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Error Estimates for a Mixed Hybridized Finite Volume Method for 2nd Order Elliptic Problems

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Summary. In this article, we carry out the convergence analysis of the dual–mixed hybridized finite volume scheme proposed in [3] for the numerical approximation of transport problems in symmetrizable form. Using the results of [8, 5] optimal error estimates are obtained for the scalar unknown and the flux in the appropriate graph norm, while using the techniques and analysis of [1, 4] the superconvergence of the hybrid variable and of its post-processed (nonconforming) reconstruction are proved. Numerical experiments are included to support the theoretical conclusions.

1 Introduction

In this article, we consider the elliptic model problem in *mixed form*:

$$\begin{cases} \mathrm{div}\boldsymbol{\sigma} = f & \text{in } \Omega \\ a^{-1}\boldsymbol{\sigma} + \boldsymbol{\nabla} u = \mathbf{0} & \text{in } \Omega \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a convex polygonal domain, while a is a piecewise smooth function over Ω such that $a(\boldsymbol{x}) \geq a_0 > 0$ almost everywhere (a.e.) in Ω and $f \in L^2(\Omega)$ is a given function. Using the terminology of Continuum Mechanics, the scalar variable u is referred to as *displacement* while the vector-valued variable $\boldsymbol{\sigma}$ is the *flux* (or the *stress*). In [3], a dual–mixed Hybridized (DMH) finite element method with numerical integration of the local flux mass matrix has been proposed for the discretization of (1) in the case where such system represents the model for transport phenomena in symmetrizable form. The use of numerical integration allows to implement the DMH formulation as a genuine finite volume (FV) scheme for the approximation u_h of u that is a piecewise constant function over Ω. The resulting DMH-FV method enjoys the usual properties of dual-mixed approximations (interelement normal flux conservation and local self-equilibrium), and satisfies the discrete maximum principle (DMP) under the sole requirement that the finite element grid is of Delaunay type, with a considerable reduction of the computational effort compared to standard DMH formulations. In this article, we carry the error analysis of the DMH-FV method showing that it enjoys the same theoretical convergence properties as the corresponding DMH formulation. Precisely, after describing the DMH-FV scheme in Sects. 2, 3 and 4, we prove in Sect. 5 optimal error estimates for the scalar unknown and the flux in the appropriate graph norm, and the superconvergence of the hybrid variable and of its post-processed (nonconforming) reconstruction. The theoretical conclusions of Sect. 5 are then numerically validated in Sect. 6, where the model problem (1) is solved in the case of transport phenomena in symmetrized form.

2 Geometric Discretization

In view of the numerical discretization of (1), we consider a regular family of given partitions $\{\mathcal{T}_h\}$ of the domain Ω into open triangles K satisfying the usual admissibility condition (see [9], Sect. 3.1 and Def. 3.4.1). For a given $\mathcal{T}_h$, we denote by $|K|$ and h_K the area and the diameter of K, respectively, and we set $h = \max_{\mathcal{T}_h} h_K$. Let $\boldsymbol{x} = (x, y)^T$ be the position vector in Ω; then, for each $K \in \mathcal{T}_h$, we denote by $\boldsymbol{x}_q$, $q = 1, 2, 3$, the three vertices of K ordered according to a counterclockwise orientation, by $\boldsymbol{e}_q$ the edge of ∂K which is opposite to $\boldsymbol{x}_q$, by θ_q^K the angle opposite to $\boldsymbol{e}_q$ and by C_K the circumcenter of K. We denote by $|\boldsymbol{e}_q|$ the length of $\boldsymbol{e}_q$ and by $\boldsymbol{n}_q$ the outward unit normal vector along $\boldsymbol{e}_q$. Moreover, we define s_q^K as the signed distance between C_K and the midpoint M_q of $\boldsymbol{e}_q$. If $\theta_q^K < \pi/2$ then $s_q^K > 0$, while if K is obtuse in θ_q^K then $s_q^K < 0$, and C_K falls outside K. Notice also that if $\theta_q^K = \pi/2$ then $s_q^K = 0$, and C_K coincides with M_q. We denote by $\mathcal{E}_h$ the set of edges of $\mathcal{T}_h$, and by $\mathcal{E}_{h,int}$ those belonging to the interior of Ω. For each $e \in \mathcal{E}_{h,int}$, we indicate by K_e^1 and K_e^2 the pair of elements of $\mathcal{T}_h$ such that $e = \partial K_e^1 \cap \partial K_e^2$. Finally, we let $s_e = s_e^{K_e^1} + s_e^{K_e^2}$ denote the signed distance between $C_{K_e^1}$ and $C_{K_e^2}$. If $\theta_e^{K_e^1} + \theta_e^{K_e^2} < \pi$ for all $e \in \mathcal{E}_{h,int}$, then $s_e > 0$, and $\mathcal{T}_h$ is called a *Delaunay triangulation* [6]. The Delaunay condition prevents the occurrence of pairs of *obtuse* neighbouring elements in $\mathcal{T}_h$, still allowing the possibility of having single obtuse triangles in the computational grid (see [7] for algorithmic details). We assume from now on that $\mathcal{T}_h$ is a Delaunay triangulation, and we refer to [3] for the case where $\mathcal{T}_h$ is a *degenerate Delaunay triangulation* (i.e., $s_e = 0$ for some $e \in \mathcal{E}_{h,int}$).

