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Collezione dei *Quaderni di Dipartimento*, numero **QDD 64**
Inserito negli *Archivi Digitali di Dipartimento* in data 18-6-2010

STOCHASTIC SCHRÖDINGER EQUATIONS AND MEMORY

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18th June 2010

Abstract

By starting from the stochastic Schrödinger equation and quantum trajectory theory, we introduce memory effects by considering stochastic adapted coefficients. As an example of a natural non-Markovian extension of the theory of white noise quantum trajectories we use an Ornstein-Uhlenbeck coloured noise as the output driving process. Under certain conditions a random Hamiltonian evolution is recovered. Moreover, we show that our non-Markovian stochastic Schrödinger equations unravel some master equations with memory kernels.

Keywords: Stochastic Schrödinger equation; Non Markovian quantum master equation; Unravelling; Quantum trajectories; Memory kernels
PACS: 02.50.Ey, 03.65.Ta, 05.40.-a, 42.50.Lc

1 Introduction

The problem of how to describe the reduced dynamics of a quantum open system interacting with an environment is very important [1–3]. More and more applications demand to treat dissipative effects, tendency to equilibrium, decoherence,... or how to have more equilibrium states, survival of coherences and entanglement,... in spite of the interaction with the external environment. The open system dynamics is often described in terms of *quantum master equations* which give the time evolution of the density matrix of the small system. When the Markov approximation is good (no memory effects) the situation is well understood: if the generator of the dynamics has the "Lindblad structure", then the dynamics sends statistical operators into statistical operators and it is completely positive [4,5].

However, for many new applications the Markovian approximation is not applicable. Such a situation appears in several concrete physical models: strong coupled systems, entanglement and correlation in the initial state, finite reservoirs... This gives rise to the theory of *non-Markovian quantum dynamics*, for which does not exist a general theory, but many different approaches [6–23].

Non Markovian reduced dynamics are usually obtained from the total dynamics of system plus bath by projection operator techniques such as Nakajima-Zwanzig operator technique, time-convolutionless operator technique [2,10], correlated projection operator or Lindblad rate equations [16, 19]... These techniques yield in principle exact master equations for the evolution of the subsystem. For example Nakajima-Zwanzig technique gives rise to an integro-differential equation with a memory kernel involving a retarded time integration over the history of the small system. However, in most of the cases the exact evolutions remain of formal interest: no analytic expression of the solution, impossible to simulate... Usually, some approximations have to be used to obtain a manageable description. But as soon as an approximation is done, the resulting equation can violate the complete positivity property; let us stress that the complete positivity (and even positivity) is a major question in non-Markovian systems [18,22].

The easiest way to preserve complete positivity is to introduce approximate or phenomenological equations at the Hilbert space level, an approach which is useful also for numerical simulations. We can say that in this way one is developing a non-Markovian theory of *unravelling* and of "Quantum Monte Carlo methods" [2,6–9,12,24]. In the Markovian case such an approach is related to the so called stochastic Schrödinger equation, quantum trajectory theory, measurements in continuous time. It provides wide applications for optical quantum systems and description of experiments such as *photo-detection* or *heterodyne/homodyne detection* [24–31]. In the non-Markovian case, an active line of research concentrates on a similar interpretation of non-Markovian unravelling. In this context, the question is more involved (for example, when complete positivity is violated the answer is hopeless) and remains an open problem. For the usual scheme of indirect quantum measurement it has been shown that in general such an interpretation is not accessible [21,23]. Actually, only few positive answers for very special cases have been found and this question is still highly debated [12,13,20,21].

Our aim is to introduce memory at Hilbert space level, in order to guarantee at the end a completely positive dynamics, and to maintain the possibility of the measurement interpretation. Our approach is based on the introduction of stochastic coefficients depending on the past history and on the use of coloured noises [32–34].

In Sect. 2 we introduce a special case of stochastic Schrödinger equation with memory. The starting point is a generalisation of the usual theory of the linear stochastic Schrödinger equation [31–33], based upon the introduction of random coefficients. This approach introduces memory effects in the underlying dynamics. The main interest is that the complete positivity is preserved and a measurement interpretation can be developed. We present this theory only in the context of the diffusive stochastic Schrödinger equation.

In Sect. 3 we attach the problem of memory by introducing an example of *coloured bath* and we show that we obtain a model of random Hamiltonian evolution [34], while we remain in the general framework of Sect. 2.

Finally, in Sect. 4, by using Nakajima-Zwanzig projection techniques, we show that the mean states satisfy closed master equations with memory kernels, which automatically preserve complete positivity. Moreover, we can say that the stochastic Schrödinger equations of the previous sections are unravellings of these memory master equations.

2 A non Markovian stochastic Schrödinger equation

The linear stochastic Schrödinger equation (lSSE) is the starting point to construct unravelling of master equations and models of measurements in continuous time [27,31]. By introducing random coefficients in such equation, but maintaining its structure, we get memory in the dynamical equations, while complete positivity of the dynamical maps and the continuous measurement interpretation are preserved [32,33]. To simplify the theory we consider only diffusive contributions and bounded operators.

