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ADHESIVE FLEXIBLE MATERIAL STRUCTURES

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ADHESIVE FLEXIBLE MATERIAL STRUCTURES

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Abstract. We study variational problems modeling the adhesion interaction with a rigid substrate for elastic strings and rods. We produce conditions characterizing bonded and detached states as well as optimality properties with respect to loading and geometry. We show Euler equations for minimizers of the total energy outside self-contact and secondary contact points with the substrate.

Contents

Introduction

At the fundamental level of some recent fields of research such as nanoscale engineering and byophysics there is the need of a fine understanding of the behavior of thin flexible material structures involved in complex interactions. Indeed, the small scale interactions of material components, governed by surface-tension forces and adhesive forces as one-dimensional nanostructures like nanotubes, nanowires and biopolymers adhering on different material substrates, are crucial in the study of biological adhesion and the development of nanoelectronics and nanocomposites as well as MEMS and NEMS devices ([27]), e.g. super coiled DNA molecules, bacteria filaments, gecko inspired materials, actuators, etc. It has been shown that at the nanoscale,

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the peeling of long slender molecules and nanostructures from substrates involves a strong coupling between elasticity, friction, and adhesive forces. At these scales, if carbon nanotubes can adhere to each other under the influence of capillary forces, fluid-regulated forces are not the only factors that must be examined and dispersion or van der Waals forces may become more important than at larger scales, as well as the microscopic intermolecular forces ([13]) of extended media start to have a macroscopic effect on structural stability ([8],[3],[17],[18]).

The previous considerations sketch the physical framework in which we move from the mathematical perspective with the aim of establishing sufficiently general and, as far as possible, simplified mathematical models capturing the essentials of the involved phenomena. In particular, here we intend to develop the line of thoughts exposed in [19], [20], [21] (where only linear elastic behavior was considered) and [7], [23], [25] and [24] by focusing on nonlinear models of the structural behavior to the aim of settle a variational scheme in which the study of the adhesion interactions of one-dimensional non linear elastic filaments and curved rods can be carried on.

In Section 1 we study elastic models whose bulk energy is characterized by shear deformations, under the simplifying assumption that the rigid substrate boundary is a graph: we study the adhesion regime and focus the attention on the main features regulating the mechanical behavior. In particular we show that the debonded state depends on the constitutive parameters and on the length of the curve representing the substrate boundary, but it does not depend on the shape of such a curve (Theorems 1.3, 1.4).

In Section 2 we study adhesion models governed by curvature elasticity, i.e. the bulk energy density is a measure of the curvature gap between the rod and the rigid substrate.

Precisely we focus our analysis on the minimization of the functional

$$\mathcal{F}(\mathbf{u}) = \begin{cases} \dfrac{EJ}{2}\displaystyle\int_0^L |\kappa(\mathbf{u}) - \kappa(\mathbf{u}_*)|^2\, ds - W_{\mathbf{f}}(\mathbf{u}) + W_\psi(\mathbf{u}) & \text{if } \mathbf{u} \in \mathcal{A}, \\ +\infty & \text{else,} \end{cases} \tag{0.1}$$

where: $\mathbf{u}_*$ and $\mathbf{u}$ denote respectively the unloaded and loaded rod, κ is the scalar curvature, the flexural rigidity of the rod is given by the product EJ of the Young modulus E times the moment of inertia J of the cross-section of the rod, $\mathbf{f}$ is a given load, adhesion energy W_ψ and load potential $W_{\mathbf{f}}$ are expressed respectively by

$$W_\psi(\mathbf{u}) \ = \ \psi(\mathcal{H}^1(\{\mathbf{p} : \mathbf{u}(\mathbf{p}) \neq \mathbf{u}_*(\mathbf{p})\}))\,,$$

$$W_{\mathbf{f}}(\mathbf{u}) = \mathbf{f} \cdot \{\mathbf{u}(L) - \mathbf{u}_*(L)\}\,,$$

with ψ strictly increasing, $\psi(0) = 0$, $\mathcal{H}^1$ denotes the 1-d Hausdorff measure and the set $\mathcal{A}$ of admissible configurations (clamped at first end, loaded at the other end and confined in $\overline{\Omega}$) is the closure in the weak topology of $H^2((0, L); \overline{\Omega})$ of the following set A of simple curves

$$A = \big\{\mathbf{u} \in H^2((0, L); \overline{\Omega}) : \mathbf{u} \text{ injective in } [0, L],\ |\dot{\mathbf{u}}| = 1, \\ \text{and } \ \mathbf{u}(0) = \mathbf{u}_*(0),\ \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_*(0)\}\,,$$

where $\Omega \subset \mathbb{R}^2$ is an open set, the substrate is given by $\Omega \setminus \mathbb{R}^2$.

We emphasize that $\mathcal{A}$ contains also non-simple curves, nevertheless self-crossing of the rod is always forbidden in $\mathcal{A}$, while self-contact of the rod without interpenetration may take place (see Definition 2.9, Lemma 2.3, Lemma 2.11): coincidence of the tangents must hold true (up to the sign) at any multiple point (Proposition 2.2). The set $\mathcal{A}$ allows also configurations undergoing secondary contact with the rigid substrate at detached points of the rod. We analyze general conditions regulating bonded and debonded states of the rod and, in particular, we deduce precise relationship governing the case of strong adhesion (Theorem 2.15, Corollary 2.16, Remark 2.17) in which the whole rod remains bonded to the substrate. This suggests a shape optimization problem (Remark 2.18) in view of finding the support curve realizing the strongest adhesion. Several properties of functional (0.1) are proven in the last sections: in Section 3 we derive necessary conditions of minimality, precisely we deduce the Euler-Lagrange equation (3.20) of a detached solution in a general geometry; such equation retrieves *Euler elastica* equation when the substrate is flat and the rod is compressed; in Section 4 we show explicit conditions (Theorems 4.3, 4.4) for detachment of rectilinear rods by exploiting an auxiliary rescaled functional.

About motivations for taking into account only scalar curvature in functional (0.1) we refer to [1] and to a forthcoming paper [22] where justification of this assumption is deduced by a dimension reduction via scaling arguments. We refer to a forthcoming paper also for the analysis of local minimizers (related to buckling phenomenon), which is motivated by data of type described in Example 2.20 and Example 3.5 and can be performed by exploiting the Euler equation (3.20) itself.

1. Adhesion of shearable elastic strings to a rigid substrate

In this subsection we study a shearable elastic string modeling, for instance, a viscous fluid filament, bonded to a rigid substrate through a thin adhesive layer. We suppose the rigid substrate is given by the subgraph of a given scalar function $h \in C^1([0,1])$ and that the initial configuration Γ of the string is the graph of h: hence Γ is a C^1 regular curve. We denote by γ the parametrization of Γ:

$$\gamma(x) = (x, h(x)),\ \ \forall x \in [0,1], \tag{1.1}$$

$$\int_0^1 |\dot{\gamma}|\, dx = \int_0^1 \sqrt{1+\dot{h}^2}dx = L. \tag{1.2}$$

The unit normal vector to the curve is inward oriented with respect to the rigid substrate. The tangent and normal fields will be denoted respectively by $\mathbf{t}_\gamma$ and $\mathbf{n}_\gamma$ (where $\mathbf{n}_\gamma$ is the $\pi/2$ radiants clock-wise rotation of $\mathbf{t}_\gamma$). We set $\gamma(0) = \mathbf{p}_0$ and $\gamma(1) = \mathbf{p}_1$ where, due to (1.2), $\mathbf{p}_1$ belongs to the set

$$\mathbf{P} \stackrel{def}{=} \big\{\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2\ :\ |\mathbf{p} - \mathbf{p}_0| \le L\,,\ p_2 \ge h(p_1)\big\} \tag{1.3}$$

Let $\mathbf{u} : \Gamma \to \mathbb{R}^2$ be the displacement field of the string, we take the following form for the elastic (shearing) energy of the string:

$$W_e(\mathbf{u}) = \frac{k}{2}\int_\Gamma |D_{\mathbf{t}_\gamma}(\mathbf{u}\cdot\mathbf{n}_\gamma)|^2\, d\mathcal{H}^1\,, \tag{1.4}$$

where k denotes the stiffness of the string and $D_{\mathbf{t}_\gamma(}\mathbf{u}\cdot\mathbf{n}_\gamma)$ represents the tangential derivative of $\mathbf{u}\cdot\mathbf{n}_\gamma$ on Γ. The adhesion interaction of the string with the substrate offers to the energetic competition a contribution which is an increasing function of the length of the detached set, that is

$$W_\psi(\mathbf{u}) = \lambda\mathcal{H}^1(\Gamma_\mathbf{u}), \tag{1.5}$$

where

$$\Gamma_\mathbf{u} = \{p \in \Gamma \mid \mathbf{u}(p)\cdot\mathbf{n}_\gamma(p) > 0\} \tag{1.6}$$

is the detached set and

$$\lambda > 0 \tag{1.7}$$

is a given constitutive parameter. Therefore the total energy is given by the functional

$$F(\mathbf{u}) = W_e(\mathbf{u}) + W_\psi(\mathbf{u}).$$

In order to study the above functional we introduce the vector valued function $\mathbf{v} : [0,1] \to \mathbb{R}^2$ defined as $\mathbf{v} = \mathbf{u} \circ \gamma$ and set $w = \mathbf{v} \cdot \mathbf{n}_\gamma = (\mathbf{u} \circ \gamma) \cdot \mathbf{n}_\gamma$. We denote the set of parameters related to the detached set by

$$I_w = \{x \in [0,1] \mid w = (\mathbf{v}(x) - \gamma(x)) \cdot \mathbf{n}_\gamma(x) > 0\}. \tag{1.8}$$

hence $\gamma(I_w) = \Gamma_{\mathbf{v}}$. Let

$$G(w) = \frac{1}{2} \int_0^1 k \frac{|\dot{w}|^2}{|\dot{\gamma}|} \, dx + \lambda \int_{I_w} |\dot{\gamma}| \, dx. \tag{1.9}$$

