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Hardy–Rellich inequalities with boundary remainder terms and applications

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Abstract

We prove a family of Hardy-Rellich inequalities with optimal constants and additional boundary terms. These inequalities are used to study the behavior of extremal solutions to biharmonic Gelfand-type equations under Steklov boundary conditions.

1 Introduction

A well-known generalization of the first order Hardy inequality [35, 36] to the second order is due to Rellich [44] and reads

$$\int_\Omega |\Delta u|^2\, dx \geq \frac{N^2(N-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4}dx \qquad \forall\, u\in H^2_0(\Omega) \tag{1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 5)$ is a smooth bounded domain and the constant $\frac{N^2(N-4)^2}{16}$ is optimal, in the sense that it is the largest possible. In [29], the validity of (1) was extended to the space $H^2\cap H^1_0(\Omega)$ (see also previous work in [21] when Ω is a ball) and the existence of *remainder terms* was established. More precisely, among other results, it is proved there that there exist constants $C_i = C_i(\Omega) > 0$ $(i = 1, 2)$ such that

$$\int_\Omega |\Delta u|^2\, dx \geq \frac{N^2(N-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4}dx + C_1\int_\Omega \frac{u^2}{|x|^2}dx + C_2\int_\Omega u^2 dx \tag{2}$$

for all $u\in H^2\cap H^1_0(\Omega)$. Higher order versions of (1) and (2) were obtained in [17, 29, 40]. We also refer to [1, 4, 5, 33, 45] for further improvements and variants of these inequalities.
The existence of a constant $d_0 = d_0(\Omega) > 0$ such that

$$\int_\Omega |\Delta u|^2\, dx \geq d_0\int_{\partial\Omega} u_\nu^2 dS \quad \forall\, u\in H^2\cap H^1_0(\Omega), \tag{3}$$

where ν is the unit outer normal to $\partial\Omega$, is proved in [8]. The number d_0 is the first simple boundary eigenvalue of the biharmonic operator Δ^2 under the so-called Steklov boundary conditions: $u = \Delta u - du_\nu = 0$ on $\partial\Omega$ for some $d\in\mathbb{R}$. When $\Omega = B$, the unit ball, one has $d_0(B) = N$.
The first purpose of the present paper is to consider intermediate situations between inequalities (2) and (3). We seek two positive constants $h = h(\Omega)$ and $d = d(\Omega)$ such that

$$\int_\Omega |\Delta u|^2\, dx \geq h\int_\Omega \frac{u^2}{|x|^4}\, dx + d\int_{\partial\Omega} u_\nu^2\, dS \quad \forall\, u\in H^2\cap H^1_0(\Omega). \tag{4}$$

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Clearly, in (4) one expects a dependence $d = d(h)$ and, of course, one is interested in finding the optimal (largest) constant d for any $h \geq 0$. We will prove (4) in all the situations between the two extremal cases where $h = 0$ (corresponding to (3)) and $h = \frac{N^2(N-4)^2}{16}$ (corresponding to (2) with a boundary remainder term). As expected, in the limit situation where $h = \frac{N^2(N-4)^2}{16}$ there is a loss of compactness and (4) becomes a *strict* inequality for all $u \neq 0$. There is no way to further increase h, even at expenses of a smaller (possibly negative) value of d; to see this, it suffices to consider the original inequality (1) in the space $H^2_0(\Omega)$.

The second purpose of the paper is to apply inequalities (4) to the study of the regularity of *extremal solutions* to semilinear biharmonic problems such as

$$(G_\lambda) \begin{cases} \Delta^2 u = \lambda g(u),\ u \geq 0 & \text{in } \Omega \\ u = \Delta u - d u_\nu = 0 & \text{on } \partial\Omega \end{cases}$$

where λ and d are nonnegative constants, and the nonlinearity g has the following form:

$$g(u) = e^u\ , \qquad g(u) = (1+u)^p \quad (p > 1),$$
$$g(u) = \frac{1}{(1-u)^\gamma} \quad (0 \leq u < 1,\ \gamma \geq 2). \tag{5}$$

Fourth order problems as (G_λ) were studied under different boundary conditions, such as Dirichlet conditions $u = u_\nu = 0$ on $\partial\Omega$, see [3, 13, 15, 18, 19, 20, 24, 25, 26, 47], or Navier conditions $u = \Delta u = 0$ on $\partial\Omega$, see [7, 19, 20, 34]. See also [14, 38] for related nonlocal problems. We are here interested in intermediate boundary conditions, the Steklov conditions $u = \Delta u - d u_\nu = 0$ on $\partial\Omega$.

Although problem (G_λ) is the fourth order version of long-standing and well-studied second order problems [9, 11, 32, 37, 39], the proofs of the corresponding results turn out to be much more involved. The first difficulty one has to face for biharmonic problems is the lack of a general maximum principle. Under Steklov conditions, a maximum principle can be proved only for restricted values of d (see [8, 30]). In particular, the maximum principle holds for $d \in [0, d_0)$ (see (3)) in any domain Ω and also for some $d < 0$ depending on the domain. This is one reason why, for general domains Ω, we only have partial results, see Section 3.1. When Ω is a ball, more can be said. When looking for radial solutions, one may perform a phase space analysis for the corresponding system of ODEs. For second order problems the phase space is a plane. For the fourth order counterpart, the phase space is four-dimensional where the topology is more complicated and the Poincaré-Bendixson theory is not available, see [2, 3, 28]. Finally, in problems (G_λ) one usually does not succeed in finding explicit singular solutions which allow to describe the bifurcation branch (λ, u) of nontrivial solutions. However, in somehow particular situations (suitable choice of d) we can still prove fairly exhaustive results, see Section 3.2 where we take advantage of the Hardy-type inequalities (4). This paper should be intended as a further step towards a complete understanding of the mathematical phenomena related to (G_λ).

The interest in studying (G_λ) is not only mathematical. For instance when $g(u) = \frac{1}{(1-u)^2}$, it represents a model for MicroElectroMechanicalSystems, see [43] for a systematic development on the subject and also the introduction in [13, 14]. In this context, one of the main points is the study of the so-called pull-in instability, which corresponds to the situation in which the deflected profile u reaches the value $u = 1$ and generates ruptures phenomena.

This paper is organized as follows. In Section 2 we derive our family of Hardy-Rellich type inequalities. In Section 3 we state the results about (G_λ), first for general domains Ω (Section 3.1) and then for the ball (Section 3.2). The proofs of the results of Section 2 are given in Section 4, the ones of Section 3 are given in Sections 6 and 7.

2 Hardy–Rellich type inequalities

Let $\Omega \subset \mathbb{R}^N$, $N \geq 5$, be a bounded domain containing the origin and with smooth boundary and denote by B the unit ball of $\mathbb{R}^N$. For $h \geq 0$ and $d \in \mathbb{R}$, consider the following boundary eigenvalue problem

$$\begin{cases} \Delta^2 u = h \dfrac{u}{|x|^4} & \text{in } \Omega \\ u = \Delta u - d u_\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

For fixed h, we are interested in finding the values d for which (6) admits nontrivial solutions, namely $u \in H^2 \cap H^1_0(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v \, dx - d \int_{\partial\Omega} u_\nu \, v_\nu \, dS = h \int_\Omega \frac{u}{|x|^4} \, v \, dx \quad \forall \, v \in H^2 \cap H^1_0(\Omega). \tag{7}$$

Since the boundary conditions in (6) satisfy the complementing condition, by elliptic regularity any solution to (6) belongs to $C^\infty(\Omega \setminus \{0\})$, whereas up to the boundary the solution is as smooth as the boundary, see [8, Lemma 15]. The first (smallest) eigenvalue of (6) has the following variational characterization

$$d_1(h) := \inf_{H^2 \cap H^1_0(\Omega) \setminus H^2_0(\Omega)} \frac{\int_\Omega |\Delta u|^2 \, dx - h \int_\Omega \frac{u^2}{|x|^4} \, dx}{\int_{\partial\Omega} u_\nu^2 \, dS} \tag{8}$$

and we have

Proposition 1. *The infimum in* (8) *is achieved if and only if* $0 \leq h < \frac{N^2(N-4)^2}{16}$ *and, up to a multiplicative constant, the minimizer* $\overline{u}_h$ *is unique, positive, superharmonic in* Ω *and it solves* (6) *when* $d = d_1(h)$*. Moreover,* $\overline{u}_h$ *is the only eigenfunction of* (6) *having one sign.*

The proof of Proposition 1 is somehow standard and we briefly sketch it in Section 4. By Proposition 1 we deduce the first part of the next statement.

Theorem 2. *Let* $0 \leq h \leq \frac{N^2(N-4)^2}{16}$ *and let* $d_1(h)$ *be as in (8). Then* $d_1(h) \geq 0$ *and*

$$\int_\Omega |\Delta u|^2 \, dx \geq h \int_\Omega \frac{u^2}{|x|^4} \, dx + d_1(h) \int_{\partial\Omega} u_\nu^2 \, dS \quad \forall \, u \in H^2 \cap H^1_0(\Omega). \tag{9}$$

Furthermore, the map $h \mapsto d_1(h)$ *is strictly decreasing, the constant* $d_1(h)$ *is sharp and attained if and only if* $0 \leq h < \frac{N^2(N-4)^2}{16}$*. Finally, if* Ω *is strictly starshaped with respect to the origin, then* $d_1(\frac{N^2(N-4)^2}{16}) > 0$.

The last statement of Theorem 2 requires the domain Ω to be strictly starshaped. We do not know if a similar result holds without this geometric assumption. Related to this problem, one may wonder if, for a fixed $h \in [0, \frac{N^2(N-4)^2}{16}]$, there exists a constant $d_1 > 0$ which serves as the best possible for all domains Ω having a given measure. The results obtained in [12] in the case $h = 0$ suggest a negative answer since one has

$$\inf_{|\Omega|=1} d_0(\Omega) = 0,$$

where $d_0 = d_1(0)$ is the optimal constant in (3).
In the case of the unit ball, $d_1(h)$ can be explicitly determined. In order to simplify the computations, we introduce an auxiliary parameter $0 \leq \alpha \leq N - 4$ and we set

$$h(\alpha) := \frac{\alpha(\alpha + 4)(\alpha + 4 - 2N)(\alpha + 8 - 2N)}{16} \tag{10}$$

and

$$\delta_1(\alpha) := \frac{N - \alpha + \sqrt{N^2 - \alpha^2 + 2\alpha(N-4)}}{2}. \tag{11}$$

For $\alpha \in [0, N-4]$, the map $\alpha \mapsto \delta_1(\alpha)$ is strictly decreasing whereas the map $\alpha \mapsto h(\alpha)$ is strictly increasing, $h(0) = 0$ and $h(N-4) = \frac{N^2(N-4)^2}{16}$ so that $0 \le h(\alpha) \le \frac{N^2(N-4)^2}{16}$ for all $\alpha \in [0, N-4]$ and Theorem 2 applies. Moreover, (10) is invertible so that $\alpha = \alpha(h)$ is well-defined for any $0 \le h \le \frac{N^2(N-4)^2}{16}$ and, by making this dependence explicit, (11) can be used to obtain $d_1(h) = \delta_1(\alpha(h))$.