Assumption 1 (The linear stochastic Schrödinger equation)**.** Let $\mathcal{H}$ be a complex, separable Hilbert space, the space of the quantum system, and $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$ be a stochastic basis satisfying the usual hypotheses, where a d-dimensional continuous Wiener process is defined; $\mathbb{Q}$ will play the role of a reference probability measure. The lSSE we consider is

$$\mathrm{d}\psi(t) = K(t)\psi(t)\mathrm{d}t + \sum_{j=1}^{d} R_j(t)\psi(t)\mathrm{d}W_j(t), \tag{1}$$
$$\psi(0) = \psi_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{Q}; \mathcal{H}).$$

Let us denote by $\mathcal{T}(\mathcal{H})$ the trace class on $\mathcal{H}$, by $\mathcal{S}(\mathcal{H})$ the subset of the statistical operators and by $\mathcal{L}(\mathcal{H})$ the space of the linear bounded operators.

Assumption 2 (The random coefficients)**.** The coefficient in the drift has the structure

$$K(t) = -\mathrm{i}H(t) - \frac{1}{2}\sum_{j=1}^{d} R_j(t)^* R_j(t). \tag{2}$$

The coefficients $H(t)$, $R_j(t)$ are random bounded operators with $H(t) = H(t)^*$, say predictable càglàd processes in $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$.

Moreover, $\forall T > 0$, we have

$$\int_0^T \mathbb{E}_{\mathbb{Q}}\left[\|H(t)\|\right] \mathrm{d}t < +\infty, \tag{3a}$$

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{2\sum_{j=1}^{d}\int_0^T \|R_j(t)\|^2\,\mathrm{d}t\right\}\right] < +\infty. \tag{3b}$$

Theorem 1. *Under Assumptions 1, 2, the lSSE* (1) *has a pathwise unique solution. The square norm $\|\psi(t)\|^2$ is a continuous positive martingale given by*

$$\|\psi(t)\|^2 = \|\psi_0\|^2 \exp\left\{\sum_j \left[\int_0^t m_j(s)\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t m_j(s)^2\,\mathrm{d}s\right]\right\}, \tag{4}$$

$$m_j(t) := 2\,\mathrm{Re}\left\langle \hat{\psi}(t) \middle| R_j(t)\hat{\psi}(t)\right\rangle, \tag{5}$$

$$\hat{\psi}(t) := \begin{cases} \psi(t)/\left\|\psi(t)\right\|, & \text{if } \|\psi(t)\| \neq 0, \\ v \text{ (fixed unit vector)}, & \text{if } \|\psi(t)\| = 0. \end{cases} \tag{6}$$

Proof. Assumptions 1, 2 imply the Hypotheses of [32, Proposition 2.1 and Theorem 2.4], but Hypothesis 2.3.A of page 295. According to the discussion at the end of p. 297, this last hypothesis can be substituted by (3b), which implies Novikov condition, a sufficient condition for an exponential supermartingale to be a martingale. Then, all the statements hold. □

Remark 1. By expression (4) we get that on the set $\{\|\psi_0\| > 0\}$ we have $\|\psi(t)\| > 0$ $\mathbb{Q}$-a.s. This means that, if $\psi_0 \neq 0$ $\mathbb{Q}$-a.s., then the process $\hat{\psi}(t)$ (6) is almost surely defined by the normalisation of $\psi(t)$ and the arbitrary vector v does not play any role with probability one.

Remark 2. Let us define the positive, $\mathcal{T}(\mathcal{H})$-valued process

$$\sigma(t) := |\psi(t)\rangle\langle\psi(t)|. \tag{7}$$

By applying the Itô formula to $\langle\psi(t)|a\psi(t)\rangle$, $a \in \mathcal{L}(\mathcal{H})$, we get the weak-sense linear stochastic master equation (lSME)

$$\mathrm{d}\sigma(t) = \mathcal{L}(t)[\sigma(t)]\mathrm{d}t + \sum_{j=1}^{d} \mathcal{R}_j(t)[\sigma(t)]\mathrm{d}W_j(t), \tag{8}$$

$$\mathcal{R}_j(t)[\rho] := R_j(t)\rho + \rho R_j(t)^*, \tag{9}$$

$$\mathcal{L}(t)[\rho] = -\mathrm{i}[H(t), \rho] + \sum_{j=1}^{d} \left(R_j(t)\rho R_j(t)^* - \frac{1}{2}\left\{R_j(t)^* R_j(t), \rho\right\} \right); \tag{10}$$

$\mathcal{L}(t)$ is the random Liouville operator [32, Proposition 3.4].

Assumption 3 (The initial condition)**.** Let us assume that the initial condition ψ_0 is normalised, in the sense that $\mathbb{E}_{\mathbb{Q}}\left[\|\psi_0\|^2\right] = 1$. Then, $\varrho_0 := \mathbb{E}_{\mathbb{Q}}\left[|\psi_0\rangle\langle\psi_0|\right] \in \mathcal{S}(\mathcal{H})$ represents the initial statistical operator.