3 Finite Element Spaces

For $k \geq 0$ and a given set $\mathcal{S}$, we denote by $\mathbb{P}_k(\mathcal{S})$ the space of polynomials of degree $\leq k$ defined over $\mathcal{S}$. We also denote by $\mathbb{RT}_0(K) := (\mathbb{P}_0(K))^2 \oplus \mathbb{P}_0(K)\,\boldsymbol{x}$ the

Raviart–Thomas (RT) finite element space of lowest degree [10], and by $\mathcal{P}_0$ the L^2-projection over constant functions. Then, we introduce the following finite element spaces:

$$
\begin{aligned}
\mathbf{V}_h &:= \{\boldsymbol{v} \in (L^2(\Omega))^2 \,|\, \boldsymbol{v}|_K \in \mathbb{RT}_0(K)\ \forall K \in \mathcal{T}_h\} \\
W_h &:= \{w \in L^2(\Omega) \,|\, w|_K \in \mathbb{P}_0(K)\ \forall K \in \mathcal{T}_h\} \\
M_h &:= \{m \in L^2(\mathcal{E}_h) \,|\, m|_{\partial K} \in \mathcal{R}_0(\partial K) \forall K \in \mathcal{T}_h, \\
&\qquad m^{K_e^1}|_e = m^{K_e^2}|_e\ \forall e \in \mathcal{E}_{h,int},\ m|_e = 0, \forall e \in \Gamma\} \\
\Lambda_h &:= \{v_h \in L^2(\Omega) \,|\, v_h \in \mathbb{P}_1(K)\ \forall K \in \mathcal{T}_h, \\
&\qquad v_h(M_e^{K_e^1}) = v_h(M_e^{K_e^2})\ \forall e \in \mathcal{E}_{h,int},\ v_h(M_e) = 0\ \forall e \in \Gamma\},
\end{aligned}
\tag{2}
$$

where $\mathcal{R}_0(\partial K) := \{v \in L^2(\partial K) |\, v|_e \in \mathbb{P}_0(e)\ \forall e \in \partial K\}$, and $m^{K_e^1}$, $m^{K_e^2}$ are the restrictions of the generic function $m \in M_h$ on K_e^1 and K_e^2, respectively. Functions belonging to $\mathbf{V}_h$ and W_h are completely discontinuous over $\mathcal{T}_h$, while functions in M_h are single-valued on $\mathcal{E}_h$. Functions in Λ_h are discontinuous and piecewise linear over $\mathcal{T}_h$, with continuity only at the midpoint of each edge $e \in \mathcal{E}_{h,int}$.

