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Jump-diffusion unravelling of a non Markovian generalized Lindblad master equation

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The "correlated-projection technique" has been successfully applied to derive a large class of highly non Markovian dynamics, the so called non Markovian generalized Lindblad type equations or Lindblad rate equations. In this article, general unravellings are presented for these equations, described in terms of jump-diffusion stochastic differential equations for wave functions. We show also that the proposed unravelling can be interpreted in terms of measurements continuous in time, but with some conceptual restrictions. The main point in the measurement interpretation is that the structure itself of the underlying mathematical theory poses restrictions on what can be considered as observable and what is not; such restrictions can be seen as the effect of some kind of superselection rule. Finally, we develop a concrete example and we discuss possible effects on the heterodyne spectrum of a two-level system due to a structured thermal-like bath with memory.

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I. INTRODUCTION

Open quantum system theory concentrates on the study of the time evolution a quantum system in contact with an environment; in particular, this theory aims to describe phenomena such as decoherence, relaxation, emission of light, evolution of entanglement[1–5]. Starting from the Hamiltonian approach describing the coupled evolution of the quantum system and the environment, the reduced evolution of the quantum system is obtained by tracing out the degrees of freedom of the environment. This allows to describe the time evolution of the open system in terms of its density matrix $\rho_S(t)$ with the help of a quantum master equation. Invoking standard assumptions as weak coupling limit and Born-Markov approximation, one can derive the Markovian quantum master equation[3–5], with infinitesimal generator in the Lindblad form[6,7]. This approach, called the *Markovian approach*, is physically based upon the absence of memory effects in the action of the environment. This is a good and useful assumption in several physical examples, namely in quantum optics[1–5].

However, such assumptions are not valid in general and in many physically important cases the description of a reduced quantum evolution requires a non-Markovian approach involving strong and long memory effects. For example, situations with strong coupled systems, entanglement and correlation in the initial state, finite reservoirs. . . need to be described by non-Markovian dynamics. Different techniques, such as the Nakajima-Zwanzig projection technique, the time-convolutionless operator technique, random Lindblad operator, random functional equations have been developed to derive non-Markovian quantum master equations[3,4,8–14]. Recently, the concept of correlated projection technique has been used in order to describe a non-Markovian generalization of Lindblad type master equations (or Lindblad rate equations)[15–17]. This approach has been successfully applied to describe non-Markovian models: structured reservoirs, two-state systems coupled with energy bands[15–25]. . .

An active line of research concentrates on the study of the behaviour of the solutions of these equations (thermalization, return to equilibrium, decoherence,...). But even in the Markovian case, the quantum master equations remain often of a formal interest. In particular, most of the equations cannot be solved analytically and involve a large number of parameters which prevent numerical simulations. Concerning the numerical aspect, a powerful approach is the theory of "stochastic wave function unravelling". This consists in constructing a stochastic differential equation for a wave function $\psi(t)$ such that $\mathbb{E}[|\psi(t)\rangle\langle\psi(t)|] = \rho_S(t)$. Then, by taking the average of a large number of realizations of $\psi(t)$ one reproduces the solution of the master equation. This has been applied in many Markovian situations[2,4,26,27]. Concerning the non-Markovian framework, different extensions of this approach have been developed[28–30] (there is no general and common approach).

In the Markovian case the stochastic unravelling of the master equation has not only a technical usefulness, but it can be also interpreted in terms of measurements in continuous time; often the name of *quantum trajectory theory* is used[2,27]. In particular, for quantum optical systems the stochastic formulation is used to describe direct, heterodyne and homodyne detection. However, in the non-Markovian setup the notion of quantum trajectories as well as the measurement interpretation are still highly debated[8,9,28,30–33].

For the non Markovian generalization of Lindblad type master equations[15–17], only particular unravellings have been presented[34,35]. In this article, we aim to present a general approach to obtain unravellings for this type of equations and to show that in this case an interpretation of the unravelling in terms of measurements in continuous time is possible. Our approach is based upon the general technique used to unravel Markovian Lindblad equations. In particular, our results include and generalize the previous results[34]. However, we have an important conceptual difference from the Markovian case. We are assuming that the structure of the bath responsible of the non Markovian behaviour is not observable and this makes unobservable some of the components of the noises introduced in the unravelling.

The article is structured as follows. In Section II, we describe the Lindblad rate equations. In Section III, we present the jump-diffusion unravellings of these equations. In particular, we derive non Markovian generalizations of stochastic Schrödinger equations. The stochastic master equations and the measurement interpretation are given in Section IV. In Section V, we construct a concrete non Markovian model (a two level system in contact with a structured environment), we present a possible unravelling, and we show possible effects of the non Markov dynamics on the heterodyne spectrum.

II. NON-MARKOVIAN GENERALIZED LINDBLAD-TYPE MASTER EQUATIONS

In this section we introduce the non Markovian Lindblad-type master equation which we are interested in. These equations can be obtained by the application of the *correlated projection technique* and are sometimes called *Lindblad rate equations*[10,15–21,23–25]. For any separable complex Hilbert space $\mathcal{H}$ we denote by $\mathcal{L}(\mathcal{H})$ the space of the linear bounded operators on $\mathcal{H}$, by $\mathcal{T}(\mathcal{H})$ the space of trace class operators and by $\mathcal{S}(\mathcal{H})$ the set of statistical operators (a statistical operator is a trace class, positive operator with trace 1).

Let $\mathcal{H}_S$ denote the Hilbert space representing the open system. The generalized master equation we consider is the evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\eta_i(t) = -\mathrm{i}[H^i, \eta_i(t)] + \sum_{\alpha\in A}\sum_{k=1}^{n}\left(R_\alpha^{ik}\eta_k(t)R_\alpha^{ik*} - \frac{1}{2}\left\{R_\alpha^{ki*}R_\alpha^{ki}, \eta_i(t)\right\}\right) \tag{1}$$

for the vector $\big(\eta_1(t), \dots, \eta_n(t)\big)$ with components in $\mathcal{T}(\mathcal{H}_S)$. The quantities H^i, R_α^{ki} are system operators which we take to be bounded for mathematical simplicity and A is a finite set of indices.

Assumption 1. $\mathcal{H}_S$ is a complex separable Hilbert space, $H^i = H^{i*} \in \mathcal{L}(\mathcal{H}_S)$, $R_\alpha^{ki} \in \mathcal{L}(\mathcal{H}_S)$, $k, i = 1, \dots, n$, $\alpha \in A$. The initial condition of Eq. (1) has the properties

$$\eta_i(0) \in \mathcal{T}(\mathcal{H}_S), \qquad \eta_i(0) \geq 0, \qquad \sum_{i=1}^{n} \mathrm{Tr}_{\mathcal{H}_S}\left\{\eta_i(0)\right\} = 1. \tag{2}$$

Remark 1. Equation (1) preserves the properties (2) at all times[16]; then, we interprete as system state the statistical operator

$$\eta_S(t) = \sum_{i=1}^{n} \eta_i(t). \tag{3}$$

The proof of the positivity preservation property of Eq. (1) is very instructive and goes through the embedding of the dynamics $\{\eta_i(0)\} \mapsto \{\eta_i(t)\}$ into an usual Lindblad dynamics in an extended state space[16]. Let us consider the enlarged space $\mathcal{H} = \mathcal{H}_S \otimes \mathbb{C}^n$. Let $\{e_i,\ i = 1, \dots, n\}$ be a reference orthonormal basis of $\mathbb{C}^n$. Let us introduce the block diagonal operator $\tilde{\eta}(t)$ on $\mathcal{H}$ by

$$\tilde{\eta}(t) = \sum_{i=1}^{n} \eta_i(t) \otimes |e_i\rangle\langle e_i|$$

and set

$$H = \sum_{i=1}^{n} H^i \otimes |e_i\rangle\langle e_i|, \qquad S_\alpha^{ij} = R_\alpha^{ij} \otimes |e_i\rangle\langle e_j|. \tag{4}$$

Then, we get immediately from (1) the evolution equation for $\tilde\eta(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\tilde\eta(t) = \tilde{\mathcal{L}}[\tilde\eta(t)] \equiv -\mathrm{i}[H,\tilde\eta(t)] + \sum_{\alpha\in A}\sum_{i,j=1}^{n}\left(S_\alpha^{ij}\tilde\eta(t)S_\alpha^{ij*} - \frac{1}{2}\left\{S_\alpha^{ij*}S_\alpha^{ij},\tilde\eta(t)\right\}\right). \tag{5}$$

For block diagonal initial conditions the two equations (1) and (5) are completely equivalent. Let us note that the linear map $\tilde{\mathcal{L}}$ is explicitly in the Lindblad form, so that the maps $\tilde\eta(0)\mapsto\tilde\eta(t)$ and $\{\eta_i(0)\}_{i=1}^n \mapsto \{\eta_i(t)\}_{i=1}^n$ are *completely positive* (CP).

In spite of the construction above, the index i is not interpreted as a quantum degree of freedom, but as the value of a classical observable. In typical applications the index i labels the energy bands of a structured environment[15,16,21,22,34]. A vector of operators with the properties (2) can be seen as a classical/quantum state. If we set $p_i(t) = \mathrm{Tr}_{\mathcal{H}_S}\{\eta_i(t)\}$ and $\hat\eta_i(t) = \eta_i(t)/p_i(t)$, we have $\hat\eta_i(t)\in\mathcal{S}(\mathcal{H}_S)$, $p_i(t)\geq 0$, $\sum_{i=1}^n p_i(t) = 1$. In quantum information the set of probabilities and statistical operators $\{p_i(t),\hat\eta_i(t);\, i=1,\ldots,n\}$ is called an *ensemble* and it is completely equivalent to the vector $\big(\eta_1(t),\ldots,\eta_n(t)\big)$[36,37]. In this setup the system state (3) is known as average state and it does not contain the information on the classical label i. Equation (1) gives a memoryless evolution for the ensemble $\{p_i(t),\hat\eta_i(t);\, i=1,\ldots,n\}$; it is the evolution of the system state $\eta_S(t)$ which is non Markovian.

In the Markov case it is well known how to construct general unravellings of a master equation and how to give a measurement interpretation to them. So, to have an usual master equation in Lindblad form extending our non Markovian dynamics (1) is a good starting point for the whole construction. However, Eq. (5) is not the unique extension of (1) and here we give another extension which is in some sense more convenient as starting point. The possible extensions depend on having or not the condition $R_\alpha^{ij}\propto\delta_{ij}$; so, we put in evidence some diagonal terms.