Theorem 3. *For every $u \in H^2 \cap H^1_0(B)$ and $0 \le \alpha \le N-4$, there holds*

$$\int_B |\Delta u|^2 \, dx \ge h(\alpha) \int_B \frac{u^2}{|x|^4} \, dx + \delta_1(\alpha) \int_{\partial B} u_\nu^2 \, dS \tag{12}$$

where $h(\alpha)$ and $\delta_1(\alpha)$ are defined in (10) and (11). Furthermore, the best constant $\delta_1(\alpha)$ is attained if and only if $0 \le \alpha < N-4$, by multiples of the function

$$\overline{u}_\alpha(x) = |x|^{-\frac{\alpha}{2}} - |x|^{\frac{4-N+\sqrt{N^2-\alpha^2+2\alpha(N-4)}}{2}}.$$

Notice that also for $\alpha \ge N-4$ the functions $\overline{u}_\alpha$ solve (6) in $B \setminus \{0\}$, however, they fail to have finite energy since they do not belong to $H^2(B)$. When $\alpha = 0$, one has $h(0) = 0$ and $\delta_1(0) = N$, thus (12)

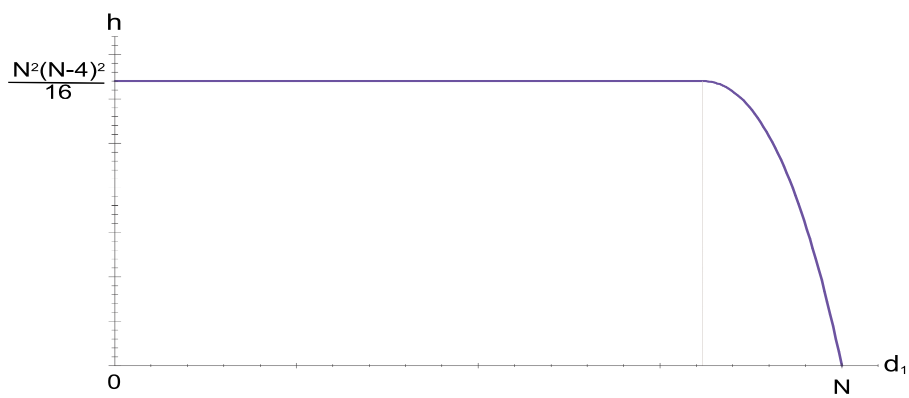

Figure 1: The relationship between $d_1(h) = \delta_1(\alpha(h))$ and h in (12).

reduces to (3). When $\alpha = N-4$, (12) becomes a Hardy-Rellich inequality with a boundary remainder term, as established in the following

Corollary 4. *For every $u \in H^2 \cap H^1_0(B)$ we have*

$$\int_B |\Delta u|^2 \, dx \ge \frac{N^2(N-4)^2}{16} \int_B \frac{u^2}{|x|^4} \, dx + \frac{4 + \sqrt{2N^2 - 8N + 16}}{2} \int_{\partial B} u_\nu^2 \, dS$$

and the boundary constant is the best possible.

3 Biharmonic problems under Steklov boundary conditions

In the sequel, when we need to specify the nonlinearity involved in (5), in place of (G_λ) we will refer to problems:

$$(E_\lambda)\begin{cases}\Delta^2 u=\lambda e^u,\ u\geq 0 & \text{in } \Omega\\ u=\Delta u-du_\nu=0 & \text{on } \partial\Omega,\end{cases}$$

$$(P_\lambda)\begin{cases}\Delta^2 u=\lambda(1+u)^p,\ u\geq 0 & \text{in } \Omega\\ u=\Delta u-du_\nu=0 & \text{on } \partial\Omega,\end{cases}$$

$$(\Gamma_\lambda)\begin{cases}\Delta^2 u=\dfrac{\lambda}{(1-u)^\gamma},\ 0\leq u\leq 1 & \text{in } \Omega\\ u=\Delta u-du_\nu=0 & \text{on } \partial\Omega.\end{cases}$$

Actually, for $\lambda\neq 0$ the solutions are *strictly* positive.

3.1 Results in general domains

Throughout this section we assume that

$$\Omega\subset\mathbb{R}^N,\ (N\geq 2)\text{ is a bounded domain with smooth boundary,}$$

$$0\leq d<d_0,$$

with d_0 as in (3). When $\Omega=B$, this assumption can be relaxed to $d<N$.
Let $\mathcal{H}_d(\Omega)$ denote the space $H^2\cap H^1_0(\Omega)$ endowed with the scalar product

$$(u,v):=\int_\Omega \Delta u\Delta v\,dx-d\int_{\partial\Omega}u_\nu v_\nu\,dS.$$

By (3) this scalar product induces a norm on $H^2\cap H^1_0(\Omega)$ which is equivalent to the norm $\|\Delta\,\cdot\,\|_2$. We also introduce the space

$$X_d:=\left\{\varphi\in C^4(\overline{\Omega});\varphi=\Delta\varphi-d\varphi_\nu=0\text{ on }\partial\Omega\right\}$$

and give some definitions.

Definition 5. *The function $u\in L^1(\Omega)$ is a* **solution** *to (G_λ) if $u\geq 0$ a.e., $g(u)\in L^1(\Omega)$ and*

$$\int_\Omega u\,\Delta^2\varphi\,dx=\lambda\int_\Omega g(u)\,\varphi\,dx\quad\forall\,\varphi\in X_d.$$

For problem (Γ_λ) we also require $u\leq 1$. Solutions to (G_λ) which belong to $\mathcal{H}_d(\Omega)$ are called **energy solutions***. If $\underline{u}_\lambda$ is a solution to (G_λ) such that for any other solution u_λ to (G_λ) one has*

$$\underline{u}_\lambda(x)\leq u_\lambda(x)\quad\textit{for a.e.}\quad x\in\Omega$$

then we say that $\underline{u}_\lambda$ is the **minimal solution** *of (G_λ).*
Moreover, a solution u to (E_λ) or (P_λ) is said to be **regular** *if $u\in L^\infty(\Omega)$,* **singular** *if $u\notin L^\infty(\Omega)$. A solution u to (Γ_λ) is said to be* **regular** *if $\|u\|_\infty<1$,* **singular** *if $\|u\|_\infty=1$.*

By elliptic regularity, it follows that regular solutions are smooth and solve (G_λ) in the classical sense. Formally, the first eigenvalue μ_1 of the linearized operator $\Delta^2 - \lambda\, g'(u_\lambda)$, at a solution u_λ to (G_λ) such that $g'(u_\lambda) \in L^1(\Omega)$, has the following variational characterization

$$\mu_1(\lambda) := \inf_{\varphi \in X_d \setminus \{0\}} \frac{\int_\Omega |\Delta\varphi|^2\, dx - d \int_{\partial\Omega} \varphi_\nu^2\, dS - \lambda \int_\Omega g'(u_\lambda)\, \varphi^2\, dx}{\int_\Omega \varphi^2\, dx}.$$

Note that it may happen that $\mu_1(\lambda) = -\infty$ although this will not occur in the "interesting cases" where u_λ is regular or not too singular.
We have the following

Theorem 6. *There exists $\lambda^* = \lambda^*(\Omega) > 0$ such that for $0 < \lambda < \lambda^*$, problems (E_λ) and (Γ_λ) have a minimal regular solution $\underline{u}_\lambda$ which is positive and stable, namely $\mu_1(\lambda) > 0$. Moreover, for almost every $x \in \Omega$ there exists*

$$u^*(x) := \lim_{\lambda \nearrow \lambda^*} \underline{u}_\lambda(x) \tag{13}$$

which is an energy solution when $\lambda = \lambda^*$. For $\lambda > \lambda^*$, there are no solutions.

Remark 7. *By arguing as in [24, Theorem 1], one can show that the same statements hold for (P_λ) provided $\Omega = B$: here a major difficulty is to establish that*

$$\lambda^* = \sup\{\lambda \geq 0 \,:\, (P_\lambda) \text{ possesses a classical solution}\}\ ,$$

which so far seems to be unknown in general domains.

The condition $\mu_1(\lambda) > 0$ yields a *strong* stability of minimal solutions. In some cases, this can be relaxed. We call a solution u_λ to (G_λ) *weakly* stable if $g'(u_\lambda) \in L^1(\Omega)$ and $\mu_1(\lambda) \geq 0$, i.e.

$$\int_\Omega |\Delta\varphi|^2\, dx - d \int_{\partial\Omega} \varphi_\nu^2\, dS \geq \lambda \int_\Omega g'(u_\lambda)\, \varphi^2\, dx \qquad \forall\, \varphi \in X_d. \tag{14}$$

Condition (14) enables us to characterize singular energy solutions. This will be the key ingredient to study the regularity of the *extremal solution* u^* for problem (G_λ) and to determine the *critical dimension.*

Theorem 8. *Let u^* be the extremal solution to (E_λ) or (Γ_λ), as given by (13). Then, we have:*

(i) u^* is weakly stable;

(ii) if u^* is regular, then it is the unique solution for $\lambda = \lambda^*$;

(iii) if $u_\lambda \in \mathcal{H}_d(\Omega)$ is a singular weakly stable solution for some $\lambda \in (0, \lambda^*]$, then $\lambda = \lambda^*$ and u^* is singular.

These statements hold for (P_λ) provided $\Omega = B$.

We expect the solution to (G_{λ^*}) to be unique (even in weak sense) although a full proof of this fact in general domains Ω seems not trivial. When $\Omega = B$, one can argue as in [18] and obtain uniqueness for (G_{λ^*}), with the restriction $1 < p \leq 2$ or $p \geq 3$ for (P_λ) since the arguments in [18] use the positivity of g^{iv}. However, the same arguments seem not to extend to general domains since one would need to know the location of the singularities of the solutions.
If $\underline{u}_\lambda$ is the minimal solution, by Theorem 6, $g'(\underline{u}_\lambda) \in L^\infty(\Omega)$ and (14) can be extended to any $\varphi \in \mathcal{H}_d(\Omega)$. On the other hand, if $u_\lambda = u^*$ and it is singular, Theorem 8-(i) ensures that it is weakly

stable, so that the right hand side in (14) is finite and, by density arguments, one has that (14) holds for all φ in the energy class $\mathcal{H}_d(\Omega)$. This is an evidence that u^* cannot be "too singular" since otherwise $g'(u_\lambda)\,\varphi^2 \notin L^1(\Omega)$ for some $\varphi \in \mathcal{H}_d(\Omega)$. The results obtained in the ball, see Section 3.2, suggest that $g'(u^*) \sim \frac{C}{|x|^4}$ as $x \to 0$, the point where u^* becomes singular.
The regularity issue in low dimensions is a consequence of embedding results and it is summarized in

Theorem 9. *The following regularity results hold:*

(i) If $2 \leq N \leq 4$, then any energy solution to (E_λ) is regular.

(ii) If $N = 2$ and $\gamma > 2$ or $N = 3$ and $\gamma \geq 6$, then any energy solution to (Γ_λ) is regular. Moreover, if $2 \leq N \leq 4$ and $\gamma \geq 2$ then the extremal solution u^* to (Γ_{λ^*}) is regular.

