2m,\gamma}$-sense and if the differential operator L is ε-close to $(-\Delta)^m$ in $C^{2m-1,\gamma}$-sense, then for every $f \in C^{0,\gamma}(\overline{\Omega})$ satisfying $f \gneqq 0$ the solution $u \in C^{2m,\gamma}(\overline{\Omega})$ to the Dirichlet problem (6.2) is strictly positive, namely

$$u > 0 \text{ in } \Omega.$$

Remark 6.4.
1. Let $E_{a,b}$ be an ellipse with half axes $a, b > 0$. In case of small eccentricity, i.e. $\frac{a}{b} \approx 1$, Green's function for Δ^2 in $E_{a,b}$ is positive. For larger eccentricity, e.g. $\frac{a}{b} \approx 1.2$, it changes sign according to the example of Garabedian [176] and the refined version by Hedenmalm, Jakobsson, and Shimorin in [226].
2. The proof of Theorem 6.3 suggests that $\varepsilon_0(m) \searrow 0$ for increasing $m \nearrow \infty$.
3. As long as one restricts to the polyharmonic operator $(-\Delta)^m$ in perturbed domains, it was shown by Sassone [358] that $C^{m,\gamma}$-closeness to the disk is sufficient. In case of the clamped plate equation this means that positivity is governed by closeness of the boundary curvature to a constant with respect to a Hölder norm. We think that also in the case of perturbed principal parts, the required closeness to the polyharmonic operator may be relaxed. But we expect that such a relaxation will require a big technical effort. In particular, all problems should be written in divergence form and one should refer to $C^{m,\gamma}$-Schauder-theory for operators in divergence form.
4. We recall that Theorem 6.3 cannot be proved by just referring to continuous dependence on data.
5. According to Jentzsch's [236] or Kreĭn-Rutman's [257] theorem, see Theorem 3.3, positivity of the Green function implies existence of a positive first eigenfunction. A somehow stronger result was proved in [212], which was already briefly mentioned in Section 3.1.3. Assume that $(\Omega_t)_{t \geq 0}$ is a C^{2m+1}-smooth family of domains with $\Omega_0 = B$. Assume further that transition from positivity of $G_{(-\Delta)^m,\Omega_t}$ to sign change may be observed and let t_g be the largest parameter such that $G_{(-\Delta)^m,\Omega_t} > 0$ for $t \in [0, t_g)$. Then for some $\varepsilon > 0$ and for all $t \in [0, t_g + \varepsilon)$, the first polyharmonic eigenvalue in Ω_t is still simple and the corresponding eigenfunction may be chosen strictly positive.

In order to prove Theorem 6.3 we proceed in three steps.

1. First, we consider $\tilde{a}_{ij} = \delta_{ij}$ and domains Ω which are close to the disk in a conformal sense. In this case the claim can be proved by using conformal maps which leave the principal part $(-\Delta)^m$ invariant. The pulled back differential equation is a lower order perturbation of the polyharmonic equation and Theorem 5.4 is applicable. See Lemma 6.5 below.
2. Next we employ a quantitative version of the Riemannian mapping theorem. Conformal maps $B \to \Omega$ enjoy a representation based on the harmonic Green function in Ω. This representation allows to apply elliptic theory in order to conclude "conformal closeness" from "differentiable closeness". See Lemma 6.6.
3. The theory of normal forms for second order elliptic operators allows to transform the leading coefficients $\tilde{a}_{ij}$ into δ_{ij} thereby giving rise to a further "small" perturbation of the domain Ω. See Lemma 6.7.

Only in two dimensions, the theory of normal forms is available and, moreover, sufficiently many conformal maps exist to deform suitable domains into the unit disk. In higher dimensions, the only conformal maps are Möbius transforms, which map balls onto balls or half spheres.