Assumption 2. Let us take $A = \{-m_1,\ldots,-1,1,\ldots,m_2\}$, and $R_{-\alpha}^{ij} = \delta_{ij}L_\alpha^i$, for $\alpha = 1,\ldots,m_1$.

With this assumption Eq. (1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\eta_i(t) = \mathcal{K}_i\big(\eta_1(t),\ldots,\eta_n(t)\big), \tag{6a}$$

$$\mathcal{K}_i\big(\tau_1,\dots,\tau_n\big) := -\mathrm{i}[H^i,\tau_i] + \sum_{\alpha=1}^{m_1}\left(L^i_\alpha \tau_i {L^i_\alpha}^* - \frac{1}{2}\left\{{L^i_\alpha}^* L^i_\alpha, \tau_i\right\}\right) + \sum_{\alpha=1}^{m_2}\sum_{k=1}^{n}\left(R^{ik}_\alpha \tau_k R^{ik*}_\alpha - \frac{1}{2}\left\{R^{ki*}_\alpha R^{ki}_\alpha, \tau_i\right\}\right). \quad (6b)$$

By using the operators (4), we define the new operators

$$V_\alpha = \sum_{i,j=1}^{n} S^{ij}_{-\alpha} \equiv \sum_{i=1}^{n} L^i_\alpha \otimes |e_i\rangle\langle e_i|, \qquad \alpha = 1,\dots,m_1\,, \quad (7a)$$

$$S^j_\beta = \sum_{i=1}^{n} S^{ij}_\beta \equiv \sum_{i=1}^{n} R^{ij}_\beta \otimes |e_i\rangle\langle e_j|, \qquad \beta = 1,\dots,m_2\,, \quad (7b)$$

and the Lindblad map $\mathcal{L}$: $\forall \tau \in \mathcal{T}(\mathcal{H})$,

$$\mathcal{L}[\tau] = -\mathrm{i}[H,\tau] + \sum_{\alpha=1}^{m_1}\left(V_\alpha \tau {V_\alpha}^* - \frac{1}{2}\left\{{V_\alpha}^* V_\alpha, \tau\right\}\right) + \sum_{\alpha=1}^{m_2}\sum_{j=1}^{n}\left(S^j_\alpha \tau {S^j_\alpha}^* - \frac{1}{2}\left\{{S^j_\alpha}^* S^j_\alpha, \tau\right\}\right). \quad (8)$$

Then, we consider the Markovian quantum master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\eta(t) = \mathcal{L}[\eta(t)], \quad (9)$$

with the initial condition

$$\eta(0) \in \mathcal{S}(\mathcal{H}), \quad \mathrm{Tr}_{\mathbb{C}^n}\left\{\eta(0)\left(\mathbb{1}\otimes|e_i\rangle\langle e_i|\right)\right\} = \eta_i(0). \quad (10)$$

Remark 2. Let us use a subscript i to denote the i-th block on the diagonal of any trace-class operator, i.e. $\tau_i = \mathrm{Tr}_{\mathbb{C}^n}\left\{\tau\left(\mathbb{1}\otimes|e_i\rangle\langle e_i|\right)\right\}$. It is easy to check that

$$\tilde{\mathcal{L}}[\tau]_i = \mathcal{L}[\tau]_i = \mathcal{K}_i(\tau_1,\dots,\tau_n), \quad (11)$$

and, so, both the master equations (5) and (9) reduce to the same Lindblad rate equation (6) for the blocks on the diagonal, while they are different for the off-diagonal blocks. Being equal at time $t=0$ due to (10), we have that the blocks on the diagonal of $\eta(t)$ are exactly the quantities $\eta_i(t)$ satisfying Eq. (6).

Another way to describe the situation is to say that there is a *superselection rule* and only block-diagonal observables are permitted. Then, statistical operators with the same blocks on the diagonal are equivalent and represent the same physical state. In this sense the two master equations (5) and (9) are physically equivalent.

It is worthwhile to note that the operator $\tilde{\mathcal{L}}$ can always be written in the form $\mathcal{L}$. It is enough to change the meaning of the subscript in the operators R_α^{ij} or L_α^i in such a way that it includes also the index i. Then, given two triples (i, j, α) and (i', j', α'), we have that $i \neq i' \Rightarrow \alpha \neq \alpha'$ (the same holds for two couples (i, α) and (i', α')). In this way, in the sums in Eqs. (7) only one term survives and $\tilde{\mathcal{L}} = \mathcal{L}$. So, there is no loss of generality in considering only the master equation (9); the other case is always included, eventually at the price of a renaming and reordering of the indices.

It is useful to formalize the framework we have presented in terms of normal states on W^*-algebras and of CP dynamics.

Remark 3. Let $\mathcal{C}\big(\mathcal{X}; \mathcal{L}(\mathcal{H}_S)\big)$ be the W^*-algebra of the functions from $\mathcal{X} = \{1, 2, \ldots, n\}$ into $\mathcal{L}(\mathcal{H}_S)$[36,38]. By natural identifications we have $\mathcal{C}(\mathcal{X}; \mathbb{C}) \simeq \mathbb{C}^n$ and $\mathcal{C}\big(\mathcal{X}; \mathcal{L}(\mathcal{H}_S)\big) \simeq \mathcal{L}(\mathcal{H}_S) \otimes \mathbb{C}^n$, so that $a \in \mathcal{C}\big(\mathcal{X}; \mathcal{L}(\mathcal{H}_S)\big)$ means $a = (a_1, \ldots, a_n)$, $a_j \in \mathcal{L}(\mathcal{H})$; then, $\|a\| = \max_{j \in \mathcal{X}} \|a_j\|$. The predual space of $\mathcal{C}\big(\mathcal{X}; \mathcal{L}(\mathcal{H}_S)\big)$ is $\mathcal{C}\big(\mathcal{X}; \mathcal{T}(\mathcal{H}_S)\big) \simeq \mathcal{T}(\mathcal{H}_S) \otimes \mathbb{C}^n$, so that $\tau \in \mathcal{C}\big(\mathcal{X}; \mathcal{T}(\mathcal{H}_S)\big)$ means $\tau = (\tau_1, \ldots, \tau_n)$, $\tau_j \in \mathcal{T}(\mathcal{H})$; then, $\|\tau\|_1 = \sum_{j \in \mathcal{X}} \|\tau_j\|_1 = \sum_{j=1}^n \mathrm{Tr}_{\mathcal{H}_S} \left\{ \sqrt{\tau_j^* \tau_j} \right\}$. In a natural way a and τ can be considered as block-diagonal elements of $\mathcal{L}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$, respectively: $a \simeq \sum_{j=1}^n a_j \otimes |e_j\rangle\langle e_j|$, $\tau \simeq \sum_{j=1}^n \tau_j \otimes |e_j\rangle\langle e_j|$.

Remark 4. Equations (8) and (9) define a CP quantum dynamical semigroup $\mathcal{T}(t)$ on $\mathcal{T}(\mathcal{H})$. Then, we define the projection $\mathcal{P} : \mathcal{T}(\mathcal{H}) \to \mathcal{C}\big(\mathcal{X}; \mathcal{T}(\mathcal{H}_S)\big) \subset \mathcal{T}(\mathcal{H})$ by $(\mathcal{P}[\tau])_j = \mathrm{Tr}_{\mathbb{C}^n}\{\tau(\mathbb{1} \otimes |e_j\rangle\langle e_j|\}$. The dynamics associated to the Lindblad rate equation (6) turns out to be $\mathcal{P} \circ \mathcal{T}(t)\big|_{\mathcal{C}\left(\mathcal{X}; \mathcal{T}(\mathcal{H}_S)\right)}$; it is CP and Markovian. Finally, we define the projection $\mathcal{P}_S : \mathcal{C}\big(\mathcal{X}; \mathcal{T}(\mathcal{H}_S)\big) \to \mathcal{T}(\mathcal{H})$ by $\mathcal{P}_S[\tau] = \sum_j \tau_j$. The CP dynamics giving the system state (3) is $\mathcal{P}_S \circ \mathcal{P} \circ \mathcal{T}(t)\big|_{\mathcal{C}\left(\mathcal{X}; \mathcal{T}(\mathcal{H}_S)\right)}$ and it is this dynamics which is non Markovian.

III. UNRAVELLING OF NON MARKOVIAN LINDBLAD-TYPE MASTER EQUATIONS

In this section, we derive a general form of jump-diffusion stochastic differential equations (SDEs) for wave functions in the enlarged space $\mathcal{H} = \mathcal{H}_S \otimes \mathbb{C}^n$ which provide unravellings of the Lindblad rate equations (6). Having at hand the usual Markovian master equation (9), we adopt the usual approach[27,39,40] of stochastic Schrödinger equations in the Markovian case. This method is based on classical stochastic calculus (see for instance Refs. 41 and 42 and 27, Appendix A) and the notion of *a posteriori* states[43–46].

The key point of the theory is the construction of a linear and a non-linear *stochastic Schrödinger equation* (SSE), connected by a normalization and a Girsanov transformation, and, then, of the linear and non-linear *stochastic master equations.* The non-linear SSE is the key starting point for numerical simulations of the solution of a master equation, while the possibility of passing to linear equations is fundamental for the possibility of giving a measurement interpretation to the whole construction without violating the rules of quantum mechanics. Finally, the non-linear stochastic master equation gives the *a posteriori states*, the conditional state to be attributed at the system at time t, knowing the results of the measurement up to time t.