(iii) If $2 \leq N \leq 4$ or $N \geq 5$ and $p \leq \frac{N+4}{N-4}$, then any energy solution to (P_λ) is regular.

We conclude this section by observing that under the assumptions of Theorem 9, besides the minimal solution, for all $\lambda \in (0, \lambda^*)$ there exists a mountain pass solution, which can be obtained by arguing as in [16, 22]. The critical case $p = \frac{N+4}{N-4}$ for (P_λ) needs a particular attention, see [31].

3.2 Results in the ball

Throughout this section we take $\Omega = B$. It is worth to compare some features of problem (G_λ) with the corresponding problem under Dirichlet boundary conditions which is the limiting case of (G_λ) as $d \to -\infty$, namely

$$(G_\lambda^\infty)\begin{cases}\Delta^2 u = \lambda g(u) & \text{in } B\\ u = u_\nu = 0 & \text{on } \partial B\end{cases}$$

where $\lambda \geq 0$ and g is as in (5).

We start by comparing the extremal parameters of problems (G_λ) and (G_λ^∞).

Theorem 10. *Let $d < N$ in (G_λ) and assume that $\lambda^*(d)$ and λ^*_∞ are the extremal parameters of problems (G_λ) and (G_λ^∞) respectively. The map $d \mapsto \lambda^*(d)$ is non increasing. In particular,*

$$\lambda^*(d) \leq \lambda^*_\infty \quad \textit{for any } d \in (-\infty, N).$$

The monotonicity of the map $d \mapsto \lambda^*(d)$ is reached by means of comparison arguments which are strictly connected with the validity of the positivity preserving property of the operator Δ^2 under Steklov boundary conditions. When $\Omega = B$ the positivity preserving property holds for any $d < N$, whereas for general domains Ω, it holds only for restricted values of d, see [30].
Let us now make precise what we mean by critical dimension.

Definition 11. *We say that $N_d^* \in \mathbb{N}$ is the critical dimension for problem (G_λ) (resp. N_∞^* for problem (G_λ^∞)) if, for every $N < N_d^*$ (resp. $N < N_\infty^*$), the extremal solution u^* is regular, whereas for $N \geq N_d^*$ (resp. $N \geq N_\infty^*$) the extremal solution is singular.*

The critical dimension for (G_λ) depends on d and one would like to have the exact value of N_d^* for any d. This seems out of reach at the moment and, for each nonlinearity in (5), we fix $d = d_*$, where

$$d_* := \frac{v''(1) + (N-1)v'(1)}{v'(1)}$$

and $v = v(r)$ is the unique entire singular (sign-changing) solution of the corresponding equation in $\mathbb{R}^N$ which vanishes on ∂B. This choice allows explicit computations and simplifies some arguments.

Since the asymptotic behavior of singular solutions is independent of d, we believe that similar results hold for any d; see Lemmata 24, 27 and 30. In the following we set $N^* := N^*_{d^*}$ and we complement the statements (i), (ii) and (iii) of Theorem 9 with the regularity results in large dimensions.

Theorem 12. *Consider problem (E_λ) and let $d = d_* := N - 2$.*

(i) If $5 \le N \le 13$, then $\lambda^* > 8(N-2)(N-4)$ and the extremal solution is regular.

(ii) If $N \ge 14$, then $\lambda^* = 8(N-2)(N-4)$ and the extremal solution is singular and given by $u^*(x) = -4\log|x|$.

Remark 13. *Theorem 12 yields the critical dimension for problem (E_λ), namely $N^* = 14$. We recall that from [3, 18, 42] one has $N^*_\infty = 13$.*

To deal with (Γ_λ), for all $\gamma \ge 2$ we introduce the integer number

$$N^*(\gamma) := \min\left\{ N \in \mathbb{N} : N \ge \frac{\sqrt{89\gamma^2 + 54\gamma + 1} + 17\gamma + 3}{2(\gamma+1)} \right\}.$$

Notice that $\gamma \mapsto N^*(\gamma)$ is increasing and $10 \le N^*(\gamma) \le 14$, for all $\gamma \ge 2$. Furthermore, we set

$$\lambda_{N,\gamma} := \frac{8}{(\gamma+1)^4}(\gamma-1)((N-2)\gamma + N + 2)((N-4)\gamma + N). \tag{15}$$

Theorem 14. *Consider problem (Γ_λ) and let $d = d_*(\gamma) := N - \frac{2(\gamma-1)}{\gamma+1}$.*

(i) If $5 \le N \le N^*(\gamma) - 1$ and $\gamma \ge 3$, then $\lambda^* > \lambda_{N,\gamma}$ and the extremal solution is regular.

(ii) If $N \ge N^*(\gamma)$, then $\lambda^* = \lambda_{N,\gamma}$ and the extremal solution is singular and given by $u^*(x) = 1 - |x|^{\frac{4}{\gamma+1}}$.

Remark 15. *It is shown in [15] the critical dimension $N^*_\infty(2) = 9$ for (Γ^∞_λ) in the case $\gamma = 2$. Theorem 14 allows to conjecture that $N^*(2) = 10$.*

We conclude with the positive power case. In [28], for $N \ge 13$, a limiting value for p is defined, namely $p_c(N) \in \left(\frac{N+4}{N-4}, +\infty\right)$ which is the unique solution of the equation:

$$\frac{4\,p_c}{p_c - 1}\left(\frac{4}{p_c-1} + 2\right)\left(N - 2 - \frac{4}{p_c-1}\right)\left(N - 4 - \frac{4}{p_c-1}\right) = \frac{N^2(N-4)^2}{16}.$$

If $N \ge 13$ and $p < p_c$, the extremal solution of (P^∞_λ) is regular, see [25, Theorem 3]. Thus it is natural to conjecture that $N^*_\infty = 13$. This suggests to take $p > p_c$ in what follows. Since $33 > p_c(N)$ for all $N \ge 13$, we fix $p = 33$. In fact, by numerical evidence we expect that statements (i) and (ii) below hold true for all $p > p_c$.

Theorem 16. *Consider problem (P_λ). Let $p = 33$ and $d = d_*(33) := N - \frac{17}{8}$.*

(i) If $5 \le N \le 13$, then $\lambda^* > \frac{1088N^2 - 6800N + 9537}{4096}$ and the extremal solution is regular.

(ii) If $N \ge 14$, then $\lambda^* = \frac{1088N^2 - 6800N + 9537}{4096}$ and the extremal solution is singular and given by $u^*(x) = |x|^{-\frac{1}{8}} - 1$.

Theorem 16 tells that $N^*(33) = 14$ and numerical computations seem to show that the same holds whenever p is sufficiently large. Together with Remarks 13 and 15, this supports the following intriguing

Conjecture:

$$\boxed{N^* = N^*_\infty + 1.}$$

Clearly, this conjecture strongly depends on the nonlinearity involved in (G_λ). How general can we take g? If the conjecture were true, it would be a further reason to consider d_* as a special parameter.

4 Proof of Theorem 2

We first briefly sketch the *proof of Proposition 1*, referring to the literature for the details. Recalling (2), we deduce that for $0 \le h < \frac{N^2(N-4)^2}{16}$ the functional

$$I(u) := \int_\Omega |\Delta u|^2\, dx - h \int_\Omega \frac{u^2}{|x|^4}\, dx \qquad u \in H^2 \cap H^1_0(\Omega)$$

is coercive. Furthermore, I is bounded from below and weakly lower semicontinuous. Thus, the existence of a minimizer for I on the manifold

$$M := \left\{ u \in H^2 \cap H^1_0(\Omega) : \int_{\partial\Omega} u_\nu^2\, dS = 1 \right\}$$

follows by compactness of the map $u \in H^2(\Omega) \mapsto u_\nu|_{\partial\Omega} \in L^2(\partial\Omega)$. The Euler-Lagrange equation related to I is given by (6) with $d = d_1(h)$.
Concerning the positivity of a minimizer, one may argue as in [8, Lemma 16]. Uniqueness of the minimizer up to a multiple, then follows arguing by contradiction. To show that $\overline{u}_h$ is the only eigenfunction not changing sign one may follow [23, Lemma 2.2].
Finally, if the infimum in (8) were achieved in the borderline case $h = \frac{N^2(N-4)^2}{16}$, then there would exists a positive strictly superharmonic function $u \in H^2 \cap H^1_0(\Omega)$ which solves problem (6) with $h = \frac{N^2(N-4)^2}{16}$ and $d = d_1(\frac{N^2(N-4)^2}{16})$. Let B_R be a ball of radius $0 < R < 1$ centered at the origin and such that $B_R \subset \Omega$. For any $\delta > 1$ consider the function

$$\varphi_\delta(x) := |x|^{-(N-4)/2}(\log(1/|x|))^{-\delta/2}.$$

Then $\varphi_\delta \in H^2(B_R)$ and, arguing as in [1, Theorem 2.2], by letting $\delta \to 1^+$ one finds that

$$u(x) \ge m\varphi_1(x) \quad \text{a.e. in } B_{R'},$$

a contradiction since $\int_{B_{R'}} \frac{\varphi_1^2}{|x|^4}\, dx = +\infty$, while $\int_\Omega \frac{u^2}{|x|^4}\, dx$ is finite by (2).

The first statements of Theorem 2 are straightforward consequences of Proposition 1. The only part to be proved is the positivity of $d_1(h_N)$ in strictly starshaped domains Ω, where $h_N = \frac{N^2(N-4)^2}{16}$.
For all $h < h_N$ let u_h be the unique superharmonic minimizer for $d_1(h)$ satisfying

$$\int_\Omega |\Delta u_h|^2\, dx = 1.$$

Then, the sequence $\{u_h\}$ is bounded in $H^2 \cap H^1_0(\Omega)$ and, up to a subsequence,

$$\exists \overline{u} \in H^2 \cap H^1_0(\Omega) \quad \text{s.t.} \quad u_h \rightharpoonup \overline{u}.$$

In order to prove that

$$d_1(h_N) > 0, \tag{16}$$

we argue by contradiction. We prove the following

Lemma 17. *If $d_1(h_N) = 0$, then $\lim_{h\to h_N} d_1(h) \,=\, 0$ and $\overline{u} = 0$.*

Proof. By assumption we know that for all $\varepsilon > 0$ there exists $u_\varepsilon \in H^2 \cap H^1_0(\Omega) \setminus H^2_0(\Omega)$ such that

$$\frac{\int_\Omega |\Delta u_\varepsilon|^2\,dx - h_N \int_\Omega \frac{u_\varepsilon^2}{|x|^4}\,dx}{\int_{\partial\Omega} (u_\varepsilon)^2_\nu\,dS} \;<\; \varepsilon.$$

Then, for all $h < h_N$ we have

$$\begin{aligned} 0 < d_1(h) &\le \frac{\int_\Omega |\Delta u_\varepsilon|^2\,dx - h_N \int_\Omega \frac{u_\varepsilon^2}{|x|^4}\,dx}{\int_{\partial\Omega} (u_\varepsilon)^2_\nu\,dS} \;+\; (h_N - h)\frac{\int_\Omega \frac{u_\varepsilon^2}{|x|^4}\,dx}{\int_{\partial\Omega} (u_\varepsilon)^2_\nu\,dS} \\ &< \varepsilon + C_\varepsilon(h_N - h). \end{aligned}$$