6.1.1 Domain perturbations

Lemma 6.5. *There exists $\varepsilon_1 = \varepsilon_1(m) > 0$ such that the following statement holds true. Let Ω be a simply connected bounded $C^{2m,\gamma}$-smooth domain. For the differential operator L in (6.3), we assume that $\tilde{a}_{ij} = \delta_{ij}$. Moreover, let $h : \overline{B} \to \overline{\Omega}$ be a biholomorphic map with $h \in C^{2m,\gamma}(\overline{B})$, $h^{-1} \in C^{2m,\gamma}(\overline{\Omega})$.*

If $\|h - Id\|_{C^{2m-1}(\overline{B})} \le \varepsilon_1$ and $\|a_\beta\|_{C^0(\overline{B})} \le \varepsilon_1$ for all $|\beta| \le 2m-1$, then the Green function $G_{L,\Omega}$ for the boundary value problem (6.2) in Ω exists and is positive.

Proof. In the disk B, the corresponding result is given in Theorem 5.4. In order to apply this theorem also to the boundary value problem (6.2) in Ω, it has to be "pulled back" to the disk. The crucial point is that conformal maps leave the principal part $(-\Delta)^m$ invariant and yield only additional terms of lower order.

Let $\varepsilon := \max\left\{\max_{|\beta|\le 2m-1} \|a_\beta\|_{C^0(\overline{\Omega})}, \|h - Id\|_{C^{2m-1}(\overline{B})}\right\}$ be sufficiently small. For the pulled back solution $v : \overline{B} \to \mathbb{R}$, $v(x) := u(h(x))$, using

$$\Delta v(x) = \frac{1}{2}\,|\nabla h(x)|^2 \big((\Delta u) \circ h\big)(x),$$

the boundary value problem

$$\begin{cases} \left(-\dfrac{2}{|\nabla h|^2}\Delta\right)^m v + \displaystyle\sum_{|\beta|\le 2m-1} \hat{a}_\beta D^\beta v = f \circ h & \text{in } B, \\ D^\alpha v|_{\partial B} = 0 & \text{for } |\alpha| \le m-1, \end{cases}$$

has to be considered with suitable coefficients $\hat{a}_\beta \in C^{0,\gamma}(\overline{B})$, $\|\hat{a}_\beta\|_{C^0(\overline{B})} = O(\varepsilon)$. Computing $\left(-\frac{2}{|\nabla h|^2}\Delta\right)^m$ yields additional coefficients $D^\mu\left(\frac{1}{|\nabla h|^2}\right)$ with $0 < |\mu| \leq 2m-2$ for the lower order terms. The leading term becomes $\left(\frac{2}{|\nabla h|^2}\right)^m(-\Delta)^m v$. Here, $\|\frac{2}{|\nabla h|^2} - 1\|_{C^{2m-2}(\overline{B})} = O(\varepsilon)$. So, we obtain the boundary value problem

$$\begin{cases} (-\Delta)^m v + \sum\limits_{|\beta|\leq 2m-1} \tilde{a}_\beta D^\beta v = \tilde{f} & \text{in } B, \\ D^\alpha v|_{\partial B} = 0 & \text{for } |\alpha| \leq m-1, \end{cases}$$

with $\tilde{f} := \left(\frac{|\nabla h|^2}{2}\right)^m f \circ h$ and suitable coefficients $\tilde{a}_\beta \in C^{0,\gamma}(\overline{B})$, which obey the estimate $\|\tilde{a}_\beta\|_{C^0(\overline{B})} = O(\varepsilon)$. Obviously, $f \gneqq 0$ in Ω is equivalent to $\tilde{f} \gneqq 0$ in B. Hence, for sufficiently small ε all statements of Theorem 5.4 carry over to the boundary value problem (6.2). $\square$

The Riemannian mapping theorem, combined with the Kellogg-Warschawski theorem, see e.g. [344], shows existence of conformal maps which satisfy the qualitative assumptions of Lemma 6.5. Observe that here the assumptions on the domain in Lemma 6.5 are to be used. However, even in very simple domains it may be extremely difficult to give an explicit expression for the conformal map $h : B \to \Omega$ and even more difficult to check explicitly the smallness condition imposed on $\|h - Id\|_{C^{2m-1}(\overline{B})}$. For ellipses such maps were constructed in [366] by means of elliptic functions.

So, Lemma 6.5 is not very useful yet. However, the next lemma gives a general abstract result that "differentiable closeness" always implies "conformal closeness".