A. The linear stochastic Schrödinger equation

We consider a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$, satisfying the *usual hypotheses* [27, Appendix A]. On this space, we consider $d_1 + d_2 \times n$ independent standard Wiener processes W_α, W_β^j ($\alpha = 1, \ldots, d_1 \leq m_1$; $\beta = 1, \ldots, d_2 \leq m_2$; $j = 1, \ldots, n$) and $(m_1 - d_1) + (m_2 - d_2) \times n$ independent standard Poisson point processes N_α of intensity $\lambda_\alpha > 0$ and N_β^j of intensity $\lambda_\beta^j > 0$ ($\alpha = d_1 + 1, \ldots, m_1$; $\beta = d_2 + 1, \ldots, m_2$; $j = 1, \ldots, n$), also independent of the Wiener processes. All these processes are adapted and $W_\alpha^k(t)$, $N_\alpha(t) - \lambda_\alpha t$ and $N_\alpha^j(t) - \lambda_\alpha^j t$ are $(\mathcal{F}_t)$-martingales, under the reference probability $\mathbb{Q}$[41,42]. The trajectories of the Wiener processes are taken to be continuous and the trajectories of the Poisson processes continuous from the right. We set also

$$\lambda = \sum_{\alpha=d_1+1}^{m_1} \lambda_\alpha + \sum_{\alpha=d_2+1}^{m_2} \sum_{j=1}^{n} \lambda_\alpha^j \,.$$

Now, on $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$, we consider the following SDE for an $\mathcal{H}$-valued process:

$$\begin{aligned} \mathrm{d}\zeta(t) = {} & \left(K + \frac{\lambda}{2}\right) \zeta(t_-)\mathrm{d}t + \sum_{\alpha=1}^{d_1} V_\alpha \zeta(t_-)\mathrm{d}W_\alpha(t) + \sum_{\alpha=1}^{d_2} \sum_{k=1}^{n} S_\alpha^k \zeta(t_-)\mathrm{d}W_\alpha^k(t) \\ & + \sum_{\alpha=d_1+1}^{m_1} \left(\frac{1}{\sqrt{\lambda_\alpha}} V_\alpha - \mathbb{1}\right) \zeta(t_-)\mathrm{d}N_\alpha(t) + \sum_{\alpha=d_2+1}^{m_2} \sum_{k=1}^{n} \left(\frac{1}{\sqrt{\lambda_\alpha^k}} S_\alpha^k - \mathbb{1}\right) \zeta(t_-)\mathrm{d}N_\alpha^k(t), \end{aligned} \tag{12}$$

where the operator in the drift part is given by

$$K = -\mathrm{i}H - \frac{1}{2} \sum_{\alpha=1}^{m_1} {V_\alpha}^* V_\alpha - \frac{1}{2} \sum_{\alpha=1}^{m_2} \sum_{k=1}^{n} {S_\alpha^k}^* S_\alpha^k = \sum_{j=1}^{n} K^j \otimes |e_j\rangle\langle e_j|,$$

$$K^j = -\mathrm{i}H^j - \frac{1}{2}\sum_{\alpha=1}^{m_1} {L_\alpha^j}^* L_\alpha^j - \frac{1}{2}\sum_{\alpha=1}^{m_2}\sum_{k=1}^{n} {R_\alpha^{kj}}^* R_\alpha^{kj} \,.$$

By using the decomposition $\zeta(t) = \sum_{j=1}^n \zeta_j(t)\otimes e_j$, we get the equivalent system of SDEs

$$\begin{aligned}\mathrm{d}\zeta_j(t) = \left(K^j + \frac{\lambda}{2}\right)\zeta_j(t_-)\mathrm{d}t + \sum_{\alpha=1}^{d_1} L_\alpha^j \zeta_j(t_-)\mathrm{d}W_\alpha(t) + \sum_{\alpha=d_1+1}^{m_1}\left(\frac{1}{\sqrt{\lambda_\alpha}}L_\alpha^j - \mathbb{1}\right)\zeta_j(t_-)\mathrm{d}N_\alpha(t)\\ + \sum_{\alpha=1}^{d_2}\sum_{k=1}^{n} R_\alpha^{jk}\zeta_k(t_-)\mathrm{d}W_\alpha^k(t) + \sum_{\alpha=d_2+1}^{m_2}\sum_{k=1}^{n}\left(\frac{1}{\sqrt{\lambda_\alpha^k}}R_\alpha^{jk}\zeta_k(t_-) - \zeta_j(t_-)\right)\mathrm{d}N_\alpha^k(t). \quad (13)\end{aligned}$$

As usual the solutions of SDEs with jumps are taken to be continuous from the right with left limits (càdlàg processes); the notation t_- means the left limit.

Remark 5. If some of the operators S in the jump part is zero, we eliminate its contribution by taking the corresponding Poisson process with zero intensity, so that it is almost surely 0 for all times. In other words, if we have $S_\alpha^k = 0$ for some k and some $\alpha > d_2$, we take $\lambda_\alpha^k \downarrow 0$.

Assumption 3. We take a random normalized initial condition: $\zeta(0) = \zeta^0 = \sum_{i=1}^n \zeta_i^0 \otimes e_i$, ζ^0 is $\mathcal{F}_0$-measurable, $\mathbb{E}_\mathbb{Q}\left[\left\|\zeta^0\right\|^2\right] \equiv \sum_{i=1}^n \mathbb{E}_\mathbb{Q}\left[\left\|\zeta_i^0\right\|^2\right] = 1$. To reproduce the initial condition (2) we ask also $\mathbb{E}_\mathbb{Q}\left[|\zeta^0\rangle\langle\zeta^0|\right] = \eta(0)$. Mean values of random operators are defined in weak sense.

Equation (12) is a particular case of the equations studied in Refs. 30 and 40, so, we refer to those papers for the properties of its solution, while all the results could be obtained by standard arguments in stochastic calculus and the Itô formula for continuous and jump processes summarized by the *Itô table*

$$\begin{aligned}\mathrm{d}W_\alpha(t)\mathrm{d}W_\beta(t) &= \delta_{\alpha\beta}\mathrm{d}t, & \mathrm{d}W_\alpha^k(t)\mathrm{d}W_\beta^l(t) &= \delta_{\alpha\beta}\delta_{kl}\mathrm{d}t,\\ \mathrm{d}N_\alpha(t)\mathrm{d}N_\beta(t) &= \delta_{\alpha\beta}\mathrm{d}N_\alpha(t), & \mathrm{d}N_\alpha^i(t)\mathrm{d}N_\beta^j(t) &= \delta_{\alpha\beta}\delta_{ij}\mathrm{d}N_\alpha^i(t);\end{aligned} \quad (14)$$

all the other products are vanishing.

Theorem 1 ([30, Prop. 2.1, Theor. 2.4, Prop. 3.2]; [40, Theor. 1.1, Theor. 1.2])**.** *Under Assumptions 1 and 3, the SDE* (12) *admits a unique (up to $\mathbb{Q}$-equivalence) solution $\zeta(t)$, $t \geq 0$. Moreover, the mean state $\mathbb{E}_\mathbb{Q}[|\zeta(t)\rangle\langle\zeta(t)|]$ satisfies the master equation* (9).

Finally, under the probability $\mathbb{Q}$*, the process* $p(t) := \|\zeta(t)\|^2 \equiv \sum_{i=1}^{n} \|\zeta_i(t)\|^2$ *is a non-negative* $(\mathcal{F}_t)$*-martingale with* $\mathbb{Q}$*-mean* 1 *and it satisfies the Doléans SDE*

$$\begin{aligned}\mathrm{d}p(t) = p(t_-)\Bigg\{&\sum_{\alpha=1}^{d_1} v_\alpha(t)\mathrm{d}W_\alpha(t) + \sum_{\alpha=d_1+1}^{m_1}\left(\frac{I_\alpha(t)}{\lambda_\alpha}-1\right)\Big(\mathrm{d}N_\alpha(t)-\lambda_\alpha\mathrm{d}t\Big)\\ &+\sum_{\alpha=1}^{d_2}\sum_{k=1}^{n} v_\alpha^k(t)\mathrm{d}W_\alpha^k(t) + \sum_{\alpha=d_2+1}^{m_2}\sum_{k=1}^{n}\left(\frac{I_\alpha^k(t)}{\lambda_\alpha^k}-1\right)\Big(\mathrm{d}N_\alpha^k(t)-\lambda_\alpha^k\mathrm{d}t\Big)\Bigg\},\end{aligned} \tag{15}$$

where

$$v_\alpha(t) = 2\,\mathrm{Re}\left\langle \psi(t_-)\middle| V_\alpha\psi(t_-)\right\rangle \equiv 2\sum_{j=1}^{n}\mathrm{Re}\left\langle \psi_j(t_-)\middle| L_\alpha^j\psi_j(t_-)\right\rangle, \tag{16a}$$

$$v_\alpha^k(t) = 2\,\mathrm{Re}\left\langle \psi(t_-)\middle| S_\alpha^k\psi(t_-)\right\rangle \equiv 2\sum_{j=1}^{n}\mathrm{Re}\left\langle \psi_j(t_-)\middle| R_\alpha^{jk}\psi_k(t_-)\right\rangle, \tag{16b}$$

$$I_\beta(t) = \|V_\beta\psi(t_-)\|^2 \equiv \sum_{j=1}^{n}\left\|L_\beta^j\psi_j(t_-)\right\|^2, \tag{16c}$$

$$I_\beta^k(t) = \left\|S_\beta^k\psi(t_-)\right\|^2 \equiv \sum_{j=1}^{n}\left\|R_\beta^{jk}\psi_k(t_-)\right\|^2. \tag{16d}$$

The process

$$\psi(t) = \sum_{i=1}^{n}\psi_i(t)\otimes e_i \tag{17a}$$

is defined by

$$\begin{cases}\psi_k(t) = \dfrac{\zeta_k(t)}{\|\zeta(t)\|}, & \textit{if } \|\zeta(t)\|\neq 0,\\ \psi_k(t) = \psi, & \textit{if } \|\zeta(t)\| = 0,\end{cases} \tag{17b}$$

where $\psi \in \mathcal{H}_S$ *is a fixed vector of norm* $1/\sqrt{n}$*.*

Remark 6 (A first unravelling)*.* By the theorem above, $\mathbb{E}_{\mathbb{Q}}[|\zeta(t)\rangle\langle\zeta(t)|]$ satisfies the master equation (9) with initial condition $\eta(0)$ (Assumption 3). So, we have $\eta(t) = \mathbb{E}_{\mathbb{Q}}[|\zeta(t)\rangle\langle\zeta(t)|]$, $\forall t \geq 0$, and, by the discussion below Eq. (10), we get

$$\eta_i(t) = \mathbb{E}_{\mathbb{Q}}[|\zeta_i(t)\rangle\langle\zeta_i(t)|], \qquad i = 1,\ldots,n, \quad t\geq 0, \tag{18}$$

which shows that $\zeta(t)$ is a pure-state unravelling of the solution of the Lindblad rate equation (6).

