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SURVIVAL, EXTINCTION AND APPROXIMATION OF DISCRETE-TIME BRANCHING RANDOM WALKS

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Abstract. We consider a general discrete-time branching random walk on a countable set X. We relate local and global survival with suitable inequalities involving the first-moment matrix M of the process. In particular we prove that, while the local behavior is characterized by M, the global behavior cannot be completely described in terms of properties involving M alone. Moreover we show that locally surviving branching random walks can be approximated by sequences of spatially confined and stochastically dominated branching random walks which eventually survive locally if the (possibly finite) state space is large enough. An analogous result can be achieved by approximating a branching random walk by a sequence of multitype contact processes and allowing a sufficiently large number of particles per site. We compare these results with the ones obtained in the continuous-time case and we give some examples and counterexamples.

Keywords: branching random walk, branching process, percolation, multitype contact process.
AMS subject classification: 60J05, 60J80.

1. Introduction

The theory of branching random walks (BRWs from now on) has a long history dating back to the earlier works on branching processes (see [5] for the original work of Galton and Watson and [8]). In the last 20 years much effort has been put in the study of continuous-time BRWs (see [10, 11, 12, 15, 16] just to name a few). Many interesting phenomena have been detected regarding the distinction between local and global survival (see for instance [18, 20, 1, 2]) or the relation between multitype contact processes and BRWs (see [3]) and some papers have explored the subject of continuous-time BRWs on random environments (see for instance [7, 21]).

In recent years there has been a growing interest on discrete-time BRWs on deterministic graphs and on random environments (see [4, 9, 13, 14, 17]). Indeed, any continuous-time BRW admits a discrete-time counterpart with the same behavior (see Section 2.2), thus, under a certain point of view, discrete-time BRWs generalize continuous-time BRWs. Discrete-time BRWs have been explicitly studied in some papers and implicitly used in some others (see for instance [1, 2]). The aim of this paper is threefold: we want to study the global and local behavior of discrete-time BRWs, the possibility of approximating BRW with a sequence of "spatially confined" and stochastically dominated BRWs and, finally, the approximation of a BRW by means of a sequence of *multitype contact processes* (that we call *truncated BRWs*). The results of this paper generalize those of [1, 2, 3] not only because the class of discrete-time BRWs extends the class of continuous-time BRWs but also since some of the theorems are stronger and require weaker hypotheses.

The importance of studying fairly simple (and non-interacting) models such as BRWs relies on the fact that they are fundamental tools in the journey to understanding more sophisticated models: for instance, they are frequently used as comparison to prove survival or extinction of different particle systems (using the well-known technique called *coupling*).

Here is the outline of the paper. In Section 2 we define discrete-time BRWs and discuss their main properties. In Section 2.2 we briefly introduce continuous-time BRWs and we construct their discrete-time counterparts. In Section 3 we give the technical definitions and we state some basic results. Section 4 is devoted to the study of local and global survival. The main result (Theorem 4.1) characterizes local survival by means of the first-moment matrix M of the process (see Section 2.1), and global survival using a (possibly infinite dimensional) generating function associated to the BRW. We show that, in general, global survival cannot be characterized in term of the first-moment matrix alone (see Example 4.4), nevertheless some functional inequalities (involving only the first-moment matrix M) must hold in case of global survival (see Theorem 4.1). We introduce a class of fairly regular BRWs (which includes BRWs on quasi-transitive graphs and BRWs on regular graphs) for which we can give a complete characterization of global survival in terms of the matrix M. In Section 5 we first generalize a Theorem due to Sarymshakov and Seneta (see [19, Theorem 6.8]) and then we use this result to obtain an approximation of a general BRW by means of a sequence of spatially confined BRWs (Theorem 5.3). Here we show in particular that, if we have a surviving process, then by confining it to a sufficiently large (possibly finite) subgraph the resulting BRW survives as well. At the end of the section we give some examples and counterexamples. Section 6 deals with the approximation of the BRW with a sequence of truncated BRWs which are, in fact, multitype contact processes. The key to obtain such a result is the comparison of our process with a suitable oriented percolation (as explained in Section 6.1). The strategy is then applied to some classes of regular BRWs in Theorem 6.5 (concerning local behavior) and Theorem 6.7 (concerning global behavior). Finally in Section 7 we briefly discuss some open questions and possible future developments.