Lemma 6.6. *For any $\varepsilon_1 > 0$ there exists $\varepsilon_2 = \varepsilon_2(m, \varepsilon_1) > 0$ such that for $0 \leq \varepsilon \leq \varepsilon_2$ we have:*

If the $C^{2m,\gamma}$-smooth domain Ω is ε-close to B in C^{2m}-sense, then there exists a biholomorphic map $h : B \to \Omega$, $h \in C^{2m,\gamma}(\overline{B})$, $h^{-1} \in C^{2m,\gamma}(\overline{\Omega})$ with

$$\|h - Id\|_{C^{2m-1}(\overline{B})} \leq \varepsilon_1.$$

Proof. Let $g : \overline{B} \to \overline{\Omega}$ be a map according to Definition 6.1 with $\varepsilon := \|g - Id\|_{C^{2m}(\overline{B})}$. In what follows we always assume $\varepsilon \geq 0$ to be sufficiently small. In particular, Ω is then simply connected and bounded, and $0 \in \Omega$.

According to [112], see also [383, Sect. 4.2], a biholomorphic map $h : B \to \Omega$ such that $h \in C^{2m,\gamma}(\overline{B})$, $h^{-1} \in C^{2m,\gamma}(\overline{\Omega})$ may be constructed as follows.

Let $G_{-\Delta,\Omega}$ be the Green function of $-\Delta$ in Ω under Dirichlet boundary conditions. For $x \in \overline{\Omega}$, we set

$$w(x) := 2\pi G_{-\Delta,\Omega}(x, 0)$$

and introduce the conjugate harmonic function

$$w^*(x) := \int_{1/2}^{x} \left(-\frac{\partial}{\partial \xi_2} w(\xi)\, d\xi_1 + \frac{\partial}{\partial \xi_1} w(\xi)\, d\xi_2 \right).$$

Here, the integral is taken along any curve from the complex number $\frac{1}{2}$ to $x = x_1 + ix_2$ in $\Omega \setminus \{0\}$. The function w^* is well-defined up to integer multiples of 2π. Identifying $\mathbb{R}^2$ and $\mathbb{C}$, by means of

$$h^{-1}(x) := \exp\left(-w(x) - iw^*(x)\right), \quad x \in \overline{\Omega},$$

we obtain a well-defined holomorphic map $\overline{\Omega} \to \overline{B}$ enjoying the required qualitative properties. Moreover, $h^{-1}(0) = 0$ and $\frac{1}{2}$ is mapped onto the positive real half axis.

The Green function $G_{-\Delta,\Omega}$ is given by

$$G_{-\Delta,\Omega}(x,0) = -\frac{1}{2\pi}\left(\log|x| - r(x)\right), \quad x \in \overline{\Omega},$$

where $r : \overline{\Omega} \to \mathbb{R}$ solves the boundary value problem

$$\begin{cases} \Delta r = 0 & \text{in } \Omega, \\ r(x) = \varphi(x) \text{ on } \partial\Omega, & \varphi(x) := \log|x|. \end{cases}$$

It is sufficient to show that

$$\|r\|_{C^{2m-1}(\overline{\Omega})} = O(\varepsilon), \tag{6.4}$$

because by virtue of

$$h^{-1}(x) = x \cdot \exp\left(-r(x) - ir^*(x)\right), \quad x \in \overline{\Omega},$$

one obtains $\|h^{-1} - Id\|_{C^{2m-1}(\overline{\Omega})} = O(\varepsilon)$ and finally $\|h - Id\|_{C^{2m-1}(\overline{B})} = O(\varepsilon)$. Here, the estimate $\|r\|_{C^0(\overline{\Omega})} = O(\varepsilon)$ is an obvious consequence of the maximum principle.

We assume first that $\varphi|_{\partial\Omega}$ may be extended by $\hat{\varphi} \in C^{2m}(\overline{\Omega})$ in such a way that

$$\|\hat{\varphi}\|_{C^{2m}(\overline{\Omega})} = O(\varepsilon) \tag{6.5}$$

holds true. The Schauder estimates of Theorem 2.19 then give $\|r\|_{C^{2m-1,\gamma}(\overline{\Omega})} = O(\varepsilon)$ and (6.4) is proved. Here, one should observe that thanks to the ε–closeness of Ω to B in C^{2m}-sense and $m \geq 2$, for all small enough $\varepsilon > 0$ the estimation constants can be chosen independently of ε.

