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APPROXIMATING INFINITE DELAY WITH FINITE DELAY

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Abstract. Equations with infinite delay commonly face the philosophical objection of being "unphysical", since a memory of infinite duration conflicts with reality. Indeed, besides common sense, experimental observations on concrete physical models tell that effects from the far past cannot possibly influence the current dynamics of a given system. On the other hand, infinite delay arises quite naturally in the mathematical description of several relevant phenomena. In this note, we propose a possible conceptual solution, showing that infinite delay can be recovered as a limiting case of finite delay on a large time-scale, along with a quantitative control of the discrepancy.

1. Introduction

The mathematical description of materials with memory is an important issue which attracted the attention of several authors. The origins of modern viscoelasticity and, more generally, of the so-called hereditary systems, trace back to the works of Boltzmann and Volterra [1, 2, 22, 23], who first introduced the notion of memory in connection with the analysis of elastic bodies. The key assumption in the hereditary theory of elasticity is that the deformation of the mechanical system is due both to the instantaneous stress and to the past stresses. Such a behavior is modelled by the so-called *equations with memory*, influenced by the past values of the variables in play (cf. [10, 21]). In abstract form, the equation of linear viscoelasticity at a given time $t > 0$ ($t = 0$ being understood as the initial time) reads

$$(\boldsymbol{\mathcal{E}}) \qquad \ddot{u}(t) + A\Big[u(t) - \int_0^\infty \mu(s)u(t-s)\,\mathrm{d}s\Big] = 0.$$

Here, A is a densely defined selfadjoint strictly positive linear operator on a real Hilbert space H, while the *memory kernel* $\mu : \mathbb{R}^+ \to [0,\infty)$ is a nonincreasing summable function of total mass

$$\kappa = \int_0^\infty \mu(s)\,\mathrm{d}s \in (0,1).$$

Equation $\boldsymbol{\mathcal{E}}$ is supplemented with the "initial conditions"

$$u(0) = \alpha, \qquad \dot{u}(0) = \beta, \qquad u(-s)_{|s>0} = \gamma(s),$$

where α, β and the function γ, representing the *past history* of u, are prescribed data. Indeed, the convolution integral in $\boldsymbol{\mathcal{E}}$ requires u to be assigned for all negative times, where need not solve the equation.

The longterm memory appearing in the model raised some criticism in the scientific community from the very beginning, due to the conceptual difficulty in accepting the idea

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of a past history defined on an infinite time-interval. Volterra [22, 23] circumvented the problem by merely assuming a vanishing past history before some fixed time (conventionally, the initial time $t = 0$), obtaining the particular instance of $\boldsymbol{\mathcal{E}}$

$$\ddot{u}(t) + A\Big[u(t) - \int_0^t \mu(s)u(t-s)\,\mathrm{d}s\Big] = 0, \tag{1.1}$$

whose solutions are uniquely determined by the knowledge of $u(0)$ and $\dot{u}(0)$. In the Sixties, Coleman and Mizel [4, 5] answered the infinite delay question through the notion of *fading memory*; namely, they introduced the further assumption that the values of the deformation history of a material in the far past produce negligible effects on the value of the present stress. Still, the philosophical objection of a memory having infinite duration remains open. A natural way out is considering the somehow more physical model

$$\ddot{u}(t) + A\Big[u(t) - \int_0^{\frac{1}{\varepsilon}} \mu(s)u(t-s)\,\mathrm{d}s\Big] = 0, \quad \varepsilon > 0,$$

where the memory kernel acts only on the finite time-interval $(0, \frac{1}{\varepsilon})$. Introducing the *truncated kernel*

$$\widehat{\mu}_\varepsilon(s) = \mu(s)\chi_{(0,\frac{1}{\varepsilon})}(s),$$

the above equation can be given the equivalent formulation

$$\ddot{u}(t) + A\Big[u(t) - \int_0^\infty \widehat{\mu}_\varepsilon(s)u(t-s)\,\mathrm{d}s\Big] = 0. \tag{1.2}$$

It is then interesting to clarify whether or not (1.2) provides a satisfactory approximation of $\boldsymbol{\mathcal{E}}$ for small values of ε. This is exactly the aim of the present paper. More generally, we will consider a class of perturbation kernels μ_ε, depending on a parameter $\varepsilon > 0$, converging (in a suitable sense) from below to the limiting kernel μ when $\varepsilon \to 0$. As a particular case, we will recover the truncated kernels $\widehat{\mu}_\varepsilon$. We should mention that dealing with this kind of perturbations is rather nontrivial, since minimal changes in the memory kernel may dramatically affect the behavior of the corresponding solutions (see [17] for related examples).

