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WAVE EQUATIONS WITH MEMORY: THE MINIMAL STATE APPROACH

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Abstract. Recently, in [6, 12], a new theoretical scheme has been developed in order to study equations with memory, the so–called minimal state approach. The aim of this work is to provide the technical body needed to study the asymptotic behavior of semilinear integrodifferential equations of hyperbolic type in the novel framework.

1. Introduction

In many physical phenomena (e.g. viscoelasticity, heat flow in real conductors, population dynamics, phase separation) the actual evolution of the system is influenced by the past values of one or more variables in play. A correct modeling of this phenomenon naturally leads to differential equations, the so-called *equations with memory*, where a memory term arises as the time convolution of the unknown function against a suitable memory kernel. The nonlocal character of such models represents an intrinsic difficulty in the analysis of equations with memory, that have been poorly understood for many decades. Nowadays we know that an effective way to circumvent this difficulty is trying to translate the integro–differential problem into an ordinary differential equation generating a dynamical system on some abstract space, where one can exploit the powerful toolbox of semigroups theory.

In the literature, this strategy traces back to C. Dafermos in the seventies [9] and constitutes the core of the classical *history approach*. It is based on the introduction of an auxiliary variable, ruled out by its own equation, which contains all the information about the unknown function up to the actual time, its *past history*. In recent years, an alternative scheme has been proposed in [12] to investigate equations with memory, the so-called *minimal state approach*. The introduction of the new theoretical scheme is motivated by an objection raised in Dafermos' framework, where it might happen that two different past histories lead to the same solution, hence they are indistinguishable from the point of view of the dynamics. As an attempt to overcome this weakness, in the state approach a different additional variable, rather then the past history, is employed to describe the initial state of the system. This is based on the novel notion of *minimal state*,

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which satisfies the desired property that different initial states entail different evolutions (see also [10, 11]).

Many progresses in the analysis of equations with memory have been made thanks to Dafermos' scheme, see e.g. [1, 2, 4, 7, 8, 14, 15] just to mention some recent contributions. Then, a natural point in order to investigate the effectiveness of the minimal state approach is trying to prove corresponding results within the novel framework.

A first contribution in this direction is given by [6], where an asymptotic theory for the nonlinear model of viscoelasticity (see (2.1) below) is developed. This is based on earlier contributions in the past history framework [7, 14] where the asymptotic properties of the semigroup $\hat{S}(t)$ generated by (2.1) in Dafermos' scheme has been analyzed in full details. Indeed, [6] investigates the relationship between $\hat{S}(t)$ and the corresponding semigroup $S(t)$ acting on a new extended phase space $\mathcal{H}$ according to the state approach. As a consequence, leaning on the existence of the global attractor for $\hat{S}(t)$, the authors obtain by comparison the existence of a regular global attractor for $S(t)$ in the new scheme.

The goal of this work is to keep further the development of the minimal state approach by providing the technical body which is needed to handle equations with memory in the novel abstract framework, without going through the history approach.

In this paper we discuss two of the main ingredients which allow to exploit the machinery of dynamical systems in the new extended phase space, namely, a general compactness result directly applicable to its subsets, and a family of auxiliary functionals suitable to recover energy estimates for the semigroup. As an application we furnish a direct proof of the existence of a regular attractor for $S(t)$, but the tools here devised are quite general and suitable to be applied and adapted to a large variety of models.

1.1. **Plan of the paper.** We first present the hyperbolic nonlinear model with memory under investigation. Then, after stating the general assumptions on the nonlinearities involved in the equation and on the memory kernel, in Section 3 we recall the abstract functional setting needed to treat the model in the minimal state framework and the main results concerning with the asymptotic behavior of $S(t)$. Section 4 provides a general compactness theorem for a class of functional spaces including $\mathcal{H}$. The subsequent Section 5 is devoted to construct suitable energy functionals; some of the proofs are postponed in the Appendix at the end of the paper. In Section 6 we finally show how to exploit the whole machinery to provide a direct proof of the existence of a global attractor of optimal regularity for $S(t)$.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. We consider the strictly positive operator $A = -\Delta$ acting on $L^2(\Omega)$ with domain $\mathrm{dom}(A) = H^2(\Omega) \cap H^1_0(\Omega)$. For $r \in \mathbb{R}$, we define the scale of compactly nested Hilbert spaces

$$\mathrm{H}^r = \mathrm{dom}(A^{r/2}), \quad \langle u, v\rangle_r = \langle A^{r/2}u, A^{r/2}v\rangle_{L^2(\Omega)}, \quad \|u\|_r = \|A^{r/2}u\|_{L^2(\Omega)}.$$

We will always omit the index r whenever $r = 0$. The symbol $\langle\cdot,\cdot\rangle$ will also stand for the duality product between H^r and its dual space H^{-r}. We recall the relations

$$\mathrm{H}^{-1} = H^{-1}(\Omega), \quad \mathrm{H} = L^2(\Omega), \quad \mathrm{H}^1 = H^1_0(\Omega), \quad \mathrm{H}^2 = H^2(\Omega) \cap H^1_0(\Omega),$$

along with the generalized Poincaré inequalities

$$\sqrt{\lambda_1}\,\|u\|_r \leq \|u\|_{r+1}, \quad \forall u \in \mathrm{H}^{r+1},$$

where $\lambda_1 > 0$ is the first eigenvalue of A.

The model. We consider the hyperbolic equation with memory arising in the theory of isothermal viscoelasticity [13, 19]

$$\ddot{u} + A\Big[\alpha u - \int_0^\infty \mu(s)u(t-s)\mathrm{d}s\Big] + g(u) = f, \tag{2.1}$$

where $u = u(\boldsymbol{x}, t) : \Omega \times \mathbb{R} \to \mathbb{R}$ represents the displacement of an elastic body occupying the region Ω. Here, g is a nonlinear smooth function, whose typical form is an odd polynomial with positive leading coefficient (see (2.6)–(2.7) below), f is an external forcing and α a positive constant.
The boundary-value problem (2.1) is supplemented with the initial conditions

$$u(0) = u_0, \quad \partial_t u(0) = v_0, \tag{2.2}$$

where u_0, v_0 are prescribed data. Besides, in order to compute the convolution term in (2.1), it is usually assumed the knowledge of the values of u for all past times, namely

$$u(-s)|_{s>0} = \phi_0(s), \tag{2.3}$$

where the *past history* function ϕ_0 on $\mathbb{R}^+$ is a given datum. We shall return on this point later.

Calling

$$F_0(t) = \int_0^\infty \mu(t+s)\phi_0(s)ds,$$

equation (2.1) can be rewritten as

$$\ddot{u} + A\Big[\alpha u - \int_0^t \mu(s)u(t-s)\mathrm{d}s - F_0(t)\Big] + g(u) = f. \tag{2.4}$$

Accordingly, given $(u_0, v_0) \in \mathrm{H}^1 \times \mathrm{H}$, and $\phi_0 : \mathbb{R}^+ \to \mathrm{H}^1$ such that $F_0(t) \in \mathrm{H}^1$ for a.e. $t > 0$, we say that a function $u \in \mathcal{C}([0,\infty), \mathrm{H}^1) \cap \mathcal{C}^1([0,\infty), \mathrm{H})$ is a weak solution to (2.1)–(2.3) if $u(0) = u_0$, $\dot{u}(0) = v_0$ and, for every $w \in \mathrm{H}^1$ and a.e. $t > 0$,

$$\langle \ddot{u}(t), w\rangle + \alpha\langle u(t), w\rangle_1 - \int_0^t \mu(s)\langle u(t-s), w\rangle_1 \mathrm{d}s - \langle F_0(t), w\rangle_1 + \langle g(u(t)), w\rangle = \langle f, w\rangle.$$

Assumptions on μ. The memory kernel μ is supposed to be a (nonnegative) nonincreasing and summable function on $\mathbb{R}^+ = (0,\infty)$, with total mass

$$\int_0^\infty \mu(s)\mathrm{d}s \in (0, \alpha),$$

mapping nullsets into nullsets. In order to simplify the discussion, we assume that the discontinuity points of μ, if any, form an increasing sequence $\{\sigma_n\}$. In particular, μ is piecewise absolutely continuous, and thus differentiable almost everywhere with $\mu' \leq$

0, albeit possibly unbounded about zero. Without loss of generality, we take μ right-continuous, and we denote the jump amplitudes at the (left) discontinuity points σ_n by

$$\mu_n = \mu(\sigma_n^-) - \mu(\sigma_n) > 0,$$

so that

$$\mu(s) = -\int_0^\infty \mu'(s+\sigma)\mathrm{d}\sigma + \sum_{s<\sigma_n} \mu_n. \tag{2.5}$$

Besides, we set

$$m(\tau) = \int_\tau^\infty \mu(s)\mathrm{d}s, \quad \tau \geq 0.$$

For simplicity, we agree to put $\alpha - m(0) = 1$.

Assumptions on g and f. The external force $f = f(\boldsymbol{x})$ belongs to $L^2(\Omega)$, while the nonlinearity $g \in \mathcal{C}^1(\mathbb{R})$, with $g(0) = 0$, fulfills the growth and dissipativity assumptions

$$|g'(u) - g'(v)| \leq c\big|u - v|(1 + |u| + |v|\big), \tag{2.6}$$

$$\liminf_{|u|\to\infty} \frac{g(u)}{u} > -\lambda_1. \tag{2.7}$$

3. A Dynamical System in the Minimal State Space Framework

In this section we collect the main definitions and basic results contained in [6, 12], concerning with the minimal state framework.