2. The dynamics: discrete and continuous time

2.1. **Discrete-time branching random walks.** We start with the construction of the general discrete BRW (see also [2] where it is called *infinite-type branching process*) on a set X which is at most countable. To this aim we consider a general family $\mu = \{\mu_x\}_{x \in X}$ of probability measures on $S_X := \{f : X \in \mathbb{N} : \sum_y f(y) < \infty\}$. The updating rule is the following: a particle at a site $x \in X$ lives one unit of time, then, with probability μ_x, a function $f : X \to \mathbb{N}$ is chosen and the original particle is replaced by $f(y)$ particles at y, for all $y \in X$; this is done independently for all the particles.

Here is another equivalent dynamics: let us we define the function $H : S_X \to \mathbb{N}$ as $H(f) := \sum_{x \in X} f(x)$ and denote by ρ_x the measure on $\mathbb{N}$ defined by $\rho_x(\cdot) := \mu_x(H^{-1}(\cdot))$; this is the law of the (random) number of children of every particle living at x. For each particle (independently) we pick a number n at random, according to the law ρ_x, and then we choose at random a function $f \in H^{-1}(n)$ with probability $\mu_x(f)/\rho_x(n) \equiv \mu_x(f)/\sum_{g\in H^{-1}(n)} \mu_x(g)$.

More precisely, given a family $\{f_{i,n,x}\}_{i,n\in\mathbb{N},x\in X}$ of independent S_X-valued random variable such that, for every $x \in X$, $\{f_{i,n,x}\}_{i,n\in\mathbb{N}}$ have the common law μ_x, then the discrete-time BRW $\{\eta_n\}_{n\in\mathbb{N}}$ is defined iteratively as follow

$$\eta_{n+1}(x) = \sum_{y\in X} \sum_{i=1}^{\eta_n(y)} f_{i,n,y}(x) = \sum_{y\in X} \sum_{j=0}^{\infty} \mathbb{1}_{\{\eta_n(y)=j\}} \sum_{i=1}^{j} f_{i,n,y}(x) \tag{2.1}$$

starting from an initial condition η_0.

One can show that this frame includes every *infinite-type* BRW as well. Indeed, consider a process where each particle on X is labeled with some $i \in I$ (where I is an at-most-countable set) and it can possibly have offsprings carrying different labels: this process can be seen as a discrete BRW on $X \times I$. Moreover, our model is time-homogeneous, nevertheless any *time-inhomogeneous* BRW on X can be seen as a particular case of a discrete time BRW on $X \times \mathbb{N}$ (simply by coupling time and space).

Define the expected number of particles from x to y (that is, the expected number of children that a particle living at x can send to y) by $m_{xy} := \sum_{f\in S_X} f(y)\mu_x(f)$ and suppose that $\sup_{x\in X} \sum_{y\in X} m_{xy} < +\infty$; most of the results of this paper still hold without this hypothesis, nevertheless it allows us to avoid dealing with an infinite expected number of offsprings.

We denote the *first-moment* matrix by $M = (m_{xy})_{x,y\in X}$ and by $m^{(n)}_{xy}$ the entries of the matrix M^n. We call *diffusion matrix* the matrix P with entries $p(x,y) = m_{xy}/\sum_{w\in X} m_{xw}$.