Obviously, we have $F(\mathbf{u}) = G(w)$ hence we are interested in minimizing the functional G in the admissible set

$$W_\gamma = \{w \in H^1((0,1)) : w \geq 0, \ w(0) = 0, w(1) = \overline{w}\} \tag{1.10}$$

where

$$\overline{w} \overset{def}{=} (\mathbf{v}(1) - \mathbf{p}_1) \cdot \mathbf{n}_\gamma(1). \tag{1.11}$$

Theorem 1.1. *Assume (1.7)-(1.9). Then the functional G achieves a non-negative minimum in the set W_γ, whenever we fix the C^1 regular graph γ.*

Proof. Notice that $\min |\dot{\gamma}| \geq 1$ since γ is $C^1(0,1)$ regular. Then the elastic term in (1.9) is lower semicontinuous in $H^1(0,1)$. If $\varphi_k \rightharpoonup \varphi$ in $H^1(0,1)$ then φ_k uniformly converges to φ hence $\mathbf{1}_{I_{w_k}} \to \mathbf{1}_{I_w}$ in $L^1(0,1)$ and therefore

$$\int_{I_{w_k}} |\dot{\gamma}| \, dx \to \int_{I_w} |\dot{\gamma}| \, dx \tag{1.12}$$

thus proving semicontinuity of G. Since G is coercive and nonnegative, we get the thesis by applying a standard compactness argument. □

Theorem 1.2. *Assume (1.7)-(1.9). Assume that $w \in \operatorname{argmin}_{W_\gamma} G$. Then there exists a unique $\xi \in [0,1]$ such that $I_w = (\xi, 1]$ and*

$$G(w) = \frac{1}{2} \int_\xi^1 k \frac{|\dot{w}|^2}{|\dot{\gamma}|} \, dx + \lambda \int_\xi^L |\dot{\gamma}| \, dx.$$

Proof. We can repeat the same analysis which is contained in the proof of Proposition 2.3 in [19]. □

Theorem 1.3. *Assume (1.7)-(1.9). Then*

$$\min G = \min_\xi \left\{ \frac{1}{2} k \bar{w}^2 \left(\int_\xi^1 |\dot{\gamma}| \, dx \right)^{-1} + \lambda \int_\xi^1 |\dot{\gamma}| \, dx \right\}. \tag{1.13}$$

Proof. By Proposition 1.2 we get easily

$$\min G = \min_{\xi} \min \{J(\xi, w) : w \in W_\gamma\} \tag{1.14}$$

where

$$J(\xi, w) = \frac{1}{2} \int_\xi^1 k \frac{|\dot{w}|^2}{|\dot{\gamma}|}\, dx + \lambda \int_\xi^1 |\dot{\gamma}|\, dx. \tag{1.15}$$

If $w_o \in \operatorname{argmin}_w J(\xi, w)$ then Euler equation in $(\xi, 1)$ yield

$$\dot{w}_o = \overline{w} |\dot{\gamma}| \left(\int_\xi^1 |\dot{\gamma}|\, dx \right)^{-1} \tag{1.16}$$

in $(\xi, 1]$ and by substituting in (1.15) we get easily (1.13). □

Theorem 1.4. *Assumptions (1.7)-(1.9) entail that only one of the following two alternatives hold true.*
If

$$\frac{k\overline{w}^2}{2\lambda} < L^2, \tag{1.17}$$

then the detachment parameter ξ is the unique solution in $(0, L)$ of

$$\int_\xi^1 |\dot{\gamma}|\, dx = \frac{\overline{w}\sqrt{k}}{\sqrt{2\lambda}} \tag{1.18}$$

and

$$\min G = \overline{w}\sqrt{2\lambda k}. \tag{1.19}$$

If

$$\frac{k\overline{w}^2}{2\lambda} \geq L^2, \tag{1.20}$$

then the detachment point is $\xi = 0$, i.e. we obtain a complete debonding, and

$$\min G = \frac{\overline{w}^2 k}{2L} + \lambda L. \tag{1.21}$$

Proof. The theorem follows by applying the standard optimality conditions to the function

$$\xi \to \left\{ \frac{1}{2} k \overline{w}^2 \left(\int_\xi^1 |\dot{\gamma}|\, dx \right)^{-1} + \lambda \int_\xi^1 |\dot{\gamma}|\, dx \right\}. \tag{1.22}$$

□

2. Adhesion of elastic rods to a rigid substrate

We focus our attention on the adhesion of an Euler rod which is glued to a rigid substrate clamped at one end and loaded at the other one: the aim of this section is to give some condition on the load in order to avoid the detachment. We study the adhesion phenomenon in the context of non linear elasticity by considering the bulk energy density as the curvature gap between the rod and the support.
We denote the standard basis of $\mathbb{R}^2$ by $\{\mathbf{e}_1, \mathbf{e}_2\}$ and the clockwise rotation of $\pi/2$ by

$$\mathbb{W} = \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1. \tag{2.1}$$

We assume: Ω is a bounded C^2-regular bounded open subset of $\mathbb{R}^2$; the bonded configuration Γ of the rod is a (not necessarily flat) portion of $\partial\Omega$ such that $\mathcal{H}^1(\Gamma) = L > 0$, here Γ is the unstressed configuration of the elastic rod; Ω is the region where the obstacle allows the rod to undergo deformations; ψ is the cost function to detach a unit length of the road. We assume

$$\psi : [0, L) \to [0, \infty) \ \text{strictly increasing}, \ \ \psi(0) = 0. \tag{2.2}$$

We shall use the notation

$$\mathbf{n}_{\mathbf{u}} = \mathbb{W}\dot{\mathbf{u}} \ \ \forall \mathbf{u} \in H^2((0, L), \mathbb{R}^2).$$

We introduce a parametrization $\mathbf{u}_*$ of Γ with respect to the arc length and the related regularity assumptionsas follows

$$\mathbf{u}_* \in H^2((0, L); \partial\Omega), \qquad \mathbf{u}_* \ \text{ is injective in } [0, L]\,. \tag{2.3}$$

The above parametrization is choosen in such a way that $\dot{\mathbf{u}}_*$ provides the standard positive orientation of the boundary $\partial\Omega$ and

$$\mathbf{n}_\Omega = \mathbb{W}\dot{\mathbf{u}}_* = \mathbf{n}_{\mathbf{u}_*}$$

is the unit outward vector normal to $\partial\Omega$. We describe the admissible region Ω as the sublevel of a given function φ :

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 : \ \varphi(\mathbf{x}) \leq 0\} \tag{2.4}$$

$$\varphi \in C^2(\mathbb{R}^2, \mathbb{R}); \ \lim_{|\mathbf{x}| \to +\infty} \varphi(\mathbf{x}) = +\infty; \ \{\nabla\varphi(\mathbf{x}) = 0\} \cap \{\varphi(\mathbf{x}) = 0\} = \emptyset. \tag{2.5}$$

Let $\mathbf{f} \in \mathbb{R}^2$ be a given concentrated load acting at the end point of the rod. For every $\mathbf{u} \in H^2(0, L; \overline{\Omega})$ such that $|\dot{\mathbf{u}}| = 1$ a.e. in $[0, L]$ we define the scalar curvature

$$\kappa(\mathbf{u}) = |\ddot{\mathbf{u}}| = \ddot{\mathbf{u}} \cdot \mathbf{n}_{\mathbf{u}} = \ddot{\mathbf{u}} \cdot \mathbb{W}\dot{\mathbf{u}}. \tag{2.6}$$

It is worth noticing that for any such $\mathbf{u}$ we have $\ddot{\mathbf{u}} \cdot \dot{\mathbf{u}} = 0$ a.e. hence $\ddot{\mathbf{u}} = \kappa(\mathbf{u})\mathbb{W}\dot{\mathbf{u}}$. The elastic rod is clamped at $s = 0$ and confined in $\overline{\Omega}$. We define the set $\mathcal{A}$ of the admissible configurations of the rod via

$$A = \big\{\mathbf{u} \in H^2((0,L);\overline{\Omega}) : \ \mathbf{u} \text{ injective}, \ |\dot{\mathbf{u}}| = 1 \\ \text{and} \ \ \mathbf{u}(0) = \mathbf{u}_*(0), \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_*(0)\} \tag{2.7}$$

$$\mathcal{A} \text{ is the closure of } A \text{ in the weak topology of } H^2((0,L);\Omega). \tag{2.8}$$

We notice that the bonded configuration $\mathbf{u}_*$ belongs to $\mathcal{A}$.
The total energy of the rod in adhesion contact with the support and subject to a given load acting at the endpoint $s = L$ is given by the functional

$$\mathcal{F}(\mathbf{u}) = \begin{cases} \dfrac{EJ}{2}\displaystyle\int_0^L |\kappa(\mathbf{u}) - \kappa(\mathbf{u}_*)|^2 ds - W_{\mathbf{f}}(\mathbf{u}) + W_\psi(\mathbf{u}), & \text{if } \mathbf{u} \in \mathcal{A} \\ +\infty & \text{otherwise} \end{cases} \tag{2.9}$$

where

$$W_\psi(\mathbf{u}) \ = \ \psi(\mathcal{H}^1(\{\mathbf{p} : \mathbf{u}(\mathbf{p}) \neq \mathbf{u}_*(\mathbf{p})\})), \tag{2.10}$$

$$W_{\mathbf{f}}(\mathbf{u}) = \mathbf{f} \cdot \{\mathbf{u}(L) - \mathbf{u}_*(L)\}, \quad \mathbf{f} \in \mathbb{R}^2. \tag{2.11}$$

Theorem 2.1. *Assume (2.2)-(2.11).*
Then the functional $\mathcal{F}$ admits minimizers.