3.1. **The functional setting.** We define the new memory kernel

$$\nu(\tau) = \begin{cases} 1/\mu(\tau), & \tau \in (0, s_\infty), \\ \lim\limits_{s\to 0} 1/\mu(s), & \tau = 0, \end{cases}$$

where

$$s_\infty = \sup\big\{s \in \mathbb{R}^+ : \mu(s) > 0\big\}.$$

To provide a unitary picture for finite delay ($s_\infty < \infty$) and infinite delay ($s_\infty = \infty$), given any function $h = h(\tau)$, we agree to put $h(\tau) = 0$ whenever $s_\infty < \tau < \infty$.
Notice that, In view of the assumptions on μ, the kernel ν is nondecreasing and piecewise absolutely continuous, with nonnegative derivative (defined a.e.)

$$\nu'(\tau) = -\mu'(\tau)/[\mu(\tau)]^2.$$

- For $r \in \mathbb{R}$, we define the *state space* (again, r is omitted if $r = 0$)

$$\mathcal{S}^r = L^2_\nu(\mathbb{R}^+, \mathrm{H}^{r+1}),$$

namely, the space of L^2-functions on $\mathbb{R}^+$ with values in H^{r+1} with respect to the measure $\nu(\tau)\mathrm{d}\tau$, which is a Hilbert space endowed with inner product

$$\langle \xi_1, \xi_2\rangle_{\mathcal{S}^r} = \int_0^\infty \nu(\tau)\langle \xi_1(\tau), \xi_2(\tau)\rangle_{r+1}\mathrm{d}\tau.$$

Note that, if $\xi \in \mathcal{S}^r$, then $\xi \in L^1(\mathbb{R}^+, \mathrm{H}^{r+1})$ and

$$\int_0^\infty \|\xi(\tau)\|_{r+1}\mathrm{d}\tau = \int_0^\infty \sqrt{\mu(\tau)}\,\sqrt{\nu(\tau)}\,\|\xi(\tau)\|_{r+1}\mathrm{d}\tau \leq \sqrt{m(0)}\,\|\xi\|_{\mathcal{S}^r}. \tag{3.1}$$

• We denote by

$$P : \mathrm{dom}(P) \subset \mathcal{S} \to \mathcal{S}$$

the infinitesimal generator of the strongly continuous semigroup of left translations on $\mathcal{S}$, namely,

$$P\xi = D\xi, \quad \mathrm{dom}(P) = \big\{\xi \in \mathcal{S} : D\xi \in \mathcal{S}\big\}$$

with D the distributional derivative. Note that, due to (3.1), there hold

$$\mathrm{dom}(P) \subset W^{1,1}(\mathbb{R}^+, \mathrm{H}^1) \subset \mathcal{C}_0([0,\infty), \mathrm{H}^1), \tag{3.2}$$

where $\mathcal{C}_0$ is the space of continuous functions vanishing at infinity. Hence we have that $\|\xi(\tau)\|_1 \to 0$ as $\tau \to s_\infty$, whenever $\xi \in \mathrm{dom}(P)$.

If $\xi \in \mathrm{dom}(P)$, we have the relation

$$2\langle P\xi, \xi\rangle_{\mathcal{S}} = -\int_0^\infty \nu'(\tau)\|\xi(\tau)\|_1^2\mathrm{d}\tau - \nu(0)\|\xi(0)\|_1^2 - \sum \nu_n\|\xi(\sigma_n)\|_1^2 \leq 0, \tag{3.3}$$

where $\nu_n = \nu(\sigma_n) - \nu(\sigma_n^-) > 0$.

• We finally define the *extended state spaces* as the product Hilbert spaces

$$\mathcal{H}^r = \mathrm{H}^{r+1} \times \mathrm{H}^r \times \mathcal{S}^r.$$

normed by

$$\|(u, v, \eta)\|_{\mathcal{H}^r}^2 = \|u\|_{\mathrm{H}^{r+1}}^2 + \|v\|_{\mathrm{H}^r}^2 + \|\xi\|_{\mathcal{S}^r}^2.$$

In particular,

$$\mathcal{H} = \mathrm{H}^1 \times \mathrm{H} \times \mathcal{S},$$

will be the new phase–space where we shall reformulate the original problem (2.1).

3.2. **The equation in the state framework.** We consider, for $t > 0$, the system of two variables $u = u(t)$ and $\xi = \xi^t(\tau)$

$$\begin{cases} \ddot{u} + A\Big[u + \displaystyle\int_0^\infty \xi(\tau)\mathrm{d}\tau\Big] + g(u) = f, \\ \dot{\xi} = P\xi + \mu\dot{u}, \end{cases} \tag{3.4}$$

where ξ represents the so–called *minimal state variable.* It is said to be *minimal* in the sense that the knowledge of $u(t)$ for all $t \geq 0$ uniquely determines ξ^t, see [6, Remark 4.4].

As shown in [6, 12], system (3.4) generates a strongly continuous semigroup of solutions $S(t) : \mathcal{H} \to \mathcal{H}$. Thus, for every $t \geq 0$ and every $z = (u_0, v_0, \xi_0) \in \mathcal{H}$,

$$S(t)z = (u(t), \dot{u}(t), \xi^t)$$

is the unique weak solution at time t to (3.4) with initial datum z, whose third component fulfills the representation formula

$$\xi^t(\tau) = \xi_0(t + \tau) + \int_0^t \mu(\tau + s)\dot{u}(t - s)\mathrm{d}s. \tag{3.5}$$

An integration by parts along with (2.5) yields the equivalent relation

$$\begin{aligned}\xi^t(\tau) &= \xi_0(t+\tau) + \mu(\tau)u(t) - \mu(t+\tau)u_0 \\ &\quad + \int_0^t \mu'(\tau+s)u(t-s)\mathrm{d}s - \sum_{\tau<\sigma_n\leq t+\tau} \mu_n u(t+\tau-\sigma_n),\end{aligned} \tag{3.6}$$

see [6, Remark 4.3]. The correspondence between the new system (3.4) and the original problem (2.1) is given in the following proposition (see [6, Proposition 5.3]), stating that the system constitutes the correct reformulation of (2.1) in the state framework.

Proposition 3.1. *Let* $(u_0, v_0) \in \mathrm{H}^1 \times \mathrm{H}$ *and let* $\phi_0 : \mathbb{R}^+ \to \mathrm{H}^1$ *be such that*

$$F_0(t) = \int_0^\infty \mu(t+s)\phi_0(s)\mathrm{d}s$$

belongs to H^1 *for a.e.* $t > 0$. *Assume in addition that* $DF_0 \in \mathcal{S}$. *A function* u *is a weak solution to* (2.1) *with initial conditions* (u_0, v_0, F_0) *if and only if*

$$(u(t), \dot{u}(t), \xi^t) = S(t)(u_0, v_0, \xi_0),$$

where ξ^t *is given by* (3.5) *and* $\xi_0 = \mu u_0 + DF_0$.

Accordingly, we name F_0 *state function*, and we shall interpret F_0, rather then ϕ_0, as the correct initial datum accounting for the past evolution of u, so identifying all the initial past histories ϕ_0 leading to the same solution. Indeed, it is apparent from (2.4) that F_0 contains all the information needed to capture the future dynamics of the system.

From now on we shall restrict our attention to initial state functions F_0 with $DF_0 \in \mathcal{S}$, so that the solutions of the original equation are in correspondence with the first component of the semigroup $S(t)$ on $\mathcal{H}$. In particular, having well-posedness in the extended state space gives an existence and uniqueness result for (2.1). Besides, the asymptotic behavior of its solutions is described by the long term dynamics of $S(t)$.

3.3. **Asymptotic behavior.** The long term dynamics of a dissipative semigroup is well described by the so–called *global attractor*. We recall that this is the unique compact set $\mathfrak{A} \subset \mathcal{H}$ fully invariant and attracting for the semigroup (see e.g. [3, 16, 20]). Namely, $S(t)\mathfrak{A} = \mathfrak{A}$, for every $t \geq 0$, and

$$\lim_{t\to\infty} \mathrm{dist}_{\mathcal{H}}(S(t)\mathfrak{B}, \mathfrak{A}) = 0,$$

for every bounded set $\mathfrak{B} \subset \mathcal{H}$, where $\mathrm{dist}_{\mathcal{H}}$ is the usual Hausdorff semidistance in $\mathcal{H}$. In this respect, the main result concerning with the semigroup associated with (2.1) is the following ([6, Section 7]):

Theorem 3.2. *Assume that* μ *satisfies the further conditions*

$$\mu(\tau+s) \leq \Theta \mathrm{e}^{-\theta\tau}\mu(s) \quad \textit{and} \quad \mu'(s) < 0, \tag{3.7}$$

for some $\Theta \geq 1$ *and* $\theta > 0$, *every* $\tau \geq 0$ *and (almost) every* $s \in (0, s_\infty)$. *Then,* $S(t)$ *possesses a global attractor* $\mathfrak{A}$ *bounded in* $\mathcal{H}^1$.

The next sections will be devoted to provide a direct proof of this result.

4. A compactness result

In order to prove the existence of a global attractor for $S(t)$ some compactness tools are needed. Notice that, although the inclusion $\mathrm{H}^{r+1} \subset \mathrm{H}$ is compact for all $r > 0$, the injection $\mathcal{S}^r \subset \mathcal{S}$ is not compact in general (cf. [18] for a counterexample). The aim of this section is to provide an abstract compactness result for subsets of $\mathcal{S}$.

For $i = -1, 0, 1$, let $Y^i = \mathrm{H}^{r_i}$ for some $r_i \in \mathbb{R}$ satisfying

$$Y^1 \Subset Y^0 \subset Y^{-1}$$

(where $\Subset$ stands for a compact embedding). Let us recall that all the above embeddings are dense and continuous and that the following interpolation inequality holds

$$\|y\|_{Y^0} \leq k_0 \|y\|^{\theta}_{Y^{-1}} \|y\|^{1-\theta}_{Y^1} \qquad \forall\, y \in Y^1, \tag{4.1}$$

for some $k_0 > 0$ and $\theta \in [0, 1)$,depending on r_i. Finally, set

$$\mathcal{Y}^i = L^2_\nu(\mathbb{R}^+, Y^i)$$

and define the Banach space

$$\mathcal{T} = \Big\{\xi \in \mathcal{Y}^1 : D\xi \in \mathcal{Y}^{-1}\Big\}, \qquad \|\xi\|^2_{\mathcal{T}} = \|\xi\|^2_{\mathcal{Y}^1} + \|D\xi\|^2_{\mathcal{Y}^{-1}}.$$

The following compactness result holds.