Remark 3. Under the previous assumptions $p(t) := \|\psi(t)\|^2$ is a positive, mean-one martingale and, $\forall T > 0$, we can define the new probability law on $(\Omega, \mathcal{F}_T)$

$$\forall F \in \mathcal{F}_T \qquad \mathbb{P}^T(F) := \mathbb{E}_{\mathbb{Q}}[p(T)1_F]. \tag{11}$$

By the martingale property these new probabilities are consistent in the sense that, for $0 \leq s < t$ and $F \in \mathcal{F}_s$, we have $\mathbb{P}^t(F) = \mathbb{P}^s(F)$.

The new probabilities are interpreted as the physical ones, the law of the output of the time continuous measurement. Let us stress that it is possible to express the physical probabilities in agreement with the axiomatic formulation of quantum mechanics by introducing positive operator valued measures and completely positive instruments [32,33].

Remark 4. By Girsanov theorem, the d-dimensional process

$$\widehat{W}_j(t) := W_j(t) - \int_0^t m_j(s)\,\mathrm{d}s, \qquad j = 1, \dots, d, \quad t \in [0, T], \tag{12}$$

is a standard Wiener process under the physical probability $\mathbb{P}^T$ [32, Proposition 2.5, Remark 2.6].

By adding further sufficient conditions two more important equations can be obtained.

Assumption 4 ([32, Hypotheses 2.3.A])**.** Let us assume that we have

$$\sup_{\omega\in\Omega} \int_0^t \Big\| \sum_{j=1}^d R_j(s,\omega)^* R_j(s,\omega) \Big\| \mathrm{d}s < +\infty. \tag{13}$$

Theorem 2. *Let Assumptions 1–4 hold. Under the physical probability the normalized state $\hat{\psi}(t)$, introduced in Eq.* (6)*, satisfies the non-linear* stochastic Schrödinger equation *(SSE)*

$$\begin{aligned} \mathrm{d}\hat{\psi}(t) = \sum_j \Big[R_j(t) - \mathrm{Re}\, n_j \Big(t, \hat{\psi}(t)\Big) \Big] \hat{\psi}(t)\, \mathrm{d}\widehat{W}_j(t) + K(t)\hat{\psi}(t)\,\mathrm{d}t \\ + \sum_j \bigg[\Big(\mathrm{Re}\, n_j \Big(t, \hat{\psi}(t)\Big)\Big) R_j(t) - \frac{1}{2} \Big(\mathrm{Re}\, n_j \Big(t, \hat{\psi}(t)\Big)\Big)^2 \bigg] \hat{\psi}(t)\,\mathrm{d}t\,, \end{aligned} \tag{14}$$

where $n_j(t,x) := \langle x | R_j(t) x\rangle$, $\forall t \in [0, +\infty)$, $j = 1, \dots, d$, $x \in \mathcal{H}$.

*Moreover, the process (*a priori or average states*) defined by*

$$\eta(t) := \mathbb{E}_{\mathbb{Q}}[\sigma(t)], \quad \textit{or} \quad \mathrm{Tr}\,\{a\eta(t)\} = \mathbb{E}_{\mathbb{Q}}\left[\langle \psi(t) | a\psi(t)\rangle\right] \quad \forall a \in \mathcal{L}(\mathcal{H}) \tag{15}$$

satisfies the master equation

$$\eta(t) = \varrho_0 + \int_0^t \mathbb{E}_{\mathbb{Q}}\left[\mathcal{L}(s)[\sigma(s)]\right] \mathrm{d}s. \tag{16}$$

Proof. One can check that all the hypotheses of [32, Theorem 2.7] hold. Then, the SSE for $\hat{\psi}(t)$ follows.

As in the proof of [32, Propositions 3.2], one can prove that the stochastic integral in the lSME (8) has zero mean value. Then, Eq. (16) follows. □

Note that, by the definition of the physical probabilities, we have also

$$\mathrm{Tr}\,\{a\eta(t)\} = \mathbb{E}_{\mathbb{P}^T}\Big[\langle \hat{\psi}(t) | a\hat{\psi}(t)\rangle\Big]\,, \quad \forall a \in \mathcal{L}(\mathcal{H}), \quad \forall t,\, T : 0 \le t \le T. \tag{17}$$

The SSE (14) is the starting point for numerical simulations; the key point is that norm of its solution $\hat{\psi}(t)$ is constantly equal to one. We underline that Eq. (16) is not a closed equation for the mean state of the system. In the last section we shall see how to obtain, in principle, a closed equation for the *a priori* states of the quantum system.

The finite dimensional case

If we assume a finite dimensional Hilbert space and we strengthen the conditions on the coefficients, we obtain a more rich theory. We discuss here below the situation [33].

Assumption 5. 1. The Hilbert space of the quantum system is finite dimensional, say $\mathcal{H} := \mathbb{C}^n$. We write $M_n(\mathbb{C})$ for the space of the linear operators on $\mathcal{H}$ into itself ($n \times n$ complex matrices).

2. The coefficient processes R_j and H are $M_n(\mathbb{C})$-valued and progressive with respect to the reference filtration.