Proof. Since the rod is clamped in $s = 0$ then $W_{\mathbf{f}}$ is bounded and hence $\mathcal{F}$ is bounded from below and coercive. Then every minimizing sequence, say $(\mathbf{u}_n)_{n\in\mathbb{N}}$, is bounded in $H^2(0,L;\mathbb{R}^2)$ hence, up to subsequences, both $\dot{\mathbf{u}}_n$ and $\mathbf{u}_n$ are uniformly convergent in $[0,L]$. Lower semi-continuity of $\mathcal{F}$ yields the expected result. □

In the sequel we shall use the short notation $\operatorname{argmin} \mathcal{F}$ in place of $\operatorname*{argmin}_{\mathcal{A}} \mathcal{F}$.

For every $\tau \in [0,L)$ we introduce the following sets

$$A_\tau = \big\{\mathbf{u} \in H^2((\tau,L);\overline{\Omega}) : \ \mathbf{u} \ \text{ injective in } [\tau,L], \ |\dot{\mathbf{u}}| = 1, \\ \text{and} \ \ \mathbf{u}(\tau) = \mathbf{u}_*(\tau), \ \dot{\mathbf{u}}(\tau) = \dot{\mathbf{u}}_*(\tau)\},$$

$$\mathcal{A}_\tau \ = \ \text{closure of } A_\tau \text{ in the weak topology of } H^2((\tau,L);\Omega).$$

The set $\mathcal{A}_0$ will be shortly denoted by $\mathcal{A}$.
The curves in $\mathcal{A}_\tau$ may lack injectivity, nevertheless the self contact is allowed only without crossing, as it is clarified in the sequel by Definitions 2.7, 2.11, Lemmas 2.2, 2.11 and Theorem 2.13.

Lemma 2.2. *Assume (2.4)-(2.5), (2.7)-(2.8),* $\mathbf{u} \in \mathcal{A}_\tau \setminus A_\tau$ *and there exist* $0 \le s_1 < s_2 < L$ *such that* $\mathbf{u}(s_1) = \mathbf{u}(s_2)$ *. Then* $|\dot{\mathbf{u}}(s_1) \cdot \dot{\mathbf{u}}(s_2)| = 1$.

Proof. Let $\mathbf{u} \in \mathcal{A}_\tau \setminus A_\tau$ and $0 \le s_1 < s_2 < L$ such that $\mathbf{u}(s_1) = \mathbf{u}(s_2)$. If we assume that $|\dot{\mathbf{u}}(s_1) \cdot \dot{\mathbf{u}}(s_2)| < 1$ then by setting $B_\pm(\mathbf{u}(s_1), \delta) = \{\mathbf{x} \in B(\mathbf{u}(s_1), \delta) : \mathbf{x} \cdot \mathbf{n}_\mathbf{u} \lesseqgtr 0\}$, we find that $\mathbf{u}(s)$ belongs to $B_-(\mathbf{u}(s_1), \delta)$ (resp to $B_+(\mathbf{u}(s_1), \delta)$) in a small right (resp. left) neighborhood of s_2. By recalling now that $\mathbf{u}$ is the weak limit (in H^2) of simple curves we get a contradiction. □

Lemma 2.3. *Assume (2.2)-(2.11) and* $\mathbf{u} \in \operatorname{argmin} \mathcal{F}$ *.*
Then, either $\mathbf{u} \equiv \mathbf{u}_*$ *or there exists a unique* $\xi_\mathbf{u} \in [0, L)$ *such that*

$$\mathbf{u}(s) = \mathbf{u}_*(s) \ \ \forall s \in [0, \xi_\mathbf{u}] , \qquad \mathbf{u}(s) \neq \mathbf{u}_*(s) \ \ \forall s \in (\xi_\mathbf{u}, L] . \tag{2.12}$$

Proof. Let $K = \{s \in [0, L] : \mathbf{u}(s) = \mathbf{u}_*(s)\}$. Since $\varphi(\mathbf{u}(s)) \ge 0$ then $\dot{\mathbf{u}} = \dot{\mathbf{u}}_*$ on K and if we assume by contradiction that there exists $(\alpha, \beta) \subset [0, L] \setminus K$ with $\alpha, \beta \in K$ then by choosing $\bar{\mathbf{u}} = \mathbf{u}$ in $[0, L] \setminus [\alpha, \beta]$ and $\bar{\mathbf{u}} = \mathbf{u}_*$ otherwise we get $\mathcal{F}(\bar{\mathbf{u}}) < \mathcal{F}(\mathbf{u})$ thus contradicting minimality of $\mathbf{u}$. □

Definition 2.4. *We will denote by* $\widetilde{\mathcal{A}}$ *the subset of* $\mathbf{u} \in \mathcal{A}$ *such that there exists a value* $\xi_\mathbf{u} \in [0, L)$ *with*

$$\mathbf{u}(s) = \mathbf{u}_*(s) \ \ \forall s \in [0, \xi] , \qquad \mathbf{u}(s) \neq \mathbf{u}_*(s) \ \ \forall (\xi, L] .$$

Remark 2.5. By virtue of Lemma 2.3 we get $\operatorname{argmin}_\mathcal{A} \mathcal{F} \subset \widetilde{\mathcal{A}}$ and

$$\min_{\mathcal{A}} \mathcal{F} \ = \ \min_{\widetilde{\mathcal{A}}} \mathcal{F} \ = \ \min_{\xi} \left\{ \psi(L - \xi) + \min_{\mathbf{u} \in \mathcal{A}_\xi} \mathcal{F}_\xi(\mathbf{u}) \right\} \tag{2.13}$$

where

$$\mathcal{F}_\xi(\mathbf{u}) \ = \ \frac{EJ}{2} \int_\xi^L |\kappa(\mathbf{u}) - \kappa(\mathbf{u}_*)|^2 \, ds - \int_\xi^L \mathbf{f} \cdot (\dot{\mathbf{u}} - \dot{\mathbf{u}}_*) \, ds. \tag{2.14}$$

Definition 2.6. *For every* $\mathbf{u} \in \widetilde{\mathcal{A}}$ *s.t* $\mathbf{u} \not\equiv \mathbf{u}_*$ *, the value* $\xi_\mathbf{u}$ *given by Definition 2.4 is called* detachment parameter *of* $\mathbf{u}$ *while* $\mathbf{u}(\xi_\mathbf{u})$ *is called the* detachment point *of* $\mathbf{u}$.
$\xi_\mathbf{u}$ *is the unique* $\xi \in [0, L]$ *with* $\mathbf{u}(s) = \mathbf{u}_*(s)$ *for every* $s \in [0, \xi]$ *and* $\mathbf{u}(s) \neq \mathbf{u}_*(s)$ *in* $(\xi, L]$.
We emphasize that this value $\xi_\mathbf{u}$ *coincide with the one introduced by Lemma 2.3. This is the reason why they are labelled in the same way.*
The detachment parameter $\xi_\mathbf{u}$ *is shortly denoted by* ξ *whenever there is no risk of confusion.*

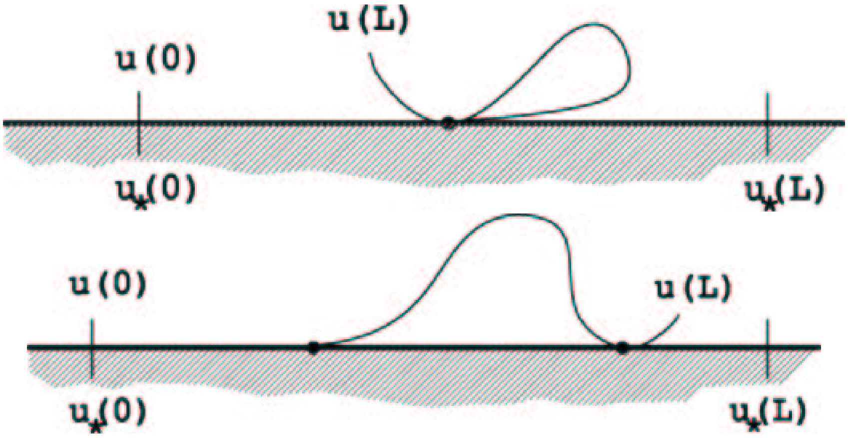

FIGURE 2.1. Examples of secondary contact points.

Definition 2.7. *We say that* $\mathbf{x} \in \partial\Omega$ *is a secondary contact point of* $\mathbf{u} \in \widetilde{\mathcal{A}}$ *with the substrate if there exist* $s \in (\xi_{\mathbf{u}}, L]$ *with* $\mathbf{x} = \mathbf{u}(s)$ *and* $\varphi(\mathbf{x}) = 0$.

Remark 2.8. Lemma 2.3 does not exclude secondary contact points.

Definition 2.9. *We say that* $\mathbf{x} \in \overline{\Omega}$ *is self-contact point of* $\mathbf{u} \in \mathcal{A}$ *if there exist* $s, \tilde{s} \in [0, L]$ *with* $\tilde{s} \neq s$ *and* $\mathbf{x} = \mathbf{u}_*(\tilde{s}) = \mathbf{u}(s) \neq \mathbf{u}_*(s)$.
We notice that self-contact points may have multiplicity bigger than 2.

Remark 2.10. The property $\mathbf{u} \in \mathcal{A}$ does not exclude self-contact points $\mathbf{x}$, moreover Proposition 2.2 entails that all oriented tangent vectors at $\mathbf{x}$ coincide up to the sign if $\mathbf{x} \neq \mathbf{u}(L)$. Nevertheless crossing is forbidden for $\mathbf{u}$ the set of admissible configurations $\mathcal{A}$ even if self-contact takes places, as it is clarified by the following statements.
Moreover, Lemma 2.3 allows for $\mathbf{u} \in \operatorname{argmin} \mathcal{F}$ the existence of a secondary contact or self contact point $\mathbf{x} = \mathbf{u}(s)$ with $\xi_{\mathbf{u}} < s \leq L$.