Proposition 4.1. *Let $\mathcal{K} \subset \mathcal{T}$ be bounded and such that*

$$\sup_{\xi \in \mathcal{K}} \|\xi(s)\|^2_{Y^{-1}} \leq f(s), \qquad a.e.\ s \in \mathbb{R}^+$$

for some function $f \in L^1_\nu(\mathbb{R}^+)$. Then $\mathcal{K}$ is precompact in $\mathcal{Y}^0$.

Proof. By assumption, $\mathcal{K}$ is bounded in $L^2_\nu([S_0, S_1], Y^1) \cap H^1_\nu([S_0, S_1], Y^{-1})$, for any $0 < S_0 < S_1 < s_\infty$, with bound independent of S_0, S_1. Set

$$\phi(s) = \int_0^s \nu(\tau)\,\mathrm{d}\tau, \qquad \text{for } s \in [0, s_\infty).$$

Let $\psi = \phi^{-1}$ defined on $\mathbb{R}^+$, and consider the set

$$\mathcal{K}_\psi = \{\xi \circ \psi : \xi \in \mathcal{K}\}.$$

For any $\xi \circ \psi \in \mathcal{K}_\psi$ there holds

$$\|\xi \circ \psi\|^2_{L^2([\phi(S_0), \phi(S_1)], Y^1)} = \int_{S_0}^{S_1} \nu(s) \|\xi(s)\|^2_{Y^1} ds,$$

and

$$\begin{aligned}\|(\xi \circ \psi)'\|^2_{L^2([\phi(S_0), \phi(S_1)], Y^{-1})} &= \int_{\phi(S_0)}^{\phi(S_1)} |\psi'(s)|^2 \|D\xi(\psi(s))\|^2_{Y^{-1}} ds \\ &\leq \frac{1}{(\nu(S_0))^2} \int_{S_0}^{S_1} \nu(s) \|D\xi(s)\|^2_{Y^{-1}} ds.\end{aligned}$$

Hence, $\mathcal{K}_\psi$ is bounded in

$$L^2([\phi(S_0), \phi(S_1)], Y^1) \cap H^1([\phi(S_0), \phi(S_1)], Y^{-1}) \hookrightarrow L^2([\phi(S_0), \phi(S_1)], Y^0)$$

with compact embedding (see, e.g. [17, pag. 57]). Let ξ_n a sequence in $\mathcal{K}$. Then there exists $\eta \in L^2([\phi(S_0), \phi(S_1)], Y^0)$ such that, up to a subsequence,

$$\xi_n \circ \psi \to \eta \qquad \text{in } L^2([\phi(S_0), \phi(S_1)], Y^0).$$

On the other end the function $\xi = \eta \circ \phi \in L^2_\nu([S_0, S_1], Y^0)$, and the above convergence implies

$$\xi_n \to \xi \qquad \text{in } L^2_\nu([S_0, S_1], Y^0).$$

Using a classical diagonalization method, it is possible to find a subsequence, still denoted ξ_n, converging to some $\xi \in L^2_\nu([S_0, S_1], Y^0)$, for any $0 < S_0 < S_1 < s_\infty$. Since $\mathcal{K}$ is bounded in $L^2_\nu(\mathbb{R}^+, Y^1)$ and $\nu(s) = 0$ in (s_∞, ∞), then $\xi \in L^2_\nu(\mathbb{R}^+, Y^0)$. We are left to show that $\xi_n \to \xi$ in $L^2_\nu(\mathbb{R}^+, Y^0)$. To this aim, it is sufficient to show that

$$\lim_{S_0 \to 0,\ S_1 \to s_\infty} \bigg(\sup_{\xi \in \mathcal{K}} \Big[\int_0^{S_0} \nu(s) \|\xi(s)\|^2_{Y^0} ds + \int_{S_1}^{s_\infty} \nu(s) \|\xi(s)\|^2_{Y^0} ds \Big] \bigg) = 0. \tag{4.2}$$

Then, exploiting (4.1), for any measurable $E \subset \mathbb{R}^+$ and any $\xi \in \mathcal{K}$, we obtain

$$\begin{aligned}\int_E \nu(s) \|\xi(s)\|^2_{Y^0} ds &\le k_0^2 \Big(\int_E \nu(s) \|\xi(s)\|^2_{Y^1} ds \Big)^\theta \Big(\int_E \nu(s) \|\xi(s)\|^2_{Y^{-1}} ds \Big)^{1-\theta} \\ &\le C \Big(\int_E \nu(s) \|\xi(s)\|^2_{Y^{-1}} ds \Big)^{1-\theta}\end{aligned}$$

for some C dependent only on $\mathcal{K}$. This provides

$$\sup_{\xi \in \mathcal{K}} \int_E \nu(s) \|\xi(s)\|^2_{Y^0} ds \le C \Big(\int_E \nu(s) f(s) ds \Big)^{1-\theta}$$

which, since $f \in L^1_\nu(\mathbb{R}^+)$, proves (4.2). □

As a consequence of this abstract tool, we can prove the following compactness lemma which is particularly useful when dealing with solutions of differential systems with memory in the minimal state framework.

Lemma 4.2. *For $r > 0$, let $\mathcal{U}$ be bounded in $L^\infty(\mathbb{R}^+, \mathrm{H}^{r+1}) \cap W^{1,\infty}(\mathbb{R}^+, \mathrm{H}^r)$. Define*

$$\mathcal{K} = \textstyle\bigcup_{u \in \mathcal{U}} \bigcup_{t \ge 0} \xi,$$

where $\xi = \xi^t_u$ solves the equation $\dot{\xi} = P\xi + \mu \dot{u}$ with null initial datum. Then, $\mathcal{K}$ is precompact in $\mathcal{S}$.

Proof. We are going to show that $\mathcal{K}$ is bounded in $\mathcal{T} = \{\xi \in \mathcal{S}^r : D\xi \in \mathcal{S}^{-1}\}$ and that

$$\sup_{\xi \in \mathcal{K}} \|\xi(\tau)\|^2 \le C \mu^2(\tau), \qquad a.e.\ \tau \in \mathbb{R}^+. \tag{4.3}$$

Hence by direct application of Proposition 4.1 for the choice $Y^{-1} = \mathrm{H}$, $Y^0 = \mathrm{H}^1$ and $Y^1 = \mathrm{H}^{r+1}$, we will get that $\mathcal{K}$ is precompact in the corresponding $\mathcal{Y}^0 = \mathcal{S}$.

In what follows, $C \ge 0$ denotes a generic constant possibly depending on $\mathcal{U}$. For $\xi \in \mathcal{K}$, taking into account that $\xi^0 = 0$ and exploiting (2.5), it is easy to show that the representation formula (3.5) is equivalent to (see e.g. Lemma 2.1 in [6])

$$\xi^t(\tau) = \int_\tau^\infty \Big(- \int_0^t \mu'(y+s) \dot{u}(t-s) \mathrm{d}s + \sum_{y < \sigma_n \le t+y} \mu_n \dot{u}(t + y - \sigma_n) \Big) dy,$$

for some $u \in \mathcal{U}$ and $t \geq 0$. So, using (2.5) again, we find the inequality

$$\|D\xi(\tau)\|_r \leq C\mu(\tau).$$

In a similar manner, writing ξ as in (3.6) leads to

$$\|\xi(\tau)\|_{r+1} \leq C\mu(\tau).$$

This directly proves (4.3) and provides the required boundedness, since

$$\|\xi\|_{\mathcal{S}^r} + \|D\xi\|_{\mathcal{S}^{-1}} \leq \|\xi\|_{\mathcal{S}^r} + C\|D\xi\|_{\mathcal{S}^{r-1}} \leq C.$$

□

4.1. **Compact embeddings.** In this section, along the line of [15], we introduce a class of Banach spaces which are compactly embedded in $\mathcal{Y}^0$. To this aim, given $x > 0$, we define the tail function of $\xi \in \mathcal{Y}^{-1}$ as

$$\mathbb{T}_\xi(x) = \int_{I_x} \nu(\tau)\|\xi(\tau)\|^2_{Y^{-1}}\, d\tau, \qquad x \geq x_\infty,$$

where $x_\infty = \max\{1, 2/s_\infty\}$ and

$$I_x = \begin{cases} (0, \frac{1}{x}) \cup (s_\infty - \frac{1}{x}, s_\infty), & s_\infty < \infty, \\ (0, \frac{1}{x}) \cup (x, \infty), & s_\infty = \infty. \end{cases}$$

Given any increasing function $g : [x_\infty, \infty) \to \mathbb{R}^+$ such that $\lim_{x\to\infty} g(x) = \infty$, we define the Banach space

$$\mathcal{T}_g = \Big\{\xi \in \mathcal{Y}^1 : D\xi \in \mathcal{Y}^{-1},\ \sup_{x \geq x_\infty} g(x)\mathbb{T}_\xi(x) < \infty\Big\} \subset \mathcal{T} \subset \mathcal{Y}^0,$$

endowed with the norm

$$\|\xi\|^2_{\mathcal{T}_g} = \|\xi\|^2_{\mathcal{Y}^1} + \|D\xi\|^2_{\mathcal{Y}^{-1}} + \sup_{x \geq x_\infty} g(x)\mathbb{T}_\xi(x).$$

Then, there holds

Lemma 4.3. *The continuous embedding $\mathcal{T}_g \Subset \mathcal{Y}^0$ is compact. Besides, closed balls of $\mathcal{T}_g$ are compact in $\mathcal{Y}^0$.*