Remark 7 ([40, Theor. 1.2]; [42, Theor. 29.2]). The solution of the Doléans SDE (15) is

$$
\begin{aligned}
p(t) = \left\|\zeta^0\right\|^2 \exp\Bigg\{ & \sum_{\alpha=1}^{d_1} \left(\int_0^t v_\alpha(s) \mathrm{d}W_\alpha(s) - \frac{1}{2} \int_0^t v_\alpha(s)^2 \mathrm{d}s \right) \\
& + \sum_{\alpha=1}^{d_2} \sum_{k=1}^{n} \left(\int_0^t v_\alpha^k(s) \mathrm{d}W_\alpha^k(s) - \frac{1}{2} \int_0^t v_\alpha^k(s)^2 \mathrm{d}s \right) \Bigg\} \\
\times \prod_{\beta=d_1+1}^{m_1} & \left\{ \exp\left[\int_0^t \left(\lambda_\beta - I_\beta(s)\right) \mathrm{d}s \right] \prod_{r\in(0,t]} \left[1 + \left(\frac{I_\beta(r)}{\lambda_\beta} - 1 \right) \Delta N_\beta(r) \right] \right\} \\
\times \prod_{\beta=m_1+1}^{m} \prod_{\ell=1}^{n} & \left\{ \exp\left[\int_0^t \left(\lambda_\beta^\ell - I_\beta^\ell(s)\right) \mathrm{d}s \right] \prod_{r\in(0,t]} \left[1 + \left(\frac{I_\beta^\ell(r)}{\lambda_\beta^\ell} - 1 \right) \Delta N_\beta^\ell(r) \right] \right\},
\end{aligned}
$$

where $\Delta N_\beta(r,\omega) = N_\beta(r,\omega) - N_\beta(r_-,\omega)$, $\Delta N_\beta^\ell(r,\omega) = N_\beta^\ell(r,\omega) - N_\beta^\ell(r_-,\omega)$. By the fact that a Poisson process has only a finite number of jumps in a compact interval, for every ω only a finite number of factors contributes to the product over r in the representation above.

Note that, if for some t, ω, β, ℓ one has $I_\beta^\ell(t,\omega) = 0$ and $\Delta N_\beta^\ell(t,\omega) = 1$, then $p(T,\omega) = 0$, $\forall T > t$. Similarly, $I_\beta(t,\omega) = 0$ and $\Delta N_\beta(t,\omega) = 1$ imply $p(T,\omega) = 0$, $\forall T > t$.

B. The generalized stochastic Schrödinger equation

The final aim is to derive an equation for the normalized process (17). This is based upon Itô stochastic calculus again and a Girsanov-type change of measure.

Remark 8 (The change of probability measure). For for every $T > 0$, we define the *physical probability* $\mathbb{P}^T$ over $(\Omega, \mathcal{F}_T)$ by

$$
\mathbb{P}^T(A) = \mathbb{E}_{\mathbb{Q}}\left[1_A p(T)\right] \equiv \int_A \|\zeta(T,\omega)\|^2 \mathbb{Q}(\mathrm{d}\omega), \qquad \forall A \in \mathcal{F}_T\,. \tag{19}
$$

Note that $\mathbb{P}^T$ depends also on ζ^0, which we assume to be normalized in the sense of Assumption 3. The martingale property given in Theorem 1 ensures that the family of probabilities $\{\mathbb{P}^T,\, T > 0\}$ is consistent, that is

$$
0 < t < T, \quad A \in \mathcal{F}_t \qquad \Rightarrow \qquad \mathbb{P}^T(A) = \mathbb{P}^t(A). \tag{20}
$$

To obtain from (20) the existence of a unique probability in the infinite horizon limit $T \to +\infty$ is a delicate problem and can be guaranteed only with respect to some subfiltration composed by Borel standard σ-algebras [27, Section A.5.5].

It is important to note that the denominator $\|\zeta(t)\|$ in the definition of the processes $\psi_k(t)$ could indeed vanish as stated in Remark 7. But, by the construction in Remark 8, this happens with probability zero with respect to the new probability $\mathbb{P}^T$, while this is not guaranteed under the reference probability $\mathbb{Q}$.

The important consequences of this change of measure are the modification of the characteristics of the driving processes $N_\alpha^k(t)$ and $W_\alpha^j(t)$ (due to some extension of the Girsanov theorem to the diffusive/jump case[41]) and the fact that $\psi(t)$ satisfies a non linear SDE, the *stochastic Schrödinger equation*[27,43–46].

Theorem 2 ([30, Prop. 2.5, Theor. 2.7]; [40, Prop. 1.1, Theor. 1.3])**.** *Under the probability $\mathbb{P}^T$, the processes $\hat{W}_\alpha$, $\hat{W}_\beta^k$, $t \in [0,T]$, $\alpha = 1,\ldots,d_1$, $\beta = 1,\ldots,d_2$, $k = 1,\ldots,n$, defined by*

$$\hat{W}_\alpha(t) = W_\alpha(t) - \int_0^t v_\alpha(s)\,\mathrm{d}s, \qquad \hat{W}_\beta^k(t) = W_\beta^k(t) - \int_0^t v_\beta^k(s)\,\mathrm{d}s, \tag{21}$$

are independent standard Wiener processes and the processes $N_\alpha(t)$, $N_\beta^k(t)$, $t \in [0,T]$, $\alpha = d_1+1,\ldots,m_1$, $\beta = d_2+1,\ldots,m_2$, $k = 1,\ldots,n$, are counting processes of stochastic intensities $I_\alpha(t)$ and $I_\beta^k(t)$, respectively.

Again under the probability $\mathbb{P}^T$, the components of the process $\psi(t)$ satisfy in the time interval $[0,T]$ the SDE

$$\begin{aligned}
\mathrm{d}\psi_j(t) = V_j\big(\psi_1(t_-),\ldots,\psi_n(t_-)\big)\mathrm{d}t + \sum_{\alpha=1}^{d_1}\Big(L_\alpha^j - \frac{1}{2}\,v_\alpha(t)\Big)\psi_j(t_-)\mathrm{d}\hat{W}_\alpha(t)& \\
+ \sum_{\alpha=1}^{d_2}\sum_{k=1}^{n}\Big(R_\alpha^{jk}\psi_k(t_-) - \frac{1}{2}\,v_\alpha^k(t)\psi_j(t_-)\Big)\mathrm{d}\hat{W}_\alpha^k(t)& \\
+ \sum_{\alpha=d_1+1}^{m_1}\left(\frac{L_\alpha^j}{\sqrt{I_\alpha(t)}} - 1\right)\psi_j(t_-)\,\mathrm{d}N_\alpha(t) + \sum_{\alpha=d_2+1}^{m_2}\sum_{k=1}^{n}\left(\frac{R_\alpha^{jk}\psi_k(t_-)}{\sqrt{I_\alpha^k(t)}} - \psi_j(t_-)\right)\mathrm{d}N_\alpha^k(t),&
\end{aligned} \tag{22a}$$

where

$$\begin{aligned}
V_j(\psi_1(t_-),\ldots,\psi_n(t_-)) = K^j\psi_j(t_-) + \frac{1}{2}\sum_{\alpha=d_1+1}^{m_1} I_\alpha(t)\psi_j(t_-)& \\
+ \frac{1}{2}\sum_{\alpha=d_2+1}^{m_2}\sum_{k=1}^{n} I_\alpha^k(t)\psi_j(t_-) + \frac{1}{2}\sum_{\alpha=1}^{d_1} v_\alpha(t)\left(L_\alpha^j - \frac{1}{4}\,v_\alpha(t)\right)\psi_j(t_-)& \\
+ \frac{1}{2}\sum_{\alpha=1}^{d_2}\sum_{k=1}^{n} v_\alpha^k(t)\left(R_\alpha^{jk}\psi_k(t_-) - \frac{1}{4}\,v_\alpha^k(t)\psi_j(t_-)\right).&
\end{aligned} \tag{22b}$$

Note that the SDE (22) is non-linear in $\psi(t)$, because the quantities $I_\alpha(t)$, $I_\alpha^k(t)$, $v_\alpha(t)$, $v_\alpha^k(t)$ are bilinear in $\psi(t)$ itself. Moreover, to consider (22) as a closed equation for $\psi(t)$ poses interesting mathematical problems on the definition of *solution* and on the meaning of uniqueness because the law of the driving noises N_α^k depends on the solution $\psi(t)$ itself through the stochastic intensities I_α, I_α^k[47].

Proposition 3 (A normalized unravelling)**.** *The solution of the Lindblad rate equation* (6) *can be expressed as the following mean with respect to the physical probability*

$$\eta_i(t) = \mathbb{E}_{\mathbb{P}^T}[|\psi_i(t)\rangle\langle\psi_i(t)|], \qquad i = 1, \ldots, n, \quad T \geq t \geq 0. \tag{23}$$

Proof. Let us introduce the set $A_t = \{\omega \in \Omega : \|\zeta(t,\omega)\| = 0\}$. Then, by the definitions of $p(t)$ and $\psi(t)$ given in Theorem 1, we have

$$|\zeta_i(t)\rangle\langle\zeta_i(t)| = 1_{A_t^c}|\zeta_i(t)\rangle\langle\zeta_i(t)| = 1_{A_t^c}p(t)|\psi_i(t)\rangle\langle\psi_i(t)| = p(t)|\psi_i(t)\rangle\langle\psi_i(t)|.$$

By taking the $\mathbb{Q}$-expectation and by taking into account Eq. (18) and the definition of the new probability, we get

$$\eta_i(t) = \mathbb{E}_{\mathbb{Q}}[|\zeta_i(t)\rangle\langle\zeta_i(t)|] = \mathbb{E}_{\mathbb{Q}}[p(t)|\psi_i(t)\rangle\langle\psi_i(t)|] = \mathbb{E}_{\mathbb{P}^t}[|\psi_i(t)\rangle\langle\psi_i(t)|].$$

Finally, by the consistency property (20), we get (23). □

This proposition gives an unravelling of the Lindblad rate equation (6) based on the components of the *normalized* vector $\psi(t)$. When $d_1 = m_1 = 0$ and $d_2 = 0$, we recover the pure jump unravelling proposed in Ref. 34. If the aim is only to simulate Eq. (6), a normalized pure state unravelling is much more efficient than a non-normalized one such as (18)[4]. The simulation techniques based on (22) with $d_1 = d_2 = 0$ correspond to the Monte-Carlo wave function method started in Ref. 26, while the case $d_1 = m_1$ and $d_2 = m_2$ gives rise to simulations of diffusive type as in Refs. 48–50. From the point of view of simulations, the fact that the starting point was Eq. (9), and not Eq. (5), has produced a more convenient unravelling with less noises (no dependence on the label j).