Lemma 2.11. *Assume (2.4)-(2.5) and (2.7)-(2.8). If* $\mathbf{x} = \mathbf{u}(s) = \mathbf{u}(\widetilde{s})$ *is an isolated self-contact point of* $\mathbf{u} \in \mathcal{A}$, *i.e.* $s, \widetilde{s} \in [0, L]\,,\ \ s \neq \widetilde{s}\,,$ *and*

$$\exists\, \delta > 0 \ \ \textit{s.t. } \mathbf{x} \textit{ is the only self contact point of } \mathbf{u} \textit{ in } B_\delta(\mathbf{x})\,,$$

then there is $\varepsilon > 0$ *s.t. the two curves obtained by restricting the parametrization of* $\mathbf{u}$ *to* $(s - \varepsilon, s + \varepsilon)$ *and* $(\widetilde{s} - \varepsilon, \widetilde{s} + \varepsilon)$ *do not cross each other.*

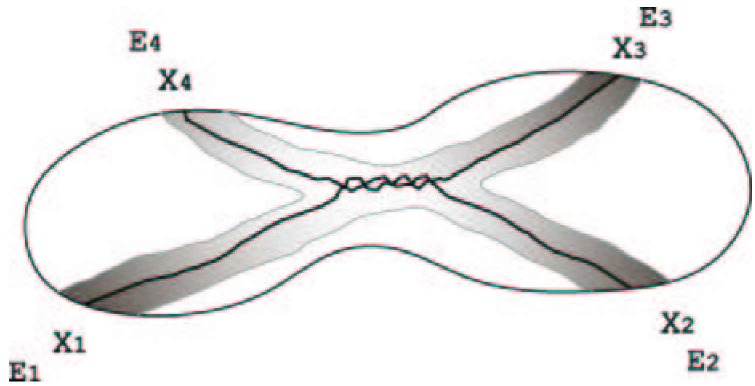

FIGURE 2.2. Example of self-crossing.

Proof. By definition of the admissible set $\mathcal{A}$ there is a sequence of simple curves $\mathbf{u}_k$ which is H^2 weakly convergent (hence uniformly) to $\mathbf{u}$. By contradiction: if the two curves cross each other at an isolated self-contact point $\mathbf{x}$ takes place, then for small enough $\delta > 0$ and $0 < r < \delta$ there is a $\delta-$tubular neighborhood U of $\mathbf{u}([0, L])$ s.t. $U \cap B_r(\mathbf{x})$ has an X shaped topology. For k big enough, the curve $\mathbf{u}_k$ must be contained in $U \cap B_r(\mathbf{x})$ and follow the orientation of $\mathbf{u}$, hence $\mathbf{u}_k$ cannot be simple, which is a contradiction. □

In general self-crossing of an C^∞ curve may take place in a more complicate situation than the case of an isolated self-contact point: along a nontrivial portion of a curve or even a Cantor-like set. In order to show that this never happens in $\mathcal{A}$, first we introduce a suitable definition Definition 2.12), then we get the conclusion by adapting the proof of the simpler case.

Definition 2.12. *We say that* $\mathbf{u}$ *undergoes self-crossing if there are five distinct points* $\mathbf{x}$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ *and a simply connected, bounded* C^1 *neighborhood* V *of* $\mathbf{x}$ *s.t.* $V \subset \Omega$, $\mathbf{x}_j$ *belong to* ∂V *and are ordered according to their indexes* j *along* ∂V, $\mathbf{x}_j = \mathbf{u}(s_j)$, $j = 1, 2, 3, 4$, $\mathbf{x} = \mathbf{u}(s) = \mathbf{u}(\widetilde{s})$, $0 \leq s_1 < s < s_3 < s_2 < \widetilde{s} < s_4 \leq L$ *and* $\dot{\mathbf{u}}(s_1)$, $\dot{\mathbf{u}}(s_2)$ *are the inward normal at* ∂V *at* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *respectively, while* $\dot{\mathbf{u}}(s_3)$, $\dot{\mathbf{u}}(s_4)$ *are the outward normal at* ∂V *at* $\mathbf{x}_3$ *and* $\mathbf{x}_4$ *respectively and both* $\mathbf{u}((s_1, s_3)), \mathbf{u}((s_2, s_4)) \subset V$.

Theorem 2.13. *Assume* $\mathbf{u} \in \mathcal{A}$ *and (2.4)-(2.5),(2.7)-(2.8) hold true. Then, referring to Definition 2.12,* $\mathbf{u}$ *never undergoes self-crossing.*

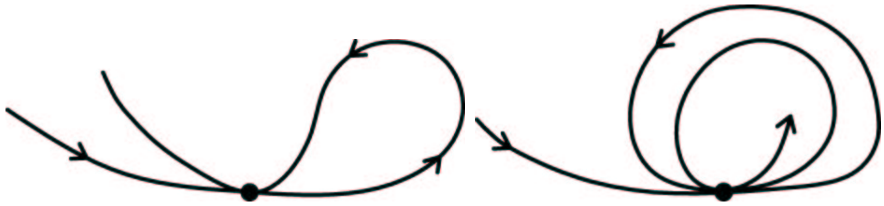

FIGURE 2.3. Examples of self-contact points.

Proof. By definition of the admissible set $\mathcal{A}$ there is a sequence of simple curves $\mathbf{u}_k$ which is H^2 weakly convergent (hence uniformly) to $\mathbf{u}$. By contradiction: if $\mathbf{u}$ undergoes self-crossing then, according to Definition 2.12 and the notation therein, for small enough $\delta > 0$ there is a $\delta-$tubular neighborhood U of $\mathbf{u}([s_1, s_3]) \cup \mathbf{u}([s_2, s_4])$ s.t. $U \cap \partial V$ has four disjoint components E_j, with $\mathbf{x}_j = \mathbf{u}(s_j) \in E_j$, which can be chosen such that $\mathbf{n}_V(\mathbf{x}) \cdot \dot{\mathbf{u}}(s_j) > 3/4$, for every $\mathbf{x} \in E_j$. For k big enough, the curve $\mathbf{u}_k$ must hit E_j in such a way that $\mathbf{u}_k$ hit E_j at $\mathbf{y} = \mathbf{u}_k(\sigma_j)$ with $\dot{\mathbf{u}}_k(\sigma_j) \cdot \mathbf{u}(s_j) > 3/4$ and $\mathbf{u}_k((\sigma_1, \sigma_2)) \cup \mathbf{u}_k((\sigma_3, \sigma_4)) \subset V$. Then $\mathbf{u}_k([\sigma_1, \sigma_3])$ disconnects V and $\mathbf{x}_2, \mathbf{x}_4$ are in different connected components, hence $\mathbf{u}_k$ cannot be simple, which is a contradiction. □

Remark 2.14. We emphasize that Definition 2.12 neither entails that $\mathbf{x}$ is an isolated crossing point of $\mathbf{u}$, nor that there is any other isolated crossing point of $\mathbf{u}$ in V. Definition 2.12 simply tells that two branches of $\mathbf{u}$ cross each other in V.

We are now in a position to prove the following statement.

Theorem 2.15. *(Strong Adhesion)*
Assume (2.2)-(2.11), $1 \leq p \leq \infty$, *and*

$$\psi(\tau) \geq \frac{|\mathbf{f}|^2}{6EJ}\tau^3 + \frac{2\,|\mathbf{f}|\,\|\kappa(\mathbf{u}_*)\|_{L^p(L-\tau,L)}}{(p/p-1)^{p/p-1}}\,\tau^{2-\frac{1}{p}} \quad \forall \tau \in [0, L] \tag{2.15}$$

and let $\mathbf{u} \in \operatorname{argmin} \mathcal{F}$. *Then* $\mathbf{u} \equiv \mathbf{u}_*$.

Proof. An integration by parts and the condition $\dot{\mathbf{u}}(\xi) = \dot{\mathbf{u}}_*(\xi)$ show that

$$\int_\xi^L \mathbf{f} \cdot (\dot{\mathbf{u}} - \dot{\mathbf{u}}_*)\,ds = \int_\xi^L (L - s)\mathbf{f} \cdot (\ddot{\mathbf{u}} - \ddot{\mathbf{u}}_*)\,ds \tag{2.16}$$

and since

$$\mathbf{f}\cdot(\ddot{\mathbf{u}}-\ddot{\mathbf{u}}_*) = (\kappa(\mathbf{u})-\kappa(\mathbf{u}_*))\mathbf{f}\cdot\mathbb{W}\dot{\mathbf{u}} + \kappa(\mathbf{u}_*)\mathbf{f}\cdot(\mathbb{W}\dot{\mathbf{u}}-\mathbb{W}\dot{\mathbf{u}}_*) \tag{2.17}$$

by using (2.14) we get

$$\psi(L-\xi)+\mathcal{F}_\xi(\mathbf{u}) \geq$$

$$\geq \psi(L-\xi) + \int_\xi^L \left| \left(\sqrt{\frac{EJ}{2}}\,\kappa(\mathbf{u})-\kappa(\mathbf{u}_*)\right) - \frac{L-s}{\sqrt{2EJ}}\,\mathbf{f}\cdot\mathbb{W}\dot{\mathbf{u}}\right|^2 ds +$$

$$-\frac{|\mathbf{f}|^2}{2EJ}\int_\xi^L (L-s)^2\,ds - \int_\xi^L (L-s)\,\kappa(\mathbf{u}_*)\;\mathbf{f}\cdot(\mathbb{W}\dot{\mathbf{u}}-\mathbb{W}\dot{\mathbf{u}}_*)\,ds \geq$$

$$\geq \psi(L-\xi) - \frac{|\mathbf{f}|^2(L-\xi)^3}{6EJ} - \frac{2(L-\xi)^{2-\frac{1}{p}}}{(p/p-1)^{p/p-1}}|\mathbf{f}|\|\kappa(\mathbf{u}_*)\|_{L^p(\xi,L)} \geq$$

$$\geq 0 = \mathcal{F}(\mathbf{u}_*)\quad \forall \xi\in(0,L]$$

and the proof is achieved. □

Corollary 2.16. *We assume (2.2)-(2.11),*

$$\psi(\tau) \;=\; \mu\,\tau^{2-\frac{1}{p}}\quad \forall\tau\in[0,L]\,,\quad \mu>0 \tag{2.18}$$

and

$$|\mathbf{f}| \leq \min\left\{\frac{\sqrt{6\mu\,EJ}}{L^{\frac{p+1}{2}}},\; \frac{\mu(p/p-1)^{p/p-1}}{2\|\kappa(\mathbf{u}_*)\|_{L^p(0,L)}}\right\}\,. \tag{2.19}$$

Then $\mathbf{u}=\mathbf{u}_*$.