4 The DMH-FV Method

The DMH-FV Galerkin approximation of problem (1) consists of finding $(\boldsymbol{\sigma}_h,\, u_h,\, \lambda_h) \in (\mathbf{V}_h \times W_h \times M_h)$ such that:

$$
\begin{cases}
(A\,\boldsymbol{\sigma}_h,\, \boldsymbol{\tau}_h)_{\mathcal{T}_h,h} - (u_h,\, \mathrm{div}\boldsymbol{\tau}_h)_{\mathcal{T}_h} + \langle \lambda_h,\, \boldsymbol{\tau}_h \cdot \boldsymbol{n}\rangle_{\mathcal{E}_h} = 0 & \forall \boldsymbol{\tau}_h \in \mathbf{V}_h \\
(\mathrm{div}\boldsymbol{\sigma}_h\, q_h)_{\mathcal{T}_h} = (f,\, q_h)_{\mathcal{T}_h} & \forall q_h \in W_h \\
\langle \boldsymbol{\sigma}_h \cdot \boldsymbol{n},\, \mu_h\rangle_{\mathcal{E}_h} = 0 & \forall \mu_h \in M_h,
\end{cases}
\tag{3}
$$

where $A := a^{-1}$ and where we denote by $(\cdot,\,\cdot)_{\mathcal{T}_h}$, $(\cdot,\,\cdot)_{\mathcal{T}_h,h}$ and $\langle\cdot,\,\cdot\rangle_{\mathcal{E}_h}$ the elementwise L^2 inner product over $\mathcal{T}_h$, its approximation using a numerical integration formula over each element $K \in \mathcal{T}_h$ yet unspecified, and the edgewise L^2 inner product over $\mathcal{E}_h$, respectively. The equations in (3) have the following interpretation: $(3)_1$ expresses the approximate local constitutive law; $(3)_2$ expresses the approximate local self-equilibrium; $(3)_3$ expresses the approximate continuity of $\boldsymbol{\sigma} \cdot \boldsymbol{n}$ across each interelement edge. To construct the DMH-FV discretization, we assume, only for ease of presentation, that $\mathcal{T}_h$ is *strictly acute*, i.e., $\theta_q^K < \pi/2$ for each $K \in \mathcal{T}_h$ and $q = 1, 2, 3$. This implies that $s_q^K > 0$ $q = 1, 2, 3$ for each $K \in \mathcal{T}_h$. For each $K \in \mathcal{T}_h$, we denote by $\{\boldsymbol{\tau}_j\}_{j=1}^3$ the basis for $\mathbb{RT}_0(K)$ and set

$$
\boldsymbol{\sigma}_h^K(\boldsymbol{x}) = \sum_{j=1}^{3} \Phi_j^K\, \boldsymbol{\tau}_j(\boldsymbol{x}) \qquad \boldsymbol{x} \in K,
\tag{4}
$$

where the degree of freedom Φ_j^K is the flux of $\boldsymbol{\sigma}_h^K$ across edge $\boldsymbol{e}_j$, $j = 1,2,3$. Then, we consider the following quadrature formula

$$(A\,\boldsymbol{\tau}_j,\,\boldsymbol{\tau}_i)_K \simeq (A\,\boldsymbol{\tau}_j,\,\boldsymbol{\tau}_i)_{K,h} := \frac{1}{2}\overline{A}_i^K \cot(\theta_i^K)\delta_{ij} = \overline{A}_i^K \,\frac{s_i^K}{|\boldsymbol{e}_i|}\delta_{ij} \quad i,j=1,2,3, \tag{5}$$

where $\overline{A}_i^K := \int_{C_K}^{M_i} A^K(\zeta)\mathrm{d}\zeta/|s_i^K|$.

Proposition 1. *Assume that $a|_K \in W^{1,\infty}(K)$ for each $K \in \mathcal{T}_h$. Then, there exists a positive constant C_K depending only on the regularity of $\mathcal{T}_h$ such that $\forall \boldsymbol{p}, \boldsymbol{q} \in \mathbb{RT}_0(K)$ we have*

$$\Big|(A\,\boldsymbol{p},\,\boldsymbol{q})_K - (A\,\boldsymbol{p},\,\boldsymbol{q})_{K,h}\Big| \le C_K \|A\|_{W^{1,\infty}(K)} h_K \|\boldsymbol{p}\|_{L^2(K)} \|\boldsymbol{q}\|_{L^2(K)}. \tag{6}$$

Proof. We first need to check that (6) holds in the case $A = 1$. This follows by inspection on the analysis of [2] and noting that the supremum in (12) of [2] can be taken on the larger set $(L^2(K))^2 \supset H(\mathrm{div}; K)$. Then, the estimate (6) easily follows by proceeding as in [5] pgg. 375–376.