IV. MEASUREMENTS AND STOCHASTIC MASTER EQUATIONS

In this section we face the problem of the measurement interpretation of the unravelling we have constructed. We introduce the notions of *instruments* and *a posteriori* states and we derive the non Markovian generalization of the *stochastic master equations*.

A. Outputs and noises

In the theory of measurements in continuous time[27,30,39,40] it is assumed that the output of the measurement is given by some components of the driving noises appearing in the SDE (Eq. (12) or (13) in our case); the law of the output in $[0,T]$ is the physical probability (19). Not all the components of W and N have to contribute to the output. The role of some of the components of the noises could be only to perform the unravelling of some dissipative term.

Let us examine first the components $W_\alpha^k(t)$, $\alpha = 1,\ldots,d_2$, $k=1,\ldots,n$. For $t\in[0,T]$, under the physical probability $\mathbb{P}^T$, from (21) we get $W_\alpha^k(t) = \hat{W}_\alpha^k(t) + \int_0^t v_\alpha^k(s)\,\mathrm{d}s$; but, as one sees from Eq. (16b), $v_\alpha^k(s)$ mixes different components of $\psi(t)$ and cannot be an observable, because it does not respect the superselection rule. In particular the mean value of $W_\alpha^k(t)$ turns out to be $\mathbb{E}_{\mathbb{P}^T}\left[W_\alpha^k(t)\right] = \int_0^t \mathbb{E}_{\mathbb{P}^T}\left[v_\alpha^k(s)\right]\mathrm{d}s$ with

$$\mathbb{E}_{\mathbb{P}^T}\left[v_\alpha^k(t)\right] = \mathbb{E}_{\mathbb{P}^t}\left[v_\alpha^k(t)\right] = 2\sum_{j=1}^n \operatorname{Re}\operatorname{Tr}_{\mathcal{H}}\left\{\left(R_\alpha^{jk}\otimes|e_j\rangle\langle e_k|\right)\eta(t)\right\}$$

and it involves the unphysical non-diagonal blocks $\operatorname{Tr}_{\mathbb{C}^n}\left\{(\mathbb{1}\otimes|e_j\rangle\langle e_k|)\,\eta(t)\right\}$. So, $W_\alpha^k(t)$ cannot contribute to the output.

No problem of this kind arises for the other processes, as one sees from Eqs. (16). The stochastic intensities $I_\alpha(t)$, $I_\alpha^k(t)$ and the processes $v_\alpha(t)$ do not mix different components of $\psi(t)$. However, if the counting process N_α^k is detected we gain information on the block contributing to the emission (the block k), as one sees for instance from the mean intensity

$$\mathbb{E}_{\mathbb{P}^t}\left[I_\alpha^k(t)\right] = \sum_{j=1}^n \operatorname{Tr}_{\mathcal{H}_S}\left\{R_\alpha^{jk*}R_\alpha^{jk}\eta_k(t)\right\}.$$

If we assume that the index k is not physically observable coherently with the fact that the system state is the sum (3), the process N_α^k is not observable by itself. However, there is no obstruction in considering as physically observable the counting process

$$M_\alpha(t) := \sum_{k=1}^n N_\alpha^k(t), \qquad \alpha = d_2+1,\ldots,m_2, \tag{24}$$

whose stochastic intensity, under the physical probability, is $\sum_{k=1}^n I_\alpha^k(t)$. No problem arises on the observability of the other counting processes N_β ($\beta = d_1+1,\ldots,m_1$), whose stochastic intensity under the physical probability is $I_\beta(t)$.

Let us stress that, under the reference probability $\mathbb{Q}$, M_α is a Poisson process of intensity

$$\Lambda_\alpha = \sum_{k=1}^{n} \lambda_\alpha^k\,, \qquad \alpha = d_2+1,\ldots,m_2\,. \tag{25}$$

Let us consider finally the processes W_α. At least in quantum optical systems, observations with a "diffusive" character come out from heterodyne or homodyne detection and the involved operators must have an explicit time dependence due to the presence of the *local oscillator* [27, Chapt. 7]. We assume a very smooth time dependence which does not cause any essential change in the previous results.

Assumption 4. For $\alpha = 1,\ldots,d_1$, we assume the operators L_α^j to be time dependent and given by

$$L_\alpha^j(t) = \overline{h_\alpha^j(t)}\,\hat{L}_\alpha^j\,, \qquad \hat{L}_\alpha^j \in \mathcal{L}(\mathcal{H}_S), \quad \left|h_\alpha^j(t)\right| = 1;$$

the complex functions $h_\alpha^j(t)$ are continuous from the left.

No time dependence is introduced into the master equations of Section II. The explicit time dependence involves only the terms with $\mathrm{d}W_\alpha$ in Eqs. (12), (13), (22a) and the third term in the right hand side of (22b); moreover, from Eq. (16a), we get

$$v_\alpha(t) = 2\,\mathrm{Re} \sum_{j=1}^{n} \overline{h_\alpha^j(t)} \big\langle \psi_j(t_-) \big| \hat{L}_\alpha^j \psi_j(t_-)\big\rangle.$$

The key result of the previous discussion is that, due to the mathematical structure and the meaning of the discrete label in the states, only the processes W_α ($\alpha = 1,\ldots,d_1$), N_β ($\beta = d_1+1,\ldots,m_1$), M_γ ($\gamma = d_2+1,\ldots,m_2$) can be considered as possible components of the output. However, some of the components could represent pure noises, not observed quantities. So, we assume that only the first components are observed.

Remark 9. Let us take $d_1' \le d_1$, $m_1' \le m_1$, $m_2' \le m_2$. We assume that the observed outputs are W_α with $1 \le \alpha \le d_1'$, N_β with $d_1+1 \le \beta \le m_1'$, M_γ with $d_2+1 \le \gamma \le m_2'$. If some set of indices is empty, no component of the corresponding process is observed. The law of the output is the physical probability (19). Finally we denote by $\{\mathcal{G}_t,\, t \ge 0\}$ the augmented natural filtration generated by the set of the observed processes.

B. The linear stochastic master equations and the instruments

It is possible to have only the observed processes as driving noises in the dynamical equations, but for this we need to work with density matrices and trace-class operators. Let us introduce the positive trace-class operators

$$\sigma(t) := \mathbb{E}_{\mathbb{Q}}\big[|\zeta(t)\rangle\langle\zeta(t)|\big|\mathcal{G}_t\big], \qquad \sigma_i(t) = \mathbb{E}_{\mathbb{Q}}\big[|\zeta_i(t)\rangle\langle\zeta_i(t)|\big|\mathcal{G}_t\big]. \tag{26}$$

This means to take the mean on the non-observed components of the noises. Let us recall that $\zeta(0)$ is connected to the initial condition $\eta(0)$ (given in Eq. (10)) by Assumption 3. By the fact that $\mathcal{G}_0$ is trivial we get

$$\sigma(0) = \eta(0) \in \mathcal{S}(\mathcal{H}), \qquad \sigma_i(0) = \eta_i(0). \tag{27}$$

Proposition 4. *In the stochastic basis $(\Omega, \mathcal{F}, \mathcal{G}_t, \mathbb{Q})$, the operator valued process $\sigma(t)$ satisfies the linear stochastic master equation*

$$\begin{aligned}
\mathrm{d}\sigma(t) = \mathcal{L}[\sigma(t_-)]\mathrm{d}t &+ \sum_{\alpha=1}^{d_1'} \left(V_\alpha(t)\sigma(t_-) + \sigma(t_-)V_\alpha(t)^*\right) \mathrm{d}W_\alpha(t) \\
&+ \sum_{\alpha=d_1+1}^{m_1'} \left(\frac{V_\alpha \sigma(t_-) V_\alpha^*}{\lambda_\alpha} - \sigma(t_-)\right) \left(\mathrm{d}N_\alpha(t) - \lambda_\alpha \mathrm{d}t\right) \\
&+ \sum_{\alpha=d_2+1}^{m_2'} \left(\sum_{k=1}^{n} \frac{S_\alpha^k \sigma(t_-) S_\alpha^{k*}}{\Lambda_\alpha} - \sigma(t_-)\right) \left(\mathrm{d}M_\alpha(t) - \Lambda_\alpha \mathrm{d}t\right),
\end{aligned} \tag{28}$$

where $\mathcal{K}_i\big(\rho_1, \ldots, \rho_n\big)$ is defined by Eq. (6b)*, $\mathcal{L}$ by* (8)*, Λ_α by* (25)*,*

$$V_\alpha(t) = \sum_{i=1}^{n} \overline{h_\alpha^i(t)}\, \hat{L}_\alpha^i \otimes |e_i\rangle\langle e_i|, \qquad \alpha = 1, \ldots, d_1.$$

Given the initial condition, Eq. (28) *has pathwise unique solution. For the components (the blocks on the diagonal) Eq.* (28) *reduces to*

$$\begin{aligned}
\mathrm{d}\sigma_j(t) = \mathcal{K}_j\big(\sigma_1(t_-), \ldots, \sigma_n(t_-)\big)\mathrm{d}t &+ \sum_{\alpha=1}^{d_1'} \Big(\overline{h_\alpha^j(t)}\, \hat{L}_\alpha^j \sigma_j(t_-) + h_\alpha^j(t) \sigma_j(t_-) \hat{L}_\alpha^{j*}\Big) \mathrm{d}W_\alpha(t) \\
&+ \sum_{\alpha=d_1+1}^{m_1'} \left(\frac{L_\alpha^j \sigma_j(t_-) L_\alpha^{j*}}{\lambda_\alpha} - \sigma_j(t_-)\right) \left(\mathrm{d}N_\alpha(t) - \lambda_\alpha \mathrm{d}t\right) \\
&+ \sum_{\alpha=d_2+1}^{m_2'} \left(\sum_{k=1}^{n} \frac{R_\alpha^{jk} \sigma_k(t_-) R_\alpha^{jk*}}{\Lambda_\alpha} - \sigma_j(t_-)\right) \left(\mathrm{d}M_\alpha(t) - \Lambda_\alpha \mathrm{d}t\right).
\end{aligned} \tag{29}$$

Proof. By applying the Itô formula to $|\zeta(t)\rangle\langle\zeta(t)|$ and, then, by taking the conditional expectation, we get the linear stochastic master equation for $\sigma(t)$ as explained in [30, Sect. 4.2]. Existence and uniqueness of the solution of Eq. (28) is given in [30, Prop. 3.4]. Equation (29) is obtained by direct computations. □

Remark 10. Let us consider now the physical probability introduced in Remark 8. The probability density of the restriction of $\mathbb{P}^t$ to $\mathcal{G}_t$ with respect to the reference measure $\mathbb{Q}$ on $\mathcal{G}_t$ is

$$p_{\mathcal{G}}(t) = \mathbb{E}_{\mathbb{Q}}[p(t)|\mathcal{G}_t] = \mathrm{Tr}_{\mathcal{H}}\{\sigma(t)\} \equiv \sum_{j=1}^{n} \mathrm{Tr}_{\mathcal{H}_S}\{\sigma_j(t)\}. \tag{30}$$

The density $p_{\mathcal{G}}$ is a $\mathcal{G}$-martingale under $\mathbb{Q}$ and the restrictions of the physical probabilities are consistent.