Hence it remains to show that extensions $\hat{\varphi}$ of $\varphi|_{\partial\Omega}$ satisfying (6.5) indeed exist. For this purpose, only the "tangential derivatives" of $\varphi|_{\partial\Omega}$ have to be estimated. This means that it is enough to consider the boundary data being parametrised with the help of the maps $g|_{\partial B} : \partial B \to \partial\Omega$ and $\mathbb{R} \ni t \mapsto (\cos t, \sin t) \in \partial B$:

$$\psi(t) := \varphi\left(g(\cos t, \sin t)\right).$$

For this map, we show that

$$\max_{j=0,\dots,2m} \max_{t\in\mathbb{R}} \left| \left(\frac{d}{dt}\right)^j \psi \right| = O(\varepsilon). \tag{6.6}$$

Indeed, for $j=0$ this is due to $\|g-Id\|_{C^0(\overline{B})} = O(\varepsilon)$ and $|\log(1+\varepsilon)| = O(\varepsilon)$. We set $\tilde{g}(t) = g(\cos t, \sin t)$, $\tilde{g}:\mathbb{R}\to\partial\Omega$. For $j\geq 1$ a general chain rule shows that

$$\begin{aligned}\left(\frac{d}{dt}\right)^j \psi &= \left(\frac{d}{dt}\right)^j (\varphi\circ\tilde{g})\\ &= \sum_{|\beta|=1}^{j} \left(\left(D^\beta\varphi\right)\circ\tilde{g}\right)\left(\sum_{\substack{p_1+\ldots+p_{|\beta|}=j\\ p_1,\dots,p_{|\beta|}\geq 1}} d_{j,\beta,\mathbf{p}} \prod_{\ell=1}^{|\beta|} \left(\frac{d}{dt}\right)^{p_\ell} \tilde{g}^{(\mu_\ell)}\right)\end{aligned}$$

with suitable coefficients $d_{j,\beta,\mathbf{p}}$, $\mathbf{p}=(p_1,\dots,p_{|\beta|})$. The coefficient μ_ℓ refers to the component of $\tilde{g}$ and is chosen as $\mu_\ell = 1$ for $\ell = 1,\dots,\beta_1$ and $\mu_\ell = 2$ for $\ell = \beta_1 + 1,\dots,|\beta| = \beta_1+\beta_2$. To show that this huge sum is indeed $O(\varepsilon)$, we observe that it is equal to 0, provided $\Omega = B$ and $g = Id$. So we put $\tilde{g}_0(t) = Id\circ(\cos t,\sin t) = (\cos t, \sin t)$ and compare corresponding terms. We obtain

$$\begin{aligned}\left(\frac{d}{dt}\right)^j \psi \;=\; &\sum_{|\beta|=1}^{j} \left(\left((D^\beta\varphi)\circ\tilde{g} - (D^\beta\varphi)\circ\tilde{g}_0\right) + (D^\beta\varphi)\circ\tilde{g}_0\right)\\ \times &\left(\sum_{\substack{p_1+\ldots+p_{|\beta|}=j\\ p_1,\dots,p_{|\beta|}\geq 1}} d_{j,\beta,\mathbf{p}} \prod_{\ell=1}^{|\beta|}\left(\left(\left(\frac{d}{dt}\right)^{p_\ell}\tilde{g}^{(\mu_\ell)} - \left(\frac{d}{dt}\right)^{p_\ell}\tilde{g}_0^{(\mu_\ell)}\right) + \left(\frac{d}{dt}\right)^{p_\ell}\tilde{g}_0^{(\mu_\ell)}\right)\right).\end{aligned}$$

As already mentioned, thanks to $\varphi(\tilde{g}_0(t)) = \log|(\cos t,\sin t)| \equiv 0$ the sum taken over all the terms which contain only $\tilde{g}_0$ and no differences, equals 0. In the remaining sum, each term contains at least one factor of the type

$$(D^\beta\varphi)\circ\tilde{g} - (D^\beta\varphi)\circ\tilde{g}_0 \qquad \text{or} \qquad \left(\frac{d}{dt}\right)^{p_\ell}\left(\tilde{g}^{(\mu_\ell)} - \tilde{g}_0^{(\mu_\ell)}\right).$$