Proof. We have to show first that, given any bounded subset $\mathcal{K} \subset \mathcal{T}_g$, then $\mathcal{K}$ is precompact in $\mathcal{Y}^0$. This can be proven reasoning exactly as in the proof of Proposition 4.1, the only thing to show being the validity of (4.2) therein. To this aim, notice that, since $\mathcal{K}$ is bounded in $\mathcal{T}_g$ and $g(x)$ is unbounded as $x \to \infty$, then

$$\lim_{x\to\infty} \sup_{\xi \in \mathcal{K}} \mathbb{T}_\xi(x) = 0.$$

Hence (4.2) follows by

$$\begin{aligned} \sup_{\xi\in\mathcal{K}} \int_{I_x} \nu(s)\|\xi(s)\|^2_{Y^0} ds &\leq \sup_{\xi\in\mathcal{K}} k_0^2 \Big(\int_{I_x} \nu(s)\|\xi(s)\|^2_{Y^1} ds\Big)^\theta \Big(\int_{I_x} \nu(s)\|\xi(s)\|^2_{Y^{-1}} ds\Big)^{1-\theta} \\ &\leq C \sup_{\xi\in\mathcal{K}} \Big(\mathbb{T}_\xi(x)\Big)^{1-\theta} \end{aligned}$$

for some C only depending on $\mathcal{K}$.
To finish the proof, we have just to prove that closed balls of $\mathcal{T}_g$ are closed in $\mathcal{Y}^0$. Obviously, it is enough to consider balls centered at zero. Hence, given $\xi_n \in \mathcal{T}_g$ such that $\|\xi_n\|_{\mathcal{T}_g} \leq r$ for some $r > 0$ and $\xi_n \to \xi$ in $\mathcal{Y}^0$, we are left to prove that $\xi \in \mathcal{T}_g$ and $\|\xi\|_{\mathcal{T}_g} \leq r$. To this aim, notice first ξ_n is bounded in the reflexive Banach space $\mathcal{T}$ hence (up to a subsequence) ξ_n converges weakly to η in $\mathcal{T}$. In particular, the weak-lower semi continuity of the $\mathcal{T}$–norm ensures

$$\|\xi\|^2_{\mathcal{Y}^1} + \|D\xi\|^2_{\mathcal{Y}^{-1}} \leq \liminf_{n\to\infty} \big(\|\xi_n\|^2_{\mathcal{Y}^1} + \|D\xi_n\|^2_{\mathcal{Y}^{-1}}\big).$$

Furthermore, by the convergence in $\mathcal{Y}^0$ we get, for every fixed $x > x_\infty$,

$$g(x)\mathbb{T}_\xi(x) = \lim_{n\to\infty} g(x)\mathbb{T}_{\xi_n}(x) \leq \liminf_{n\to\infty}[\sup_{y>x_\infty} g(y)\mathbb{T}_{\xi_n}(y)], \tag{4.4}$$

which provides

$$\sup_{x>x_\infty} g(x)\mathbb{T}_\xi(x) \leq \liminf_{n\to\infty}[\sup_{x>x_\infty} g(x)\mathbb{T}_{\xi_n}(x)]. \tag{4.5}$$

Collecting (4.4) and (4.5) we conclude the proof. □

We conclude the section noticing that, by (4.3), the set $\mathcal{K}$ in Lemma 4.2 is bounded in $\mathcal{T}_g$ with $g(x) = (\int_{I_x} \mu(s)\mathrm{d}s)^{-1}$.

5. Some auxiliary functionals

We consider the family of nonhomogeneous linear systems

$$\begin{cases} \ddot{u} + A\Big[u + \displaystyle\int_0^\infty \xi(\tau)\mathrm{d}\tau\Big] + \gamma = 0, \\ \dot{\xi} = P\xi + \mu\dot{u}. \end{cases} \tag{5.1}$$

for some $\gamma = \gamma(t)$. Aim of this section is to construct suitable auxiliary functionals and to prove some differential inequalities holding for any sufficiently regular global solution to (5.1). The main result reads as follows:

Proposition 5.1. *For every $\varepsilon > 0$ small and every $r \in [0,1]$, there is a function $\Lambda^r_\varepsilon : \mathcal{H}^r \to \mathbb{R}$ such that*

$$\frac{1}{2}\|z\|^2_{\mathcal{H}^r} \leq \Lambda^r_\varepsilon(z) \leq \frac{3}{2}\|z\|^2_{\mathcal{H}^r}, \quad \forall z \in \mathcal{H}^r, \tag{5.2}$$

and the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda^r_\varepsilon(Z) + c_\varepsilon\Lambda^r_\varepsilon(Z) + 2\langle\gamma,\dot{u}\rangle_r + 2c_\varepsilon\langle\gamma,u\rangle_r \leq c_\varepsilon\sqrt{\varepsilon}\,\|\gamma\|^2_{r-1} \tag{5.3}$$

holds for any sufficiently regular solution $Z(t) = (u(t),\dot{u}(t),\xi^t)$ to (5.1) and some $c_\varepsilon > 0$ independent on Z and γ such that $\lim_{\varepsilon\to 0} c_\varepsilon = 0$.

The proof of the proposition, which will play a crucial role in the next section when proving higher order energy estimates for the semigroup $S(t)$, is based on some auxiliary functionals that we are going to define.

For any $\delta > 0$, we consider the sets

$$P_\delta = \{s \in \mathbb{R} : \mu' + \delta\mu > 0\} \qquad \text{and} \qquad N_\delta = \{s \in \mathbb{R} : \mu' + \delta\mu \le 0\}.$$

As we are assuming that $\mu'(s) < 0$ for almost every s it is apparent that the probability measure

$$\hat{\mu}(P_\delta) = \frac{1}{m(0)} \int_{P_\delta} \mu(s) ds$$

vanishes as $\delta \to 0$.
For $r \in \mathbb{R}$ and $\xi \in \mathcal{S}^r$ we denote

$$\mathcal{P}_\delta^r[\xi] = \int_{P_\delta} \nu(s) \|\xi(s)\|_{r+1}^2 \mathrm{d}s \qquad \text{and} \qquad \mathcal{N}_\delta^r[\xi] = \int_{N_\delta} \nu(s) \|\xi(s)\|_{r+1}^2 \mathrm{d}s.$$

Notice that, since $P_\delta^r \cup \mathcal{N}_\delta^r = \mathbb{R}^+$ (possibly up to a nullset), it holds $\mathcal{P}_\delta^r[\xi] + \mathcal{N}_\delta^r[\xi] = \|\xi\|_{\mathcal{S}^r}^2$. Choosing $\kappa > 0$ such that $m(\kappa) > 0$, we define $\rho(\tau) = \min\{\tau/\kappa, 1\}$ and, for $z = (u, v, \xi) \in \mathcal{H}^r$ with $r \in [0, 1]$ we introduce the functionals

$$\Phi_1^r(z) = -\frac{1}{m(\kappa)} \int_0^\infty \rho(\tau) \langle v, \xi(\tau) \rangle_r \mathrm{d}\tau,$$

$$\Phi_2^r(z) = \langle v, u \rangle_r,$$

$$\Phi_3^r(z) = \int_0^\infty \left(\int_0^\tau \nu(s) \chi_{P_\delta}(s) \mathrm{d}s \right) \|\xi(\tau) - \mu(\tau) u\|_{r+1}^2 \mathrm{d}\tau.$$

Notice that (3.7) implies, for any $\tau \in \mathbb{R}^+$ and some $C > 0$,

$$\int_0^\tau \nu(s) \mathrm{d}s \le C \nu(\tau^-), \tag{5.4}$$

hence, taking advantage of (3.1), it is readily seen that for some $C > 0$,

$$0 \le |\Phi_1^r(z)| + |\Phi_2^r(z)| + \Phi_3^r(z) \le C \|z\|_{\mathcal{H}^r}^2. \tag{5.5}$$

Then, if $Z(t) = (u(t), \dot{u}(t), \xi^t)$ is any sufficiently regular solution to (5.1), the following inequalities hold:

Lemma 5.2. *For any $a \in (0, 1)$ and any $\delta > 0$*

$$\begin{aligned} \frac{d}{dt} \Phi_1^r(Z) \le & -(1-a) \|\dot{u}\|_r^2 + a \|u\|_{r+1}^2 + \frac{c_1}{a} \hat{\mu}(P_\delta) \mathcal{P}_\delta^r[\xi] + \frac{c_1}{a} \mathcal{N}_\delta^r[\xi] \\ & + \frac{1}{m(\kappa)} \int_0^\infty \rho(\tau) \langle \gamma, \xi(\tau) \rangle_r \mathrm{d}\tau, \end{aligned}$$

with $c_1 = \frac{m(0)}{2m(\kappa)^2} + \frac{2m(0)}{m(\kappa)} + \frac{m(0)}{2\lambda_1 \kappa^2 m(\kappa)^2}$.

Lemma 5.3. *For any $a \in (0, 1)$ and any $\delta > 0$*

$$\frac{d}{dt} \Phi_2^r(Z) \le -(1-a) \|u\|_{r+1}^2 + \|\dot{u}\|_r^2 + \frac{m(0)}{2a} \hat{\mu}(P_\delta) \mathcal{P}_\delta^r[\xi] + \frac{m(0)}{2a} \mathcal{N}_\delta^r[\xi] - \langle \gamma, u \rangle_r.$$

Lemma 5.4. *For any $a \in (0,1)$ and any $\delta > 0$*

$$\begin{aligned}\frac{d}{dt}\Phi_3^r(Z) \leq & a\|u\|^2_{r+1} - \Big(1-\hat\mu(P_\delta)\frac{4m(0)}{a}\Big)\mathcal{P}^r_\delta[\xi] \\ & + \frac{c_3}{a}\Big(\int_0^\infty \nu'(s)\|\xi(s)\|^2_{r+1}ds + \sum \nu_n\|\xi(\sigma_n)\|^2_{r+1}\Big),\end{aligned}$$

with $c_3 = \max\big\{4\mu(\sigma_1^-), 4\int_0^\infty s\mu(s)\mathrm{d}s\big\}$.