3. The coefficients satisfy the following conditions: for every $T > 0$ there exist two positive constants $M(T)$ and $L(T)$ such that

$$\begin{aligned} &\sup_{\omega\in\Omega}\sup_{t\in[0,T]}\left\|\sum_{j=1}^{d} R_j(t,\omega)^* R_j(t,\omega)\right\| \leq L(T) < \infty, \\ &\sup_{\omega\in\Omega}\sup_{t\in[0,T]}\|H(t,\omega)\| \leq M(T) < \infty. \end{aligned} \tag{18}$$

Under Assumption 5 it is possible to prove existence and pathwise uniqueness of the solution of the lSSE just modifying classical results for existence and uniqueness of the solution for stochastic differential equation with deterministic coefficients.

Moreover, it is possible to prove that the solution of lSSE fulfil some L^p estimate: in this point the finite dimension of the Hilbert space plays a fundamental role because the bounds we obtain for the process $\psi(t)$ involve constants depending on n.

Obviously, in this context the martingale property of the norm of the solution is still valid and so one can define the consistent family of physical probabilities. It is also possible to introduce the propagator of the lSSE, that is the two times $M_n(\mathbb{C})$-valued stochastic process $A(t, s)$ such that $A(t, s)\psi(s) = \psi(t)$, for all $t, s \geq 0$ s.t. $s \leq t$. We are able to obtain a stochastic differential equation (with pathwise unique solution) for the propagator and, by means of it, to prove that the propagator takes almost surely values in the space of the invertible matrices. The L^p estimates for $\psi(t)$ are useful to obtain L^p estimates on $A(t,s)$. Furthermore, the propagator satisfies the typical composition law of an evolution: $A(t,s) = A(t,r)A(r,s)$ for all $t,r,s \geq 0$ s.t. $s \leq r \leq t$.

The almost sure invertibility of the propagator guarantees that the process $\hat{\psi}(t)$ can be almost surely defined and that this process satisfies, under the physical probabilities, a non linear SSE, similar to Eq. (14). It is possible to prove that in this case the SSE has a pathwise unique solution.

When we go on extending the theory to the space of the statistical operators, we can take as initial condition a random statistical operator or a deterministic one. We define the process $\sigma(t)$ as in Eq. (7) and, by using the Itô formula, we obtain an equation formally similar to the lSME, but in this case we are able to prove the uniqueness of its solution given the initial statistical operator ϱ_0 (the existence comes out by construction). In this way we can say that the lSME is the evolution equation of the quantum system, when the initial condition is a deterministic (or even random) statistical operator ϱ_0. We can introduce the propagator of the lSME, which is a two times-linear map valued stochastic process, say $\Lambda(t,s)$, such that $\Lambda(t,0)[\varrho_0] = \sigma(t)$, $\Lambda(t,s) = \Lambda(t,r) \circ \Lambda(r,s)$, for all $t,r,s \geq 0$ s.t. $s \leq r \leq t$ and $\Lambda(t,s)[\tau] = A(t,s)\tau A(t,s)^*$. From the last expression of the propagator of the lSME, it comes out that this is a completely positive map valued process.

Also in this case we can introduce a consistent family of physical probabilities. Indeed, the process $\mathrm{Tr}\{\sigma(t)\}$ is an exponential mean-one martingale that can be used to define the new probability laws, as we did in the Hilbert space.

It is then possible to define the normalisation of $\sigma(t)$ with respect to its trace,

$$\varrho(t) = \frac{\sigma(t)}{\mathrm{Tr}\{\sigma(t)\}}$$

and, under the physical probabilities, we have the following non linear equation for $\varrho(t)$, with pathwise unique solution

$$\begin{cases} \mathrm{d}\varrho(t) = \mathcal{L}(t)[\varrho(t)]\mathrm{d}t + \displaystyle\sum_{j=1}^{d} \{\mathcal{R}_j(t)[\varrho(t)] - v_j(t)\varrho(t)\}\, \mathrm{d}\widehat{W}_j(t)\,, \qquad t \geq 0 \\ \varrho(0) = \varrho_0\,, \end{cases} \tag{19}$$

where $v_j(t) := \mathrm{Tr}\{(R_j(t)+R_j(t)^*)\varrho(t)\}$, and $\widehat{W}(t)$ is a Wiener process under the physical probabilities defined by

$$\widehat{W}_j(t) := W_j(t) - \int_0^t v_j(s)\mathrm{d}s\,, \quad \forall j = 1, \dots, d\,.$$

3 Random Hamiltonian

In the previous section, we have presented a non Markovian generalisation of the usual diffusive lSSE by using random coefficients to introduce memory. In this section we adopt an alternative strategy and we start with a usual lSSE with non random coefficients, but driven by a coloured noise; in this way the memory is encoded in the driving noise of the lSSE, not in the coefficients. As we shall see, this model too turns out to be a particular case of the general theory presented in Section 2. Moreover, the new lSSE will be norm-preserving and will represent a quantum system evolving under a random Hamiltonian dynamics, while the Hamiltonian is very singular and produces dissipation.