For $\varepsilon\to 0$ each of these factors is $O(\varepsilon)$, and the remaining factors are uniformly bounded independently of ε. This proves (6.6) and the claim of the lemma. □

6.1.2 Perturbations of the principal part

We define

$$L_0 u := -\sum_{i,j=1}^{2} \tilde{a}_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}, \quad \tilde{a}_{ij} = \tilde{a}_{ji} \in C^{2m-1,\gamma}(\overline{\Omega}), \tag{6.7}$$

the second order elliptic operator whose m-th power forms the principal part of the operator L in (6.3) under investigation. By means of a suitable coordinate transformation $(x_1,x_2) \mapsto (\xi_1,\xi_2)$, $\Omega \to \Omega^\natural$, (6.7) can be reduced to normal form

$$\tilde{L}_0 v = -A(\xi)\Delta v - B_1(\xi)\frac{\partial v}{\partial \xi_1} - B_2(\xi)\frac{\partial v}{\partial \xi_2},$$

see e.g. [177, pp.66-68]. In this way, the operator L is transformed into an operator $\tilde{L}$ where Lemma 6.5 becomes applicable. We check that $\tilde{L}$ remains "close" to $(-\Delta)^m$ and $\Omega^\natural$ "close" to B, if the same holds for L and Ω, respectively. The new coordinates $\xi_1 = \varphi(x_1,x_2)$, $\xi_2 = \psi(x_1,x_2)$ satisfy the Beltrami system in Ω, namely

$$\frac{\partial \varphi}{\partial x_1} = \frac{\tilde{a}_{21}\psi_{x_1} + \tilde{a}_{22}\psi_{x_2}}{\sqrt{\tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}^2}}, \quad \frac{\partial \varphi}{\partial x_2} = -\frac{\tilde{a}_{11}\psi_{x_1} + \tilde{a}_{12}\psi_{x_2}}{\sqrt{\tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}^2}}. \tag{6.8}$$

Assume that we have already found a bijective, at least twice differentiable transformation

$$\Phi = (\varphi,\psi) : \Omega \to \Omega^\natural. \tag{6.9}$$

Then as in [177] one finds that

$$L_0 u = \left(\tilde{L}_0 v\right) \circ \Phi, \tag{6.10}$$

where we put

$$\begin{cases} v(\xi_1,\xi_2) &= u \circ \Phi^{-1}(\xi_1,\xi_2), \\ A\left(\Phi(x)\right) &= \tilde{a}_{11}(x)\varphi_{x_1}^2 + 2\tilde{a}_{12}(x)\varphi_{x_1}\varphi_{x_2} + \tilde{a}_{22}(x)\varphi_{x_2}^2 \\ &= \tilde{a}_{11}(x)\psi_{x_1}^2 + 2\tilde{a}_{12}(x)\psi_{x_1}\psi_{x_2} + \tilde{a}_{22}(x)\psi_{x_2}^2 > 0, \\ B_1\left(\Phi(x)\right) &= \tilde{a}_{11}(x)\varphi_{x_1x_1} + 2\tilde{a}_{12}(x)\varphi_{x_1x_2} + \tilde{a}_{22}(x)\varphi_{x_2x_2}, \\ B_2\left(\Phi(x)\right) &= \tilde{a}_{11}(x)\psi_{x_1x_1} + 2\tilde{a}_{12}(x)\psi_{x_1x_2} + \tilde{a}_{22}(x)\psi_{x_2x_2}. \end{cases} \tag{6.11}$$

We determine ψ as solution of the boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial x_1}\left(\dfrac{\tilde{a}_{11}\psi_{x_1} + \tilde{a}_{12}\psi_{x_2}}{\sqrt{\tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}^2}}\right) + \dfrac{\partial}{\partial x_2}\left(\dfrac{\tilde{a}_{21}\psi_{x_1} + \tilde{a}_{22}\psi_{x_2}}{\sqrt{\tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}^2}}\right) = 0 & \text{in } \Omega, \\ \psi(x) = x_2 & \text{on } \partial\Omega, \end{cases} \tag{6.12}$$

and then construct φ with the help of the Beltrami equations (6.8) and the normalisation $\varphi(0) = 0$.