Since the proof of these lemmas is technical and rather involved, we postpone it into the Appendix and we directly go to the proof of the main Proposition 5.1.

Proof of Proposition 5.1. We preliminary observe that, if $z = (u, v, \xi) \in \mathcal{H}^r$ and $\delta > 0$ there holds

$$\mathcal{N}^r_\delta[\xi] \leq \frac{1}{\delta}\int_0^\infty \nu'(s)\|\xi(s)\|^2_{r+1}\mathrm{d}s. \tag{5.6}$$

Besides, we have

$$\begin{aligned}\Big(\int_0^\infty \|\xi(\tau)\|_{r+1}\mathrm{d}\tau\Big)^2 &\leq m(0)\Big(\sqrt{\hat\mu(P_\delta)\mathcal{P}^r_\delta[\xi]} + \sqrt{(1-\hat\mu(P_\delta))\mathcal{N}^r_\delta[\xi]}\Big)^2 \\ &\leq 2m(0)\big(\hat\mu(P_\delta)\mathcal{P}^r_\delta[\xi] + \mathcal{N}^r_\delta[\xi]\big).\end{aligned} \tag{5.7}$$

Let $\varepsilon > 0$, choose $\delta = \delta(\varepsilon) > 0$ such that

$$\hat\mu(P_\delta) = \min\left\{\frac{\sqrt{\varepsilon}m(\kappa)^2}{32m(0)}, \frac{1}{4(64c_1 + 144m(0))}\right\} \tag{5.8}$$

(with c_1 as in Lemma 5.2), and let $Z(t) = (u(t), \dot u(t), \xi^t)$ be a sufficiently regular solution to (5.1). Applying Lemma 5.2, 5.3, 5.4 with $a = 1/16$ we find

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(4\Phi_1^r(Z) + 2\Phi_2^r(Z) + 2\Phi_3^r(Z)\big) + \frac{3}{2}\|u\|^2_{r+1} + \frac{3}{2}\|\dot u\|^2_r + 2\langle\gamma, u\rangle_r \\ &\quad\leq -\Big(2 - \hat\mu(P_\delta)(64c_1 + 144m(0))\Big)\mathcal{P}^r_\delta[\xi] + (64c_1 + 16m(0))\mathcal{N}^r_\delta[\xi] \\ &\quad+ \frac{4}{m(\kappa)}\int_0^\infty \rho(\tau)\langle\gamma, \xi(\tau)\rangle_r\mathrm{d}\tau + 32c_3\Big(\int_0^\infty \nu'(s)\|\xi(s)\|^2_{r+1}ds + \sum \nu_n\|\xi(\sigma_n)\|^2_{r+1}\Big).\end{aligned}$$

Hence, in light of (5.7) and (5.8), it holds

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(4\Phi_1^r(Z) + 2\Phi_2^r(Z) + 2\Phi_3^r(Z)\big) + \frac{3}{2}\|u\|^2_{r+1} + \frac{3}{2}\|\dot u\|^2_r + 2\langle\gamma, u\rangle_r \\ &\quad - 32c_3\Big(\int_0^\infty \nu'(s)\|\xi(s)\|^2_{r+1}ds + \sum \nu_n\|\xi(\sigma_n)\|^2_{r+1}\Big) \\ &\quad\leq -\frac{7}{4}\mathcal{P}^r_\delta[\xi] + (64c_1 + 16m(0))\mathcal{N}^r_\delta[\xi] + \frac{4}{m(\kappa)}\|\gamma\|_{r-1}\int_0^\infty \|\xi(\tau)\|_{r+1}\mathrm{d}\tau \\ &\quad\leq -\frac{3}{2}\|\xi\|^2_{\mathcal{S}^r} + \Big(\frac{3}{2} + 64c_1 + 16m(0) + \frac{8m(0)}{\sqrt{\varepsilon}m(k)^2}\Big)\mathcal{N}^r_\delta[\xi] + \sqrt{\varepsilon}\|\gamma\|^2_{r-1}.\end{aligned}$$

Besides, by (3.3) we easily obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Z\|^2_{\mathcal{H}^r} + 2\langle\gamma,\dot u\rangle_r = -\int_0^\infty \nu'(s)\|\xi(s)\|^2_{r+1}ds - \nu(0)\|\xi(0)\|^2_{r+1} - \sum \nu_n\|\xi(\sigma_n)\|^2_{r+1}.$$

Then, if we set

$$\Lambda^r_\varepsilon(z) = \|z\|^2_{\mathcal{H}^r} + \varepsilon\delta\big\{4\Phi^r_1(z) + 2\Phi^r_2(z) + 2\Phi^r_3(z)\big\},$$

collecting all the above inequalities and exploiting (5.6), we are led to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Lambda^r_\varepsilon(Z) + \delta\varepsilon\frac{3}{2}\|Z\|^2_{\mathcal{H}^r} + 2\langle\gamma,\dot u\rangle_r + 2\delta\varepsilon\langle\gamma,u\rangle_r \\
&\quad\le \delta\varepsilon\sqrt{\varepsilon}\,\|\gamma\|^2_{r-1} + c\delta\sqrt{\varepsilon}\mathcal{N}^r_\delta[\xi] - (1-c\delta\varepsilon)\Big(\int_0^\infty \nu'(s)\|\xi(s)\|^2_{r+1}ds + \sum\nu_n\|\xi(\sigma_n)\|^2_{r+1}\Big) \\
&\quad\le \delta\varepsilon\sqrt{\varepsilon}\,\|\gamma\|^2_{r-1} - (1-c\delta\varepsilon - c\sqrt{\varepsilon})\Big(\int_0^\infty \nu'(s)\|\xi(s)\|^2_{r+1}ds + \sum\nu_n\|\xi(\sigma_n)\|^2_{r+1}\Big),
\end{aligned}$$

for some $c > 0$ independent of ε, Z and γ. This proves (5.3) for ε small enough, and since it is apparent from (5.5) that (5.2) holds (up to possibly reducing ε), the proof is finished. □

Remark 5.5. Inequality (5.3) can be equivalently written as

$$\frac{\mathrm{d}}{\mathrm{d}t}E^r_\varepsilon(Z) + c_\varepsilon E^r_\varepsilon(Z) \le 2\langle\dot\gamma,u\rangle_r + c_\varepsilon\sqrt{\varepsilon}\,\|\gamma\|^2_{r-1} + c_\varepsilon c, \quad c\in\mathbb{R}, \tag{5.9}$$

where

$$E^r_\varepsilon(Z) = \Lambda^r_\varepsilon(Z) + 2\langle\gamma,u\rangle_r + c.$$

6. Proof of Theorem 3.2

We start by showing that the new model (3.4) is a gradient system, which allows to characterize the global attractor $\mathfrak{A}$ as the unstable set of the stationary points of (3.4), see [6, Section 7].

6.1. **The gradient system structure.** Let us call

$$\mathcal{E} = \big\{z_\star\in\mathcal{H} : S(t)z_\star = z_\star,\ \forall t\ge 0\big\}$$

the set of equilibria of $S(t)$, made of all vectors $z_\star = (u_\star,0,0)$, with $u_\star$ solution to the elliptic equation

$$Au_\star + g(u_\star) = f.$$

In light of (2.7), the set $\mathcal{E}$ is nonempty and bounded in the more regular space $\mathcal{H}^1$.
Let us recall that $S(t)$ is a *gradient system* if there exists $\mathcal{L}\in\mathcal{C}(\mathcal{H},\mathbb{R})$, called a *Lyapunov function*, satisfying the following properties:

(i) $\mathcal{L}(z)\to\infty$ if and only if $\|z\|_{\mathcal{H}}\to\infty$;
(ii) $\mathcal{L}(S(t)z)$ is nonincreasing for any $z\in\mathcal{H}$;
(iii) if $\mathcal{L}(S(t)z) = \mathcal{L}(z)$ for all $t>0$, then $z\in\mathcal{E}$.

Proposition 6.1. *Assume that* (3.7) *holds. Then, the function*

$$\mathcal{L}(z) = \|z\|^2_{\mathcal{H}} + 2\langle G(u),1\rangle - 2\langle f,u\rangle, \qquad z=(u,v,\xi)\in\mathcal{H}$$

with $G(u) = \int_0^u g(y)\mathrm{d}y$, *is a Lyapunov functional.*

Proof. The continuity of $\mathcal{L}$ is apparent. Besides, on account of (2.6)-(2.7),

$$\varpi\|z\|_{\mathcal{H}}^2 - c \leq \mathcal{L}(z) \leq c\|z\|_{\mathcal{H}}^4 + c,$$

for some $\varpi > 0$ depending only on the limit (2.7), and some $c > 0$. Next, we verify that $\mathcal{L}$ is decreasing along the trajectories of $S(t)$. Indeed, working within a suitable regularization scheme, in light of (3.3) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(S(t)z) + \int_0^\infty \nu'(s)\|\xi^t(s)\|_1^2 ds + \nu(0)\|\xi^t(0)\|_1^2 + \sum \nu_n\|\xi^t(\sigma_n)\|_1^2 \leq 0, \tag{6.1}$$

yielding (ii). Furthermore, if $\mathcal{L}$ is constant along a trajectory of $S(t)$, by (6.1) we learn in particular that $\int_0^\infty \nu'(s)\|\xi^t(s)\|_1^2 ds$ and since $\nu' > 0$ almost everywhere, then $\xi^t = 0$. Plugging this information in (3.4), we conclude that $S(t)z$ is constant in time and that $z \in \mathcal{E}$. □

As a byproduct we have the following uniform–in–time estimate

Corollary 6.2. *Assume that* (3.7) *holds. Then, for any $R \geq 0$, there exists a positive constant $C = C(R)$ such that, whenever $\|z\|_{\mathcal{H}} \leq R$,*

$$\sup_{t\geq 0} \|S(t)z\|_{\mathcal{H}} \leq C.$$

Since $S(t)$ is a gradient system with a bounded set of equilibria, exploiting a general argument (cf. [7, 16]), the existence of the global attractor is proved if we show that, for any $R \geq 0$, there exists a compact set $\mathfrak{C} = \mathfrak{C}(R) \subset \mathcal{H}$ such that

$$\lim_{t\to\infty} \mathrm{dist}_{\mathcal{H}}(S(t)\mathfrak{B}_R, \mathfrak{C}) = 0, \tag{6.2}$$

where $\mathfrak{B}_R = \{z \in \mathcal{H} : \|z\|_{\mathcal{H}} \leq R\}$. Furthermore, $\mathfrak{A} \subset \mathfrak{C}$ for some R large enough.