As our aim is just to explore some possibility, we keep things simple and we consider a one-dimensional driving noise and two non-random, bounded operators A and B on $\mathcal{H}$ in the drift and in the diffusive terms. The starting point is then the basic linear stochastic Schrödinger equation

$$\mathrm{d}\psi(t) = A\psi(t)\mathrm{d}t + B\psi(t)\mathrm{d}X(t). \tag{20}$$

The simplest choice of a coloured noise is the stationary Ornstein-Uhlenbeck process defined by

$$X(t) = \mathrm{e}^{-\gamma t}Z + \int_0^t \mathrm{e}^{-\gamma(t-s)}\mathrm{d}W(s), \qquad \gamma > 0.$$

where $(W(t))$ is a one dimensional Wiener process, defined on the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ and Z is an $\mathcal{F}_0$-measurable, normal random variable with mean 0 and variance $1/(2\gamma)$. The Ornstein-Uhlenbeck process $(X(t))$ is a Gaussian process with zero mean and correlation function

$$\mathbb{E}_{\mathbb{Q}}[X(t)X(s)] = \frac{\mathrm{e}^{-\gamma|t-s|}}{2\gamma}\,; \tag{21}$$

it satisfies the stochastic differential equation

$$\mathrm{d}X(t) = -\gamma X(t)\mathrm{d}t + \mathrm{d}W(t), \qquad X(0) = Z. \tag{22}$$

Formally, Eq. (20) is driven by the derivative of the Ornstein-Uhlenbeck process (heuristically, $\mathrm{d}X(t) = \dot{X}(t)\mathrm{d}t$), whose two-time correlation is no more a delta, as in the case of white noise, but it is formally given by $\mathbb{E}_{\mathbb{Q}}[\dot{X}(t)\dot{X}(s)] = \delta(t-s) - \frac{\gamma}{2}\,\mathrm{e}^{-\gamma|t-s|}$. Note that the Markovian regime is recovered in the limit $\gamma \downarrow 0$.

It is then straightforward that Eq. (20) can be rewritten in the form

$$\mathrm{d}\psi(t) = \big(A - \gamma X(t)B\big)\psi(t)\mathrm{d}t + B\psi(t)\mathrm{d}W(t), \tag{23}$$

on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$. The initial condition is assumed to satisfy Assumption 3. Assumption 1 is satisfied with $d = 1$, $K(t) = A - \gamma X(t)B$, $R(t) = B$.

The key point of the construction of Section 2 and of its interpretation is the fact that $\|\psi(t)\|^2$ is a martingale. To this end we compute its stochastic differential by using the Itô rules and we get

$$\begin{aligned}\mathrm{d}\langle\psi(t)|\psi(t)\rangle &= \langle \mathrm{d}\psi(t)|\psi(t)\rangle + \langle\psi(t)|\mathrm{d}\psi(t)\rangle + \langle \mathrm{d}\psi(t)|\mathrm{d}\psi(t)\rangle \\ &= \langle\psi(t)|\left[A^* + A - \gamma X(t)\left(B^* + B\right) + B^*B\right]\psi(t)\rangle\mathrm{d}t \\ &\qquad + \langle\psi(t)|\big(B^* + B\big)\psi(t)\rangle\mathrm{d}W(t).\end{aligned} \tag{24}$$

Then, the process $(\|\psi(t)\|^2)$ can be a martingale only if the term in front of $\mathrm{d}t$ (the drift term) is equal to zero. This imposes that

$$A^* + A - \gamma X(t)\left(B^* + B\right) + B^*B = 0, \qquad \forall t. \tag{25}$$

By taking the mean of this equation we get $A^* + A + B^*B = 0$; then, we need also $B^* + B$. These conditions impose that there are two self-adjoint operators L and H_0 such that $B = -\mathrm{i}L$ and $A = -\mathrm{i}H_0 - \frac{1}{2}L^2$. As a consequence the initial equation (20) becomes

$$\mathrm{d}\psi(t) = \left[-\mathrm{i}\left(H_0 - \gamma X(t)L\right) - \frac{1}{2}\,L^2\right]\psi(t)\mathrm{d}t - \mathrm{i}L\psi(t)\mathrm{d}W(t)\,. \tag{26}$$

Now, being $X(t)$ a continuous adapted process and $H_0 = {H_0}^* \in \mathcal{L}(\mathcal{H})$, $L = L^* \in \mathcal{L}(\mathcal{H})$, also Assumption 2 holds with

$$H(t) = H_0 - \gamma X(t)L, \quad R(t) = -\mathrm{i}L, \quad K(t) = -\mathrm{i}\big(H_0 - \gamma X(t)L\big) - \frac{1}{2}\,L^2.$$

Moreover, we have $\mathbb{E}_{\mathbb{Q}}[|X(t)|] \leq \sqrt{\mathbb{E}_{\mathbb{Q}}[X(t)^2]} \leq 1/\sqrt{2\gamma}$,

$$\mathbb{E}_{\mathbb{Q}}[\|H(t)\|] \leq \|H_0\| + \gamma\,\|L\|\,\mathbb{E}_{\mathbb{Q}}[|X(t)|] \leq \|H_0\| + \sqrt{\frac{\gamma}{2}}\,\|L\|\,,$$

which implies condition (3a). Condition (3b) and Assumption 4 are trivially satisfied because $R(t)$ is non random, time independent and bounded.