In this special situation, the required results for the transformation Φ can be easily proved directly.

Lemma 6.7. *Let $\varepsilon_2 > 0$. Then there exists $\varepsilon_3 = \varepsilon_3(m,\varepsilon_2)$ such that for $0 \le \varepsilon \le \varepsilon_3$ the following holds true.*

Assume that the domain Ω is $C^{2m,\gamma}$-smooth and ε-close to B in C^{2m}-sense. Let the operator L of (6.3) be ε-close to $(-\Delta)^m$ in $C^{2m-1,\gamma}$-sense. Then we have for the transformation $\Phi : \Omega \to \Omega^\natural = \Phi(\Omega)$ defined in (6.8), (6.9) and (6.12), that

- *Φ is bijective, $\Phi \in C^{2m,\gamma}(\overline{\Omega})$, $\Phi^{-1} \in C^{2m,\gamma}(\overline{\Omega^\natural})$,*
- *$\Omega^\natural$ is ε_2-close to B in C^{2m}-sense.*

Putting $v := u \circ \Phi^{-1}$, see (6.11), the boundary value problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ D^\alpha u|_{\partial\Omega} = 0 & \text{for } |\alpha| \le m-1, \end{cases}$$

is transformed into

$$\begin{cases} \hat{L}v = A^{-m}\,(f \circ \Phi^{-1}) & \text{in } \Omega^\natural, \\ D^\alpha v|_{\partial\Omega^\natural} = 0 & \text{for } |\alpha| \le m-1. \end{cases}$$

Here $\hat{L}v = (-\Delta)^m v + \sum_{|\beta|\le 2m-1} \hat{a}_\beta(\,.\,)D^\beta v$ with suitable coefficients $\hat{a}_\beta \in C^{0,\gamma}(\overline{\Omega^\natural})$ such that for all $|\beta| \le 2m-1$ the smallness condition

$$\|\hat{a}_\beta\|_{C^0(\overline{\Omega^\natural})} \le \varepsilon_2$$

is satisfied.

Proof. We may assume ε to be sufficiently small and in particular Ω to be bounded and uniformly convex. First we consider the boundary value problem (6.12), which is uniformly elliptic thanks to $\|\tilde{a}_{ij} - \delta_{ij}\|_{C^{2m-1,\gamma}(\overline{\Omega})} \le \varepsilon$ with coefficients in the space $C^{2m-1,\gamma}(\overline{\Omega})$. Since Ω is $C^{2m,\gamma}$-smooth, elliptic theory (see Theorem 2.19) shows the existence of a solution $\psi \in C^{2m,\gamma}(\overline{\Omega})$ to (6.12). At the same time, this differential equation is the integrability condition for (6.8) in the convex domain Ω. This shows that a solution $\varphi \in C^{2m,\gamma}(\overline{\Omega})$ of the Beltrami system (6.8) with $\varphi(0) = 0$ exists.

Next we investigate $\Phi = (\varphi, \psi)$ quantitatively. For this purpose we consider the auxiliary function $\Psi(x) := \psi(x) - x_2$ that solves the boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial x_1}\left(\dfrac{\tilde{a}_{11}\Psi_{x_1} + \tilde{a}_{12}\Psi_{x_2}}{\sqrt{\tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}^2}}\right) + \dfrac{\partial}{\partial x_2}\left(\dfrac{\tilde{a}_{21}\Psi_{x_1} + \tilde{a}_{22}\Psi_{x_2}}{\sqrt{\tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}^2}}\right) \\ \quad = -\dfrac{\partial}{\partial x_1}\left(\dfrac{\tilde{a}_{12}}{\sqrt{\tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}^2}}\right) - \dfrac{\partial}{\partial x_2}\left(\dfrac{\tilde{a}_{22}}{\sqrt{\tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}^2}}\right) =: F(x_1, x_2) \text{ in } \Omega, \\ \Psi|_{\partial\Omega} = 0, \end{cases} \tag{6.13}$$

where

$$F = O(\varepsilon) \text{ in } C^{2m-2,\gamma}(\overline{\Omega}).$$

Schauder estimates for higher order norms as in Theorem 2.19 yield

$$\begin{aligned}\|\psi - x_2\|_{C^{2m,\gamma}(\overline{\Omega})} = \|\Psi\|_{C^{2m,\gamma}(\overline{\Omega})} \le C\|F\|_{C^{2m-2,\gamma}(\overline{\Omega})},\\ \|\psi - x_2\|_{C^{2m}(\overline{\Omega})} \le C\|F\|_{C^{2m-2,\gamma}(\overline{\Omega})}.\end{aligned} \tag{6.14}$$