Proof. Assumptions (2.18) and (2.19) together entail (2.15).
Hence by Theorem 2.15 we have $\mathbf{u}=\mathbf{u}_*$. □

Remark 2.17. We underline the dependence of the right hand side of (2.19) on the physical and geometrical characteristics of the structure: in particular the dependence on the ratio EJ/L^α ($\alpha>1$) which is crucial in the study of elastic stability, while the dependence on the ratio $\mu/\|\kappa(u_*)\|_{L^p(0,L)}$ says that the constitutive property of adhesion material and the substrate curvature determine the overall adhesion strength.

Remark 2.18. The right-hand side of (2.19) can be thought of as a measure of the global adhesion strength of the rod glued in the configuration $\mathbf{u}_*$. This perspective leads to the formulation of the following optimization problem: *find a curve maximizing the global adhesion strength among the closed curves* Γ *which enclose a connected region with fixed area*", via the minimization of functionals of type

$$\Gamma \longmapsto \int_\Gamma (c + \kappa^p)\, d\mathcal{H}^1 . \tag{2.20}$$

Similar minimization problems are studied also in image segmentation and image inpainting: we refer to [5], where the relaxed formulation of (2.20) in the class of varifolds is studied.

When the force field has the same direction of the inner normal to the rigid substrate, then intuition suggests that minimizers coincide with the fully bonded rod, since admissible deformations are allowed to stay only in the complementary region of the rigid obstacle. Indeed this is not true in general: precisely the following statement shows that, if Ω is convex (say, the substrate is concave), then this intuition is correct; on the other hand Example 2.20 shows that, if Ω is concave then it fails to be true.

Proposition 2.19. *Assume (2.2)-(2.11),* $\mathbf{u} \in \operatorname{argmin} \mathcal{F}$, φ *is a convex function and*

$$\mathbf{f} = \lambda\, \nabla\varphi(\mathbf{u}_*(L)), \qquad \lambda > 0 . \tag{2.21}$$

Then $\mathbf{u} \equiv \mathbf{u}_*$.

Proof. We have that, by using convexity of φ and recalling $\varphi(\mathbf{u}(L)) \leq 0 = \varphi(\mathbf{u}_*(L))$. By contradiction, if $\mathbf{u} \neq \mathbf{u}_*$ then $W_\psi(\mathbf{u}) > 0$, hence

$$\begin{aligned}
\mathcal{F}(\mathbf{u}) &> \frac{EJ}{2} \int_0^L |\kappa(\mathbf{u}) - \kappa(\mathbf{u}_*)|^2\, ds - W_{\mathbf{f}}(\mathbf{u}) = \\
&= \frac{EJ}{2} \int_0^L |\kappa(\mathbf{u}) - \kappa(\mathbf{u}_*)|^2 ds - \lambda \nabla\varphi(\mathbf{u}_*(L)) \cdot (\mathbf{u}(L) - \mathbf{u}_*(L)) \geq \\
&\geq \frac{EJ}{2} \int_0^L |\kappa(\mathbf{u}) - \kappa(\mathbf{u}_*)|^2 ds - \lambda(\varphi(\mathbf{u}(L) - \varphi(\mathbf{u}_*(L)) \geq 0 \\
&= \mathcal{F}(\mathbf{u}_*) .
\end{aligned} \tag{2.22}$$

□

Unfortunately the above result is true only in the case the admissible deformations take place in a convex set, as we can show in the following

Example 2.20. We choose $\varphi(\mathbf{x}) = 1 - |\mathbf{x}|^2$, $\Omega = \mathbb{R}^2 \setminus \overline{B_1(\mathbf{0})}$, $\mathbf{u}_*(s) = (\cos s, \sin s)$, $s \in [-\frac{\pi}{2}, \pi]$, and $\mathbf{f} = f\mathbf{e}_1$. By assuming $\dot{\mathbf{v}}(s) = \mathbf{e}_1$ with $\mathbf{v}(-\frac{\pi}{2}) = \mathbf{u}_*(-\frac{\pi}{2}) = -\mathbf{e_2}$ we get $\varphi(\mathbf{v}(s)) \le 0$ with strict inequality for $-\pi/2 < s \le L$. Then, by taking f sufficiently large, the energy of $\mathbf{v}$ becomes strictly negative, therefore $\mathbf{u}_*$ cannot be a minimizer:

$$\mathcal{F}(\mathbf{v}) = \frac{3}{4}\pi EJ - f\left(\frac{1}{2} + \frac{3}{2}\pi\right) + \psi\left(\frac{3}{2}\pi\right) \; < \; 0 \; = \; \mathcal{F}(\mathbf{u}_*)\,.$$

The previous example suggests that a more accurate description of the problem requires a careful analysis of the local minimizers besides the study of global minimizers which we are considering in the present work.

3. Euler equations for a detached rod

In this section we assume a general geometry of the substrate as described by (2.4),(2.5) and look for conditions fulfilled by an optimal configuration $\mathbf{u}$ in case of detachment state.

We fix $\mathbf{u} \in \operatorname{argmin} \mathcal{F}$ according to this case, then, by Lemma 2.3, we can assume its detachment parameter $\xi = \xi_{\mathbf{u}}$ is such that

$$0 \le \xi < L, \tag{3.1}$$

$$\mathbf{u}(s) \equiv \mathbf{u}_*(s) \;\; \forall s \in [0, \xi], \quad \mathbf{u}(s) \ne \mathbf{u}_*(s) \forall s \in]\xi, L]. \tag{3.2}$$

Let $\mathbb{M} \in SO(2)$ be an orthogonal matrix, then by Euler Formula there is $\vartheta \in [-\pi, \pi)$ s.t. $\mathbb{M}$ represents a rotation of angle ϑ in $\mathbb{R}^2$:

$$\mathbb{M} = \mathbb{M}(\vartheta) = \cos(\vartheta)\mathbb{I} + \sin(\vartheta)\mathbb{W}\,, \tag{3.3}$$

where $\mathbb{W}$ is given by (2.1) and $\mathbb{I}$ is the identity matrix.

We can represent any admissible configuration $\mathbf{v} \in \mathcal{A}$ of the rod as follows

$$\dot{\mathbf{v}} \; = \; \mathbb{M}(\vartheta_{\mathbf{v}})\,\dot{\mathbf{u}}_*\,, \qquad \vartheta_{\mathbf{v}} = \vartheta_{\mathbf{v}}(s), \tag{3.4}$$

by selecting a branch $\vartheta_{\mathbf{v}}$ of the multi-valued function $\Theta_{\mathbf{v}}$ (oriented angle between $\mathbf{v}$ and $\mathbf{u}_*$) such that

$$\vartheta_{\mathbf{v}}(s) \in H^1(0, L), \qquad \mathbb{M} \in C^\infty(\mathbb{R}, SO(2)). \tag{3.5}$$

The restriction of $\mathbf{u}$ to the interval $[\xi_{\mathbf{u}}, L]$ minimizes

$$\mathcal{F}_{\xi_{\mathbf{u}}}(\mathbf{v}) = \frac{EJ}{2}\int_{\xi_{\mathbf{u}}}^{L} |\kappa(\mathbf{v}) - \kappa(\mathbf{u}_*)|^2\, ds - \int_{\xi_{\mathbf{u}}}^{L} \mathbf{f}\cdot(\dot{\mathbf{v}} - \dot{\mathbf{u}}_*)\, ds \tag{3.6}$$

among $\mathbf{v}$ in

$$\mathcal{A}_\xi = \{\mathbf{v} \in \mathcal{A} : \ \mathbf{v}(s) = \mathbf{u}(s) = \mathbf{u}_*(s) \ \forall s \in [0, \xi_{\mathbf{u}}]\}. \tag{3.7}$$

To the aim of deducing necessary conditions of minimality we have to study variations of $\mathcal{F}_\xi$ around a curve $\mathbf{u}$, whose restriction in $[\xi, L]$ is a global minimizer in $\mathcal{A}_\xi$.. In order to perform these variations correctly, if $\mathbf{u}$ undergoes self-contact and/or secondary contact with the substrate, the variations can be made only in the last interval avoiding these interactions.

Theorem 3.1. *(Euler-Lagrange equations)*
Assume (2.2)-(2.5), (2.9)-(2.11), $\mathbf{u}$ *belongs to* $\operatorname{argmin} \mathcal{F}$ *and* $\xi = \xi_{\mathbf{u}} \in [0, L]$ *is the detachment parameter of* $\mathbf{u}$.
Moreover, by referring to Definitions 2.7 and setting

$$\begin{aligned} \widetilde{\xi}_{\mathbf{u}} = \max\{\xi, s, t\}, \quad &\textit{over } \xi, s, t \in [0, L] \ \textit{s.t.} \ \ \xi = \xi_{\mathbf{u}}\,, \ \textit{and} \\ \mathbf{u}(s) \ \textit{is a secondary contact point, } &\mathbf{u}(t) \ \textit{is a self-contact point,} \end{aligned} \tag{3.8}$$

assume

$$\widetilde{\xi}_{\mathbf{u}} < L; \tag{3.9}$$

Then ϑ_u *fulfils the following relationship:*

$$\ddot{\vartheta}(s) \ = \frac{1}{EJ}\,\mathbf{f} \cdot \{(\sin\vartheta(s)\mathbf{I} - \cos\vartheta(s)\mathbb{W}) \ \dot{\mathbf{u}}_*(s)\}, \quad s \in (\widetilde{\xi}_{\mathbf{u}}, L), \tag{3.10}$$

$$\vartheta(\widetilde{\xi}_{\mathbf{u}}) \ = \ 0\,, \quad \dot{\vartheta}(L) = 0\,. \tag{3.11}$$