As all Assumptions 1–4 hold, also all statements of Theorems 1, 2 and Remarks 1–4 hold. In particular the lSSE (26) has a pathwise unique solution.

What is peculiar of the present model is that Eqs. (24)–(26) give $\|\psi(t)\|^2 = \|\psi(0)\|^2$ or that the probability densities are independent of time, $p(t) = p(0)$, cf. Eqs. (5) and (12), which give $m(t) = 0$ and $\widehat{W}(t) = W(t)$. We have also, from Eq. (6), $\hat{\psi}(t) = \psi(t)/\|\psi(0)\|$, if $\|\psi(0)\| \neq 0$, and, from Remark 3, $\mathbb{P}^t(F) = \mathbb{Q}(F)$, for all events $F \in \mathcal{F}_t$, independent of $\mathcal{F}_0$. As a consequence the change of probability has no effect (the new probability is equal to the initial for events independent of $\mathcal{F}_0$). In other terms, no information has been extracted from the measurement interpretation.

Moreover, the property $\|\psi(t)\| = \|\psi_0\|$ is in agreement with a purely Hamiltonian evolution. More precisely, let $\overleftarrow{\mathrm{T}}\exp\{\cdots\}$ denotes the time ordered exponential; then, the formal solution of Eq. (26) is given by

$$\psi(t) = \overleftarrow{\mathrm{T}}\exp\left\{-\mathrm{i}\int_0^t (H_0 - \gamma X(s)L)\,\mathrm{d}s - \mathrm{i}\int_0^t L\,\mathrm{d}W(s)\right\}\psi_0.$$

The evolution of the quantum system is then completely determined by the time-dependent, random Hamiltonian

$$\hat{H}_t = H_0 + \left(\dot{W}(t) - \gamma X(t)\right)L.$$

Let us stress that it is a formal expression, due to the presence of $\dot{W}(t)$.

This shows that the usual measurement interpretation of (20) *coloured* with an Ornstein-Uhlenbeck process gives raise to a random Hamiltonian evolution. As announced, we recover the framework of the evolution of a closed system incorporating a random environment characterised in terms of an Ornstein-Uhlenbeck noise.

One can investigate the evolution of the corresponding density matrices. To this end, we consider the pure state process $(\sigma(t))$ defined by $\sigma(t) = |\psi(t)\rangle\langle\psi(t)|$, Eq. (7). By using Itô rules, the process $(\sigma(t))$ satisfies the stochastic differential equation (SDE)

$$\mathrm{d}\sigma(t) = -\mathrm{i}[H_0 - \gamma X(t)L, \sigma(t)]\mathrm{d}t - \mathrm{i}[L, \sigma(t)]\mathrm{d}W(t) - \frac{1}{2}[L, [L, \sigma(t)]]\mathrm{d}t, \quad (27)$$

which is, of course, equivalent to (26) with random Liouville operator (10) given by

$$\mathcal{L}(t) = -\mathrm{i}[H_0 - \gamma X(t)L, \cdot]\mathrm{d}t - \frac{1}{2}[L, [L, \cdot]]\mathrm{d}t.$$

Let us stress that the presence of the Ornstein-Uhlenbeck process implies that the solution $(\sigma(t))$ of Eq. (27) is not a Markov process.

Taking the expectation, we get the evolution of the mean $\eta(t) = \mathbb{E}_{\mathbb{Q}}[\sigma(t)]$, Eq. (15), which turns out to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\eta(t) = -\mathrm{i}[H_0, \eta(t)] - \frac{1}{2}\,[L, [L, \eta(t)]] + \mathrm{i}\gamma\big[L, \mathbb{E}_{\mathbb{Q}}[X(t)\sigma(t)]\big]. \tag{28}$$

Note that it is not a closed master equation for the mean state $\eta(t)$. Actually, we have derived a model with memory for the mean state. Indeed, the term $\mathrm{i}\gamma\big[L, \mathbb{E}_{\mathbb{Q}}[X(t)\sigma(t)]\big]$ introduces non-Markovian memory effects in the dynamics. Moreover, Eq. (26) is an unravelling of the master equation (28).

4 Projection techniques and closed master equations with memory

As we have seen in Eq. (16), the *a priori* states or average states $\eta(t) = \mathbb{E}_{\mathbb{Q}}[\sigma(t)] = \mathbb{E}_{\mathbb{P}^t}[\rho(t)]$ satisfy the equation $\dot{\eta}(t) = \mathbb{E}_{\mathbb{Q}}\left[\mathcal{L}(t)[\sigma(t)]\right]$, which is not closed because both $\mathcal{L}(t)$ and $\sigma(t)$ are random. However, at least heuristically, some kind of generalised master equations can be obtained by using the Nakajima-Zwanzig projection technique [2, Section 9.1.2].