In the axiomatic formulation of a quantum theory, measurements are represented by instruments, which give the probabilities and the states after the measurement (*a posteriori* states). As in Refs. 30 and 40, we put

$$\mathcal{I}_t(F)[\eta(0)] = \mathbb{E}_{\mathbb{Q}}[1_F \sigma(t)], \qquad F \in \mathcal{G}_t\,, \quad \eta(0) \in \mathcal{S}(\mathcal{H}). \tag{31}$$

By linearity we extend $\mathcal{I}_t(F)$ to the whole $\mathcal{T}(\mathcal{H})$ and we get an *instrument* with value space $(\Omega, \mathcal{G}_t)$, which means that $\mathcal{I}_t(F)$ is a CP map from $\mathcal{T}(\mathcal{H})$ into itself for all $F \in \mathcal{G}_t$, it is a strongly σ-additive measure as a function of F and $\mathcal{I}_t(\Omega)$ is trace-preserving.

Remark 11. Let us particularize the definition of instrument in the enlarged space to our case. We define

$$\mathcal{I}_t^i(F)[\eta_1(0), \ldots, \eta_n(0)] = \mathbb{E}_{\mathbb{Q}}[1_F \sigma_i(t)], \qquad F \in \mathcal{G}_t, \tag{32}$$

for all $\eta(0)$ satisfying the superselection rules. With the notations of Remarks 3 and 4 we have $\big(\mathcal{I}_t^1(F), \ldots, \mathcal{I}_t^n(F)\big) = \mathcal{P} \circ \mathcal{I}_t(F)\big|_{\mathcal{C}\left(\mathcal{X};\mathcal{T}(\mathcal{H}_S)\right)}$. This is an instrument with the same value space as before, but made up of maps on $\mathcal{C}\big(\mathcal{X};\mathcal{T}(\mathcal{H}_S)\big)$. Finally, by defining

$$\mathcal{I}_t^S(F) = \sum_{j=1}^{n} \mathcal{I}_t^j(F), \tag{33}$$

we get an instrument with value space $(\Omega, \mathcal{G}_t)$ made up of CP maps from $\mathcal{C}\big(\mathcal{X};\mathcal{T}(\mathcal{H}_S)\big)$ into $\mathcal{T}(\mathcal{H}_S)$. Moreover, the connection with the various dynamical maps introduced in Remark 4 is given by $\mathcal{I}_t(\Omega) = \mathcal{T}(t)$,

$$\big(\mathcal{I}_t^1(\Omega), \ldots, \mathcal{I}_t^n(\Omega)\big) = \mathcal{P} \circ \mathcal{T}(t)\big|_{\mathcal{C}\left(\mathcal{X};\mathcal{T}(\mathcal{H}_S)\right)}, \qquad \mathcal{I}_t^S(\Omega) = \mathcal{P}_S \circ \mathcal{P} \circ \mathcal{T}(t)\big|_{\mathcal{C}\left(\mathcal{X};\mathcal{T}(\mathcal{H}_S)\right)}.$$

The instruments give the physical probabilities once one has the pre-measurement state. In our case we have, $\forall F \in \mathcal{G}_t$,

$$\mathbb{P}^t(F) = \mathrm{Tr}_{\mathcal{H}}\{\mathcal{I}_t(F)[\eta(0)]\} = \sum_{i=1}^{n} \mathrm{Tr}_{\mathcal{H}_S}\{\mathcal{I}_t^i(F)[\eta_1(0), \ldots]\} = \mathrm{Tr}_{\mathcal{H}_S}\{\mathcal{I}_t^S(F)[\eta_1(0), \ldots]\}. \tag{34}$$

This equation says that the probabilities $\mathbb{P}^t$, introduced before by starting from some stochastic differential equation, can be obtained also from instruments; so, the axiomatic structure of a quantum theory is respected and the interpretation as physical probabilities is justified.

C. The a posteriori states and the stochastic master equation

The instruments give also the *a posteriori states*, the conditional states after the measurement. Let us recall the definition in the case of $\mathcal{I}_t$; in the other cases the definition is analogue. The *a posteriori* state for the instrument $\mathcal{I}_t$ and the pre-measurement state $\eta(0)$ is the $\mathcal{S}(\mathcal{H})$-valued random variable $\rho(t)$ such that

$$\mathcal{I}_t(F)[\eta(0)] = \mathbb{E}_{\mathbb{P}^t}[1_F \rho(t)], \qquad \forall F \in \mathcal{G}_t.$$

By taking into account that the density of $\mathbb{P}^t$ with respect to $\mathbb{Q}$ is the trace of $\sigma(t)$ and how $\mathcal{I}_t(F)[\eta(0)]$ is defined in terms of $\sigma(t)$, we get easily

$$\rho(t) = \frac{\sigma(t)}{\mathrm{Tr}_{\mathcal{H}}\{\sigma(t)\}} = \mathbb{E}_{\mathbb{P}^t}\left[|\psi(t)\rangle\langle\psi(t)| \big| \mathcal{G}_t \right].$$

The components of $\rho(t)$, which are

$$\rho_i(t) = \mathbb{E}_{\mathbb{P}^t}\left[|\psi_i(t)\rangle\langle\psi_i(t)| \big| \mathcal{G}_t \right] = \frac{\sigma_i(t)}{\mathrm{Tr}_{\mathcal{H}}\{\sigma(t)\}}, \quad i = 1, \ldots, n,$$

give the *a posteriori* states for $\mathcal{I}_t^i$:

$$\mathcal{I}_t^i(F)[\eta_1(0), \ldots, \eta_n(0)] = \mathbb{E}_{\mathbb{P}^t}[1_F \rho_i(t)]. \tag{35}$$

Note that we have also $\mathbb{E}_{\mathbb{P}^t}[\rho_i(t)] = \eta_i(t)$. Finally, by taking the sum over i in Eq. (35), we get the *a posteriori* states $\rho_S(t) = \sum_i \rho_i(t)$ for $\mathcal{I}_t^S$.

On the other side, the states $\eta(t)$, $\eta_i(t)$ are called the *a priori states*, due to the fact that these states are the averages of the *a posteriori* states and that they are interpreted as the states to be assigned to the system at time t when the result of the observation is not taken into account.

Remark 12 (The stochastic master equation). For $\alpha = 1, \ldots, d_1'$, $\beta = d_1 + 1, \ldots, m_1'$, $\gamma = d_2 + 1, \ldots, m_2'$, let us define

$$m_\alpha(t) =: \mathbb{E}_{\mathbb{P}^t}\left[v_\alpha(t) \big| \mathcal{G}_t \right] = 2\,\mathrm{Re} \sum_{j=1}^n \overline{h_\alpha^j(t)}\, \mathrm{Tr}_{\mathcal{H}_S}\left\{ \hat{L}_\alpha^j \rho_j(t_-) \right\}, \tag{36}$$

$$J^1_\beta(t) := \mathbb{E}_{\mathbb{P}^t}\left[I_\beta(t)\big|\mathcal{G}_t\right] = \sum_{j=1}^n \mathrm{Tr}_{\mathcal{H}_S}\left\{{L^j_\beta}^* L^j_\beta \rho_j(t_-)\right\}, \tag{37}$$

$$J^2_\gamma(t) := \sum_{k=1}^n \mathbb{E}_{\mathbb{P}^t}\left[I^k_\gamma(t)\big|\mathcal{G}_t\right] = \sum_{j,k=1}^n \mathrm{Tr}_{\mathcal{H}_S}\left\{R^{jk*}_\gamma R^{jk}_\gamma \rho_k(t_-)\right\}. \tag{38}$$

Then, by stochastic calculus, under the new probability $\mathbb{P}^T$ and for $t \in [0,T]$, we get the equation for $\rho(t)$ [30, Rem. 3.6] and, then, the *stochastic master equation* for the components

$$\begin{aligned}
\mathrm{d}\rho_j(t) &= \mathcal{K}_j\big(\rho_1(t_-),\dots,\rho_n(t_-)\big)\mathrm{d}t \\
&\quad + \sum_{\alpha=1}^{d'_1}\Big(\overline{h^j_\alpha(t)}\,\hat{L}^j_\alpha\rho_j(t_-) + h^j_\alpha(t)\rho_j(t_-)\hat{L}^{j*}_\alpha - m_\alpha(t)\rho_j(t_-)\Big)\,\mathrm{d}\hat{W}_\alpha(t) \\
&\quad + \sum_{\beta=d_1+1}^{m'_1}\left(\frac{L^j_\beta\rho_j(t_-)L^{j*}_\beta}{J^1_\beta(t)} - \rho_j(t_-)\right)\left(\mathrm{d}N_\beta(t) - J^1_\beta(t)\mathrm{d}t\right) \\
&\quad + \sum_{\gamma=d_2+1}^{m'_2}\left(\sum_{k=1}^n \frac{R^{jk}_\gamma\rho_k(t_-)R^{jk*}_\gamma}{J^2_\gamma(t)} - \rho_j(t_-)\right)\big(\mathrm{d}M_\gamma(t) - J^2_\gamma(t)\mathrm{d}t\big).
\end{aligned} \tag{39}$$

The processes $\hat{W}_\alpha$ are independent standard Wiener processes, $N_\beta(t)$ is a counting process of stochastic intensity $J^1_\beta(t)$ and $M_\gamma(t)$ is a counting process of stochastic intensity $J^2_\gamma(t)$.

V. A TWO-LEVEL SYSTEM IN A STRUCTURED BATH

To give a simple, but concrete example of the theory we have developed and to have a first idea of the effects on physically measurable quantities, here we study a model of a two level atom in contact with a non-trivial structured reservoir and we compute the heterodyne spectrum of its emitted light. This is a modification of a model[15,16,34] which could represent the dynamics of a single qubit in a non Markovian environment or the dynamics of an optically active molecule, as the fluorophore system, in a local nano-environment[24].