Let us fix $R \geq 0$, and select any $z \in \mathfrak{B}_R$. In what follows, the *generic* constant $C \geq 0$ will depend on R, but not on the particular $z \in \mathfrak{B}_R$. We write

$$S(t)z = (u(t), \dot u(t), \xi^t) = S_0(t)z + S_1(t)z,$$

where

$$S_0(t)x = (v(t), \dot v(t), \zeta^t) \quad \text{and} \quad S_1(t)x = (w(t), \dot w(t), \psi^t)$$

solve the systems

$$\begin{cases} \ddot v + A\Big[v + \displaystyle\int_0^\infty \zeta(s)\mathrm{d}s\Big] + g(v) + kv = 0, \\ \dot\zeta = P\zeta + \mu\dot v, \end{cases} \tag{6.3}$$

and

$$\begin{cases} \ddot w + A\Big[w + \displaystyle\int_0^\infty \psi(s)\mathrm{d}s\Big] + g(u) - g(v) - kv = f, \\ \dot\psi = P\psi + \mu\dot w, \end{cases} \tag{6.4}$$

with $k \geq 0$ to be suitably chosen, and initial data

$$S_0(0)z = z \quad \text{and} \quad S_1(0)z = 0.$$

Since (6.3) is a gradient system, and recalling Corollary 6.2, we derive the uniform bounds

$$\sup_{t\geq 0}\big[\|S(t)z\|_{\mathcal{H}}+\|S_0(t)z\|_{\mathcal{H}}+\|S_1(t)z\|_{\mathcal{H}}\big]\leq C. \tag{6.5}$$

Lemma 6.3. *There exists $\beta=\beta(R)>0$ such that*

$$\|S_0(t)z\|_{\mathcal{H}}\leq C\mathrm{e}^{-\beta t}.$$

Proof. We set $G_0(v)=G(v)+\frac{1}{2}kv^2$, and we choose k large enough such that

$$\langle g(v)+kv,v\rangle\geq 0 \quad\text{and}\quad \langle G_0(v),1\rangle\geq 0.$$

This is possible thanks to (2.7) and the assumption $g(0)=0$. Applying Lemma 5.1 for $r=0$ and $\gamma=g(v)+kv$, and setting

$$E(t)=\Lambda^0_\varepsilon(S_0(t)z)+2\langle G_0(v(t)),1\rangle,$$

we infer from (2.6) and (6.5) the controls

$$\frac{1}{2}\|S_0(t)z\|^2_{\mathcal{H}}\leq E(t)\leq C\|S_0(t)z\|^2_{\mathcal{H}},$$

along with the differential inequality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t)+\frac{c_\varepsilon}{2}\|S_0(t)z\|^2_{\mathcal{H}}&\leq c_\varepsilon\sqrt{\varepsilon}\,\|g(v(t))+kv(t)\|^2_{-1}-2c_\varepsilon\langle g(v(t))+kv(t),v(t)\rangle\\&\leq c_\varepsilon\sqrt{\varepsilon}\,\|g(v(t))+kv(t)\|^2_{-1}.\end{aligned}$$

Using the straightforward estimate

$$\|g(v)+kv\|_{-1}\leq C\|v\|_1,$$

up to taking $\varepsilon=\varepsilon(R)$ small enough, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)+\frac{c_\varepsilon}{4}\|S_0(t)z\|^2_{\mathcal{H}}\leq 0.$$

The Gronwall lemma completes the argument. □

To keep further our analysis we need a dissipation integral. Namely,

Lemma 6.4. *For any $R\geq 0$, whenever the initial datum z of system* (3.4) *satisfies $\|z\|_{\mathcal{H}}\leq R$,*

$$\int_{t_1}^{t_2}\|\dot u(t)\|^2\mathrm{d}t\leq\varepsilon(t_2-t_1)+C,\qquad\forall\ t_2>t_1\geq 0\quad\textit{and}\quad\forall\ \varepsilon>0,$$

for some $C=C(R,\varepsilon)$.

Proof. By Lemma 5.2 and (5.7), for any $a\in(0,1/2)$ there holds

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1(S(t)z)+\frac{1}{2}\|\dot u\|^2\\&\quad\leq a\|u\|^2_1+\frac{c_1}{a}\hat\mu(P_\delta)\mathcal{P}_\delta[\xi]+\frac{c_1}{a}\mathcal{N}_\delta[\xi]+\frac{1}{m(\kappa)}\|g(u)-f\|_{-1}\int_0^\infty\|\xi(\tau)\|_1\mathrm{d}\tau\\&\quad\leq a\|u\|^2_1+\Big(\frac{c_1}{a}+\frac{m(0)}{2m(\kappa)^2a}\Big)\hat\mu(P_\delta)\|\xi\|^2_{\mathcal{H}}+\Big(\frac{c_1}{a}+\frac{m(0)}{2m(\kappa)^2a}\Big)\mathcal{N}_\delta[\xi]+a\|g(u)-f\|^2_{-1}.\end{aligned}$$

Applying Corollary 6.2 and choosing a and δ small enough we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1(S(t)z) + \frac{1}{2}\|\dot{u}\|^2 \leq \frac{\varepsilon}{2} + C\mathcal{N}_\delta[\xi],$$

for some $C = C(R,\varepsilon)$. Besides, exploiting estimate (5.6) in (6.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(S(t)z) + \delta\mathcal{N}_\delta[\xi^t] \leq 0.$$

Collecting the two last inequalities gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Phi_1(S(t)z) + \frac{C}{\delta}\mathcal{L}(S(t)z)\Big) + \frac{1}{2}\|\dot{u}\|^2 \leq \frac{\varepsilon}{2},$$

and the claim follows by an integration over (t_1,t_2), observing that, for a fixed R, $\Phi_1(S(t)z)$ and $\mathcal{L}(S(t)z)$ are bounded functions of t. $\square$

Lemma 6.5. *There exists $M = M(R) > 0$ such that*

$$\sup_{t\geq 0}\|S_1(t)z\|_{\mathcal{H}^{1/3}} \leq M.$$

Proof. We apply Lemma 5.1 for $r = 1/3$ and $\gamma = g(u) - g(v) - kv - f$. Calling this time

$$E(t) = \Lambda_\varepsilon^{1/3}(S_1(t)z) + 2\langle g(u) - g(v) - kv - f, A^{1/3}w\rangle + c,$$

for some $c \geq 0$, we get (cf. (5.9))

$$\frac{\mathrm{d}}{\mathrm{d}t}E + c_\varepsilon E \leq I_1 + I_2 + I_3 + c_\varepsilon c, \tag{6.6}$$

having set

$$\begin{aligned}
&I_1 = 2\langle [g'(u) - g'(v)]\dot{u}, A^{1/3}w\rangle,\\
&I_2 = 2\langle [g'(v) - g'(0)]\dot{w}, A^{1/3}w\rangle,\\
&I_3 = 2\langle g'(0)\dot{w}, A^{1/3}w\rangle - 2k\langle \dot{v}, A^{1/3}w\rangle + c_\varepsilon\sqrt{\varepsilon}\,\|g(u) - g(v) - kv - f\|^2_{-2/3}.
\end{aligned}$$

It is also clear from (2.6) and (6.5) that

$$\|g(u) - g(v) - kv - f\| \leq C.$$

Accordingly, by means of (5.2), we can choose $c = c(R)$ large enough such that

$$\frac{1}{4}\|S_1(t)z\|^2_{\mathcal{H}^{1/3}} \leq E(t) \leq 2\|S_1(t)z\|^2_{\mathcal{H}^{1/3}} + 2c. \tag{6.7}$$

By the assumptions on g, the bounds (6.5) and (6.7), and the continuous embeddings $\mathrm{H}^{(3p-6)/2p} \subset L^p(\Omega)$, we draw the estimates (note that ε is fixed)

$$\begin{aligned}
I_1 &\leq C\big(1 + \|u\|_{L^6} + \|v\|_{L^6}\big)\|\dot{u}\|\|w\|_{L^{18}}\|A^{1/3}w\|_{L^{18/5}} \leq C\|\dot{u}\|\|w\|^2_{4/3} \leq \frac{c_\varepsilon}{6}E + C\|\dot{u}\|^2E,\\
I_2 &\leq C\big(\|v\|_{L^6} + \|v\|^2_{L^6}\big)\|\dot{w}\|_{L^{18/7}}\|A^{1/3}w\|_{L^{18/5}} \leq C\|v\|_1\|\dot{w}\|_{1/3}\|w\|_{4/3} \leq \frac{c_\varepsilon}{6}E + C\|v\|_1^2E,\\
I_3 &\leq C\|\dot{w}\|E^{1/2} + C\|\dot{v}\|E^{1/2} + C \leq \frac{c_\varepsilon}{6}E + C.
\end{aligned}$$