From equation (2.1), it is straightforward to prove that the expected number of particles, starting from an initial condition η_0, satisfies the recurrence equation $\mathbb{E}^{\eta_0}(\eta_{n+1}(x)) = (\mathbb{E}^{\eta_0}(\eta_n)M)(x) = \sum_{y\in X} m_{yx}\mathbb{E}^{\eta_0}(\eta_n(y))$ hence

$$\mathbb{E}^{\eta_0}(\eta_n(x)) = \sum_{y\in X} m^{(n)}_{yx}\mathbb{E}^{\eta_0}(\eta_0(y)). \tag{2.2}$$

Moreover, the family of probability measures, $\{\mu_x\}_x$ induces in a natural way a graph structure on X that we denote by (X, E_μ) where $E_\mu := \{(x,y) : m_{xy} > 0\} \equiv \{(x,y) : \exists f \in S_X, \mu_x(f) > 0, f(y) > 0\}$. Roughly speaking, (x,y) is and edge if and only if a particle living at x can send a child at y with positive probability (from now on *wpp*). We say that there is a path from x to y, and we write $x \to y$, if it is possible to find a finite sequence $\{x_i\}_{i=0}^n$ such that $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in E_\mu$ for all $i = 0, \ldots, n-1$.

Recall that the matrix $M = (m_{xy})_{x,y\in X}$ is said to be *irreducible* if and only if the graph (X, E_μ) is *connected*, otherwise we call it *reducible*. We denote by $\deg(x)$ the degree of a vertex x, that is, the cardinality of the set $\{y \in X : (x,y) \in E_\mu\}$. We denote by $d(x,y)$ the "distance" from x to y (not necessarily symmetric) by $d(x,y) = \inf\{n \in \mathbb{N} : x \to y\}$ (the distance $d(A,B)$ between two subsets of X is defined, as usual, as $\inf_{x\in A, y\in B} d(x,y) \equiv \min_{x\in A, y\in B} d(x,y)$).

The survival of the colony can be local or global: we say that the colony survives locally wpp at $y \in X$ starting from $x \in X$ if

$$\mathbb{P}(\limsup_{n\to\infty} \eta_n(y) > 0 | \eta_0 = \delta_x) > 0;$$

we say that it survives globally wpp starting from x if

$$\mathbb{P}\Big(\sum_{x\in X} \eta_n(x) > 0, \forall n \in N | \eta_0 = \delta_x\Big) > 0.$$

From now on when we talk about survival, "wpp" will be tacitly understood. Often we will say simply that local survival occurs "starting from x" or "at x": in this case we mean that $x = y$.

Clearly local survival implies global survival and, if $x \to y$ (that is, there exists a path in (X, E_μ) from x to y), then local survival at x implies local survival at y starting from x. Analogously, if $x \to y$ then global survival starting from y implies global survival starting from x. Moreover if $x \to y$ and $y \to x$ then local (resp. global) survival starting from x is equivalent to local (resp. global) survival starting from y. In particular, if M is irreducible then the process survives locally (resp. globally) at one vertex if and only if it survives locally (resp. globally) at every vertex.

A particular (but meaningful) discrete-time BRW is described by the following updating rule: a particle at site x lives one unit of time, and is replaced by a random number of children, with law ρ_x. The children are dispersed independently on the sites of the graph, according to a stochastic matrix P. Note that this rule is a particular case of the general one, since here one simply chooses $\mu_x(f)$ to be

$$\mu_x(f) = \rho_x\left(\sum_y f(y)\right) \frac{\sum_y f(y)!}{\prod_y f(y)!} \prod_y (p(x,y))^{f(y)}. \tag{2.3}$$

Clearly in this case

$$m_{xy} = p(x,y)\bar{\rho}_x \tag{2.4}$$

where $\bar{\rho}_x := \sum_{i\in\mathbb{N}} i\rho_x(i)$ is the expected number of children of a particle living at x.

2.2. **Continuous-time branching random walks.** Continuous-time BRWs have been studies extensively by many authors; in this section we show that there is a correspondence between continuous-time BRWs and discrete-time BRWs which preserves both local and global behaviors.

In continuous time each particle has an exponentially distributed random lifetime with parameter 1. The breeding mechanisms can be regulated by putting on each edge xy and for each particle at x, a clock with $Exp(\lambda k_{xy})$-distributed intervals (where $\lambda > 0$), each time the clock rings the particle breeds in y. Equivalently one can associate to each particle at x a clock with $Exp(\lambda k(x))$-distributed intervals ($k(x) = \sum_y k_{xy}$): each time the clock rings the particle breeds and the offspring is placed at random according to a stochastic matrix P (where $p(x,y) = k_{xy}/k(x)$).