By letting $h \to h_N$ we obtain that

$$\lim_{h\to h_N} d_1(h) \;\le\; \varepsilon$$

which proves the first statement by arbitrariness of ε.
Let $h < h_N$. By using $\overline{u}$ as test function in (7) we obtain

$$\int_\Omega \Delta u_h \Delta \overline{u}\,dx - d_1(h) \int_{\partial\Omega} (u_h)_\nu\, \overline{u}_\nu\,dS = h \int_\Omega \frac{u_h \overline{u}}{|x|^4}\,dx\,.$$

Then, by letting $h \to h_N$ (and recalling that $d_1(h) = o(1)$) we infer that

$$\int_\Omega |\Delta \overline{u}|^2\,dx = h_N \int_\Omega \frac{\overline{u}^2}{|x|^4}\,dx\,.$$

Since the Hardy inequality (1) is strict for $u \neq 0$, this implies $\overline{u} = 0$. $\square$

Lemma 17 shows that if (16) is false, then

$$u_h \rightharpoonup 0\;, \qquad \int_{\partial\Omega} (u_h)^2_\nu\,dS \to 0\;, \qquad \text{as } h \to h_N,$$

the latter following from compactness of the map $u \in H^2(\Omega) \mapsto u_\nu|_{\partial\Omega} \in L^2(\partial\Omega)$. Let us set

$$\varepsilon_h := \int_{\partial\Omega} (u_h)^2_\nu\,dS\;,$$

so that

$$\varepsilon_h = o(1) \quad \text{as } h \to h_N.$$

Using this notation we have

$$o(1) = d_1(h) = \frac{1 - h\int_\Omega \frac{u_h^2}{|x|^4}\,dx}{\varepsilon_h} \ge \frac{1 - h_N \int_\Omega \frac{u_h^2}{|x|^4}\,dx}{\varepsilon_h}$$

so that, by using (2), we infer

$$\int_\Omega \frac{u_h^2}{|x|^2}\,dx = o(\varepsilon_h). \tag{17}$$

Therefore, using (7) and integrating twice by parts, we get

$$\begin{aligned} o(\varepsilon_h) &= \int_\Omega \frac{u_h^2}{|x|^2}\,dx = \int_\Omega (|x|^2 u_h)\,\frac{u_h}{|x|^4}\,dx = \frac{1}{h}\int_\Omega (|x|^2 u_h)\,\Delta^2 u_h\,dx \\ &= \frac{1}{h}\int_\Omega \Delta(|x|^2 u_h)\,\Delta u_h\,dx - \frac{1}{h}\int_{\partial\Omega} |x|^2 \Delta u_h (u_h)_\nu\,dS \\ &= \frac{1}{h}\int_\Omega \Delta u_h \left(2N u_h + 4x\cdot\nabla u_h + |x|^2\Delta u_h\right) dx + o(\varepsilon_h) \end{aligned}$$

where, in the last step, we used Lemma 17 and the estimate

$$0 < \frac{d_1(h)}{h}\int_{\partial\Omega} |x|^2 (u_h)_\nu^2\,dS \le \frac{d_1(h)\varepsilon_h}{h}\max_{\partial\Omega}|x|^2 = o(\varepsilon_h).$$

A further integration by parts shows that

$$\int_\Omega \Delta u_h\,(x\cdot\nabla u_h)\,dx = \frac{N-2}{2}\int_\Omega |\nabla u_h|^2\,dx + \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)(u_h)_\nu^2\,dS,$$

see e.g. [40, formula (1.3)]. Therefore, using

$$\int_\Omega |\nabla u_h|^2\,dx = -\int_\Omega u_h \Delta u_h\,dx,$$

we find

$$o(\varepsilon_h) = 4\int_\Omega u_h\Delta u_h\,dx + \int_\Omega |x|^2|\Delta u_h|^2\,dx + 2\int_{\partial\Omega}(x\cdot\nu)(u_h)_\nu^2\,dS. \tag{18}$$

Note that, since u_h is positive and superharmonic, $u_h\Delta u_h < 0$ in Ω. However, by Young's inequality, for any $\delta > 0$ we have the following estimate

$$\left|\int_\Omega u_h\Delta u_h\,dx\right| = \int_\Omega \frac{|u_h|}{|x|}\,|x|\,|\Delta u_h|\,dx \le \delta\int_\Omega |x|^2|\Delta u_h|^2\,dx + \frac{1}{4\delta}\int_\Omega \frac{|u_h|^2}{|x|^2}\,dx.$$

By inserting this estimate (with $\delta = 1/8$) into (18) and recalling (17) yields

$$o(\varepsilon_h) \ge \int_\Omega |x|^2|\Delta u_h|^2\,dx + 4\int_{\partial\Omega}(x\cdot\nu)(u_h)_\nu^2\,dS.$$

Since Ω is strictly starshaped, we have $\min_{\partial\Omega}(x\cdot\nu) = \gamma > 0$ so that the latter estimate gives

$$o(\varepsilon_h) \ge 4\gamma\int_{\partial\Omega}(u_h)_\nu^2\,dS = 4\gamma\varepsilon_h,$$

a contradiction.

5 Proof of Theorem 3 and Corollary 4

From Proposition 1 we know that, up to multiples, the first eigenfunction of problem (6) is unique. Since $\Omega = B$, we infer that it must be a radial function (otherwise by exploiting the invariance by rotations, we would contradict uniqueness). We are so led to seek radial solutions of (6). By setting $|x| = r \in [0,1)$, this conveys in considering the following boundary eigenvalue problem:

$$u^{iv} + \frac{2(N-1)}{r}u''' + \frac{(N-1)(N-3)}{r^2}u'' - \frac{(N-1)(N-3)}{r^3}u' = \frac{h(\alpha)u}{r^4} \tag{19}$$

$$u(1) = u''(1) + (N-1-d)u'(1) = 0, \tag{20}$$

with $h(\alpha)$ as in (10). Looking for power type solutions $u(r) = r^\beta$, we are led to annihilating the polynomial

$$\beta \mapsto \beta^4 + 2(N-4)\beta^3 + (N^2 - 10N + 20)\beta^2 - 2(N^2 - 6N + 8)\beta - h(\alpha)$$

which admits the following real roots:

$$\beta_1 = -\frac{\alpha}{2}, \quad \beta_2 = 4 - N + \frac{\alpha}{2}, \quad \beta_3 = \frac{4 - N - \gamma_N(\alpha)}{2}, \quad \beta_4 = \frac{4 - N + \gamma_N(\alpha)}{2}$$

where $\gamma_N(\alpha) := \sqrt{N^2 - \alpha^2 + 2\alpha(N-4)}$. Therefore, the solutions of (19) are given by

$$u(r) = c_1 r^{-\frac{\alpha}{2}} + c_2\, r^{4-N+\frac{\alpha}{2}} + c_3 r^{\frac{4-N-\gamma_N(\alpha)}{2}} + c_4 r^{\frac{4-N+\gamma_N(\alpha)}{2}}$$

for any $c_i \in \mathbb{R}$ and $\alpha \in [0, N-4]$. Since we are interested in solutions belonging to $H^2 \cap H^1_0(B)$, necessarily $c_2 = c_3 = 0$, whereas $r^{-\frac{\alpha}{2}} \in H^2 \cap H^1_0(B)$ if and only if $0 \le \alpha < N - 4$. For every $0 \le \alpha \le N-4$, we have $\gamma_N(\alpha) \ge N$, so that the function $r^{\frac{4-N+\gamma_N(\alpha)}{2}}$ belongs to $H^2 \cap H^1_0(B)$ for any α in this range.
On the other hand, by imposing the first condition in (20), we get $c_1 = -c_4$ and the candidate eigenfunction is, up to a multiplicative constant,

$$\overline{u}_\alpha(x) = |x|^{-\frac{\alpha}{2}} - |x|^{\frac{4-N+\gamma_N(\alpha)}{2}}$$

which satisfies the second boundary condition in (20) provided $d = \delta_1(\alpha) = \frac{N-\alpha+\gamma_N(\alpha)}{2}$, see (11).
We conclude by observing that from Proposition 1, the unique positive eigenfunction of problem (6) is the one corresponding to the first eigenvalue $d_1(h) = \delta_1(\alpha(h))$.

6 Proofs of the results in general domains

Let us first recall from [8] the following positivity preserving property

Lemma 18. *Let $0 \le d < d_0$, with d_0 as in (3), and let $u \in L^1(\Omega)$ be a solution of*

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \Delta u - d u_\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^1(\Omega)$ is such that $f \ge 0$ a.e. Then $u \ge 0$ a.e. in Ω. Moreover, one has either $u \equiv 0$ or $u > 0$ a.e. in Ω and $u_\nu < 0$ on $\partial\Omega$.
If $\Omega = B$ the same results hold for any $d < N$.

Proof. The statement in Lemma 18 is slightly stronger than [8, Lemma 18] where it is assumed that $u, f \in L^2(\Omega)$. However, the case $u \in L^1(\Omega)$ can be covered by means of an approximation argument, see [3, 9]. Note that u_ν is well-defined on $\partial\Omega$ in view of L^1-elliptic regularity, see [6]. $\square$

Next we prove a weak version of Lemma 18.