2. Setting of the Problem

In order to carry out this project, we consider in place of $\boldsymbol{\mathcal{E}}$ the family of equations

$$\ddot{u}_\varepsilon(t) + A\Big[u_\varepsilon(t) - \int_0^\infty \mu_\varepsilon(s)u_\varepsilon(t-s)\,\mathrm{d}s\Big] = 0, \tag{$\boldsymbol{\mathcal{E}_\varepsilon}$}$$

depending on $\varepsilon \geq 0$ small. For $\varepsilon = 0$, we agree to denote $u_0 = u$ and $\mu_0 = \mu$, recovering the original equation $\boldsymbol{\mathcal{E}}$.

Within a natural notion of convergence $\mu_\varepsilon \to \mu$, the goal is establishing the uniform-in-time closeness of the corresponding trajectories of $\boldsymbol{\mathcal{E}_\varepsilon}$ to those of $\boldsymbol{\mathcal{E}}$ originating from identical initial data. Accordingly, the same initial conditions

$$u_\varepsilon(0) = \alpha, \qquad \dot{u}_\varepsilon(0) = \beta, \qquad u_\varepsilon(-s)_{|s>0} = \gamma(s) \tag{2.1}$$

are understood to hold for every $\varepsilon \geq 0$.

2.1. **Assumptions on the memory kernels.** Along the whole paper, the kernels μ_ε are required to comply with the following assumptions.

- $\mu_\varepsilon : \mathbb{R}^+ \to [0,\infty)$ is nonincreasing, piecewise absolutely continuous and summable with total mass
$$\kappa_\varepsilon = \int_0^\infty \mu_\varepsilon(s)\,\mathrm{d}s \in (0,1).$$
- The inequality
$$\mu_\varepsilon(s) \leq \mu(s)$$
holds for every $\varepsilon > 0$ and almost every $s > 0$. In particular, this entails
$$0 < \kappa_\varepsilon \leq \kappa < 1.$$

Therefore, the kernels are differentiable almost everywhere with $\mu_\varepsilon' \leq 0$. Nonetheless, they are allowed to possess (even infinitely many) discontinuities (i.e. jumps), and can be unbounded in a neighborhood of zero.

2.2. **Functional setting.** Given a Banach space X, we call $L(X)$ the space of bounded linear operators on X. We denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and the inner product in H, respectively. For $\imath = 1,2$, we introduce the higher-order Hilbert spaces

$$H^\imath = \mathrm{dom}(A^{\frac{\imath}{2}}), \qquad \langle u,v\rangle_\imath = \langle A^{\frac{\imath}{2}}u, A^{\frac{\imath}{2}}v\rangle, \qquad \|u\|_\imath = \|A^{\frac{\imath}{2}}u\|.$$

Then, for any $\varepsilon \geq 0$, we define the L^2-space (we will omit the index ε whenever zero)

$$\mathcal{M}_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+; H^1), \qquad \langle \eta,\xi\rangle_{\mathcal{M}_\varepsilon} = \int_0^\infty \mu_\varepsilon(s)\langle\eta(s),\xi(s)\rangle_1\,\mathrm{d}s,$$

along with the product space

$$\mathcal{H}_\varepsilon = H^1 \times H \times \mathcal{M}_\varepsilon$$

endowed with the norm

$$\|(u,v,\eta)\|^2_{\mathcal{H}_\varepsilon} = (1-\kappa_\varepsilon)\|u\|_1^2 + \|v\|^2 + \|\eta\|^2_{\mathcal{M}_\varepsilon}.$$

In light of the assumptions, for any $\varepsilon > 0$ we have the continuous inclusion

$$\mathcal{M} \subset \mathcal{M}_\varepsilon \quad \Rightarrow \quad \mathcal{H} \subset \mathcal{H}_\varepsilon$$

with norm inequality

$$\|z\|_{\mathcal{H}_\varepsilon} \leq \lambda\|z\|_{\mathcal{H}}, \quad \forall z \in \mathcal{H},$$

where, here and in the sequel, we agree to set $\lambda = \frac{1}{\sqrt{1-\kappa}}$.