The formal construction of a BRW in continuous time is based on the action of a semigroup with infinitesimal generator

$$\mathcal{L}f(\eta) := \sum_{x\in X} \eta(x)\Big(\partial_x^- f(\eta) + \lambda \sum_{y\in X} k_{xy}\, \partial_y^+ f(\eta)\Big), \tag{2.5}$$

where $\partial_x^{\pm} f(\eta) := f(\eta \pm \delta_x) - f(\eta)$.

Disregarding the time scale, the continuous time BRW can be seen as a discrete time one, in the sense that each continuous-time BRW has a discrete counterpart and they both survive or both die (locally or globally). This is the construction of the discrete counterpart of the continuous-time BRW with infinitesimal generator given by equation (2.5). The original particles represent the generation 0 of the discrete-time BRW; the generation $n+1$ (for all $n \geq 0$) is obtained by considering the children of all the particles of generation n (along with their positions). Clearly the progeny of the discrete-time BRW with this choice of f is infinite if and only if the progeny of the original continuous-time BRW is. In this sense the theory of continuous-time BRWs, as long as we consider the computation of the probability of survival (both local and global) is a particular case of the theory of discrete-time BRWs.

Elementary calculations show that each particle living at x, before dying, has a random number of offsprings given by equation (2.3) where

$$\rho_x(i) = \frac{1}{1+\lambda k(x)} \left(\frac{\lambda k(x)}{1+\lambda k(x)}\right)^i, \qquad p(x,y) = \frac{k_{xy}}{k(x)}, \tag{2.6}$$

and this is the law of the discrete counterpart. Using equation 2.4, it is straightforward to show that $m_{xy} = \lambda k_{xy}$.

Given $x_0 \in X$, two critical parameters are associated to the continuous-time BRW: the global (or weak) survival critical parameter $\lambda_w(x_0)$ and the local (or strong) survival one $\lambda_s(x_0)$. They are defined as

$$\begin{aligned} \lambda_w(x_0) &:= \inf\{\lambda > 0 : \mathbb{P}^{\delta_{x_0}}\left(\exists t : \eta_t = \mathbf{0}\right) < 1\} \\ \lambda_s(x_0) &:= \inf\{\lambda > 0 : \mathbb{P}^{\delta_{x_0}}\left(\exists \bar{t} : \eta_t(x_0) = 0,\, \forall t \geq \bar{t}\right) < 1\}, \end{aligned} \tag{2.7}$$

where $\mathbf{0}$ is the configuration with no particles at all sites and $\mathbb{P}^{\delta_{x_0}}$ is the law of the process which starts with one individual in x_0. If the graph (X, E_μ) is connected then these values do not depend on the initial configuration, provided that this configuration is finite (that is, it has only a finite number of individuals), nor on the choice of x_0. See [1] and [2] for a deep discussion on the values of $\lambda_w(x_0)$ and $\lambda_s(x_0)$.

3. Technical definitions

In this section we give some technical definitions and we state some basic facts which are widely used in the rest of the paper.

3.1. **Reproduction trails.** A fundamental tool which is useful throughout the whole paper is the *reproduction trail*; this allows us to give an alternative construction of the BRW. We fix an injective map $\phi : X \times X \times \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$. Let the family $\{f_{i,n,x}\}_{i,n\in\mathbb{N},x\in X}$ be as in Section 2.1 and let η_0 be the initial value. For any fixed realization of the process we call *reproduction trail* to $(x, n) \in X \times \mathbb{N}$ a sequence

$$(x_0, i_0, 1), (x_1, i_1, j_1), \dots, (x_n, i_n, j_n) \tag{3.8}$$

such that $-\eta_0(x_0) \le i_0 < 0$, $0 < j_l \le f_{i_{l-1},l-1,x_{l-1}}(x_l)$ and $\phi(x_{l-1}, x_l, i_{l-1}, j_l) = i_l$ (where $0 < l \le n$). The interpretation is the following: i_n is the identification number of the particle, which lives at x_n at time n and is the j_n-th offspring of its parent. The sequence $\{x_0, x_1, \dots, x_n\}$ is the path induced by the trail or, sometimes, we say that the trail is based on this path. Given any element (x_l, i_l, j_l) of the trail (3.8), we say that the particle identified by i_n is a descendant of generation $n - l$ of the particle identified by i_l and the trail joining them is $(x_l, i_l, j_l), \dots, (x_n, i_n, j_n)$. We say also that the trail of the particle i_n is a prolongation of the trail of the particle i_l.