Here one should observe that the $C^{2m-1,\gamma}(\overline{\Omega})$-norms of the coefficients in (6.13) are bounded independently of ε; the ellipticity constants are uniformly close to 1. Finally, by means of the uniform C^{2m}-closeness of the domains to the disk B we may choose an estimation constant in (6.14) being independent of Ω. Taking also (6.8) into account we conclude that

$$\|\Phi - Id\|_{C^{2m}(\overline{\Omega})} = O(\varepsilon), \tag{6.15}$$

thereby proving the bijectivity of Φ, the qualitative statements on Φ^{-1} and $\Omega^\natural = \Phi(\Omega)$, as well as

$$\|\Phi^{-1} - Id\|_{C^{2m}(\overline{\Omega^\natural})} = O(\varepsilon). \tag{6.16}$$

We still have to study the properties of the transformed differential operator $\hat{L}$. From (6.10) it follows that

$$\begin{aligned} Lu &= L_0^m u + \sum_{|\beta|\le 2m-1} a_\beta D^\beta u \\ &= \left\{ \tilde{L}_0^m v + \sum_{|\beta|\le 2m-1} \left(a_\beta \circ \Phi^{-1}\right) \left(D^\beta (v\circ\Phi)\right)\circ\Phi^{-1} \right\}\circ\Phi \\ &= \left\{ \tilde{L}_0^m v + \sum_{|\beta|\le 2m-1} \tilde{a}_\beta D^\beta v \right\}\circ\Phi =: \left(A^m\, \hat{L}v\right)\circ\Phi. \end{aligned}$$

Here the new coefficients $\tilde{a}_\beta$ contain additional derivatives of Φ of order at most $(2m-1)$ and hence $\|\tilde{a}_\beta\|_{C^0(\overline{\Omega^\natural})} = O(\varepsilon)$. Finally, $\tilde{L}_0 v = -A\Delta v - B_1 \frac{\partial v}{\partial \xi_1} - B_2 \frac{\partial v}{\partial \xi_2}$, so we still need to show that

$$\|A-1\|_{C^{2m-2}(\overline{\Omega^\natural})} = O(\varepsilon), \qquad \|B_j\|_{C^{2m-2}(\overline{\Omega^\natural})} = O(\varepsilon).$$

Observing the definition (6.11) of A, B_1, B_2, this follows from the properties (6.15) and (6.16) of Φ and the assumptions on the coefficients $\tilde{a}_{ij}$. □

Proof of Theorem 6.3. It follows by combining Lemmas 6.5-6.7. □

Remark 6.8. Similarly as in Section 5.2, also here one has results on the qualitative boundary behaviour of solutions. Using the theory of maps described above and referring to Theorem 5.7 instead of Theorem 5.4, the claim of Theorem 5.7 on the m-th order boundary derivatives of the solution remains true also under the assumptions of Theorem 6.3, while the Dirichlet boundary data have to be prescribed homogeneously.

On the other hand, if one wants to study the influence of the sign of $D^{m-1}u|_{\partial\Omega}$ on the sign of the solution in Ω, while the first $(m-2)$ derivatives on $\partial\Omega$ are prescribed homogeneously, one has to ensure that the assumptions of Theorem 6.3 are satisfied by the (formally) adjoint operator L^*. This means that if we assume that Ω is close to B in C^{2m}-sense, L close to $(-\Delta)^m$ in C^{2m}-sense, $a_\beta \in C^{|\beta|}(\overline{\Omega})$ and $\|a_\beta\|_{C^{|\beta|}(\overline{\Omega})}$ small, then the conclusions of Theorem 5.6 remain true.