Remark 2.1. If the support of μ_ε is strictly contained in the support of μ, the inclusion $\mathcal{M} \subset \mathcal{M}_\varepsilon$ is to be correctly interpreted as $\mathcal{M}_{|\mathrm{supp}(\mu_\varepsilon)} \subset \mathcal{M}_\varepsilon$.

3. Statements of the Results

For every $\varepsilon \geq 0$ and any given set of initial data $(\alpha,\beta,\gamma)\in\mathcal{H}_\varepsilon$, equation $\boldsymbol{\mathcal{E}_\varepsilon}$ with initial conditions (2.1) is known to possess a unique weak solution (see e.g. [3, 17])

$$u_\varepsilon \in \mathcal{C}([0,\infty),H^1)\cap\mathcal{C}^1([0,\infty),H).$$

The closeness of $\boldsymbol{\mathcal{E}_\varepsilon}$ to the limiting equation $\boldsymbol{\mathcal{E}}$ will be established in terms of the difference

$$\omega(\varepsilon) = \int_0^\infty [\mu(s)-\mu_\varepsilon(s)]\,\mathrm{d}s = \kappa - \kappa_\varepsilon.$$

To this aim, further assumptions on the memory kernels are needed. In fact, the convergence result is obtained by requiring suitable decay properties

(i) either for the limiting kernel μ,
(ii) or for the family of kernels μ_ε for $\varepsilon > 0$.

As far as (i) is concerned, we have the following theorem.

Theorem 3.1. *Let there exist a set of positive measure $P\subset\mathbb{R}^+$ and two constants $K\geq 1$ and $\delta>0$ such that*

$$\mu'(s) < 0, \quad \forall s\in P, \tag{3.1}$$

and

$$\mu(t+s)\leq K\mathrm{e}^{-\delta t}\mu(s), \tag{3.2}$$

for every $t\geq 0$ and almost every $s>0$. If

$$\lim_{\varepsilon\to 0}\omega(\varepsilon) = 0, \tag{3.3}$$

the uniform-in-time convergence

$$\lim_{\varepsilon\to 0}\Big[\sup_{t\geq 0}\big\{\|u(t)-u_\varepsilon(t)\|_1 + \|\dot u(t)-\dot u_\varepsilon(t)\|\big\}\Big] = 0 \tag{3.4}$$

holds for every set of initial data $(\alpha,\beta,\gamma)\in\mathcal{H}$.

Remark 3.2. The recent work [18] shows that (3.1)-(3.2) imply the (uniform) exponential decay of the solutions to $\boldsymbol{\mathcal{E}}$, being necessary as well when H is infinite dimensional. If $K=1$, it is easily verified that (3.2) boils down to the well-known sufficient (and much stronger) condition of exponential stability

$$\mu'(s)+\delta\mu(s)\leq 0 \qquad \text{for a.e. } s>0,$$

widely adopted in the literature (e.g. [3, 7, 8, 9, 12, 13, 14, 15, 16]).

The next result deals with (ii).

Theorem 3.3. *For any $\varepsilon>0$, let there exist $\delta_\varepsilon>0$ such that*

$$\mu_\varepsilon'(s)+\delta_\varepsilon\mu_\varepsilon(s)\leq 0 \qquad \textit{for a.e. } s>0. \tag{3.5}$$

If

$$\lim_{\varepsilon\to 0}\frac{1+\delta_\varepsilon}{\delta_\varepsilon}\sqrt{\omega(\varepsilon)} = 0, \tag{3.6}$$

the convergence (3.4) *holds for every set of initial data $(\alpha,\beta,\gamma)\in\mathcal{H}$.*

For the Volterra equation (1.1), Theorem 3.3 can be improved.

Theorem 3.4. *Assuming* (3.5) *true, the convergence* (3.4) *occurs for every set of initial data in* $\mathcal{H}$ *of the form* $(\alpha, \beta, 0)$ *whenever*

$$\lim_{\varepsilon\to 0} \frac{1+\delta_\varepsilon}{\delta_\varepsilon}\,\omega(\varepsilon) = 0, \tag{3.7}$$

in place of the more restrictive (3.6).

One might ask if the convergence (3.4) can be rendered uniform with respect to bounded set of initial data. This issue will be addressed in the final §6. Before giving the proofs of the theorems, carried out in §5, we briefly discuss some features of the convergence results stated above.