Proof. We have the representation

$$\dot{\mathbf{u}} \ = \ \mathbb{M}(\vartheta_{\mathbf{u}})\,\dot{\mathbf{u}}_*\,, \qquad \vartheta_{\mathbf{u}} = \vartheta_{\mathbf{u}}(s) \tag{3.12}$$

By the definition of $\widetilde{\xi}_{\mathbf{u}}$ we know that

$$\mathbf{u}(s) \in \Omega \qquad \forall s \in (\widetilde{\xi}_{\mathbf{u}}, L]\,. \tag{3.13}$$

$$s \to \mathbf{u}(s) \text{ is injective in } (\widetilde{\xi}_{\mathbf{u}}, L]\,. \tag{3.14}$$

Then, for any $\delta > 0$ and

$$\eta \in \{h \in C^2([0, L]) \ \mathrm{spt\,h} \subset [\widetilde{\xi}_{\mathbf{u}} + \delta, \mathrm{L}]\}\,, \tag{3.15}$$

there is $\varepsilon_0 > 0$ s.t., by denoting $\mathbf{v}_\varepsilon(s)$ the unique function which fulfills

$$\mathbf{v}_\varepsilon(0) = \mathbf{u}_*(0)\,, \qquad \dot{\mathbf{v}}_\varepsilon(0) = \dot{\mathbf{u}}_*(0)\,, \qquad \dot{\mathbf{v}}_\varepsilon(s) = \mathbb{M}(\varepsilon\eta)\dot{\mathbf{u}}(s)\,,$$

we have

$$\mathbf{v}_\varepsilon \in \mathcal{A}, \quad \mathbf{v}_\varepsilon \text{ simple in } [\widetilde{\xi}_{\mathbf{u}} + \delta, L], \quad \varphi(\mathbf{v}_\varepsilon) \le \overline{\varphi} < 0 \qquad \forall \varepsilon \ : \ -\varepsilon_0 < \varepsilon < \varepsilon_0$$

say $\mathbf{v}_\varepsilon$ is an admissible configuration having neither self-contact nor secondary-contact in $[\xi_{\mathbf{u}}, L]$. As in (3.5) we choose $\vartheta_{\mathbf{u}}$, shortly denoted by $\vartheta = \vartheta(s)$. Then, by $\mathbb{M}(\vartheta + \varepsilon\eta) = \mathbb{M}(\vartheta)\mathbb{M}(\varepsilon\eta)$, we get

$$\dot{\mathbf{v}}_\varepsilon(s) \;=\; \mathbb{M}(\vartheta_{\mathbf{u}}(s) + \varepsilon\eta(s))\dot{\mathbf{u}}_*(s) \;=\; \mathbb{M}(\varepsilon\eta(s))\dot{\mathbf{u}}(s). \tag{3.16}$$

By (2.6) we have

$$\begin{aligned}\kappa(\mathbf{v}_\varepsilon) &= \ddot{\mathbf{v}}_\varepsilon \cdot \mathbb{W}\dot{\mathbf{v}}_\varepsilon = \\ &= (\,\mathbb{M}(\vartheta_{\mathbf{v}_\varepsilon})\,\ddot{\mathbf{u}}_*) \cdot (\,\mathbb{W}\,\mathbb{M}(\vartheta_{\mathbf{v}_\varepsilon})\,\dot{\mathbf{u}}_*) + \\ &\qquad + \dot{\vartheta}_{\mathbf{v}_\varepsilon}\,(\,\mathbb{M}'(\vartheta_{\mathbf{v}_\varepsilon})\,\dot{\mathbf{u}}_*) \cdot (\,\mathbb{W}\,\mathbb{M}(\vartheta_{\mathbf{v}_\varepsilon})\,\dot{\mathbf{u}}_*) \\ &= \kappa(\mathbf{u}_*) \,+\, \dot{\vartheta}_{\mathbf{v}_\varepsilon}\,(\,\mathbb{M}(\vartheta_{\mathbf{v}_\varepsilon} - \pi/2)\,\dot{\mathbf{u}}_*) \cdot (\,\mathbb{M}(\vartheta_{\mathbf{v}_\varepsilon} - \pi/2)\,\dot{\mathbf{u}}_*)\;,\end{aligned}$$

hence

$$\kappa(\mathbf{v}_\varepsilon) \;=\; \kappa(\mathbf{u}_*) \,+\, \dot{\vartheta}_{\mathbf{v}_\varepsilon}\,. \tag{3.17}$$

By taking into account (3.16) and (2.6) we evaluate the functional (3.6) at $\mathbf{v}_\varepsilon$, we get

$$\mathcal{F}(\mathbf{v}_\varepsilon) = I_{\widetilde{\xi}_{\mathbf{u}}}(\vartheta_{\mathbf{u}} + \varepsilon\eta) \tag{3.18}$$

where the functional $I_{\xi_{\mathbf{u}}}$ of the angular function is defined as follows:

$$I_{\widetilde{\xi}_{\mathbf{u}}}(\vartheta) \overset{def}{=} \int_{\widetilde{\xi}_{\mathbf{u}}}^{L} \left\{\dot{\vartheta}^2 - \mathbf{f} \cdot \big((\cos\vartheta(s) - 1)\mathbb{I} + \sin\vartheta(s)\mathbb{W}\big)\dot{\mathbf{u}}_*\right\} ds. \tag{3.19}$$

With a standard first variation argument we impose

$$\frac{d}{d\varepsilon} I_{\widetilde{\xi}_{\mathbf{u}}}(\vartheta + \varepsilon\eta)|_{\varepsilon=0} = 0 \qquad \forall\eta \;\; \text{as in (3.15)}\,.$$

Hence, by taking into account that $\mathrm{spt}\eta \subset [\widetilde{\xi} + \delta, \mathrm{L}]$ and $\varphi(\mathbf{u}(L)) < 0$, we get (3.10),(3.11).

The computation of the first variation is correct under the available regularity assumption (see (3.5)) that $\vartheta, \eta \in H^1(0, L)$, since $\mathbb{M}$ is an analytic function with bounded derivatives in $\mathbb{R}$. □

Remark 3.2. The right-hand side of (3.10) is equal to

$$\begin{aligned}&\frac{1}{EJ}\mathbf{f} \cdot \mathbf{u}_*(s)\, \sin(\vartheta(s)) - \cos(\vartheta(s))\, \|\mathbf{f} \wedge \mathbf{u}_*(s)\| = \\ &\quad = \frac{1}{EJ}|\mathbf{f}|\big(\sin(\vartheta(s))\cos((\varphi_*(s)) - \cos(\vartheta(s))\sin(\varphi_*(s))\big)\,,\end{aligned}$$

where $\varphi_*(s)$ denote the positively-oriented angle between $\mathbf{f}$ and $\mathbf{u}_*(s)$. Hence the Euler equation (3.10) reads as follows:

$$\ddot{\vartheta}(s) \;=\; \frac{1}{EJ}\,|\mathbf{f}|\,\sin\bigl(\vartheta(s)-\varphi_*(s)\bigr)\,, \quad s\in(\widetilde{\xi}_{\mathbf{u}},L)\,. \tag{3.20}$$

we emphasize that whenever $\varphi_*(s)\equiv k\pi,\ k\in\mathbb{Z}$, by equation (3.20) we retrieve the well known *Euler elastica* equation.

Remark 3.3. Among solutions ϑ of (3.20) we have to select only the ones such that, by defining $\mathbf{v}_\vartheta$ as follows

$$\mathbf{v}_\vartheta(s)=\begin{cases}\mathbf{u}_*(s)\ \text{ if }\ \ s\in[0,\widetilde{\xi}_{\mathbf{u}}]\\[2ex] \mathbf{u}_*(\widetilde{\xi}_{\mathbf{u}})+\displaystyle\int_{\widetilde{\xi}_{\mathbf{u}}}^{s}\mathbb{M}(\vartheta(\sigma))\dot{\mathbf{u}}_*(\sigma)d\sigma\ \text{ if }\ s\in(\widetilde{\xi}_{\mathbf{u}},L],\end{cases} \tag{3.21}$$

we obtain that $\mathbf{v}_\vartheta\in\mathcal{A}$ and $\mathbf{v}_\vartheta$ does not undergo neither self-contact nor secondary contact points in $(\widetilde{\xi}_{\mathbf{u}},L]$, that is

$$\mathbf{v}_\vartheta(s)\in\Omega\ \ \forall s\in(\widetilde{\xi}_{\mathbf{u}},L]\quad\text{and}\ \ \mathbf{v}_\vartheta\ \text{ is injective in }[\widetilde{\xi}_{\mathbf{u}},L]. \tag{3.22}$$

Corollary 3.4. *(Compliance)*
Assume (2.2)-(2.11), (3.8),(3.9) hold true, $\mathbf{u}$ *belongs to* $\operatorname{argmin}\mathcal{F}$ *and* $\widetilde{\xi}_{\mathbf{u}}\in[0,L]$ *is the detachment parameter of* $\mathbf{u}$. *Then*

$$\begin{aligned}\int_{\widetilde{\xi}_{\mathbf{u}}}^{L}\dot{\vartheta}^2\,ds=&\ \frac{1}{EJ}\int_{\widetilde{\xi}_{\mathbf{u}}}^{L}\mathbf{f}\cdot\left[(\cos\vartheta_{\mathbf{u}}(s)\mathbb{W}-\sin\vartheta_{\mathbf{u}}(s)\mathbb{I})\,\dot{\mathbf{u}}_*\right]\vartheta_{\mathbf{u}}(s)\,ds\\ &=-\frac{|\mathbf{f}|}{EJ}\int_{\widetilde{\xi}_{\mathbf{u}}}^{L}\vartheta_{\mathbf{u}}(s)\sin(\vartheta_{\mathbf{u}}(s)-\varphi_*(s))\,ds,\end{aligned} \tag{3.23}$$

and

$$\mathcal{F}(\mathbf{u})=I_{\widetilde{\xi}_{\mathbf{u}}}(\vartheta_{\mathbf{u}})=$$

$$=-\frac{|\mathbf{f}|}{EJ}\int_{\widetilde{\xi}_{\mathbf{u}}}^{L}\left\{\frac{1}{2}\sin(\vartheta_{\mathbf{u}}(s)-\varphi_*(s))+\vartheta_{\mathbf{u}}(s)[\cos(\vartheta_{\mathbf{u}}(s)-\varphi_*(s))-\cos\varphi_*(s)]\right\}ds. \tag{3.24}$$

Proof. By multiplying for ϑ both the terms in (3.10), after integrating and taking into account (3.11) and (3.20) we get (3.23). After a simple substitution (3.24) follows. □

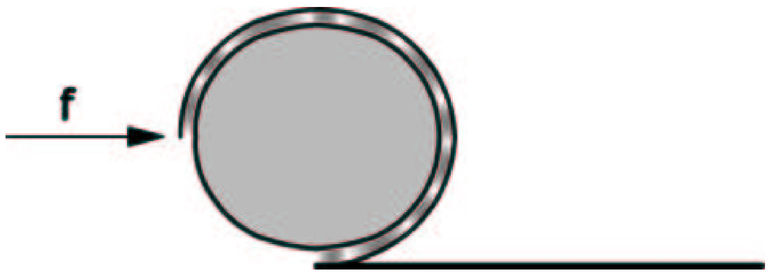

FIGURE 3.1. Example 3.5.