Lemma 19. *Let $0 \leq d < d_0$ and $v \in \mathcal{H}_d(\Omega)$ be such that $(v, \varphi) \geq 0$ for all $\varphi \in \mathcal{H}_d(\Omega)$, $\varphi \geq 0$. Then, $v \geq 0$ a.e. in Ω.*

Proof. Integrating twice by parts the inequality

$$\int_\Omega \Delta v \Delta\varphi \, dx - d\int_{\partial\Omega} v_\nu \varphi_\nu \, dS \geq 0 \quad \forall\, \varphi \in C^4(\overline{\Omega}) \cap \mathcal{H}_d(\Omega)$$

one obtains

$$\int_\Omega v\Delta^2\varphi \, dx + \int_{\partial\Omega} v_\nu[\Delta\varphi - d\varphi_\nu] \, dS \geq 0 \quad \forall\, \varphi \in C^4(\overline{\Omega}) \cap \mathcal{H}_d(\Omega). \tag{21}$$

Let $\eta \in C_c^\infty(\Omega)$, $\eta \geq 0$ and let $\widetilde{\varphi}$ be the unique classical solution to

$$\begin{cases} \Delta^2\widetilde{\varphi} = \eta & \text{in } \Omega \\ \widetilde{\varphi} = 0 & \text{on } \partial\Omega \\ \Delta\widetilde{\varphi} - d\widetilde{\varphi}_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 18, one has $\widetilde{\varphi} \geq 0$ and thus we can take $\widetilde{\varphi}$ as a test function in (21) to get

$$\int_\Omega v\eta \, dx \geq 0$$

for all $\eta \in C_c^\infty(\Omega)$, $\eta \geq 0$, which yields the claim. $\square$

In the following we say that $u_\lambda \in L^1(\Omega)$ is a **super-solution** of (G_λ) if $g(u_\lambda) \in L^1(\Omega)$ and

$$\int_\Omega u_\lambda \Delta^2\varphi \, dx \geq \lambda \int_\Omega g(u_\lambda)\varphi \, dx \quad \forall\, \varphi \in X_d\,, \varphi \geq 0. \tag{22}$$

We have

Lemma 20. *Let $u_\lambda \in \mathcal{H}_d(\Omega)$ be a weakly stable energy solution of (G_λ), then*

$$\int_\Omega \Delta u_\lambda \Delta\varphi \, dx - d\int_{\partial\Omega} (u_\lambda)_\nu \varphi_\nu \, dS = \lambda \int_\Omega g(u_\lambda)\,\varphi \, dx \quad \forall\, \varphi \in \mathcal{H}_d(\Omega).$$

Let $v_\lambda \in \mathcal{H}_d(\Omega)$ be an energy super-solution of (G_λ), then

$$\int_\Omega \Delta v_\lambda \Delta\varphi \, dx - d\int_{\partial\Omega} (v_\lambda)_\nu \varphi_\nu \, dS \geq \lambda \int_\Omega g(v_\lambda)\,\varphi \, dx \quad \forall\, \varphi \in \mathcal{H}_d(\Omega)\,, \varphi \geq 0.$$

Proof. By definition, $g(u_\lambda) \in L^1(\Omega)$ and

$$\int_\Omega u_\lambda \Delta^2\varphi \, dx = \lambda \int_\Omega g(u_\lambda)\varphi \, dx \quad \forall\, \varphi \in X_d. \tag{23}$$

Furthermore, $u_\lambda \in \mathcal{H}_d(\Omega)$ is weakly stable so that $g'(u_\lambda) \in L^1(\Omega)$ and, by Fatou's Lemma combined with a density argument, we get

$$\int_\Omega |\Delta\varphi|^2 \, dx - d\int_{\partial\Omega} \varphi_\nu^2 \, dS \geq \lambda \int_\Omega g'(u_\lambda)\,\varphi^2 \, dx \qquad \forall\, \varphi \in \mathcal{H}_d(\Omega). \tag{24}$$

On the other hand, for any $\varphi \in \mathcal{H}_d(\Omega)$, there exist $\{\varphi_n\}_n \subset X_d$ such that $\varphi_n \to \varphi$ in $\mathcal{H}_d(\Omega)$. Then, inequality (24), written with $\varphi_n - \varphi$ as test function, yields

$$\int_\Omega g'(u_\lambda)\,(\varphi_n - \varphi)^2\,dx \to 0. \tag{25}$$

We claim that there exists $C_\lambda > 0$ finite, such that

$$\left|\int_\Omega g(u_\lambda)\,\varphi_n\,dx - \int_\Omega g(u_\lambda)\,\varphi\,dx\right| \le C_\lambda \left(\int_\Omega g'(u_\lambda)\ (\varphi_n - \varphi)^2\,dx\right)^{1/2}. \tag{26}$$

We postpone the proof of (26) to the end. Combining (26) with (25), we deduce that $\int_\Omega g(u_\lambda)\,\varphi_n\,dx \to \int_\Omega g(u_\lambda)\,\varphi\,dx$. On the other hand, integrating by parts the left hand side of (23), written with φ_n as test function, we get

$$\int_\Omega \Delta u_\lambda \Delta\varphi_n\,dx - d\int_{\partial\Omega} (u_\lambda)_\nu (\varphi_n)_\nu\,dS = \lambda\int_\Omega g(u_\lambda)\,\varphi_n\,dx$$

and the statement follows by letting $n \to +\infty$.

Proof of (26). We observe that for (E_λ) there holds $g(s) = g'(s)$, for (P_λ) there holds $g(s) = (1/p)(1+s)g'(s)$ and for (Γ_λ) there holds $g(s) = (1/\gamma)(1-s)g'(s)$. By this, for (E_λ) we get

$$\begin{aligned}\left|\int_\Omega g(u_\lambda)\,\varphi_n\,dx - \int_\Omega g(u_\lambda)\,\varphi\,dx\right| &= \left|\int_\Omega g'(u_\lambda)\,(\varphi_n - \varphi)\,dx\right| \\ &\le \left(\int_\Omega g'(u_\lambda)\,dx\right)^{1/2}\left(\int_\Omega g'(u_\lambda)\ (\varphi_n - \varphi)^2\,dx\right)^{1/2} \\ &=: C_\lambda \left(\int_\Omega g'(u_\lambda)\ (\varphi_n - \varphi)^2\,dx\right)^{1/2}.\end{aligned}$$

For (P_λ) we get

$$\begin{aligned}\left|\int_\Omega g(u_\lambda)\,\varphi_n\,dx - \int_\Omega g(u_\lambda)\,\varphi\,dx\right| &= \frac{1}{p}\left|\int_\Omega (1+u_\lambda)g'(u_\lambda)\,(\varphi_n - \varphi)\,dx\right| \\ &\le \frac{1}{p}\left(\int_\Omega (1+u_\lambda)^2 g'(u_\lambda)\,dx\right)^{1/2}\left(\int_\Omega g'(u_\lambda)\ (\varphi_n - \varphi)^2\,dx\right)^{1/2} \\ &=: C_\lambda \left(\int_\Omega g'(u_\lambda)\ (\varphi_n - \varphi)^2\,dx\right)^{1/2}.\end{aligned}$$

For (Γ_λ) we get

$$\begin{aligned}\left|\int_\Omega g(u_\lambda)\,\varphi_n\,dx - \int_\Omega g(u_\lambda)\,\varphi\,dx\right| &= \frac{1}{\gamma}\left|\int_\Omega (1-u_\lambda)g'(u_\lambda)\,(\varphi_n - \varphi)\,dx\right| \\ &\le \frac{1}{\gamma}\left(\int_\Omega (1-u_\lambda)^2 g'(u_\lambda)\,dx\right)^{1/2}\left(\int_\Omega g'(u_\lambda)\ (\varphi_n - \varphi)^2\,dx\right)^{1/2} \\ &=: C_\lambda \left(\int_\Omega g'(u_\lambda)\ (\varphi_n - \varphi)^2\,dx\right)^{1/2}.\end{aligned}$$

It turns out that C_λ is always finite because $g'(u_\lambda) \in L^1(\Omega)$ and, by (24), $\int_\Omega u_\lambda^2\,g'(u_\lambda)\,dx < \infty$.
If $v_\lambda \in \mathcal{H}_d(\Omega)$ is an energy super-solution, again, a density argument combined with Fatou's Lemma enables us to use test functions $\varphi \in \mathcal{H}_d(\Omega)$ in (22) and obtain the statement.

□

Now, we prove a comparison principle.

Proposition 21. *Let $u, U \in \mathcal{H}_d(\Omega)$ be such that u is a weakly stable energy solution of (G_λ) and U is an energy super-solution of (G_λ). Then, $u(x) \le U(x)$ a.e. in Ω.*

Proof. Set $v := u - U$. By means of the Moreau decomposition in dual cones for the biharmonic operator (see [27]), there exist $v_1, v_2 \in \mathcal{H}_d(\Omega)$ such that $v = v_1 + v_2$ with $v_1 \ge 0$ and $v_1 \perp v_2$, $(v_2, \varphi) \le 0$ for all $\varphi \in \mathcal{H}_d(\Omega)$, $\varphi \ge 0$. In particular, from Lemma 19 we get $v_2 \le 0$ and eventually $v_1 \ge v$. Notice that, by Lemma 20, for $\varphi \in \mathcal{H}_d(\Omega)$, $\varphi \ge 0$ we have

$$\int_\Omega \Delta v \Delta\varphi \, dx - d\int_{\partial\Omega} v_\nu \varphi_\nu \, dS \le \lambda \int_\Omega [g(u) - g(U)]\, \varphi \, dx. \tag{27}$$

By testing (27) with $\varphi = v_1$ and exploiting the stability assumption together with the orthogonality condition $(v_1, v_2) = 0$, we get

$$\begin{aligned}
\lambda \int_\Omega g'(u) v_1^2 \, dx &\le \int_\Omega |\Delta v_1|^2 \, dx - d\int_{\partial\Omega} (v_1)_\nu^2 \, dS \\
&= \int_\Omega \Delta v \Delta v_1 \, dx - d\int_{\partial\Omega} v_\nu (v_1)_\nu \, dS \\
&\le \lambda \int_\Omega [g(u) - g(U)]\, v_1 \, dx\,.
\end{aligned}$$

Since $v_1 \ge v$, we obtain

$$0 \le \int_\Omega \big[g(u) - g(U) - (u - U) g'(u)\big]\, v_1 \, dx. \tag{28}$$

Moreover, by convexity of the map $s \mapsto g(s)$, the right hand side in (28) is non positive, thus we conclude $v_1 = 0$ and the claim follows. $\square$

Next we show that the set of λ's for which the problem admits a regular solution is an interval.

Lemma 22. *Let u_λ be a solution of (E_λ) (resp. (Γ_λ)) for $\lambda < \lambda^*$. Then, for all $0 < \varepsilon \le 1$, problem $(E_{(1-\varepsilon)\lambda})$ (resp. $(\Gamma_{(1-\varepsilon)\lambda})$) possesses a regular solution. The statement holds for (P_λ) provided $\Omega = B$.*

Proof. It can be achieved buying the line of [3, 13, 24] where the result was established respectively for exponential, singular and power types nonlinearities under Dirichlet boundary conditions. Those arguments, though delicate, are not affected by changing the boundary conditions. The key-ingredient being the positivity preserving property which is here ensured by Lemma 18. $\square$

Proof of Theorem 6. For any $\varepsilon > 0$, consider the problem

$$\begin{cases} \Delta^2 u = \varepsilon & \text{in } \Omega \\ u = \Delta u - d u_\nu = 0 & \text{on } \partial\Omega\,, \end{cases}$$

which admits a classical solution $u \in X_d(\Omega)$ and, provided ε is small, we may assume $\|u\|_\infty < 1$. Taking $\overline{\lambda} = \varepsilon / g(\|u\|_\infty)$, one has that u is a super-solution of problem (G_λ) for any $\lambda \in (0, \overline{\lambda})$. From u one builds up a super-solution to (G_λ) for λ sufficiently small and since $u = 0$ is a sub-solution, the method of sub- super-solutions yields a classical solution to (G_λ) which can be exploited to start a monotone iteration scheme to obtain the first claim; see [8, Theorem 6, Lemma 21] for (E_λ) and [13, Proposition 2.1] for (Γ_λ). Positivity follows from Lemma 18 whereas the stability is achieved by means of the comparison principle provided by Proposition 21, see also [3, Proposition 37] where it is explained why super-solutions are needed in the statement of Proposition 21. The proof that the extremal function u^*, as defined in (13), solves (G_{λ^*}) and lies in $\mathcal{H}_d(\Omega)$ can be achieved as in [8, Lemma 22]. Now let

$$\begin{aligned}\lambda_* &:= \sup\{\lambda \geq 0 \,:\, (E_\lambda)\text{ resp. } (\Gamma_\lambda) \text{ possesses a regular solution}\}\\ \lambda^* &:= \sup\{\lambda \geq 0 \,:\, (E_\lambda)\text{ resp. } (\Gamma_\lambda) \text{ possesses a solution}\}.\end{aligned}$$

The nonexistence of solutions for $\lambda > \lambda^*$ follows by establishing that $\lambda_* = \lambda^*$. On one hand one has $\lambda_* \leq \lambda^*$, on the other hand if $\lambda_* < \lambda^*$ by means of Lemma 22 we contradict the definition of λ_*; thus, necessarily $\lambda_* = \lambda^*$.