4. Remarks and Comments

I. The comparison between the theoretical model $\boldsymbol{\mathcal{E}}$ and its finite delay approximation (1.2) follows directly by considering the equations $\boldsymbol{\mathcal{E}}_\varepsilon$ corresponding to the truncated kernels $\widehat{\mu}_\varepsilon$. In which case,

$$\omega(\varepsilon) = \int_{\frac{1}{\varepsilon}}^{\infty} \mu(s)\,\mathrm{d}s.$$

Thus (3.3) is automatically satisfied, whereas the quantity δ_ε appearing in (3.5) reads

$$\delta_\varepsilon = -\operatorname{ess\,inf}\big\{\mu'(s)/\mu(s) : s \in (0, \tfrac{1}{\varepsilon})\big\}.$$

II. The sole convergence $\mu_\varepsilon \to \mu$ in the L^1-norm is not enough in order to establish approximation results. Even more, any kind of convergence (without additional hypotheses) may not suffice in ensuring the closeness of trajectories in the longterm. To see that, just consider the one-step kernel

$$\mu(s) = \chi_{(0,x)}(s), \quad x \in (0,1). \tag{4.1}$$

According to [3], there exist particular values of x (depending on the eigenvalues of A) for which the corresponding equation $\boldsymbol{\mathcal{E}}$ admits periodic solutions. In other words, the system exhibits a purely elastic behavior. On the other hand, a minimal change in the kernel shape eliminates this resonant phenomenon, and all the solutions to the perturbed equation converge to zero.

III. In connection with Theorems 3.3 and 3.4, we remark that condition (3.5), which implies the exponential decay of solutions, is assumed only for $\varepsilon > 0$. Indeed, it is possible to approximate a kernel μ, whose associated solutions do not decay exponentially, with a family μ_ε satisfying (3.5)-(3.6). For example, the truncated kernels $\widehat{\mu}_\varepsilon$ obtained from

$$\mu(s) = \frac{1}{(1+s)^p}, \quad p > 3,$$

meet the hypotheses of Theorem 3.3 with

$$\delta_\varepsilon = \frac{p\varepsilon}{1+\varepsilon}.$$

For this choice of μ the solutions to $\boldsymbol{\mathcal{E}}$ do not decay exponentially (cf. [6, 11, 17]).

IV. For the family of Volterra equations corresponding to $\widehat{\mu}_\varepsilon$, we observe that Theorem 3.4 allows to consider kernels

$$\mu(s) \sim \frac{1}{s^p} \quad (s \to \infty)$$

with $p > 2$, whereas Theorem 3.3 requires a polynomial decay rate $p > 3$. The result is somehow optimal, since a decay rate $p \leq 2$ is rather meaningless from the physical point of view. This is due to the fact that, in most concrete situations (e.g. viscoelasticity), the kernel μ is formally obtained as the derivative of a summable convex kernel.

V. Assumption (3.7) in Theorem 3.4 is sharp and cannot be weakened by asking, for instance,

$$\lim_{\varepsilon\to 0} \frac{1+\delta_\varepsilon}{\delta_\varepsilon}\,[\omega(\varepsilon)]^q = 0, \quad q > 1.$$

Indeed, consider the kernel (4.1) for a resonant value x, and define the approximating kernels

$$\mu_\varepsilon(s) = \mathrm{e}^{-\varepsilon s}\mu(s).$$

Then $\boldsymbol{\mathcal{E}}$ admits solutions departing from initial data $(\alpha, \beta, 0)$ that do not decay, contrary to the solutions to $\boldsymbol{\mathcal{E}_\varepsilon}$ (see [6]). At the same time, (3.5) is fulfilled with $\delta_\varepsilon = \varepsilon$ and

$$\lim_{\varepsilon\to 0} \frac{1+\varepsilon}{\varepsilon}\,[\omega(\varepsilon)]^q = \lim_{\varepsilon\to 0} \frac{1+\varepsilon}{\varepsilon}\Big(x + \frac{e^{-\varepsilon x}-1}{\varepsilon}\Big)^q = 0.$$

5. Proofs

5.1. **The history framework.** In view of our scopes, it is more convenient viewing $\boldsymbol{\mathcal{E}_\varepsilon}$ as an ODE on a Hilbert space accounting for the past history of the variable u_ε. Following a celebrated idea of C.M. Dafermos [8], this is done by adding an auxiliary variable $\eta_\varepsilon = \eta^t_\varepsilon(s)$, *formally* defined as

$$\eta^t_\varepsilon(s) = u_\varepsilon(t) - u_\varepsilon(t-s), \quad t \geq 0,\ s > 0.$$