A slight modification of Example 2.20 provides a simple explicit solution of the nonlinear equation (3.20) fulfilling boundary conditions (3.11), as shown by the following example.

Example 3.5. We choose $\varphi(\mathbf{x}) = x_1^4 + ((|x_2|-1)^+)^4 - 1$, say

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 \,|\, |x_1| > 1 \text{ if } |x_2| < 1, x_1^4 + |x_2 - 1|^4 > 1 \text{ if } |x_2| > 1\},$$

$\mathbf{u}_*$ denotes the arc-length parametrization of the portion of the boundary $\partial\Omega$ connecting $(0,-2)$ and $(-1,0)$ whose length is L, $\mathbf{u}_*(0) = (0,-2)$ and $\mathbf{u}_*(L) = (-1,0)$ and $\mathbf{f} = f\mathbf{e}_1$.
In such geometry we find explicitely a complete detached solution which do satify (3.20) and (3.11), given by

$$\mathbf{w}(s) = (s,-1), \quad s \in [0,L].$$

Indeed we have $\widetilde{\xi}_{\mathbf{w}} = 0$ and $\vartheta_{\mathbf{w}}(s) = \varphi_*(s)$ for every s. Then, by taking f sufficiently large we have $\mathcal{F}(\mathbf{w}) < \mathcal{F}(\mathbf{u}_*)$. It is easy to verify that $\mathbf{u}_*$ is a strict local minimizer for $\mathcal{F}$ in the weak topology of $H^2(0,L;\overline{\Omega})$, moreover $\mathbf{u}_*$ seems to be the *physical* solution since $\mathbf{u}_*$ cannot snap to $\mathbf{w}$ without overleaping a potential wall. It seems reasonable also that $\mathbf{w}$ is a global minimizer for $\mathcal{F}$ in $\mathcal{A}$, though we are not able to prove this point.

4. Explicit conditions for detachment from a flat substrate

In this section on we focus our attention on rectilinear beams. We assume

$$\Omega = \{\mathbf{x} = \{x_1, x_2\} \in \mathbb{R}^2 : \; x_2 \geq 0\}, \tag{4.1}$$

$$\mathbf{u}_*(s) = s\mathbf{e}_1, \; s \in [0,L]. \tag{4.2}$$

Hence (2.4),(2.5) are automatically fulfilled with the choice $\varphi(\mathbf{x}) = -x_2$. Then, by assuming again (2.2)(2.3), the functional (2.9) reads

$$\mathcal{F}(\mathbf{u}) = \begin{cases} \dfrac{EJ}{2}\displaystyle\int_0^L \kappa(\mathbf{u})^2\, ds - \mathbf{f}\cdot(\mathbf{u}(L) - L\mathbf{e}_1) + W_\psi(\mathbf{u}), \text{if } \ \mathbf{u}\in\mathcal{A} \\ \\ +\infty \quad \text{otherwise} \end{cases} \tag{4.3}$$

and we introduce now an auxiliary problem for a re-scaled version of the functional $\mathcal{F}$. First we define an auxiliary functional $\mathcal{J} : [0, L]\times H^2(0, 1; \mathbb{R}^2) \to \mathbf{R}\cup\{+\infty\}$ as follows:

$$\mathcal{J}(\xi, \mathbf{v}) = \begin{cases} \mathcal{E}(\xi, \mathbf{v}) & \text{if } \ \mathbf{v}\in\mathcal{B}, \ \ 0 \le \xi < L \\ 0 & \text{if } \ \mathbf{v}\in\mathcal{B}, \ \ \xi = L \\ +\infty & \text{otherwise} \end{cases} \tag{4.4}$$

where $\mathcal{B}$ is the closure in the weak topology of $H^2(0, 1; \mathbb{R}^2)$ of the set B, defined as follows,

$$B = \{\mathbf{u}\in H^2((0, 1); \mathbb{R}^2) : \ \mathbf{u} \text{ injective}, \ |\dot{\mathbf{u}}| = 1 \\ \text{and } \ \mathbf{u}(0) = \mathbf{u}_*(0), \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_*(0)\} \tag{4.5}$$

and

$$\mathcal{E}(\xi, \mathbf{v}) = \frac{EJ}{2(L-\xi)}\int_0^1 \kappa(\mathbf{v})^2\, ds - (L-\xi)\int_0^1 \mathbf{f}\cdot(\dot{\mathbf{v}}(s) - \mathbf{e}_1)\, ds + \psi(L-\xi)\,. \tag{4.6}$$

In order to prove that $\mathcal{J}$ admits global minimizers via direct method in the calculus of variations it is enough showing that $\mathcal{J}$ is lower semicontinuous in the product of $[0, L]$ and $H^2(0, 1; \Omega)$ endowed with euclidean and weak convergence respectively. This property is proven by the following Lemma.

Lemma 4.1. *Assume (2.2)-(2.3), (2.7),(2.8) and (4.1)-(4.6). Then for every* $\xi_n \to \xi$ *in* $[0, L]$ *and for every* $\mathbf{v}_n \to \mathbf{v}$ *weakly in* $H^2(0, 1; \Omega)$ *we have*

$$\liminf \mathcal{J}(\xi_n, \mathbf{v}_n) \ge \mathcal{J}(\xi, \mathbf{v})$$

and $\mathcal{J}$ *achieve a finite minimum over* $\{[0, L]\times\mathcal{A}\}$.

Proof. The proof is obvious when $\xi \ne L$. If $\xi = L$ we have only to prove that

$$\liminf \mathcal{J}(\xi_n, \mathbf{v}_n) \ge 0 = \mathcal{J}(L, \mathbf{v}).$$

Such relationship follows by Poincarè and Young inequalities:

$$\int_0^1 \mathbf{f}\cdot(\dot{\mathbf{v}}_n(s)-\mathbf{e}_1)\,ds \le \frac{|\mathbf{f}|^2}{EJ}+\frac{EJ}{4}\int_0^1 |\dot{\mathbf{v}}_n(s)-\mathbf{e}_1|^2\,ds \le \\ \le \frac{|\mathbf{f}|^2}{EJ}+\frac{EJ}{4}\int_0^1 |\ddot{\mathbf{v}}_n(s)|^2\,. \tag{4.7}$$

Then the thesis follows by

$$\mathcal{J}(\xi_n,\mathbf{v}_n) \ge \frac{EJ}{4(L-\xi_n)}\int_0^1 |\ddot{\mathbf{v}}_n(s)|^2\,ds-(L-\xi_n)\frac{|\mathbf{f}|^2}{EJ}+\psi(L-\xi_n)\ge \\ \ge -(L-\xi_n)\frac{|\mathbf{f}|^2}{EJ}\,. \tag{4.8}$$

□

We prove now that minimization of $\mathcal{J}$ and $\mathcal{F}$ are equivalent problems.

Theorem 4.2. *Assume (2.2)-(2.3), (2.7),(2.8), (4.1)-(4.6) hold true.*
If $\mathbf{u}\in \operatorname{argmin}\mathcal{F}$, *then*

- *If* $\mathbf{u}\equiv\mathbf{u}_*$ *then* $(L,\mathbf{v})\in\operatorname{argmin}\mathcal{J}$ *for every* $\mathbf{v}\in\mathcal{A}$ *and* $\mathcal{J}(L,\mathbf{v})=0$.
- *If there is a detachment parameter* $\xi_{\mathbf{u}}<L$, *then* $(\xi_{\mathbf{u}},\mathbf{v})\in\operatorname{argmin}\mathcal{J}$, *where* $\mathbf{v}$ *is the unique curve related to* $\mathbf{u}$ *by the following relation*

$$\mathbf{v}(t)=(L-\xi)^{-1}\mathbf{u}(\xi+t(L-\xi))\qquad \text{if}\,,0\le\xi<L,\ t\in[0,1],\ \mathbf{u}\in\mathcal{A}. \tag{4.9}$$

Conversely let $(\xi,\mathbf{v})\in\operatorname{argmin}\mathcal{J}$ *then*

$$\mathbf{u}(s)=\begin{cases} s\,\mathbf{e}_1 & \text{if } 0\le s\le\xi\\ (L-\xi)\ \mathbf{v}\left(\dfrac{s-\xi}{L-\xi}\right) & \text{if } \xi<s\le L\end{cases} \tag{4.10}$$

belongs to $\operatorname{argmin}\mathcal{F}$.
In addition if $\xi<L$ *we get* $\mathbf{u}(t)\cdot\mathbf{e}_2>0$ *in* $(0,L)$ *and* $\mathbf{v}(t)\cdot\mathbf{e}_2>0$ *in* $(0,1)$. *Notice that we can have* $\mathbf{v}(1)\cdot\mathbf{e}_2=0$ *in case of secondary contact with the substrate at the free end of the rod (e.g. when* $\mathbf{u}(L)\ne\mathbf{u}_*(L)$, $\varphi(\mathbf{u}(L))=0$*).*