Remark 1. The quantities $\|\boldsymbol{p}\|_{L^2(K)}$ and $\|\boldsymbol{q}\|_{L^2(K)}$ can obviously be bounded by $\|\boldsymbol{p}\|_{H(\mathrm{div};K)}$ and $\|\boldsymbol{q}\|_{H(\mathrm{div};K)}$, respectively. This allows to recover the analogous estimates of the quadrature error associated with the approximation (5) proved in [8, 5].

Using (5) into $(3)_1$, we obtain the following system of linear algebraic equations for the degrees of freedom $\{\Phi^K\}_{K\in\mathcal{T}_h}$, $\{u^K\}_{K\in\mathcal{T}_h}$ and $\{\lambda_i\}_{e_i\in\mathcal{E}_{h,int}}$ associated with the DMH-FV method:

$$\begin{cases} \overline{A}_i^K\,\Phi_i^K\,\dfrac{s_i^K}{|\boldsymbol{e}_i|} - u^K + \lambda_i^K = 0 & \forall K \in \mathcal{T}_h \qquad i=1,2,3 \\ \displaystyle\sum_{i=1}^{3} \Phi_i^K = f^K|K| & \forall K \in \mathcal{T}_h \\ \Phi_e^{K_e^1} + \Phi_e^{K_e^2} = 0 & \forall e \in \mathcal{E}_{h,int}, \end{cases} \tag{7}$$

where $f^K := \mathcal{P}_0 f|_K$ for each $K \in \mathcal{T}_h$. Eliminating from $(7)_1$ and $(7)_3$ the variables Φ_i^K and λ_i^K in favor of u^K, and using the fact that λ_h is single–valued on $\mathcal{E}_h$, we get the following finite volume set of equations

$$\begin{cases} -\displaystyle\sum_{i=1}^{3} \mathcal{H}_{e_i}(a)\frac{u^{K_i} - u^K}{s_i}|\boldsymbol{e}_i| = f^K|K| & \forall K \in \mathcal{T}_h \\ u^{K_i} = 0 & \forall e_i \in \Gamma, \end{cases} \tag{8}$$

where, for each edge $e \in \mathcal{E}_h$, the positive quantity $\mathcal{H}_e(a)$ is the *harmonic average* of a across the edge e defined as

$$\mathcal{H}_e(a) := \left(\frac{\int_{s_e} a^{-1}(\zeta)\,d\zeta}{s_e}\right)^{-1} = \frac{s_e}{\overline{A}_e^{K_e^1}\,s_e^{K_e^1} + \overline{A}_e^{K_e^2}\,s_e^{K_e^2}}. \tag{9}$$

Proposition 2. *System* (8) *has a unique solution. Moreover, the DMH-FV satisfies the DMP.*

Proof. The set of linear algebraic equations (8) is a special instance of equations (5.7) of [8]. Therefore, Lemma 5.2 of [8] applies to conclude that the stiffness matrix associated with the DMH-FV method is a Stieltjes M-matrix [12]. This latter property immediately implies that the DMH-FV scheme satisfies the DMP by application of Theorem 3.1, p.202 of [11].

Once system (8) is solved for the piecewise constant values of u_h over $\mathcal{T}_h$, the degrees of freedom for λ_h can be easily computed by post-processing as

$$\lambda_e = \frac{(\overline{A}_e^{K_e^1} s_e^{K_e^1})^{-1} u^{K_e^1} + (\overline{A}_i^{K_e^2} s_e^{K_e^2})^{-1} u^{K_e^2}}{(\overline{A}_e^{K_e^1} s_e^{K_e^1})^{-1} + (\overline{A}_e^{K_e^2} s_e^{K_e^2})^{-1}} \qquad \forall e \in \mathcal{E}_{h,int}, \tag{10}$$

while $\lambda_e = 0$ for each $e \in \Gamma$. Then, $\boldsymbol{\sigma}_h$ can be computed over each element $K \in \mathcal{T}_h$ by using $(7)_1$ and (4).