We consider a two-level system in contact with a two-band reservoir; so, $\mathcal{H}_S = \mathbb{C}^2$ and $n = 2$. Let σ_z, $\sigma_\pm$ be the usual Pauli matrices; then, $P_+ = \sigma_+\sigma_-$ is the projection on the excited state $\binom{1}{0}$ and $P_- = \sigma_-\sigma_+$ the projection on the ground state $\binom{0}{1}$. Here we give the mathematical model, while the physical interpretation is given when we write down the various dynamical equations. By using the notations introduced in Assumption 2 and

Section III A, the model we consider is defined by the following choices:

$$
\begin{aligned}
&d_1 = m_1 = 2, \quad d_2 = 0, \quad m_2 = 2; \qquad H^i = \frac{\omega_i}{2}\,\sigma_z\,, \quad \omega_i > 0, \quad i = 1,2;\\
&R_1^{ii} = 0, \quad R_1^{21} = \sqrt{\gamma_1}\,\sigma_-\,, \quad R_1^{12} = \sqrt{\gamma_2}\,\sigma_+\,, \qquad \gamma_i > 0, \quad i = 1,2,\\
&R_2^{ii} = 0, \quad R_2^{12} = 0, \quad R_2^{21} = \sqrt{\gamma_0\varkappa}\,\mathbb{1}, \qquad \varkappa > 0, \quad \gamma_0 > 0; \qquad 0 < \epsilon \leq 1,\\
&L_1^1(t) = L_1^2(t) = \mathrm{e}^{\mathrm{i}\nu t}\sqrt{\gamma_0\epsilon}\,\sigma_-\,, \quad L_2^1(t) = L_2^2(t) = \mathrm{e}^{\mathrm{i}\nu t}\sqrt{\gamma_0(1-\epsilon)}\,\sigma_-\,, \quad \nu \in \mathbb{R}.
\end{aligned}
\tag{40}
$$

The driving processes in the linear SDEs are the standard Wiener processes W_1, W_2 and the Poisson processes N_1^1, N_1^2, N_2^1, with intensities λ_1^1, λ_1^2, λ_2^1; all these processes are independent. According to Remark 5, take $\lambda_2^2 \downarrow 0$, so that N_2^2 is almost surely 0 and we can set $\mathrm{d}N_2^2(t) = 0$.

A. The Lindblad rate equation and the equilibrium state

First of all let us write down in the concrete case introduced above the Lindblad rate equation (6)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\eta_1(t) = \mathcal{K}_1\big(\eta_1(t),\eta_2(t)\big) \equiv \gamma_0 \left(\sigma_-\eta_1(t)\sigma_+ - \frac{1}{2}\{P_+,\eta_1(t)\}\right)&\\
+ \gamma_2\sigma_+\eta_2(t)\sigma_- - \frac{\gamma_1}{2}\{P_+,\eta_1(t)\} - \gamma_0\varkappa\eta_1(t) - \frac{\mathrm{i}\omega_1}{2}\,[\sigma_z,\eta_1(t)]\,,&
\end{aligned}
\tag{41a}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\eta_2(t) = \mathcal{K}_2\big(\eta_1(t),\eta_2(t)\big) \equiv \gamma_0 \left(\sigma_-\eta_2(t)\sigma_+ - \frac{1}{2}\{P_+,\eta_2(t)\}\right)&\\
+ \gamma_1\sigma_-\eta_1(t)\sigma_+ - \frac{\gamma_2}{2}\{P_-,\eta_2(t)\} + \gamma_0\varkappa\eta_1(t) - \frac{\mathrm{i}\omega_2}{2}\,[\sigma_z,\eta_2(t)]\,.&
\end{aligned}
\tag{41b}
$$

The model of Refs. 16 and 21 corresponds to $\gamma_0 = 0$, $\varkappa = 0$, $\omega_1 = \omega_2$; moreover, the rotating framework is used, so that the terms with ω_i disappear. In Refs. 15 and 24 the case $\omega_1 \neq \omega_2$ is allowed and it is explained by different energy shifts induced by the two bands of the environment. So, we have a two level molecule with two resonance frequencies due to the structured environment. The terms with γ_1 and γ_2 represent the molecular transitions induced by the environment and concomitant with transitions between the two bands of the nano environment.

Reference 24 studies the stimulated fluorescence light under laser excitation of the molecule; the treatment is based on the quantum regression formula. Instead, our aim is to study the spontaneously emitted light and to this end we have added the first term in both equations, the one with γ_0, which is an explicit spontaneous emission term.

To have emission without stimulation by external light, we need some thermal-like excitation. To get this effect we have added the terms with $\gamma_0\varkappa$. This is the simplest modification giving rise to a non trivial equilibrium state.

If we write Eqs. (41) in terms of the matrix elements of η_1 and η_2 we get two decoupled systems of equations: the system for the coherences (the off diagonal terms) and the system for the populations (the diagonal terms). Firstly, one checks easily that the coherences decay exponentially to zero. On the other side, the system of equations for the diagonal terms turns out to be equivalent to a 4-state, irreducible classical Markov chain. If we denote by 1^+, 1^-, 2^+, 2^- the four states, the transition rates different from zero are γ_0 for the transition $1^+ \to 1^-$, γ_1 for the transition $1^+ \to 2^-$, $\gamma_0\varkappa$ for $1^+ \to 2^+$, γ_2 for $2^- \to 1^+$, γ_0 for $2^+ \to 2^-$, $\gamma_0\varkappa$ for $1^- \to 2^-$. From the graph of this finite-state Markov chain we see that it is irreducible; then, there is a unique equilibrium distribution, computed below, and it is a global attractor.

Equilibrium state.

The Lindblad rate equations (41) admit a unique equilibrium state $\eta_i(\infty) = \lim_{t\to+\infty}\eta_i(t)$, $i = 1, 2$, which can be easily computed. It turns out to be given by

$$\eta_i(\infty) = p_i\left(z_i^+ P_+ + z_i^- P_-\right), \qquad z_i^- := 1 - z_i^+\,, \qquad z_i^+ := \frac{\varkappa_i}{1+\varkappa_i}\,,$$

$$\varkappa_1 := \varkappa, \qquad \varkappa_2 := \frac{\gamma_2\varkappa}{\gamma_1 + \gamma_0(1+\varkappa)}\,, \qquad p_1 := p, \qquad p_2 := 1 - p,$$

$$p := \frac{\gamma_2(1+\varkappa)}{\gamma_2 + \varkappa\left(\gamma_0+\gamma_2+\gamma_1\right) + \varkappa^2\left(\gamma_0+\gamma_2\right)}\,.$$

Let us note that we have $(1-p)z_2^+ = \varkappa p z_1^+$. By recalling that the system state is the sum of the components (3), we get that the average equilibrium state of the two-level system is

$$\eta_S^{\mathrm{eq}} = \eta_1(\infty) + \eta_2(\infty) = p\varkappa P_+ + (1-p\varkappa)P_-\,.$$

B. The stochastic Schrödinger equations

The lSSE (13) corresponding to the choices (40) is

$$
\begin{aligned}
\mathrm{d}\zeta_1(t) = {}& \left(K^1 + \frac{\lambda}{2}\right)\zeta_1(t_-)\mathrm{d}t - \zeta_1(t)\big(\mathrm{d}N_1^1(t) + \mathrm{d}N_2^1(t)\big) \\
& + \left(\sqrt{\frac{\gamma_2}{\lambda_2}}\,\sigma_+\zeta_2(t_-) - \zeta_1(t_-)\right)\mathrm{d}N_1^2(t) + \mathrm{e}^{\mathrm{i}\nu t}\sqrt{\gamma_0}\,\sigma_-\zeta_1(t_-)\left(\sqrt{\epsilon}\,\mathrm{d}W_1(t) + \sqrt{1-\epsilon}\,\mathrm{d}W_2(t)\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}\zeta_2(t) = {}& \left(K^2 + \frac{\lambda}{2}\right)\zeta_2(t_-)\mathrm{d}t + \left(\sqrt{\frac{\gamma_0\varkappa}{\lambda_0}}\,\zeta_1(t_-) - \zeta_2(t_-)\right)\mathrm{d}N_2^1(t) - \zeta_2(t)\mathrm{d}N_1^2(t) \\
& + \left(\sqrt{\frac{\gamma_1}{\lambda_1}}\,\sigma_-\zeta_1(t_-) - \zeta_2(t_-)\right)\mathrm{d}N_1^1(t) + \mathrm{e}^{\mathrm{i}\nu t}\sqrt{\gamma_0}\,\sigma_-\zeta_2(t_-)\left(\sqrt{\epsilon}\,\mathrm{d}W_1(t) + \sqrt{1-\epsilon}\,\mathrm{d}W_2(t)\right),
\end{aligned}
$$

where $\lambda = \lambda_1^1 + \lambda_1^2 + \lambda_2^1$ and

$$
K^1 = -\frac{\mathrm{i}\omega_1}{2}\,\sigma_z - \frac{\gamma_0+\gamma_1}{2}\,P_+ - \frac{\gamma_0\varkappa}{2}\,\mathbb{1}, \qquad K^2 = -\frac{\mathrm{i}\omega_2}{2}\,\sigma_z - \frac{\gamma_0}{2}\,P_+ - \frac{\gamma_2}{2}\,P_-.
$$

Note that the Wiener processes W_1 and W_2 appear always in the combination $\sqrt{\epsilon}\,W_1(t) + \sqrt{1-\epsilon}\,W_2(t)$, which is again a one-dimensional standard Wiener process. The reason for the introduction of two components is that the diffusive term represents the emitted light, which we have divided in two channels: channel 1, represented by W_1, contains the light reaching the heterodyne detector and channel 2, represented by W_2, contains the lost light. The proportion of lost light is $1-\epsilon$.