Proof. Let $\mathbf{u}$ be a global minimizer of $\mathcal{F}$, and $\xi\in[0,L)$ its detachment parameter. Set $\mathbf{v}(t)=(L-\xi)^{-1}\mathbf{u}(\xi+t(L-\xi))$. A direct computation shows that

$$\mathcal{F}(\mathbf{u}) = \frac{EJ}{2(L-\xi)} \int_0^1 \kappa(\mathbf{v})^2\, ds + \\ -(L-\xi)\int_0^1 \mathbf{f}\cdot(\dot{\mathbf{v}}(s)-\mathbf{e}_1)\, ds + \psi(L-\xi) \tag{4.11}$$

hence, arguing by contradiction, if $(\xi, \mathbf{v})$ were not a global minimizer of $\mathcal{J}$ then there should exist $(\xi', \mathbf{v}')$ such that

$$\mathcal{J}(\xi_n, \mathbf{v}_n) < \mathcal{J}(\xi, \mathbf{v}).$$

Then by setting

$$\mathbf{u}'(s) = (L-\xi')\mathbf{v}'\left(\frac{s-\xi'}{L-\xi'}\right)$$

we get easily

$$\mathcal{F}(\mathbf{u}') = \mathcal{J}(\xi', \mathbf{v}') < \mathcal{J}(\xi, \mathbf{v}) = \mathcal{F}(\mathbf{u})$$

thus contradicting minimality of $\mathbf{u}$. The case $\xi = L$ can be treated analogously.

In order to prove the converse we may notice that we have only to show that $\mathbf{v}_2 > 0$ in $(0,1)$ whenever $\xi < L$: if this were not true, then by proceeding as in the proof of Lemma 2.4, we may show that there exists a unique $0 < \tau < 1$ such that $\mathbf{v}(t) = t\mathbf{e}_1$ in $[0,\tau]$ and $\mathbf{v}(t) \neq t\mathbf{e}_1$ for every $t \in (\tau, 1]$. Then we may choose $0 < \delta < \tau$ and by setting $\mathbf{w}(t) = (1-\delta)^{-1}\mathbf{v}(\delta + t(1-\delta))$ we get, by taking into account that $\mathbf{v}(t) = t\mathbf{e}_1$ in $[0,\delta]$,

$$\mathcal{J}(\xi+\delta(L-\xi), \mathbf{v}) = (1-\delta)\left\{\frac{EJ}{2(L-\xi)}\int_0^1 |\ddot{\mathbf{v}}(s)|^2\, ds\right\} + \\ -(1-\delta)\left\{(L-\xi)\int_0^1 \mathbf{f}\cdot(\dot{\mathbf{v}}(s)-\mathbf{e}_1)\, ds\right\} + \psi((1-\delta)(L-\xi)) < \\ < \mathcal{J}(\xi, \mathbf{v}), \tag{4.12}$$

a contradiction that completes the proof. □

The equivalence Theorem 4.2 provides additional information on the structure for global minimizers of $\mathcal{F}$. For instance, in the present context of flat substrate, if ψ grows slowly enough then either the rod stays bonded to the substrate or it is fully detached, as stated by the following Theorem.

Theorem 4.3. *Assume (2.2)-(2.3), (2.7),(2.8), (4.1)-(4.6),* $\mathbf{u} \in \operatorname{argmin} \mathcal{F}$, $\xi = \xi_{\mathbf{u}}$ *is the related detachment parameter and*

$$\psi \in C^1([0,L]) \quad s.t. \quad t \to t^{-1}\psi(t) \quad \text{is non increasing in } (0,L]\,. \tag{4.13}$$

Then either $\mathbf{u} \equiv \mathbf{u}_*$ *or the detachment parameter fulfils* $\xi_{\mathbf{u}} = 0$.

Proof. Assume by contradiction that $\xi \in (0,L)$: then, by setting $\mathbf{v}(s) = \mathbf{u}(\xi + s(L-\xi))$, Theorem 4.2 implies that $(\xi, \mathbf{v})$ is a global minimizer of $\mathcal{J}$. Since

$$\frac{\partial \mathcal{J}}{\partial \xi}(\mathbf{u},\xi) = \frac{EJ}{2(L-\xi)^2}\int_0^1 \kappa(\mathbf{u})^2\, ds + \int_0^1 \mathbf{f}\cdot(\dot{\mathbf{u}}(s)-\mathbf{e}_1)\, ds - \psi'(L-\xi) \tag{4.14}$$

and we have

$$\frac{\partial \mathcal{J}}{\partial \xi}(\mathbf{v},\xi) = 0, \tag{4.15}$$

therefore by taking into account that the derivative of $t \to t^{-1}\psi(t)$ is negative we get $\psi \geq t\psi'$ and

$$\begin{aligned} \mathcal{J}(\xi,\mathbf{v}) = \frac{EJ}{(L-\xi)}\int_0^1 \kappa(\mathbf{v})^2\, ds + \\ +\psi(L-\xi) - (L-\xi)\psi'(L-\xi) \geq \frac{EJ}{(L-\xi)}\int_0^1 \kappa(\mathbf{v})^2\, ds \geq 0. \end{aligned} \tag{4.16}$$

Since, by Theorem 4.2, $\mathbf{v}\cdot\mathbf{e}_2 > 0$ in $(0,1]$, the last inequality in (4.16) is strict, hence

$$0 = \mathcal{J}(L,\mathbf{v}) < \mathcal{J}(\xi,\mathbf{v}) \tag{4.17}$$

thus contradicting minimality of $(\xi,\mathbf{v})$ and then minimality of $\mathbf{u}$. □

A necessary condition for a complete peeling of the rod is given by the following Theorem.

Theorem 4.4. *Assume that (2.2),(2.3), (2.7),(2.8), (4.1)-(4.6) hold true and there is a completely detached configuration is a global minimizer of* $\mathcal{F}$ *in* $\mathcal{A}$ *(say there is* $\mathbf{u} \in \operatorname{argmin} \mathcal{F}$ *with* $\xi_{\mathbf{u}} = 0$*). Then*

$$\psi'(L_-) \leq 4|\mathbf{f}|\min\left\{1, \frac{|\mathbf{f}|L^2}{3EJ}\right\}. \tag{4.18}$$

Proof. Let $(0,\mathbf{u}) \in \operatorname{argmin} \mathcal{F}$ be given. Then Theorem 4.2 entails $(0,\mathbf{v}) \in \operatorname{argmin} \mathcal{J}$, where $\mathbf{v}$ is related to $\mathbf{u}$ by (4.9), thus $\frac{\partial \mathcal{J}}{\partial \xi}(0,\mathbf{v}) \geq 0$ and so, after an integration by parts, by taking into account the condition $\dot{\mathbf{v}}(0) = \mathbf{e}_1$, we get

$$\psi'(L_-) \leq \frac{EJ}{2L^2}\int_0^1 \kappa(\mathbf{v})^2\,ds + \int_0^1 \mathbf{f}\cdot(\dot{\mathbf{v}}(s)-\mathbf{e}_1)\,ds =$$

$$= \frac{EJ}{2L^2}\int_0^1 \kappa(\mathbf{v})^2\,ds + \int_0^1 (1-s)\mathbf{f}\cdot\ddot{\mathbf{v}}(s),ds \leq \tag{4.19}$$

$$\leq \frac{EJ}{2L^2}\int_0^1 \kappa(\mathbf{v})^2\,ds + \frac{|\mathbf{f}|}{\sqrt{3}}\left\{\int_0^1 \kappa(\mathbf{v})^2\,ds\right\}^{\frac{1}{2}}$$

and by taking into account that

$$\mathcal{J}(0,\mathbf{v}) \leq \mathcal{J}(0,s\mathbf{e}_1) = \psi(L) \tag{4.20}$$

we get

$$\frac{EJ}{2L}\int_0^1 \kappa(\mathbf{v})^2\,ds - L\int_0^1 \mathbf{f}\cdot(\dot{\mathbf{v}}(s)-\mathbf{e}_1)\,ds + \psi(L) \leq \psi(L) \tag{4.21}$$

that is

$$\frac{EJ}{2L}\int_0^1 \kappa(\mathbf{v})^2\,ds \leq \frac{|\mathbf{f}|L}{\sqrt{3}}\left\{\int_0^1 \kappa(\mathbf{v})^2\,ds\right\}^{\frac{1}{2}}. \tag{4.22}$$

Therefore

$$\left\{\int_0^1 \kappa(\mathbf{v})^2\,ds\right\}^{\frac{1}{2}} \leq \frac{2L^2|\mathbf{f}|}{\sqrt{3}EJ}. \tag{4.23}$$

and by recalling (4.19) we get

$$\psi'(L_-) \leq \frac{4}{3}\frac{L^2|\mathbf{f}|^2}{EJ}. \tag{4.24}$$

On the other hand, by recalling again that

$$\psi'(L_-) \leq \frac{EJ}{2L^2}\int_0^1 \kappa(\mathbf{v})^2\,ds + \int_0^1 \mathbf{f}\cdot(\dot{\mathbf{v}}(s)-\mathbf{e}_1)\,ds$$

and

$$\frac{EJ}{2L}\int_0^1 \kappa(\mathbf{v})^2\,ds \leq L\int_0^1 \mathbf{f}\cdot(\dot{\mathbf{v}}(s)-\mathbf{e}_1)\,ds$$

we get

$$\psi'(L_-) \leq 2\int_0^1 \mathbf{f}\cdot(\dot{\mathbf{v}}(s)-\mathbf{e}_1)\,ds \leq 4|\mathbf{f}| \tag{4.25}$$

and thesis follows by gathering together (4.24) and (4.25). □

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