Then, introducing the three-component vector

$$U_\varepsilon(t) = (u_\varepsilon(t), \dot{u}_\varepsilon(t), \eta^t_\varepsilon),$$

equation $\boldsymbol{\mathcal{E}_\varepsilon}$ supplemented with initial conditions (2.1) for data $(\alpha, \beta, \gamma) \in \mathcal{H}_\varepsilon$ turns out to be completely equivalent to the linear differential equation on $\mathcal{H}_\varepsilon$

$$\frac{\mathrm{d}}{\mathrm{d}t} U_\varepsilon(t) = \mathbb{A}_\varepsilon U_\varepsilon(t) \tag{5.1}$$

subject to the initial condition

$$U_\varepsilon(0) = (\alpha, \beta, \alpha - \gamma), \tag{5.2}$$

where

$$\mathbb{A}_\varepsilon(u, v, \eta) = \big(v, -A\big[(1-\kappa_\varepsilon)u + \textstyle\int_0^\infty \mu_\varepsilon(s)\eta(s)\,\mathrm{d}s\big], -\eta' + v\big),$$

the *prime* being the distributional derivative, is a linear operator on $\mathcal{H}_\varepsilon$, whose domain is implicitly defined by its action (see [3, 17] for more details). Equation (5.1) is known to generate a linear contraction semigroup $S_\varepsilon(t)$ on $\mathcal{H}_\varepsilon$, i.e.

$$\|S_\varepsilon(t)\|_{L(\mathcal{H}_\varepsilon)} \leq 1, \quad \forall t \geq 0.$$

Besides, the third component of the solution $U_\varepsilon(t)$ to the Cauchy problem (5.1)-(5.2) admits the explicit representation formula (see [19])

$$\eta_\varepsilon^t(s) = \begin{cases} u_\varepsilon(t) - u_\varepsilon(t-s) & 0 < s \leq t, \\ u_\varepsilon(t) - \gamma(s-t) & s > t. \end{cases} \tag{5.3}$$

5.2. **Difference of solutions.** We now consider the difference $U(t) - U_\varepsilon(t)$ between the solutions to (5.1)-(5.2) corresponding to $\varepsilon = 0$ and an arbitrarily fixed $\varepsilon > 0$, respectively. Defining the three-component vector

$$F_\varepsilon(t) = (0, \int_0^\infty [\mu(s) - \mu_\varepsilon(s)]\, A[u(t) - \eta^t(s)]\, \mathrm{d}s, 0),$$

we are led by subtraction to the differential equation in $\mathcal{H}_\varepsilon$

$$\frac{\mathrm{d}}{\mathrm{d}t}[U(t) - U_\varepsilon(t)] = \mathbb{A}_\varepsilon[U(t) - U_\varepsilon(t)] + F_\varepsilon(t) \tag{5.4}$$

with initial value

$$U(0) - U_\varepsilon(0) = 0.$$

Here we used the fact that $\mathbb{A} = \mathbb{A}_\varepsilon$ on $\mathcal{H}$.

5.3. **More regular data.** For further scopes, we need to introduce the higher-regularity spaces

$$\mathcal{K}_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+; H^2) \subset \mathcal{M}_\varepsilon \qquad \text{and} \qquad \mathcal{V}_\varepsilon = H^2 \times H^1 \times \mathcal{K}_\varepsilon \subset \mathcal{H}_\varepsilon.$$

Remark 5.1. Since we are dealing with an *arbitrary* operator A, all the results stated in $\mathcal{H}_\varepsilon$ hold without changes in $\mathcal{V}_\varepsilon$. Indeed, $\mathcal{V}_\varepsilon$ is the space that naturally arises by replacing A with its square A^2.

In particular, $S(t)$ is a contraction semigroup on $\mathcal{V}$. Accordingly, for data $(\alpha, \beta, \gamma) \in \mathcal{V}$, we obtain the relation

$$\|F_\varepsilon(t)\|_{\mathcal{H}_\varepsilon} \leq \int_0^\infty [\mu(s) - \mu_\varepsilon(s)] \|u(t) - \eta^t(s)\|_2\, \mathrm{d}s, \tag{5.5}$$

which easily entails $F_\varepsilon \in L^\infty(\mathbb{R}^+; \mathcal{H}_\varepsilon)$ for every fixed ε. Therefore (cf. [20]), the solution to (5.4) can be expressed in the Duhamel integral form

$$U(t) - U_\varepsilon(t) = \int_0^t S_\varepsilon(t-y) F_\varepsilon(y)\, \mathrm{d}y. \tag{5.6}$$

5.4. **Proof of Theorem 3.1.** Firstly, we prove the convergence (3.4) for data (α, β, γ) in the more regular space $\mathcal{V}$. The general case will be subsequently established by means of an approximation argument. In what follows, we agree to set $z = (\alpha, \beta, \alpha - \gamma)$, which allows us to write

$$U_\varepsilon(t) = S_\varepsilon(t) z.$$

It is apparent that the map $(\alpha, \beta, \gamma) \mapsto z$ is an isomorphism both in $\mathcal{H}$ and in $\mathcal{V}$.