Finally, by Eqs. (22), the SSE for the normalized vectors is, under the physical probability,

$$
\begin{aligned}
\mathrm{d}\psi_1(t) = {}& V_1(\psi_1(t_-),\psi_2(t_-))\mathrm{d}t - \psi_1(t)\big(\mathrm{d}N_1^1(t) + \mathrm{d}N_2^1(t)\big) \\
& + \left(\frac{\sigma_+\psi_2(t_-)}{\|\sigma_+\psi_2(t_-)\|} - \psi_1(t_-)\right)\mathrm{d}N_1^2(t) \\
& + \sqrt{\gamma_0}\left(\mathrm{e}^{\mathrm{i}\nu t}\sigma_-\psi_1(t_-) - \frac{1}{2}\,v(t)\psi_1(t_-)\right)\left(\sqrt{\epsilon}\,\mathrm{d}\hat{W}_1(t) + \sqrt{1-\epsilon}\,\mathrm{d}\hat{W}_2(t)\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}\psi_2(t) = {}& V_2(\psi_1(t_-),\psi_2(t_-))\mathrm{d}t + \left(\frac{\psi_1(t_-)}{\|\psi_1(t_-)\|} - \psi_2(t_-)\right)\mathrm{d}N_2^1(t) \\
& + \left(\frac{\sigma_-\psi_1(t_-)}{\|\sigma_-\psi_1(t_-)\|} - \psi_2(t_-)\right)\mathrm{d}N_1^1(t) - \psi_2(t_-)\mathrm{d}N_1^2(t) \\
& + \sqrt{\gamma_0}\left(\mathrm{e}^{\mathrm{i}\nu t}\sigma_-\psi_2(t_-) - \frac{1}{2}\,v(t)\psi_2(t_-)\right)\left(\sqrt{\epsilon}\,\mathrm{d}\hat{W}_1(t) + \sqrt{1-\epsilon}\,\mathrm{d}\hat{W}_2(t)\right),
\end{aligned}
$$

where $\hat{W}$ is the new Wiener process introduced in (21), and

$$v_1(t) = \sqrt{\gamma_0\epsilon}\, v(t), \qquad v_2(t) = \sqrt{\gamma_0(1-\epsilon)}\, v(t),$$
$$v(t) = 2\sum_{k=1}^{2} \operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\nu t}\langle\psi_k(t_-)|\sigma_-\psi_k(t_-)\rangle\right), \qquad I_2^1(t) = \varkappa\gamma_0 \left\|\psi_1(t_-)\right\|^2,$$
$$I_1^1(t) = \gamma_1 \left\|\sigma_-\psi_1(t_-)\right\|^2, \qquad I_1^2(t) = \gamma_2 \left\|\sigma_+\psi_2(t_-)\right\|^2,$$

$$V_j(\psi_1(t_-),\psi_2(t_-)) = K^j\psi_j(t_-) + \frac{I_1^1(t) + I_1^2(t) + I_2^1(t)}{2}\,\psi_j(t_-) + \frac{\gamma_0}{2}\, v(t)\sigma_-\psi_j(t_-) - \frac{\gamma_0}{4}\, v(t)^2\psi_j(t_-).$$

C. The stochastic master equation and the heterodyne spectrum

In the situation we are considering the band transitions cannot be monitored. We take under observation the system by collecting part of the emitted light in an apparatus performing heterodyne detection. In this detection scheme the received light is made to interfere with some monochromatic light of frequency ν; to a certain extent, this frequency can be varied. As we have said, it is W_1 which represents the light reaching the detector; moreover, the (stochastic) output $J(t)$ of the detector is some smoothed version of W_1 [27, Chapt. 7], say

$$J(t) = \sqrt{k}\int_0^t \mathrm{e}^{-k(t-s)/2}\,\mathrm{d}W_1(s), \qquad k > 0. \tag{42}$$

To take into account that only W_1 is observed we use the notation of Remark 9 and we take $d_1' = 1$, $m_1' = d_1$, $m_2' = 0$; recall that we have $d_1 = m_1 = 2$, $d_2 = 0$, $m_2 = 2$. Then, all the sums with jump processes disappear from the stochastic master equations (29) and (39). The linear stochastic master equation (29) becomes

$$\mathrm{d}\sigma_j(t) = \mathcal{K}_j\big(\sigma_1(t),\sigma_2(t)\big)\mathrm{d}t + \sqrt{\gamma_0\epsilon}\left(\mathrm{e}^{\mathrm{i}\nu t}\sigma_-\sigma_j(t) + \mathrm{e}^{-\mathrm{i}\nu t}\sigma_j(t)\sigma_+\right)\mathrm{d}W_1(t),$$

where the $\mathcal{K}_j$ are the operators appearing in the Lindblad rate equations (41). The corresponding non linear stochastic master equation (39) for the *a posteriori* states $\rho_j(t) = \sigma_j(t)\big/\operatorname{Tr}_{\mathcal{H}_S}\{\sigma_1(t)+\sigma_2(t)\}$ turns out to be

$$\mathrm{d}\rho_j(t) = \mathcal{K}_j\big(\rho_1(t),\rho_2(t)\big)\mathrm{d}t + \sqrt{\gamma_0\epsilon}\left(\mathrm{e}^{\mathrm{i}\nu t}\sigma_-\rho_j(t) + \mathrm{e}^{-\mathrm{i}\nu t}\rho_j(t)\sigma_+ - m(t)\rho_j(t)\right)\mathrm{d}\hat{W}_1(t),$$

$$m(t) = 2\operatorname{Re}\left(\mathrm{e}^{\mathrm{i}\nu t}\operatorname{Tr}_{\mathcal{H}_S}\{\sigma_-\rho_j(t)\}\right).$$

The power of the output current produced by the detector is proportional to $J(t)^2$ and the mean power at large times is proportional to

$$P(\nu) = \lim_{t\to+\infty} \mathbb{E}_{\mathbb{P}^t}[J(t)^2]. \tag{43}$$

The limit is to be taken in the sense of the distributions in ν.

By using (42) we get

$$\mathbb{E}_{\mathbb{P}^t}[J(t)^2] = k\mathrm{e}^{-kt}\mathbb{E}_{\mathbb{P}^t}\left[\int_0^t \mathrm{e}^{ks/2}\,\mathrm{d}W_1(s)\int_0^t \mathrm{e}^{kr/2}\,\mathrm{d}W_1(r)\right]$$

So, to obtain an explicit expression for the power first of all we need to compute the second moments of the Wiener type integrals $\int_0^t \mathrm{e}^{ks/2}\,\mathrm{d}W_1(s)$ under the physical probability. The autocorrelation function of W_1, from which such a moments follow, can be obtained by differentiation of the so called *characteristic operator* (the Fourier transform of the instruments) [27, Proposition 4.16]. The formula valid for the Markov case needs only to be expressed by using the diagonal blocks; from [27, Eq. (4.47)] we get

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^t}[J(t)^2] = k\int_0^t \mathrm{e}^{-k(t-s)}\,\mathrm{d}s + 2k\gamma_0\epsilon\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\,\mathrm{e}^{-k(t-s)/2}\mathrm{e}^{-k(t-r)/2}\\ \times\sum_{i,j=1}^2 \operatorname{Tr}_{\mathcal{H}_S}\left\{\left(\mathrm{e}^{\mathrm{i}\nu s}\sigma_- + \mathrm{e}^{-\mathrm{i}\nu s}\sigma_+\right)\mathcal{T}_{ij}(s-r)\left[\mathrm{e}^{\mathrm{i}\nu s}\sigma_-\eta_j(r) + \mathrm{e}^{-\mathrm{i}\nu r}\eta_j(r)\sigma_+\right]\right\}.\end{aligned} \tag{44}$$

By $\sum_{j=1}^2 \mathcal{T}_{ij}(t)[\tau_j]$, $i=1,2$, we denote the solution of the Lindblad rate equation (41) with initial condition (τ_1,τ_2) at time 0. Then, the computations needed to obtain $P(\nu)$ are long, but similar to the ones in [27, Sect. 9.1]; we give only the final results:

$$P(\nu) = 1 + 4\pi\epsilon\Sigma(\nu), \tag{45}$$

$$\begin{aligned}\Sigma(\nu) = 2\gamma_0\,\frac{(1-p)\,z_2^+\Gamma_2 - pz_1^+w\left[\Gamma_2\left(\gamma_2-\gamma_1-2\gamma_0\varkappa\right)+4\left(\omega_2-\omega_1\right)\left(\nu-\omega_2\right)\right]}{\pi\left[4\left(\nu-\omega_2\right)^2+\Gamma_2^2\right]}\\ + 2\gamma_0\,\frac{pz_1^+\left\{\left[1+w\left(\gamma_2-\gamma_1-2\gamma_0\varkappa\right)\right]\Gamma_1+4w\left(\omega_2-\omega_1\right)\left(\nu-\omega_1\right)\right\}}{\pi\left[4\left(\nu-\omega_1\right)^2+\Gamma_1^2\right]},\end{aligned} \tag{46}$$

where p, z_j^+ have already be defined and

$$\Gamma_1 := \gamma_0+\gamma_1+2\gamma_0\varkappa+k, \qquad \Gamma_2 := \gamma_0+\gamma_2+k,$$

$$w := \frac{2\gamma_0\varkappa}{4\left(\omega_1-\omega_2\right)^2+\left(\gamma_2-\gamma_1-2\gamma_0\varkappa\right)^2}.$$

In Eq. (45) the term 1 is interpreted as the shot noise due to the local oscillator and $\Sigma(\nu)$ as the heterodyne spectrum. Note that the widths Γ_j contain some dynamical parameters and the instrumental width k.

By its definition, we have $P(\nu) \geq 0$, while the positivity of the spectrum $\Sigma(\nu)$ is not obvious. However, one can check that it is possible to rewrite $\Sigma(\nu)$ in a form from which its positivity is apparent:

$$\Sigma(\nu) = D\gamma_0\varkappa\left\{ \frac{\gamma_0(1+\varkappa)+\gamma_1+k}{4\,(\nu-\omega_1)^2+\Gamma_1^2} + \frac{\varkappa\,(\gamma_2+k)}{4\,(\nu-\omega_2)^2+\Gamma_2^2} \right. \\ \left. + \frac{\gamma_0\varkappa\,(\Gamma_1+\Gamma_2)^2}{\left[4\,(\nu-\omega_1)^2+\Gamma_1^2\right]\left[4\,(\nu-\omega_2)^2+\Gamma_2^2\right]} \right\}, \qquad (47)$$

$$D = \frac{2/\pi}{1+\varkappa\gamma_1/\gamma_2+\varkappa(1+\varkappa)(1+\gamma_0/\gamma_2)}.$$

Note that the heterodyne spectrum, for spontaneous emission in our model, contains information on the dynamics: all the dynamical parameters, due to the structured reservoir, determine the form of the spectrum. In particular, we have a double peaked structure only if ω_1 and ω_2 are sufficiently different and this difference can be only due to the band structure of the bath.

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