Therefore, setting $q = C\|\dot u\|^2 + C\|v\|_1^2$, inequality (6.6) improves to

$$\frac{\mathrm{d}}{\mathrm{d}t}E + \frac{c_\varepsilon}{2}E \le qE + C,$$

where, by virtue of Lemma 6.3 and Lemma 6.4,

$$\int_{t_1}^{t_2} q(t)\mathrm{d}t \le \frac{c_\varepsilon}{4}(t_2 - t_1) + C.$$

Since $E(0) = c$, on account of a Gronwall-type lemma (see e.g. [7]), we conclude that

$$E(t) \le CE(0)\mathrm{e}^{-\frac{c_\varepsilon}{4}t} + C \le C.$$

In turn, (6.7) yields the boundedness of $S_1(t)z$ in $\mathcal{H}^{1/3}$. □

Lemma 6.5 states in particular that the set $\mathcal{U} = \{w(\cdot) : z \in \mathfrak{B}_R\}$ is bounded in $L^\infty(\mathbb{R}^+, \mathrm{H}^{4/3}) \cap W^{1,\infty}(\mathbb{R}^+, \mathrm{H}^{1/3})$. Hence, by applying Lemma 4.2 to $\mathcal{U}$ and $r = 1/3$, we have that

$$\mathcal{K} = \bigcup_{w\in\mathcal{U}} \bigcup_{t\ge 0} \psi^t$$

with $(w, \dot w, \psi)$ solution to (6.4), is precompact in $\mathcal{S}$. As a consequence, in light of Lemma 6.3 and Lemma 6.5, the set

$$\mathfrak{C} = \big\{\omega \in \mathrm{H}^{4/3} \times \mathrm{H}^{1/3} : \|\omega\|_{\mathrm{H}^{4/3}\times\mathrm{H}^{1/3}} \le M\big\} \times \overline{\mathcal{K}} \subset \mathcal{H}^{1/3}$$

(where $\overline{\mathcal{K}}$ is the closure of $\mathcal{K}$ in $\mathcal{S}$) complies with (6.2). This finishes the proof of the existence of the global attractor $\mathfrak{A}$. As a matter of fact, since $\mathfrak{A} \subset \mathfrak{C}$ for R large, we have also established the following regularity result.

Corollary 6.6. *The attractor $\mathfrak{A}$ is bounded in $\mathcal{H}^{1/3}$.*

To prove that $\mathfrak{A}$ is bounded in $\mathcal{H}^1$, for $z \in \mathfrak{A}$, we split again the solution $S(t)z$ into the sum $S_0(t)z + S_1(t)z$, but taking now, in place of (6.3)-(6.4), the simpler decomposition

$$\begin{cases} \ddot v + A\Big[v + \displaystyle\int_0^\infty \zeta(s)\mathrm{d}s\Big] = 0, \\ \dot\zeta = P\zeta + \mu\dot v, \end{cases} \qquad \begin{cases} \ddot w + A\Big[w + \displaystyle\int_0^\infty \psi(s)\mathrm{d}s\Big] + g(u) = f, \\ \dot\psi = P\psi + \mu\dot w, \end{cases}$$

with initial data $S_0(0)z = z$ and $S_1(0)z = 0$. Relying on the properties of the attractor, and since the linear semigroup $S_0(t)$ is exponentially stable on $\mathcal{H}$ (as a particular case of Lemma 6.3), Theorem 3.2 follows from the next result and Lemma 4.2 for $r = 1$, arguing exactly as before.

Lemma 6.7. *We have the uniform bound*

$$\sup_{t\ge 0}\sup_{z\in\mathfrak{A}} \|S_1(t)z\|_{\mathcal{H}^1} < \infty.$$

Proof. We apply Lemma 5.1 for $r = 1$ and $\gamma = g(u) - f$, setting

$$E(t) = \Lambda_\varepsilon^1(S_1(t)z) - 2\langle f, Aw\rangle + c,$$

with $c \ge 0$ large enough such that

$$\frac{1}{4}\|S_1(t)z\|_{\mathcal{H}^1}^2 \le E(t) \le 2\|S_1(t)z\|_{\mathcal{H}^1}^2 + 2c.$$

Denoting by $C \geq 0$ a generic constant independent of $z \in \mathfrak{A}$ and applying (5.9) we are led to the inequality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E + c_\varepsilon E &\leq -2\langle g(u), \dot w\rangle_1 - 2c_\varepsilon \langle g(u), w\rangle_1 + c_\varepsilon \sqrt{\varepsilon}\, \|g(u) - f\|^2 + c_\varepsilon c \\ &\leq \frac{c_\varepsilon}{2}E + C\|g(u)\|_1^2 + C.\end{aligned}$$

On the other hand, as $S(t)z \in \mathfrak{A}$ and $\mathfrak{A}$ is bounded in $\mathcal{H}^{1/3}$ by Corollary 6.6, exploiting the continuous embeddings $\mathrm{H}^{4/3} \subset L^{18}(\Omega)$ and $\mathrm{H}^{1/3} \subset L^{18/7}(\Omega)$, and recalling (2.6), we deduce the bound

$$\|g(u)\|_1 \leq \|g'(u)\|_{L^9}\|A^{1/2}u\|_{L^{18/7}} \leq C\big(1 + \|u\|_{L^{18}}^2\big) \leq C,$$

yielding

$$\frac{\mathrm{d}}{\mathrm{d}t}E + \frac{c_\varepsilon}{2}E \leq C.$$

Since $E(0) = c$, an application of the standard Gronwall lemma will do. □

This finishes the proof of Theorem 3.2.

Appendix.

This appendix is devoted to prove in full details the technical lemmas stated in Section 5.

Proof of Lemma 5.2. By simple computations, Φ_1^r satisfies the differential equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1^r(Z) = -\frac{1}{m(\kappa)}\int_0^\infty \rho(\tau)\langle \ddot u, \xi(\tau)\rangle_r \mathrm{d}\tau - \frac{1}{m(\kappa)}\int_0^\infty \rho(\tau)\langle \dot u, \dot\xi(\tau)\rangle_r \mathrm{d}\tau.$$

By the first equation in (5.1) we learn that

$$\begin{aligned}-\frac{1}{m(\kappa)}\int_0^\infty \rho(\tau)\langle \ddot u, \xi(\tau)\rangle_r \mathrm{d}\tau &= \frac{1}{m(\kappa)}\int_0^\infty \rho(\tau)\langle u, \xi(\tau)\rangle_{r+1} \mathrm{d}\tau \\ &\quad + \frac{1}{m(\kappa)}\int_0^\infty \rho(\tau)\Big(\int_0^\infty \langle \xi(y), \xi(\tau)\rangle_{r+1}\mathrm{d}y\Big)\mathrm{d}\tau + \frac{1}{m(\kappa)}\int_0^\infty \rho(\tau)\langle \gamma, \xi(\tau)\rangle_r \mathrm{d}\tau \\ &\leq \frac{1}{m(\kappa)}\|u\|_{r+1}\int_0^\infty \|\xi(\tau)\|_{r+1}\mathrm{d}\tau + \frac{1}{m(\kappa)}\Big(\int_0^\infty \|\xi(\tau)\|_{r+1}\mathrm{d}\tau\Big)^2 \\ &\quad + \frac{1}{m(\kappa)}\int_0^\infty \rho(\tau)\langle \gamma, \xi(\tau)\rangle_r \mathrm{d}\tau \\ &\leq a\|u\|_{r+1}^2 + \Big(\frac{1}{4am(\kappa)^2} + \frac{1}{m(\kappa)}\Big)\Big(\int_0^\infty \|\xi(\tau)\|_{r+1}\mathrm{d}\tau\Big)^2 + \frac{1}{m(\kappa)}\int_0^\infty \rho(\tau)\langle \gamma, \xi(\tau)\rangle_r \mathrm{d}\tau.\end{aligned}$$

In light of (5.7) it follows that

$$\begin{aligned}-\frac{1}{m(\kappa)}\int_0^\infty \rho(\tau)\langle \ddot u, \xi(\tau)\rangle_r \mathrm{d}\tau &\leq a\|u\|_{r+1}^2 + \Big(\frac{m(0)}{2am(\kappa)^2} + \frac{2m(0)}{m(\kappa)}\Big)\big(\hat\mu(P_\delta)\mathcal{P}_\delta^r[\xi] + \mathcal{N}_\delta^r[\xi]\big) \\ &\quad + \frac{1}{m(\kappa)}\int_0^\infty \rho(\tau)\langle \gamma, \xi(\tau)\rangle_r \mathrm{d}\tau.\end{aligned}$$

Concerning the second term, applying (3.2) and (5.7) we obtain

$$\begin{aligned}
-\frac{1}{m(\kappa)}\int_0^\infty \rho(\tau)\langle \dot u, \dot\xi(\tau)\rangle_r \mathrm{d}\tau &= \frac{1}{\kappa m(\kappa)}\int_0^\kappa \langle \dot u, \xi(\tau)\rangle_r \mathrm{d}\tau - \frac{1}{m(\kappa)}\Big(\int_0^\infty \rho(\tau)\mu(\tau)\mathrm{d}\tau\Big)\|\dot u\|_r^2 \\
&\le \frac{1}{\sqrt{\lambda_1}\kappa m(\kappa)}\|\dot u\|_r \int_0^\infty \|\xi(\tau)\|_{r+1}\mathrm{d}\tau - \|\dot u\|_r^2 \\
&\le -(1-a)\|\dot u\|^2 + \frac{1}{4a\lambda_1\kappa^2 m(\kappa)^2}\Big(\int_0^\infty \|\xi(\tau)\|_{r+1}\mathrm{d}\tau\Big)^2 \\
&\le -(1-a)\|\dot u\|^2 + \frac{m(0)}{2a\lambda_1\kappa^2 m(\kappa)^2}\big(\hat\mu(P_\delta)\mathcal{P}_\delta^r[\xi] + \mathcal{N}_\delta^r[\xi]\big).
\end{aligned}$$

Collecting the above inequalities we end the proof. □

Proof of Lemma 5.3. Taking the time derivative of Φ_2^r we find the equality

$$\frac{d}{dt}\Phi_2^r(Z) = -\|u\|_{r+1}^2 + \|\dot u\|_r^2 - \int_0^\infty \langle u, \xi(\tau)\rangle_{r+1}\mathrm{d}\tau - \langle \gamma, u\rangle_r.$$