Lemma 5.2. *Within* (3.1)-(3.2), *there exists a constant $C > 0$ such that the inequality*

$$\|S(t)z - S_\varepsilon(t)z\|_{\mathcal{H}_\varepsilon} \leq C\Big\{\omega(\varepsilon) + \sqrt{\omega(\varepsilon)}\Big\}\|z\|_{\mathcal{V}}$$

holds for every $z \in \mathcal{V}$ and every $t > 0$.

Proof. According to [18], assumptions (3.1)-(3.2) imply the exponential stability of $S(t)$ in $\mathcal{V}$. Namely, there exist $M \geq 1$ and $\nu > 0$ such that

$$\|S(t)\|_{L(\mathcal{V})} \leq M\mathrm{e}^{-\nu t}.$$

Hence, setting for short

$$\varrho(\varepsilon) = \omega(\varepsilon) + \sqrt{\omega(\varepsilon)},$$

we infer from (5.5) and the Hölder inequality that

$$\begin{aligned}
\|F_\varepsilon(t)\|_{\mathcal{H}_\varepsilon} &\leq \omega(\varepsilon)\|u(t)\|_2 + \sqrt{\omega(\varepsilon)}\bigg(\int_0^\infty [\mu(s) - \mu_\varepsilon(s)]\|\eta^t(s)\|_2^2\,\mathrm{d}s\bigg)^{\frac{1}{2}} \\
&\leq \varrho(\varepsilon)\big(\|u(t)\|_2 + \|\eta^t\|_{\mathcal{K}}\big) \\
&\leq 2M\lambda\,\varrho(\varepsilon)\|z\|_{\mathcal{V}}\,\mathrm{e}^{-\nu t}.
\end{aligned}$$

Then, recalling that $S_\varepsilon(t)$ is a contraction semigroup on $\mathcal{H}_\varepsilon$, formula (5.6) leads to

$$\|S(t)z - S_\varepsilon(t)z\|_{\mathcal{H}_\varepsilon} \leq \int_0^t \|F_\varepsilon(y)\|_{\mathcal{H}_\varepsilon}\,\mathrm{d}y \leq \frac{2M\lambda}{\nu}\,\varrho(\varepsilon)\|z\|_{\mathcal{V}},$$

as claimed. □

In order to deal with the general case $z \in \mathcal{H}$, select any sequence $z_n \in \mathcal{V}$ converging to z in the norm of $\mathcal{H}$. Defining the map

$$\Lambda : (u, v, \eta) \mapsto (u, v, 0),$$

by virtue of the previous Lemma 5.2 we deduce the inequality

$$\begin{aligned}
\|\Lambda S(t)z - \Lambda S_\varepsilon(t)z\|_{\mathcal{H}} &\leq \|S(t)z_n - S_\varepsilon(t)z_n\|_{\mathcal{H}_\varepsilon} + \|S(t)(z - z_n)\|_{\mathcal{H}} + \|S_\varepsilon(t)(z_n - z)\|_{\mathcal{H}_\varepsilon} \\
&\leq C\Big\{\omega(\varepsilon) + \sqrt{\omega(\varepsilon)}\Big\}\|z_n\|_{\mathcal{V}} + \|z - z_n\|_{\mathcal{H}} + \|z_n - z\|_{\mathcal{H}_\varepsilon}.
\end{aligned}$$

Therefore, on account of (3.3),

$$\limsup_{\varepsilon\to 0} \|\Lambda S(t)z - \Lambda S_\varepsilon(t)z\|_{\mathcal{H}} \leq (1 + \lambda)\|z - z_n\|_{\mathcal{H}},$$

and letting $n \to \infty$ we obtain

$$\lim_{\varepsilon\to 0} \|\Lambda S(t)z - \Lambda S_\varepsilon(t)z\|_{\mathcal{H}} = 0,$$

which is exactly the thesis of Theorem 3.1.