Let us introduce the projection operators on the relevant part (the mean) and on the non relevant one:

$$\mathcal{P}[\cdots] := \mathbb{E}_{\mathbb{Q}}[\cdots], \qquad \mathcal{Q} := \mathbb{1} - \mathcal{P}.$$

Then, we have $\eta(t) = \mathcal{P}[\sigma(t)]$ and we define the non relevant part of the state, the mean Liouville operator and the difference from the mean of the Liouville operator

$$\sigma_\perp(t) := \mathcal{Q}[\sigma(t)] = \sigma(t) - \eta(t),$$

$$\mathcal{L}_{\mathrm{M}}(t) := \mathbb{E}_{\mathbb{Q}}[\mathcal{L}(t)], \qquad \Delta\mathcal{L}(t) := \mathcal{L}(t) - \mathcal{L}_{\mathrm{M}}(t).$$

By using the projection operators and the fact that the stochastic integrals have zero mean, which means $\mathcal{P}\int_0^t \mathrm{d}W_j(s)\cdots = 0$, from (8) we get the system of equations

$$\dot{\eta}(t) = \mathcal{L}_{\mathrm{M}}(t)[\eta(t)] + \mathcal{P} \circ \Delta\mathcal{L}(t)[\sigma_\perp(t)], \tag{29a}$$

$$\mathrm{d}\sigma_\perp(t) = \mathcal{Q}\circ\mathcal{L}(t)[\sigma_\perp(t)]\mathrm{d}t + \sum_{j=1}^{d}\mathcal{R}_j(t)[\sigma_\perp(t)]\mathrm{d}W_j(t) + \mathcal{Q}\circ\mathcal{L}(t)[\eta(t)]\mathrm{d}t + \sum_{j=1}^{d}\mathcal{R}_j(t)[\eta(t)]\mathrm{d}W_j(t). \tag{29b}$$

As one can check by using Itô formula, the formal solution of Eq. (29b) can be written as

$$\sigma_\perp(t) = \mathcal{Q}\circ\mathcal{V}(t,0)[\sigma_\perp(0)] + \int_0^t \mathcal{Q}\circ\mathcal{V}(t,s)\circ\Big(\mathcal{L}(s) - \sum_j \mathcal{R}_j(s)^2\Big)[\eta(s)]\mathrm{d}s + \mathcal{Q}\circ\mathcal{V}(t,0)\left[\sum_{j=1}^{d}\int_0^t \mathcal{V}(s,0)^{-1}\circ\mathcal{R}_j(s)[\eta(s)]\mathrm{d}W_j(s)\right], \tag{30}$$

where $\mathcal{V}(t,r)$ is the fundamental solution (or propagator) of the lSME (8) and satisfies the SDE

$$\mathcal{V}(t,r) = \mathbb{1} + \int_r^t \mathrm{d}s\,\mathcal{L}(s)\circ\mathcal{V}(s,r) + \sum_{j=1}^{d}\int_r^t \mathrm{d}W_j(s)\,\mathcal{R}_j(s)\circ\mathcal{V}(s,r).$$

Let us stress that if one includes $\mathcal{V}(t,0)$ into the stochastic integrals, from one side one gets the simpler expression $\mathcal{V}(t,0)\circ\mathcal{V}(s,0)^{-1} = \mathcal{V}(t,s)$. But the propagator is a stochastic process and could be non adapted. To overcome this difficulty one should use some definition of anticipating stochastic integral. So, we prefer the formulation with $\mathcal{V}(t,0)$ outside the stochastic integral, in order to have only adapted integrands.

By introducing the quantity (30) into Eq. (29a), we get the generalised master equation for the *a priori* states

$$\dot\eta(t) = J(t) + \mathcal{L}_{\mathrm{M}}(t)[\eta(t)] + \int_0^t \mathcal{K}(t,s)[\eta(s)]\mathrm{d}s + \mathbb{E}_{\mathbb{Q}}\left[\Delta\mathcal{L}(t)\circ\mathcal{Q}\circ\mathcal{V}(t,0)\left[\sum_{j=1}^{d}\int_0^t \mathcal{V}(s,0)^{-1}\circ\mathcal{R}_j(s)[\eta(s)]\mathrm{d}W_j(s)\right]\right], \tag{31}$$

where

$$J(t) := \mathbb{E}_{\mathbb{Q}}\left[\Delta\mathcal{L}(t)\circ\mathcal{Q}\circ\mathcal{V}(t,0)[\sigma_\perp(0)]\right]$$

is an inhomogeneous term which disappears if the initial state is non random, i.e. $\sigma_\perp(0)=0$, and

$$\mathcal{K}(t,s) := \mathbb{E}_{\mathbb{Q}}\left[\Delta\mathcal{L}(t)\circ\mathcal{Q}\circ\mathcal{V}(t,s)\circ\Big(\mathcal{L}(s) - \sum_j \mathcal{R}_j(s)^2\Big)\right]$$

is an integral memory kernel. Also the last term in (31) is a memory contribution, linear in η and depending on its whole trajectory up to t.