Applying (5.7) to estimate the integral by

$$\begin{aligned}
-\int_0^\infty \langle u, \xi(\tau)\rangle_{r+1}\mathrm{d}\tau &\le a\|u\|_{r+1}^2 + \frac{1}{4a}\Big(\int_0^\infty \|\xi(\tau)\|_{r+1}\mathrm{d}\tau\Big)^2 \\
&\le a\|u\|_{r+1}^2 + \frac{m(0)}{2a}\big(\hat\mu(P_\delta)\mathcal{P}_\delta^r[\xi] + \mathcal{N}_\delta^r[\xi]\big),
\end{aligned}$$

the claim follows. □

Proof of Lemma 5.4. Exploiting the second equation in system (5.1) we have

$$\begin{aligned}
\frac{d}{dt}\Phi_3^r(Z) &= \int_0^\infty \Big(\int_0^\tau \nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big) 2\langle \xi(\tau) - \mu(\tau)u, \dot\xi(\tau) - \mu(\tau)\dot u\rangle_{r+1}\mathrm{d}\tau \\
&= \int_0^\infty \Big(\int_0^\tau \nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big) 2\langle \xi(\tau) - \mu(\tau)u, P\xi(\tau)\rangle_{r+1}\mathrm{d}\tau \\
&= \int_0^\infty \Big(\int_0^\tau \nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big) \frac{d}{d\tau}\|\xi(\tau)\|_{r+1}^2\mathrm{d}\tau \\
&\quad - 2\int_0^\infty \Big(\int_0^\tau \nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big)\langle \mu(\tau)u, P\xi(\tau)\rangle_{r+1}\mathrm{d}\tau.
\end{aligned} \tag{6.8}$$

In order to control the first contribution, we integrate by parts reasoning as in [12, Lemma 7.2] to prove the existence of $\ell_m \to s_\infty$ such that

$$\lim_{m\to\infty} \nu(\ell_m)\|\xi(\ell_m)\|_{r+1}^2 = 0, \tag{6.9}$$

which holds if ξ, $D\xi \in \mathcal{S}^r$. By (5.4) this implies

$$\lim_{m\to\infty} \|\xi(\ell_m)\|_{r+1}^2 \int_0^{\ell_m} \nu(s)\chi_{P_\delta}(s)\mathrm{d}s = 0,$$

thus providing

$$\int_0^\infty \Big(\int_0^\tau \nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big)\frac{d}{d\tau}\|\xi(\tau)\|_{r+1}^2\mathrm{d}\tau = \lim_{m\to\infty}\int_0^{\ell_m}\Big(\int_0^\tau \nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big)\frac{d}{d\tau}\|\xi(\tau)\|_{r+1}^2\mathrm{d}\tau$$
$$= \lim_{m\to\infty}\Big\{\|\xi(\ell_m)\|_{r+1}^2\int_0^{\ell_m}\nu(s)\chi_{P_\delta}(s)\mathrm{d}s - \int_0^{\ell_m}\chi_{P_\delta}(\tau)\nu(\tau)\|\xi(\tau)\|_{r+1}^2 d\tau\Big\} = -\mathcal{P}_\delta^r[\xi].$$

We continue by estimating the last term in (6.8) as follows:

$$(6.10)\quad -2\int_0^\infty \Big(\int_0^\tau \nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big)\langle\mu(\tau)u, P\xi(\tau)\rangle_{r+1}\mathrm{d}\tau$$
$$= 2\int_0^\infty \Big(\int_0^\tau \nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big)\Big[\langle\mu'(\tau)u,\xi(\tau)\rangle_{r+1} - \frac{d}{d\tau}\langle\mu(\tau)u,\xi(\tau)\rangle_{r+1}\Big]\mathrm{d}\tau.$$

Recalling that $\nu'(\tau) = -\mu'(\tau)/[\mu(\tau)]^2$ for a.e. $\tau\in\mathbb{R}^+$, we have

$$2\int_0^\infty \Big(\int_0^\tau \nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big)\langle\mu'(\tau)u,\xi(\tau)\rangle_{r+1}d\tau$$
$$\le 2\|u\|_{r+1}\int_0^\infty \Big(\int_0^\tau \nu(s)\mathrm{d}s\Big)(-\mu'(\tau))\|\xi(\tau)\|_{r+1}\mathrm{d}\tau$$
$$\le 2\|u\|_{r+1}\Big(\int_0^\infty \Big(\int_0^\tau \nu(s)\mathrm{d}s\Big)^2(-\mu'(\tau))\mu^2(\tau)\mathrm{d}\tau\Big)^{1/2}\Big(\int_0^\infty \nu'(\tau)\|\xi(\tau)\|_{r+1}^2\mathrm{d}\tau\Big)^{1/2}.$$

Since by the monotonicity of ν and (2.5) it holds

$$\int_0^\infty \Big(\int_0^\tau \nu(s)\mathrm{d}s\Big)^2(-\mu'(\tau))\mu^2(\tau)\mathrm{d}\tau \le \int_0^\infty \tau^2\nu^2(\tau)\mu^2(\tau)(-\mu'(\tau))\mathrm{d}\tau$$
$$= \lim_{\ell\to s_\infty}\Big\{-\sum_{n:\sigma_n<\ell}\mu_n\sigma_n^2 - \mu(\ell^-)\ell^2 + 2\int_0^\ell \tau\mu(\tau)\mathrm{d}\tau\Big\} \le 2\int_0^\infty \tau\mu(\tau)\mathrm{d}\tau < \infty,$$

calling $c_3 = 4\int_0^\infty \tau\mu(\tau)\mathrm{d}\tau$ gives

$$2\int_0^\infty \Big(\int_0^\tau \nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big)\langle\mu'(\tau)u,\xi(\tau)\rangle_{r+1}d\tau \le \frac{a}{2}\|u\|_1^2 + \frac{c_3}{a}\int_0^\infty \nu'(\tau)\|\xi(\tau)\|_{r+1}^2 d\tau.$$

To estimate the latter term in (6.10) we integrate by parts

$$-2\int_0^\infty \Big(\int_0^\tau \nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big)\frac{d}{d\tau}\langle\mu(\tau)u,\xi(\tau)\rangle_{r+1}\mathrm{d}\tau$$
$$= \lim_{m\to\infty} -2\int_0^{\ell_m}\Big(\int_0^\tau \nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big)\frac{d}{d\tau}\langle\mu(\tau)u,\xi(\tau)\rangle_{r+1}\mathrm{d}\tau$$
$$= \lim_{m\to\infty}\Big\{-2\sum_{n:\sigma_n<\ell_m}\Big(\int_0^{\sigma_n}\nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big)\mu_n\langle u,\xi(\sigma_n)\rangle_{r+1}$$
$$-2\Big(\int_0^{\ell_m}\nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big)\mu(\ell_m)\langle u,\xi(\ell_m)\rangle_{r+1} + 2\int_0^{\ell_m}\chi_{P_\delta}(\tau)\nu(\tau)\langle\mu(\tau)u,\xi(\tau)\rangle_{r+1}\mathrm{d}\tau\Big\}$$
$$= -2\sum\Big(\int_0^{\sigma_n}\nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big)\mu_n\langle u,\xi(\sigma_n)\rangle_{r+1} + 2\int_0^\infty \chi_{P_\delta}(\tau)\langle u,\xi(\tau)\rangle_{r+1}\mathrm{d}\tau,$$

with ℓ_m as in (6.9). Applying (5.4) we obtain

$$
\begin{aligned}
-2\sum\Big(\int_0^{\sigma_n}\nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big)\mu_n\langle u,\xi(\sigma_n)\rangle_{r+1}+2\int_0^\infty\chi_{P_\delta}(\tau)\langle u,\xi(\tau)\rangle_{r+1}\mathrm{d}\tau\\
\le 2\|u\|_{r+1}\Big(\sum\nu(\sigma_n^-)\mu_n\|\xi(\sigma_n)\|_{r+1}+\int_{P_\delta}\|\xi(\tau)\|_{r+1}\mathrm{d}\tau\Big)\\
\le \frac{a}{2}\|u\|_{r+1}^2+\frac{4}{a}\Big(\sum\nu(\sigma_n^-)\mu_n\|\xi(\sigma_n)\|_{r+1}\Big)^2+\frac{4}{a}\Big(\int_{P_\delta}\|\xi(\tau)\|_{r+1}\mathrm{d}\tau\Big)^2.
\end{aligned}
$$

As

$$
\begin{aligned}
\Big(\sum\nu(\sigma_n^-)\mu_n\|\xi(\sigma_n)\|_{r+1}\Big)^2 &\le \Big(\sum\sqrt{\nu_n}\sqrt{\mu_n}\|\xi(\sigma_n)\|_{r+1}\Big)^2\\
&\le \sum\mu_n\sum\nu_n\|\xi(\sigma_n)\|_{r+1}^2\le\mu(\sigma_1^-)\sum\nu_n\|\xi(\sigma_n)\|_{r+1}^2
\end{aligned}
$$

we finally get

$$
\begin{aligned}
-2\int_0^\infty\Big(\int_0^\tau\nu(s)\chi_{P_\delta}(s)\mathrm{d}s\Big)\frac{d}{d\tau}\langle\mu(\tau)u,\xi(\tau)\rangle_{r+1}\mathrm{d}\tau\\
\le\frac{a}{2}\|u\|_{r+1}^2+\frac{4}{a}\mu(\sigma_1^-)\sum\nu_n\|\xi(\sigma_n)\|_{r+1}^2+\frac{4m(0)}{a}\hat{\mu}(P_\delta)\mathcal{P}_\delta^r[\xi].
\end{aligned}
$$

Collecting all the above inequalities we conclude the proof. ☐

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