Proof of Theorem 8. Being u^* the pointwise limit of minimal solutions, by (13) we conclude that u^* is weakly stable. To show that (E_{λ^*}) and (Γ_{λ^*}) admit a unique solution when u^* is regular, it suffices to argue as in the proof of [3, Lemma 2.6]. Hence, statements (i) and (ii) are verified.
For statement (iii), assume by contradiction that for $\lambda < \lambda^*$ there exists a singular solution $u_\lambda \in \mathcal{H}_d(\Omega)$ which is weakly stable. In this case, since there exists a minimal regular solution $\underline{u}_\lambda$, Proposition 21 would yield $\underline{u}_\lambda \equiv u_\lambda$, a contradiction as u_λ is assumed to be singular. Therefore $\lambda = \lambda^*$. In turn, by statement (ii), this also implies that u^* is singular.
The restriction $\Omega = B$ for problem (P_λ), is essentially due to the fact that, in this case, the regularity of the minimal solution is missing for general Ω, see Lemma 22.

Proof of Theorem 9. For $2 \leq N \leq 4$, statements (i) and (iii) follow by embedding theorems and elliptic regularity. Also (iii), for $N \geq 5$ and $p \leq \frac{N+4}{N-4}$, uses the same arguments but first one has to show that if u is an energy solution of (P_λ), then $u \in L^q(\Omega)$ for all $1 \leq q < \infty$. This can be achieved by the Moser iteration technique, as in [46, Lemma B2] and [10].
For (Γ_λ) the regularity statement follows if we show that $\|u\|_\infty < 1$. Suppose by contradiction that there exists an energy solution such that $u(x_0) = 1$. Then, by definition of energy solution we have that $u \in H^2(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$, by Sobolev's embedding, where $\alpha = 1/2$ when $N = 3$ and $\alpha \in (0,1)$ when $N = 2$. As a consequence we have

$$\infty > \int_\Omega \frac{1}{(1-u)^\gamma}\,dx \geq \int_\Omega \frac{C}{|x_0 - x|^{\alpha\gamma}}\,dx$$

and the last integral diverges as $\alpha\gamma \geq N$. Using also the stability property in Theorem 8, one can prove in a similar fashion the regularity of the extremal solution, see [26].

7 Proofs of the results in the ball

7.1 Proof of Theorem 10

From Lemma 18 we know that, for $d < N$, nontrivial solutions of (G_λ) are strictly positive in B with normal derivative strictly negative on the boundary. Let $\underline{u}_\lambda$ be the minimal (regular) solution corresponding to some $d = \overline{d} < N$. For $\varepsilon > 0$, consider the function $U_\varepsilon(x) = \underline{u}_\lambda(x) + \varepsilon(|x|^2 - 1)$. We have that $U_\varepsilon|_{\partial B} = 0$. Then we consider the ratio

$$R(\varepsilon) := \frac{\Delta U_\varepsilon(x)}{(U_\varepsilon(x))_\nu} = \frac{\overline{d}\,c + 2N\varepsilon}{c + 2\varepsilon}, \quad x \in \partial B,$$

where $c := (\underline{u}_\lambda)_\nu < 0$ on ∂B. We have $R(0) = \overline{d}$ and $R'(\varepsilon) = \frac{2\,c(N-d)}{(c+2\varepsilon)^2} < 0$. This means that, for every $\hat{d} < \overline{d}$ there exists $\hat{\varepsilon} \in (0, -c/2)$ such that $U_{\hat{\varepsilon}}$ satisfies the Steklov conditions for $d = \hat{d} := R(\hat{\varepsilon})$. Since, for every $\hat{d} \in (-\infty, \overline{d})$ and for every $\lambda \in (0, \lambda^*(\overline{d}))$ we have

$$\Delta^2 U_{\hat{\varepsilon}} = \lambda g(\underline{u}_\lambda) \geq \lambda g(u_{\hat{\varepsilon}}) \quad \text{in } B,$$

$U_{\hat{\varepsilon}}$ is a super-solution for problem (G_λ). By the sub- super-solutions method, this implies

$$\lambda^*(\hat{d}) \geq \lambda^*(\overline{d}).$$

Since Dirichlet boundary conditions are the limit case of Steklov boundary conditions, the last statement follows by letting $\hat{d} \to -\infty$.

7.2 Proof of Theorem 12

The iterative method which yields Theorem 6 implies that, when $\Omega = B$, the minimal solution $\underline{u}_\lambda$ is radially symmetric, as well as u^* by (13). Thus we are reduced to study the radial problem:

$$u^{iv}(r) + \frac{2(N-1)}{r}\, u'''(r) + \frac{(N-1)(N-3)}{r^2}\, u''(r) - \frac{(N-1)(N-3)}{r^3}\, u'(r) = \lambda\, e^{u(r)}, \quad r \in (0,1].$$

Performing the change of variable $t := \log r$ and setting $v(t) = u(e^t) + 4t$, we end up with the autonomous equation

$$v^{iv}(t) + 2(N-4)\, v'''(t) + (N^2 - 10N + 20)\, v''(t) - 2(N-2)(N-4)\, v'(t) = \lambda\, e^{v(t)} - \lambda_N, \qquad t \in (-\infty, 0], \tag{29}$$

where

$$\lambda_N := 8(N-2)(N-4). \tag{30}$$

We state a result essentially obtained in different steps in [2, 3, 24, 27, 28]; however for the sake of completeness we recall the main lines of the proof.

Lemma 23. *If u_s is a radial singular solution of the equation in (E_{λ_s}), then the corresponding function v_s satisfies*

$$\lim_{t\to-\infty} v_s^{(k)}(t) = 0, \quad \textit{for } k = 1, 2, 3, 4 \tag{31}$$

and

$$\lim_{t\to-\infty} v_s(t) = \log \frac{\lambda_N}{\lambda_s}, \tag{32}$$

with λ_N as in (30).

Proof. The proof consists of four steps.

Step 1: if v solves (29) and $\lim_{t\to-\infty} v(s) := \gamma$ exists, then either $\gamma = -\infty$ or $\gamma = \log \frac{\lambda_N}{\lambda}$.
By means of iterated integrations of (29), one gets rid of the case $-\infty < \gamma \neq \log \frac{\lambda_N}{\lambda}$. Then, to exclude the case $\gamma = +\infty$, one may apply the test function method developed by Mitidieri-Pohožaev [41], see [28] for the details.

Step 2: if v solves (29), then $\lambda e^v - \lambda_N$ is bounded.
We argue by contradiction: assume that the statement of Step 2 does not hold, then by Step 1 we deduce that

$$\liminf_{t\to-\infty} v(t) < \limsup_{t\to-\infty} v(t) = +\infty.$$

Hence, there exists a sequence of negative numbers $\{t_k\}_{k\geq 0}$ such that $t_k \to -\infty$, $v'(t_k) = 0$ and $v(t_k) = \gamma_k$, where $\lim_{k\to\infty} \gamma_k = +\infty$. Following an idea of [24, Lemma 1], let us define the sequence $v_k(t) := v(t - t_k)$, $t \in (-\infty, t_k)$. Since the functions v_k satisfy equation (29) for every $k \geq 0$, the functions $w_k(x) = v_k(\log|x|) - 4\log|x| - v_k(2t_k) + 8t_k$ solve the problems

$$\begin{cases} \Delta^2 w_k = \lambda e^{v_k(2t_k) - 8t_k} e^{w_k} & \text{in } B_{R_k} \\ w_k = 0 & \text{on } \partial B_{R_k} \\ (w_k)_\nu = -4\, e^{-2t_k} & \text{on } \partial B_{R_k}, \end{cases}$$

where B_{R_k} is the ball having radius $R_k := e^{2t_k}$. The rescaling $z_k(x) := w_k(R|x|)$ solve

$$\begin{cases} \Delta^2 z_k = \lambda e^{v_k(2t_k)} e^{z_k} & \text{in } B \\ z_k = 0 & \text{on } \partial B \\ (z_k)_\nu = -4 & \text{on } \partial B. \end{cases}$$

We have so determined a super-solution for a Dirichlet boundary value problem having parameter $\lambda_k = \lambda e^{v_k(2t_k)} \to +\infty$, as $k \to +\infty$. This implies the existence of a solution for every $\lambda > 0$, contradicting the results of [3, Theorem 3].

Step 3: if u_s is a radial singular solution of the equation in (E_λ), then

$$\int_{-\infty}^0 |v_s'(\tau)|^2\, d\tau < +\infty, \quad \int_{-\infty}^0 |v_s''(\tau)|^2\, d\tau < +\infty, \quad \int_{-\infty}^0 |v_s'''(\tau)|^2\, d\tau < +\infty,$$

$$\int_{-\infty}^0 |v_s^{iv}(\tau)|^2\, d\tau < +\infty\,, \quad \int_{-\infty}^0 |\lambda_s e^{v_s(\tau)} - \lambda_N|^2\, d\tau < +\infty.$$

The proof of the finiteness of the above four integrals follows by arguing as in [2, Lemma 9].

Step 4: conclusion.
From Step 3 we deduce that (31) and (32) hold up to a subsequence. By repeating, with minor changes, the same argument of [28, Proposition 7] one may check that the desired limits hold, see also [3, Theorem 6].