Our methods are not suitable to carry over further statements of Section 5.2 on the influence of $u|_{\partial B}$ on the sign of u in B to the situation of Theorem 6.3. This is because in the relevant result Theorem 5.15, only perturbations of order $(m-2)$ can be treated, while terms of order $(m-1)$ may indeed arise. However, in the special case of the polyharmonic operator, using Sassone's paper [358], we expect that a positivity result with respect to the two highest order boundary data may also be shown in domains Ω being a sufficiently small perturbation of the disk.

6.2 Small negative part of biharmonic Green's functions in two dimensions

We come back to the question raised in Section 1.2 whether the negative part of the biharmonic Green function is small in a suitable sense when compared with its positive part. In two dimensions, we have a family of domains – among which are even nonconvex ones – with positive Green's functions. These limaçons de Pascal are discussed first and serve as a basis to give a first answer to the question just mentioned.

6.2.1 The biharmonic Green function on the limaçons de Pascal

Lemma 6.5 does not supply us with a reasonable bound for the perturbation in order to have a positive Green function. Hadamard found an explicit formula for the biharmonic Green function on any limaçon. As already mentioned in Section 1.2 he claimed in [222] that these Green functions were all positive. Although this claim is wrong, his formula allowed Dall'Acqua and Sweers [120] to show that the Green functions for a sufficiently large class of limaçons are indeed positive. We define the filled limaçon by

$$\Omega_a = (-a,0) + \left\{(\rho\cos\varphi, \rho\sin\varphi) \in \mathbb{R}^2;\ 0 \le \rho < 1 + 2a\cos\varphi\right\}. \tag{6.17}$$

For $a \in \left[0, \frac{1}{2}\right]$ the boundary is defined by $\rho = 1 + 2a\cos\varphi$; for $a = 0$ it is the unit circle and for $a = \frac{1}{2}$ one finds the cardioid. In Figure 1.2 in Section 1.2, images are shown of these limaçons which are rotated by $\frac{1}{2}\pi$.

Proposition 6.9. *The biharmonic Green function for Ω_a with $a \in \left[0, \frac{1}{2}\right]$ is positive if and only if $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$.*

Before proving this result we fix some preliminaries that will give us additional information on what happens when positivity breaks down. To do so, we fix the conformal map $h_a : B = \Omega_0 \to \Omega_a$ that maps the unit disk on the perturbed domains and that keeps the horizontal axis on the horizontal axis. In complex coordinates it is defined as follows

$$h_a(\eta) = \eta + a\eta^2.$$

The Green function on Ω_a is defined through the coordinates on B; see Figure 6.1 for these curvilinear coordinates. We remark that according to Loewner [278] the only conformal maps which leave the biharmonic equation invariant are the Möbius transforms.

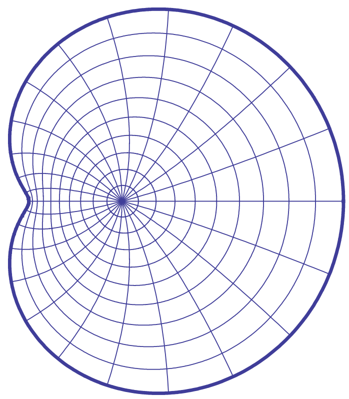

Fig. 6.1 Transformed polar coordinates corresponding to h_a.

Let us write

$$h_a(\eta) = x_1 + ix_2 \quad \text{and} \quad h_a(\xi) = y_1 + iy_2$$

and

$$r = |\eta - \xi|, \quad R = \left|1 - \eta\bar{\xi}\right| \quad \text{and} \quad s = \left|\eta + \xi + \frac{1}{a}\right|.$$

The formula Hadamard presents in [222] using these coordinates is

$$G_{\Omega_a}(x,y) = \tfrac{1}{16\pi} a^2 s^2 r^2 \left(\frac{R^2}{r^2} - 1 - \log\left(\frac{R^2}{r^2}\right) - \frac{a^2}{1-2a^2}\frac{R^2}{s^2}\frac{r^2}{R^2}\left(\frac{R^2}{r^2} - 1\right)^2\right).$$

To verify that this is indeed the biharmonic Green function for Ω_a requires some tedious calculations which can be found in [120]. By setting

$$F(\beta, q) = q - 1 - \log q - \beta \frac{(q-1)^2}{q} \tag{6.18}$$

one obtains that