Let us stress that to compute the terms appearing in Eq. (31) and to solve it is not simpler than to solve Eq. (8) and to compute the mean of the solution. The meaning of Eq. (31) is theoretical: it is a quantum master equation with memory and Eq. (14) gives an unravelling of it.

While the best way to study a concrete model is to simulate the stochastic Schrödinger equation, Eq. (31) could be the starting point for some approximation. A possibility is to approximate $\mathcal{V}(t,r)$ by the deterministic evolution generated by the mean Liouville operator:

$$\mathcal{V}_{\mathrm{M}}(t,r) = \mathbb{1} + \int_r^t \mathrm{d}s\, \mathcal{L}_{\mathrm{M}}(s) \circ \mathcal{V}_{\mathrm{M}}(s,r).$$

If we take also $\sigma_\perp(0) = 0$, we get

$$\begin{aligned}\dot{\eta}(t) \simeq \mathcal{L}_{\mathrm{M}}(t)[\eta(t)] + \int_0^t \mathcal{K}_1(t,s)[\eta(s)]\mathrm{d}s \\ + \mathbb{E}_{\mathbb{Q}}\left[\Delta\mathcal{L}(t)\left[\sum_{j=1}^d \int_0^t \mathcal{V}_{\mathrm{M}}(t,s)\circ \mathcal{R}_j(s)[\eta(s)]\mathrm{d}W_j(s)\right]\right], \end{aligned} \tag{32}$$

$$\mathcal{K}_1(t,s) := \mathbb{E}_{\mathbb{Q}}\left[\Delta\mathcal{L}(t)\circ \mathcal{V}_{\mathrm{M}}(t,s)\circ\left(\Delta\mathcal{L}(s) - \sum_j \Delta\mathcal{R}_j^2(s)\right)\right],$$

where $\Delta\mathcal{R}_j^2(s) = \mathcal{R}_j(s)^2 - \mathbb{E}_{\mathbb{Q}}[\mathcal{R}_j(s)^2]$.

For the model of the previous section we have: $\mathcal{R}_j(s) = \mathcal{R} = -\mathrm{i}[L,\cdot]$,

$$\mathcal{L}_{\mathrm{M}}(t) = \mathcal{L}_{\mathrm{M}} = -\mathrm{i}[H_0,\cdot] - \frac{1}{2}[L,[L,\cdot]], \qquad \mathcal{V}_{\mathrm{M}}(t,s) = \mathrm{e}^{\mathcal{L}_{\mathrm{M}}(t-s)},$$

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\Delta\mathcal{L}(t)\left[\sum_{j=1}^d \int_0^t \mathcal{V}_{\mathrm{M}}(t,s)\circ \mathcal{R}_j(s)[\eta(s)]\mathrm{d}W_j(s)\right]\right] \\ = -\gamma\,\mathbb{E}_{\mathbb{Q}}\left[X(t)\int_0^t \mathrm{d}W(s)\,\mathcal{R}\circ \mathrm{e}^{\mathcal{L}_{\mathrm{M}}(t-s)}\circ\mathcal{R}[\eta(s)]\right] \\ = -\gamma\int_0^t \mathrm{d}s\,\mathcal{R}\circ\mathrm{e}^{(\mathcal{L}_{\mathrm{M}}-\gamma)(t-s)}\circ\mathcal{R}[\eta(s)],\end{aligned}$$

$$\mathcal{K}_1(t,s) = \gamma^2\,\mathbb{E}_{\mathbb{Q}}[X(t)X(s)]\mathcal{R}\circ\mathrm{e}^{\mathcal{L}_{\mathrm{M}}(t-s)}\circ\mathcal{R} = \frac{\gamma}{2}\,\mathcal{R}\circ\mathrm{e}^{(\mathcal{L}_{\mathrm{M}}-\gamma)(t-s)}\circ\mathcal{R}.$$

Finally, the approximation of the non Markovian master equation turns out to be

$$\dot{\eta}(t) \simeq -\mathrm{i}[H_0, \eta(t)] - \frac{1}{2}[L, [L, \eta(t)]] \\ + \frac{\gamma}{2}\int_0^t \mathrm{d}s \left[L, \mathrm{e}^{(\mathcal{L}_{\mathrm{M}}-\gamma)(t-s)}\big[\,[L, \eta(s)]\big]\right]. \quad (33)$$

However, we have no results on the positivity preserving properties of such an approximate evolution equation, while the unravelling of the complete equation guarantees complete positivity and feasibility of numerical simulations.

Acknowledgments

PDT thanks his PhD advisor, Prof. Dr. Hans-Jurgen Engelbert, and acknowledges the financial support of the Marie Curie Initial Training Network (ITN), FP7-PEOPLE-2007-1-1-ITN, no.213841-2, Deterministic and Stochastic Controlled Systems and Applications.

CP acknowledges the financial support of the ANR "Hamiltonian and Markovian Approach of Statistical Quantum Physics" (A.N.R. BLANC no ANR-09-BLAN-0098-01).

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