Remark 2. Propositions 1 and 2 and the post-processing formula (10) still hold under the more general condition that $\mathcal{T}_h$ is a Delaunay triangulation. We refer to [3] for the details of the construction of the DMH-FV scheme under such assumption.

5 Error Estimates

In this section, we assume that the problem coefficients (and, as a consequence, the solution pair $(u, \boldsymbol{\sigma})$ of (1)) have at each step the required regularity required by the context. We also assume that exact integration is used to evaluate the right-hand side of $(3)_2$ in order to avoid dealing with the associated quadrature error. Moreover, we denote by C a positive constant, not depending on h and possibly depending on the mesh regularity constant and on the regularity of the coefficients, whose value is not necessarily the same at each occurrence.

5.1 Internal variables

Theorem 1. *There exists a positive constant C such that*

$$\|u - u_h\|_{L^2(\Omega)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathrm{div};\Omega)} \leq Ch \left(\|u\|_{H^2(\Omega)} + \|f\|_{H^1(\Omega)}\right). \tag{11}$$

Proof. The estimate (11) is an immediate consequence of Eq. (7.16) of [8] in the case $\sigma = 0$.

Let us denote by $P_h u$ the L^2-projection of u over W_h. Then, the next superconvergence result is an immediate consequence of Theorem 1 and of the analysis at pag. 186 of [4].

Theorem 2. *There exists a positive constant C such that*

$$\|P_h u - u_h\|_{L^2(\Omega)} \leq Ch^2 \left(\|u\|_{H^2(\Omega)} + \|f\|_{H^1(\Omega)}\right). \tag{12}$$

This latter result indicates that the piecewise constant values of u_h are a very good approximation of the mean value of the exact solution over each element $K \in \mathcal{T}_h$.

5.2 Hybrid variable

Let us denote by $\Pi_h u$ the L^2-projection of u over Λ_h and introduce the following mesh-dependent norm

$$|\mu_h|^2_{-1/2,h} := \sum_{e\in\mathcal{E}_h} h_e \|\mu_h\|^2_{L^2(e)} \qquad \forall \mu_h \in \Lambda_h.$$

The next result demonstrates the superconvergence of the hybrid variable λ_h to the L^2-projection $\Pi_h u$ of u over Λ_h.

Theorem 3. *There exists a positive constant C such that*

$$|\Pi_h u - \lambda_h|_{-1/2,h} \leq Ch^2 \left(\|u\|_{H^2(\Omega)} + \|f\|_{H^1(\Omega)}\right). \tag{13}$$

Proof. We closely follow the guidelines of the proof of Theorem 1.4 of [1]. For every $K \in \mathcal{T}_h$, we have

$$(A(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}), \boldsymbol{\tau}_h)_K - ((u_h - P_h u), \mathrm{div}\boldsymbol{\tau}_h)_K$$
$$+((\lambda_h - \Pi_h u), \boldsymbol{\tau}_h \cdot \boldsymbol{n})_{\partial K} - \boxed{((A\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)_K - (A\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)_{K,h})} = 0 \qquad \forall \boldsymbol{\tau}_h \in \mathbb{RT}_0(K).$$

The boxed term accounts for the quadrature error and is identically equal to zero in the analysis of [1]. Let us pick $\boldsymbol{\tau}_h \in \mathbb{RT}_0(K)$, as done in [1], in such a way that

$$\begin{cases} \boldsymbol{\tau}_h \cdot \boldsymbol{n}_e = \lambda_h - \Pi_h u & \text{on } e \in \partial K \\ \boldsymbol{\tau}_h \cdot \boldsymbol{n} = 0 & \text{on } \partial K \setminus e. \end{cases} \tag{14}$$

The above test function satisfies the following scaling properties

$$\|\boldsymbol{\tau}_h\|_{L^2(K)} \leq Ch_K^{1/2} \|\lambda_h - \Pi_h u\|_{L^2(e)}, \ \|\mathrm{div}\boldsymbol{\tau}_h\|_{L^2(K)} \leq Ch_K^{-1/2} \|\lambda_h - \Pi_h u\|_{L^2(e)}. \tag{15}$$