5.5. **Proof of Theorems 3.3 and 3.4.** Here, we lean on the exponential decay of $S_\varepsilon(t)$ ensured by the sufficient condition (3.5).

Lemma 5.3. *Let* (3.5) *hold. Assume also that*

$$\inf_{\varepsilon\in(0,1]} \kappa_\varepsilon > 0. \tag{5.7}$$

Then, for every $\varepsilon \in (0, 1]$*, there exist constants* $M \geq 1$ *and* $\nu_\varepsilon > 0$ *such that*

$$\|S_\varepsilon(t)\|_{L(\mathcal{H}_\varepsilon)} \leq M\mathrm{e}^{-\nu_\varepsilon t}.$$

Moreover,

$$\nu_\varepsilon = \frac{c\delta_\varepsilon}{1+\delta_\varepsilon},$$

for some $c > 0$ *independent of* ε.

We omit the proof of the lemma, which is basically entirely contained in [17]. We just mention that the constant M can be taken independent of ε thanks to the inequality $\mu_\varepsilon \le \mu$.

As in the previous case, we first prove the result for more regular initial data.

Lemma 5.4. *Within* (3.5)-(5.7)*, there exists a constant* $C > 0$ *such that the inequality*

$$\|S(t)z - S_\varepsilon(t)z\|_{\mathcal{H}_\varepsilon} \le \frac{C(1+\delta_\varepsilon)}{\delta_\varepsilon}\Big\{\omega(\varepsilon)\|z\|_{\mathcal{V}} + \sqrt{\omega(\varepsilon)}\,\|\gamma\|_{\mathcal{K}}\Big\}$$

holds for every $z = (\alpha, \beta, \alpha - \gamma) \in \mathcal{V}$*, every* $t > 0$ *and every* $\varepsilon \in (0, 1]$.

Proof. Making use of the representation formula (5.3) for η^t, we rewrite (5.5) in the equivalent form

$$\|F_\varepsilon(t)\|_{\mathcal{H}_\varepsilon} \le \int_0^t [\mu(s) - \mu_\varepsilon(s)]\|u(t-s)\|_2\,\mathrm{d}s + \int_t^\infty [\mu(s) - \mu_\varepsilon(s)]\|\gamma(s-t))\|_2\,\mathrm{d}s.$$

Since $S(t)$ is a contraction semigroup on $\mathcal{V}$,

$$\int_0^t [\mu(s) - \mu_\varepsilon(s)]\|u(t-s)\|_2\,\mathrm{d}s \le \lambda\omega(\varepsilon)\|z\|_{\mathcal{V}},$$

while the second term is estimated by exploiting the Hölder inequality and the monotonicity of μ as

$$\int_t^\infty [\mu(s) - \mu_\varepsilon(s)]\|\gamma(s-t))\|_2\,\mathrm{d}s \le \sqrt{\omega(\varepsilon)}\bigg(\int_t^\infty \mu(s)\|\gamma(s-t)\|_2^2\,\mathrm{d}s\bigg)^{\frac12} \le \sqrt{\omega(\varepsilon)}\,\|\gamma\|_{\mathcal{K}}.$$

In summary, we get

$$\|F_\varepsilon(t)\|_{\mathcal{H}_\varepsilon} \le \lambda\omega(\varepsilon)\|z\|_{\mathcal{V}} + \sqrt{\omega(\varepsilon)}\,\|\gamma\|_{\mathcal{K}}.$$

In light of Lemma 5.3, we deduce from (5.6) the estimate

$$\|S(t)z - S_\varepsilon(t)z\|_{\mathcal{H}_\varepsilon} \le M\lambda\Big\{\omega(\varepsilon)\|z\|_{\mathcal{V}} + \sqrt{\omega(\varepsilon)}\,\|\gamma\|_{\mathcal{K}}\Big\}\int_0^t \mathrm{e}^{-\nu_\varepsilon y}\,\mathrm{d}y,$$

yielding the desired conclusion. □

The general case $z \in \mathcal{H}$ is carried out exactly as in the proof of Theorem 3.1, with the only care of choosing an approximating sequence $z_n \in \mathcal{V}$ of the form $z_n = (\alpha_n, \beta_n, \alpha_n)$ in the proof of Theorem 3.4, so that (3.7), in place of the more restrictive (3.6), suffices to draw the required convergence.