□

For the proof of Step 3 above one needs to define the energy function associated to (29), namely

$$E(t) := \frac{1}{2}\,|v''(t)|^2 - \frac{(N^2 - 10N + 20)}{2}\,|v'(t)|^2 + \lambda\, e^{v(t)} - \lambda_N\, v(t).$$

This is also used in the proof of the next statement, which is the key ingredient to get Theorem 12. Notice that here we require just *one boundary condition.*

Lemma 24. *Let λ_N be as in (30). If u_s is a singular radial solution of*

$$\begin{cases} \Delta^2 u = \lambda_s\, e^u & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases}$$

then

$$\lim_{|x| \to 0} \left(u_s(x) + 4 \log |x|\right) = \log \frac{\lambda_N}{\lambda_s}. \tag{33}$$

Furthermore,

(*a*) *if $\lambda_s \le \lambda_N$, then $u_s(x) + 4 \log |x| \le \log \frac{\lambda_N}{\lambda_s}$;*

(*b*) *if $\lambda_s \ge \lambda_N$, then $u_s(x) + 4 \log |x| \ge \log \frac{\lambda_N}{\lambda_s}$.*

Proof. Notice that (33) comes from (32). On the other hand, inserting λ_s and v_s into the function E, by (31) and (32) we get that

$$\lim_{t \to -\infty} E(t) = f\left(\log \frac{\lambda_N}{\lambda_s}\right), \tag{34}$$

where $f(\tau) := \lambda_s e^\tau - \lambda_N \tau$. Moreover, we observe that (a) and (b), written in terms of v_s, become

$$(a) \quad v_s(t) \le \log \frac{\lambda_N}{\lambda_s} \quad \text{if } \lambda_s \le \lambda_N, \quad (b) \quad v_s(t) \ge \log \frac{\lambda_N}{\lambda_s} \quad \text{if } \lambda_s \ge \lambda_N.$$

We prove (a), the proof of (b) is similar. Let $\lambda_s \le \lambda_N$ and assume by contradiction that (a) does not hold. Then, by (32) and $v_s(0) = 0$ we infer that there exists $t_0 \in (-\infty, 0)$ such that $v_s'(t_0) = 0$ and $v_s(t_0) > \log \frac{\lambda_N}{\lambda_s}$. Hence,

$$E(t_0) = \frac{1}{2} |v_s''(t_0)|^2 + f(v_s(t_0)) \ge f(v_s(t_0)).$$

Exploiting (34), (29) and integrating by parts one gets

$$E(t_0) - f\left(\log \frac{\lambda_N}{\lambda_s}\right) = \int_{-\infty}^{t_0} E'(\tau)\, d\tau$$

$$= -2(N-4) \int_{-\infty}^{t_0} |v''(\tau)|^2\, d\tau - 2(N-2)(N-4) \int_{-\infty}^{t_0} |v'(\tau)|^2\, d\tau \le 0,$$

that is,

$$E(t_0) \le f\left(\log \frac{\lambda_N}{\lambda_s}\right) < f(v_s(t_0)).$$

The last inequality comes from the fact that f has a unique minimum point at $\tau = \log \frac{\lambda_N}{\lambda_s}$. This gives a contradiction and proves (a). $\square$

We may now characterize explicitly singular solutions.

Lemma 25. *Let $d = d_* := N - 2$ and λ_N be as in (30). If u_s is a singular radial solution to (E_{λ_s}) with $d = d_*$, then*

$$\lambda_s = \lambda_N \quad \text{and} \quad u_s(x) = -4 \log |x|.$$

Proof. Let

$$W_s(x) := u_s(x) + 4 \log |x|,$$

then W_s solves the problem

$$\begin{cases} \Delta^2 W_s = \dfrac{1}{|x|^4} \left(\lambda_s e^{W_s} - \lambda_N\right) & \text{in } B \\ W_s = \Delta W_s - d_* (W_s)_\nu = 0 & \text{on } \partial B. \end{cases}$$

Suppose first that (a) of Lemma 24 holds, then

$$\lambda_s e^{W_s} - \lambda_N \le 0.$$

Therefore, since $d_* \in (0, N)$, by Lemma 18 we infer that $W_s < 0$ in B, a contradiction to (33). In a similar fashion we also handle the case (b) and we conclude that, necessarily, $\lambda_s = \lambda_N$ and $W_s = 0$. $\square$

Proof of Theorem 12. By Lemma 25, the unique singular radial solution of problem (E_λ), when $d = d_*$, is $u_s(x) = -4 \log |x|$ corresponding to $\lambda = \lambda_N$. Therefore, u^* is singular if and only if $u^* = u_s$. Since $e^{u_s} = \frac{1}{|x|^4}$, u_s is weakly stable if

$$\int_B |\Delta \varphi|^2\, dx - d_* \int_{\partial B} \varphi_\nu^2\, dS \ge \lambda_N \int_B \frac{\varphi^2}{|x|^4}\, dx \quad \forall \varphi \in \mathcal{H}_d(B), \tag{35}$$

see (14).
If $5 \le N \le 9$, since $d_* < \delta_1(N-4)$ and $\lambda_N > \frac{N^2(N-4)^2}{16}$, we infer that (35) cannot hold, otherwise we would contradict the optimality of the Hardy-Rellich constant, see also Figure 1. Hence, u_s is not weakly stable and does not coincide with u^*, in view of Theorem 8-(i).
If $N \ge 10$, then $d_* \in (\delta_1(N-4), N)$ and (12) implies

$$\int_B |\Delta\varphi|^2\,dx - d_*\int_{\partial B}\varphi_\nu^2\,dS \ge h(2\sqrt{N-2})\int_B \frac{\varphi^2}{|x|^4}\,dx \quad \forall\,\varphi\in\mathcal{H}_d(B). \tag{36}$$

To establish when $h(2\sqrt{N-2}) \ge \lambda_N$, is equivalent to solving

$$(\sqrt{N-2}+2)(\sqrt{N-2}+2-N)(\sqrt{N-2}+4-N) \ge 8\sqrt{N-2}\,(N-4).$$

Setting $s := \sqrt{N-2}$ and requiring $s > \sqrt{2}$, the above inequality holds if and only if $N \ge \frac{17+\sqrt{89}}{2}$, that is, if $N \ge 14$. In this case, (36) implies (35) so that u_s is weakly stable and u^* is singular by Theorem 8-(iii). Then, as already noticed above, $u^* = u_s$.
If $10 \le N \le 13$ we have $h(2\sqrt{N-2}) < \lambda_N$ and, since by Theorem 3 the equality in (36) is achieved, (35) does not hold. Hence, $u^* \ne u_s$ by Theorem 8-(i) and u^* is regular.

7.3 Proof of Theorem 14

Consider the equation in (Γ_λ) in radial coordinates $r = |x| \in (0,1]$. Inspired by [28] we make the change of variable

$$v(t) = e^{\frac{4t}{\gamma+1}}(1-u(e^t)), \quad t\in(-\infty,0],$$

so that the equation in (Γ_λ) becomes the following autonomous equation:

$$v^{iv}(t) + K_3\,v'''(t) + K_2\,v''(t) + K_1\,v'(t) - \lambda_{N,\gamma}\,v(t) = -\frac{\lambda}{v^\gamma(t)}, \tag{37}$$

with $\lambda_{N,\gamma}$ as in (15). The constants K_i, $i=1,2,3$, are explicitly given by

$$K_1 = 2\left[\frac{128}{(\gamma+1)^3} + \frac{48(N-4)}{(\gamma+1)^2} + \frac{4(N^2-10N+20)}{\gamma+1} - (N-2)(N-4)\right], \tag{38}$$

$$K_2 = \frac{96}{(\gamma+1)^2} + \frac{24(N-4)}{\gamma+1} + N^2 - 10N + 20, \tag{39}$$

$$K_3 = 2\left[\frac{8}{\gamma+1} + N - 4\right]. \tag{40}$$

For $N \ge 5$, $K_2 > 0$ and $K_3 > 0$ whereas $K_1 < 0$ (independently of N) as long as $\gamma \ge 3$. To (37) we associate the energy function

$$E(t) := -\frac{\lambda_s}{(1-\gamma)v_s^{\gamma-1}(t)} + \frac{\lambda_{N,\gamma}}{2}\,v_s^2(t) - \frac{K_2}{2}\,|v_s'(t)|^2 + \frac{1}{2}\,|v_s''(t)|^2. \tag{41}$$

By following the line in the proof of Lemma 23 (with the advantage that here solutions stay a priori bounded), one obtains

Lemma 26. *If $\gamma \geq 3$ and u_s is a singular solution to the equation in (Γ_{λ_s}), then the corresponding function v_s satisfies*

$$\lim_{t\to-\infty} v_s^{(k)}(t) = 0, \quad \text{for } k=1,2,3,4 \tag{42}$$

$$\lim_{t\to-\infty} v_s(t) = \left(\frac{\lambda}{\lambda_{N,\gamma}}\right)^{\frac{1}{\gamma+1}}, \tag{43}$$

with $\lambda_{N,\gamma}$ as in (15).

We are now able to state the counterpart of Lemma 24 for (Γ_λ):

Lemma 27. *Let $\lambda_{N,\gamma}$ be as in (15). If $\gamma \geq 3$ and u_s is a singular radial solution to*

$$\begin{cases} \Delta^2 u = \lambda_s \dfrac{1}{(1-u)^\gamma} & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases}$$

then

$$\lim_{|x|\to 0}\left[u_s(x) - 1 + \left(\frac{\lambda_s}{\lambda_{N,\gamma}}\right)^{\frac{1}{\gamma+1}} |x|^{\frac{4}{\gamma+1}}\right] = 0 \tag{44}$$

and the following hold:

(*a*) *if $\lambda_s \leq \lambda_{N,\gamma}$, then $u_s(x) \leq 1 - \left(\frac{\lambda_s}{\lambda_{N,\gamma}}\right)^{\frac{1}{\gamma+1}} |x|^{\frac{4}{\gamma+1}}$;*

(*b*) *if $\lambda_s \geq \lambda_{N,\gamma}$, then $u_s(x) \geq 1 - \left(\frac{\lambda_s}{\lambda_{N,\gamma}}\right)^{\frac{1}{\gamma+1}} |x|^{\frac{4}{\gamma+1}}$.*

Proof. Notice that (43) yields (44). In order to prove (a) and (b) we use the energy function (41) written for $v = v_s$. By (42) and (43) one has

$$\lim_{t\to-\infty} E(t) = f\left(\frac{\lambda_s}{\lambda_{N,\gamma}}\right)$$

where

$$f(\tau) = -\frac{\lambda_s}{1-\gamma}\tau^{1-\gamma} + \frac{\lambda_{N,\gamma}}{2}\tau^2.$$

We prove (a) since (b) is similar. Let $\lambda_s \leq \lambda_N$, in terms of v_s, (a) reads

$$v_s \geq \left(\frac{\lambda_s}{\lambda_{N,\gamma}}\right)^{\frac{1}{\gamma+1}}.$$

Then, by (43) an $v_s(0) = 1$ we infer that there exists $t_0 \in (-\infty, 0)$ such that $v_s'(t_0) = 0$ and $v(t_0) < (\lambda_s/\lambda_{N,\gamma})^{1/\gamma+1}$. In particular, we have both

$$E(t_0) \geq f(v_s(t_0))$$

and, by integrating by parts and exploiting (37),

$$E(t_0) - f\left(\left(\frac{\lambda_s}{\lambda_{N,\gamma}}\right)^{1/(\gamma+1)}\right) = \int_{-\infty}^{t_0} E'(\tau)\,d\tau = -K_3\int_{-\infty}^{t_0} |v''(\tau)|^2\,d\tau + K_1\int_{-\infty}^{t_0} |v'(\tau)|^2\,d\tau \leq 0$$

thanks to the fact that $K_3 > 0$ and $K_1 < 0$. We infer that

$$E(t_0) \le f\left(\left(\frac{\lambda_s}{\lambda_{N,\gamma}}\right)^{1/(\gamma+1)}\right) < f(v_s(t_0))$$

since the function f possesses a unique minimum at $\tau = \left(\frac{\lambda_s}{\lambda_{N,\gamma}}\right)^{1/(\gamma+1)}$, a contradiction. $\square$

By Lemma 27 we deduce the following

Lemma 28. *Let $d_*(\gamma) := N - \frac{2(\gamma-1)}{\gamma+1}$ and $\lambda_{N,\gamma}$ be as in (15). If u_s is a singular radial solution to (Γ_{λ_s}) with $\gamma \ge 3$ and $d = d_*(\gamma)$, then*

$$\lambda_s = \lambda_{N,\gamma} \quad \textit{and} \quad u_s(x) = 1 - |x|^{\frac{4}{\gamma+1}}.$$

Proof. By setting

$$W_s(x) := u_s(x) - \left(\frac{\lambda_s}{\lambda_{N,\gamma}}\right)^{1/(\gamma+1)} \left(1 - |x|^{\frac{4}{\gamma+1}}\right),$$

the proof may be achieved following the line of that of Lemma 25.