Using (14), (15) and (6), we obtain

$$\|\lambda_h - \Pi_h u\|_{L^2(e)} \leq C\big(\|A\|_{L^\infty(K)} h_K^{1/2} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(K)} + h_K^{-1/2} \|P_h u - u_h\|_{L^2(K)}$$
$$+ \boxed{\|A\|_{W^{1,\infty}(K)} h_K^{3/2} \|\boldsymbol{\sigma}_h\|_{L^2(K)}}\big),$$

from which we get (13), by squaring both sides of the previous inequality and multiplying by the length h_e of edge e, then by using (11), (12) and the well-posedness of the DMH-FV problem (3) and, finally, by summing over all mesh elements.

Using (11), (12) and (13), and proceeding as in [1], Sect.2, Theorem 2.2, we can prove the following result.

Theorem 4. *There exists a positive constant C such that*

$$\|u - u_h^*\|_{L^2(\Omega)} \leq Ch^2 \left(\|u\|_{H^2(\Omega)} + \|f\|_{H^1(\Omega)}\right). \tag{16}$$

This latter result indicates that the piecewise linear (non-conforming) reconstruction of λ_h over $\mathcal{T}_h$ is optimally converging in the L^2 norm to the exact solution u as in the case of standard displacement–based formulations.

6 Numerical Results

In this section, we consider problem (1) in the case $a = e^{-\varphi}$, φ being a given piecewise linear function over $\mathcal{T}_h$ [**?**]. With this choice, system (1) represents the symmetrized form of the convection-diffusion model with convective term in gradient form that is widely used to describe transport phenomena in Electrochemistry and Semiconductor Device Modeling [3]. In order to carry out the numerical validation of the DMH-FV scheme, we set $\Omega \equiv (0,2) \times (0,1)$ and $\varphi(x,y) = -(2x+y)$, in such a way that the exact solution is $u = e^{-(x+3)}\ xy(x-2)(y-1)$. Fig. 1 shows u_h and the non-conforming interpolant u_h^* of λ_h, computed on a triangulation with $h = 0.1237$. Fig. 2 shows the corresponding error curves $\|u - u_h\|_{L^2(\Omega)}$, $\|P_h u - u_h\|_{L^2(\Omega)}$, $|\Pi_h u - \lambda_h|_{-1/2,h}$ and $\|u - u_h^*\|_{L^2(\Omega)}$. The obtained results are in complete agreement with the theoretical analysis of Sects. 4 and 5.

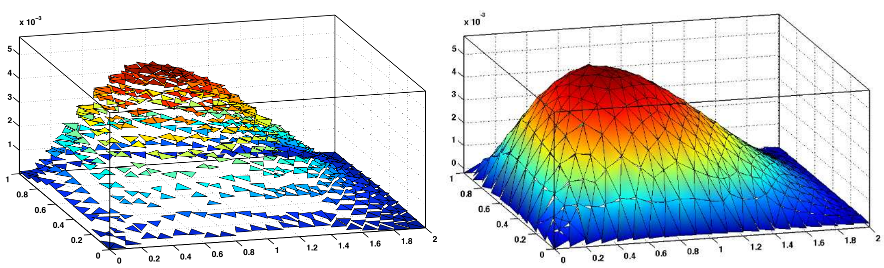

Fig. 1. Left: u_h; right: u_h^*.

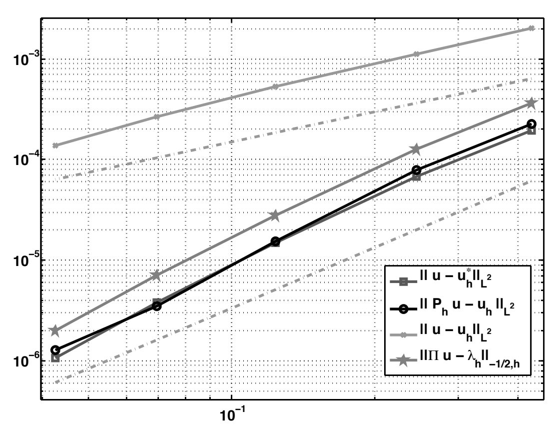

Fig. 2. Error curves as functions of the mesh size h.

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