6. Uniformity with Respect to Initial Data

We finally discuss the uniformity of the convergence (3.4) with respect to initial data. In fact, Lemma 5.2 and Lemma 5.4 tell that the conclusions of Theorems 3.1, 3.3 and 3.4 hold true uniformly with respect to bounded sets of initial data in the more regular space $\mathcal{V}$ introduced in §5.3.

A natural question is then what happens for bounded sets of initial data in the original phase-space $\mathcal{H}$. We limit ourselves to consider the more interesting case of Theorem 3.1. To this aim, we focus on the following conjecture, which improves the theorem in the desired direction.

Conjecture. *Let μ satisfy (3.1)-(3.2). Then, for every bounded set $\mathcal{B} \subset \mathcal{H}$, there exists a continuous function $\mathcal{Q} = \mathcal{Q}_{\mathcal{B}}$ vanishing at zero such that the inequality*

$$\|u(t) - u_\varepsilon(t)\|_1 + \|\dot{u}(t) - \dot{u}_\varepsilon(t)\| \leq \mathcal{Q}(\omega(\varepsilon)) \tag{6.1}$$

is satisfied for every $t \geq 0$ and every initial data $(\alpha, \beta, \gamma) \in \mathcal{B}$.

Up to redefining $\mathcal{Q}$, we preliminary observe that (6.1) implies the stronger inequality

$$\|S(t)z - S_\varepsilon(t)z\|_{\mathcal{H}_\varepsilon} \leq \mathcal{Q}(\omega(\varepsilon)), \quad z = (\alpha, \beta, \alpha - \gamma). \tag{6.2}$$

Indeed, on account of the representation formula (5.3), the third component ξ^t of the difference $S(t)z - S_\varepsilon(t)z$ reads

$$\xi_\varepsilon^t(s) = \begin{cases} u(t) - u_\varepsilon(t) - u(t-s) + u_\varepsilon(t-s) & 0 < s \leq t, \\ u(t) - u_\varepsilon(t) & s > t. \end{cases}$$

Thus, assuming (6.1), we readily obtain the uniform bound

$$\|\xi_\varepsilon^t(s)\|_{\mathcal{M}_\varepsilon} \leq 2\mathcal{Q}(\omega(\varepsilon)),$$

so establishing (6.2).

Unfortunately, such a conjecture is false in the general case. To see that, for an arbitrarily given μ complying with (3.1)-(3.2), let us consider a family μ_ε satisfying (3.3) along with the additional constraint

$$\mu(s) \leq 2\mu_\varepsilon(s), \quad \forall s > 0.$$

Lemma 6.1. *Assuming the conjecture true, the semigroups $S_\varepsilon(t)$ are exponentially stable for all $\varepsilon > 0$ sufficiently small.*

Proof. For these particular μ_ε, the reverse inclusion $\mathcal{H}_\varepsilon \subset \mathcal{H}$ holds with an embedding constant independent of ε. Hence, for an arbitrary z in the unit ball of $\mathcal{H}_\varepsilon$, exploiting (6.2) and the exponential stability of $S(t)$, we draw the inequality

$$\|S_\varepsilon(t)z\|_{\mathcal{H}_\varepsilon} \leq \|S(t)z - S_\varepsilon(t)z\|_{\mathcal{H}_\varepsilon} + \|S(t)z\|_{\mathcal{H}_\varepsilon} \leq \mathcal{Q}(\omega(\varepsilon)) + C\mathrm{e}^{-\nu t},$$

for some strictly positive constants C, ν. Indeed, it is apparent that any such z can be written in the form $z = (\alpha, \beta, \alpha - \gamma)$, for some (α, β, γ) in a bounded set $\mathcal{B} \subset \mathcal{H}$. Thus, for any $\varepsilon > 0$ sufficiently small, there exists a time $t_\varepsilon > 0$ such that

$$\|S_\varepsilon(t)\|_{L(\mathcal{H}_\varepsilon)} < 1, \quad \forall t > t_\varepsilon.$$

By a classical argument of the theory of linear semigroups (see e.g. [20]), we infer the exponential decay of $S_\varepsilon(t)$ on $\mathcal{H}_\varepsilon$. □

On the other hand, we can easily construct a family of approximating kernels μ_ε of the above kind made by step functions, i.e. with $\mu_\varepsilon' = 0$ almost everywhere. In that case, as shown in [3], the semigroups $S_\varepsilon(t)$ are never exponentially stable (at least for a quite general class of operators A), against the conclusions of the lemma.

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