$\square$

Proof of Theorem 14. By Lemma 28, the extremal solution u^* to (Γ_λ) with $d = d_*(\gamma)$ is singular if and only if $u^* = u_s$ and $\lambda = \lambda_{N,\gamma}$. Since $\frac{1}{(1-u_s)^{\gamma+1}} = \frac{1}{|x|^4}$, u_s is weakly stable if

$$\int_B |\Delta\varphi|^2\,dx - d_*(\gamma)\int_{\partial B} \varphi_\nu^2\,dS \ge \gamma\,\lambda_{N,\gamma}\int_B \frac{\varphi^2}{|x|^4}\,dx \quad \forall\,\varphi \in \mathcal{H}_d(B).$$

Let

$$F(\gamma) := \frac{2(\sqrt{3\gamma^2 - 2\gamma + 3} + 3\gamma - 1)}{\gamma + 1},$$

the map $\gamma \mapsto F(\gamma)$ is strictly increasing and $F(\gamma) \to 2(3+\sqrt{3}) \simeq 9.4$ as $\gamma \to +\infty$.
If $5 \le N \le F(\gamma)$, since $d_*(\gamma) < \delta_1(N-4)$ and $\lambda_{N,\gamma} > \frac{N^2(N-4)^2}{16}$, we immediately conclude that u_s is not weakly stable, see also the proof of Theorem 12.
If $N \ge F(\gamma)$, then $\delta_1(N-4) < d_*(\gamma) < N$ and Theorem 3 implies

$$\int_B |\Delta\varphi|^2\,dx - d_*(\gamma)\int_{\partial B} \varphi_\nu^2\,dS \ge h(\alpha_*)\int_B \frac{\varphi^2}{|x|^4}\,dx \quad \forall\,\varphi \in \mathcal{H}_d(B),$$

where

$$\alpha_* := \frac{2}{\gamma+1}\left[\sqrt{N(\gamma^2-1) - 2(\gamma-1)^2 + 4} - 2\right]$$

from which we get

$$\begin{aligned} h(\alpha_*) = \frac{1}{(\gamma+1)^4}&\left(\sqrt{N(\gamma^2-1) - 2(\gamma-1)^2 + 4} - 2\right)\\ &\times\left(\sqrt{N(\gamma^2-1) - 2(\gamma-1)^2 + 4} + 2\gamma\right)\\ &\times\left(\sqrt{N(\gamma^2-1) - 2(\gamma-1)^2 + 4} + 2\gamma - N\gamma - N\right)\\ &\times\left(\sqrt{N(\gamma^2-1) - 2(\gamma-1)^2 + 4} + 4\gamma + 2 - N\gamma - N\right). \end{aligned}$$

Therefore, when $N \geq F(\gamma)$, u_s is weakly stable provided

$$h(\alpha_*) \geq \gamma \lambda_{N,\gamma},$$

which holds as long as

$$N \geq \frac{\sqrt{89\gamma^2 + 54\gamma + 1} + 17\gamma + 3}{2(\gamma + 1)} > F(\gamma).$$

The map $\gamma \mapsto \frac{\sqrt{89\gamma^2+54\gamma+1}+17\gamma+3}{2(\gamma+1)}$ is strictly increasing and bounded from above by 14. Setting $N^*(\gamma)$ as in the statement, in view of Theorem 8, the above computations yield the thesis. See the proof of Theorem 12 for more details.

7.4 Proof of Theorem 16

The first part of the proof is performed for general $p > 1$. We make the choice $p = 33$ just at the end to simplify the computations.
By setting $v(t) = e^{-\frac{4t}{p-1}}(u(e^t) + 1)$, the radial version of (P_λ) reads

$$v^{iv}(t) + K_3\, v'''(t) + K_2\, v''(t) - K_1\, v'(t) + \lambda_{N,p}\, v(t) = \lambda\, v^p, \tag{45}$$

where $t \in (-\infty, 0]$ and

$$\lambda_{N,p} := -\lambda_{N,\gamma}|_{\gamma=-p}, \tag{46}$$

with $\lambda_{N,\gamma}$ as defined in (15). For the definition of the constants K_i, $i = 1, 2, 3$ see (38), (39) and (40) by means of the substitution $\gamma = -p$. For the sake of the proof, it is sufficient to know that $K_1 < 0$ and $K_3 > 0$. The energy function associated to (45) is

$$E(t) := \frac{\lambda}{p+1}\, v^{p+1}(t) - \frac{\lambda_{N,p}}{2}\, v^2(t) - \frac{K_2}{2}\, |v'(t)|^2 + \frac{1}{2}\, |v''(t)|^2, \quad t \in (-\infty, 0].$$

Now we recall [24, Theorem 4] obtained by studying the dynamical system associated to (45) under Dirichlet boundary conditions. Since the proof does not directly involve the boundary conditions, the result still holds in our case.

Lemma 29. *If u_s is a radial singular solution of the equation in (P_{λ_s}), then the corresponding function v_s is such that*

$$\lim_{t\to-\infty} v_s^{(k)}(t) = 0, \quad \text{for } k = 1, 2, 3, 4$$

and

$$\lim_{t\to-\infty} v_s^{p-1}(t) = \frac{\lambda_{N,p}}{\lambda_s},$$

with $\lambda_{N,p}$ as in (46).

The next step are suitable properties of singular solutions vanishing on ∂B.

Lemma 30. *Let $\lambda_{N,p}$ be as in (46). If u_s is a singular radial solution of*

$$\begin{cases} \Delta^2 u = \lambda_s\,(1+u)^p & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases}$$

then

$$\lim_{|x|\to 0} \left(u_s(x) - \left(\frac{\lambda_{N,p}}{\lambda_s}\right)^{\frac{1}{p-1}} |x|^{-\frac{4}{p-1}} + 1 \right) = 0$$

and the following hold

(a) *if* $\lambda_s \le \lambda_{N,p}$*, then* $u_s(x) \le \left(\dfrac{\lambda_{N,p}}{\lambda_s}\right)^{\frac{1}{p-1}} |x|^{-\frac{4}{p-1}} - 1$*;*

(b) *if* $\lambda_s \ge \lambda_{N,p}$*, then* $u_s(x) \ge \left(\dfrac{\lambda_{N,p}}{\lambda_s}\right)^{\frac{1}{p-1}} |x|^{-\frac{4}{p-1}} - 1$*.*

Furthermore, if u_s *is a singular radial solution of* (P_{λ_s}) *with* $d = d_*(p) := N - 2\,\frac{p+1}{p-1}$*, then*

$$\lambda_s = \lambda_{N,p} \quad \text{and} \quad u_s(x) = |x|^{-\frac{4}{p-1}} - 1.$$

Proof. The proof follows the line of that of Lemmata 24 and 25. With respect to Lemma 24, the main difference is that here, inserting $v_s(t) = e^{-\frac{4t}{p-1}}(u_s(e^t) + 1)$ into the energy function E, one has

$$\lim_{t\to-\infty} E(t) = f\left(\left(\frac{\lambda_{N,p}}{\lambda_s}\right)^{\frac{1}{p-1}}\right),$$

where $f(\tau) := \frac{\lambda_s}{p+1}\,\tau^{p+1} - \frac{\lambda_{N,p}}{2}\,\tau^2$. Then, Lemma 29 and the fact that the function f, for $\tau > 0$, has a unique minimum point at $\tau = \left(\frac{\lambda_{N,p}}{\lambda_s}\right)^{\frac{1}{p-1}}$, allow to repeat with minor changes the arguments in the proof of Lemma 24. Lemma 25 can be adapted to this case by setting

$$W_s(x) := u_s(x) - \left(\frac{\lambda_{n,p}}{\lambda_s}\right)^{1/(p-1)} \left(|x|^{-\frac{4}{p-1}} - 1\right).$$

□

Proof of Theorem 16. Fix $p = 33$, by Lemma 30 the extremal solution u^* of (P_λ) with $d = d_*(33) = N - \frac{17}{8}$ is singular if and only if

$$u^*(x) = u_s(x) = |x|^{-\frac{1}{8}} - 1 \text{ and } \lambda = \lambda_{N,33} = \frac{1088N^2 - 6800N + 9537}{4096}.$$

Since $(1 + u_s)^{32} = \frac{1}{|x|^4}$, u_s is weakly stable if

$$\int_B |\Delta\varphi|^2\,dx - d_*(33)\int_{\partial B} \varphi_\nu^2\,dS \ge 33\,\lambda_{N,33}\int_B \frac{\varphi^2}{|x|^4}\,dx \quad \forall\,\varphi \in \mathcal{H}_d(B).$$

When $5 \le N \le 9$, then $d_*(33) < \delta_1(N-4)$ and $\lambda_{N,33} > \frac{N^2(N-4)^2}{16}$, hence the same arguments applied in the proof of Theorem 12 show that u_s is not weakly stable.
If $N \ge 10$, then $d_*(33) \in (\delta_1(N-4), N)$ and Theorem 3 implies

$$\int_B |\Delta\varphi|^2\;dx - d_*(33)\int_{\partial B} \varphi_\nu^2\,dS \ge h(\alpha_{33})\int_B \frac{\varphi^2}{|x|^4}\,dx \quad \forall\,\varphi \in \mathcal{H}_d(B),$$

where $\alpha_{33} := \frac{1+\sqrt{272N-577}}{8}$. To establish when $h(\alpha_{33}) \ge 33\,\lambda_{N,33}$ is equivalent to check the inequality

$$\begin{aligned}(\sqrt{272N-577}+1)&(\sqrt{272N-577}+33)\\ &\times(\sqrt{272N-577}+33-16N)(\sqrt{272N-577}+65-16N)\\ &\qquad\qquad \ge 528(1088N^2-6800N+9537).\end{aligned}$$

Putting $s := \sqrt{272N - 577}$, we come to study the sign of the function

$$\phi(s) = s^6 - 3590s^4 + 1505353s^2 - 1501764, \quad s \geq \sqrt{511}.$$

Some computations show that, for $s \geq \sqrt{511}$, $\phi(s)$ has a unique zero at $s_0 = \sqrt{\frac{3589+\sqrt{6873865}}{2}}$ and it is positive for $s \geq s_0$. In terms of N, this means that the desired inequality is satisfied for $N \geq 14$ and, in turn, that u_s is weakly stable if and only if $N \geq 14$. Then we conclude by invoking Theorem 8 as explained in the proof of Theorem 12.

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