Roughly speaking the trail represents the past history of each single particle back to its original ancestor (the one living at time 0); we note that from the couple (n, i_n) we can trace back the entire genealogy of the particle. The random variable $\eta_n(x)$ can be defined as the number of reproduction trails to (x, n). This construction does not coincide with the one induced by the equation (2.1) but they have the same laws.

3.2. **Generating functions.** Later on we need some generating functions (both 1-dimensional and infinite dimensional). Define $T^n_x := \sum_{y\in X} m^{(n)}_{xy}$ and $\varphi^{(n)}_{xy} := \sum_{x_1,\dots,x_{n-1}\in X\setminus\{y\}} m_{xx_1}m_{x_1x_2}\cdots m_{x_{n-1}y}$ (by definition $\varphi^{(0)}_{xy} := 0$ for all $x, y \in X$). Clearly T^n_x is the expected number of elements of the n-th generation in the progeny of a particle living at x; on the other hand if we select, in the n-th generation of a particle living at x, only the particles living at y with no ancestors in the line (up to the original particle at x) living at y we have a (random) set whose expected cardinality is $\varphi^{(n)}_{xy}$.

Let us consider the following family of 1-dimensional generating functions (depending on $x, y \in X$)

$$\Gamma(x, y|\lambda) := \sum_{n=0}^{\infty} m^{(n)}_{xy}\lambda^n, \qquad \Phi(x, y|\lambda) := \sum_{n=1}^{\infty} \varphi^{(n)}_{xy}\lambda^n.$$

It is easy to prove that $\Gamma(x, x|\lambda) = \sum_{i\in\mathbb{N}} \Phi(x, x|\lambda)^i$ for all $\lambda > 0$, hence

$$\Gamma(x, x|\lambda) = \frac{1}{1 - \Phi(x, x|\lambda)}, \qquad \forall \lambda \in \mathbb{C} : |\lambda| < \left(\limsup_{n\in\mathbb{N}} \sqrt[n]{m^{(n)}_{xy}}\right)^{-1}, \tag{3.9}$$

and, clearly, we have that $\left(\limsup_{n\in\mathbb{N}} \sqrt[n]{m^{(n)}_{xy}}\right)^{-1} = \max\{\lambda \in \mathbb{R} : \Phi(x, x|\lambda) \le 1\}$ for all $x \in X$. In particular $\Phi(x, x|1) \le 1$ if and only if $\limsup_{n\in\mathbb{N}} \sqrt[n]{m^{(n)}_{xy}} \le 1$.

To the family $\{\mu_x\}_{x\in X}$ we can associate another generating function $G : [0, 1]^X \to [0, 1]^X$ which can be considered as an infinite dimensional power series (see also [2]). More precisely, for all

$z \in [0,1]^X$ the function $G(z) \in [0,1]^X$ is defined as follows

$$G(z|x) := \sum_{f \in S_X} \mu_x(f) \prod_{y \in X} z(y)^{f(y)}. \tag{3.10}$$

These generating functions will be useful in Section 4 to prove Theorem 4.1 and to discuss Examples 4.4 and 4.5.

Note that the generating function G can be explicitly computed, for instance, if equation (2.3) holds. Indeed in this case it is straightforward to show that $G(z|x) = F(Pz(x))$ where $F(y) = \sum_{n=0}^{\infty} \rho(n) y^n$ and $Pz(x) = \sum_{y \in X} p(x,y) z(y)$. In particular if $\rho(n) = \frac{1}{1+\bar{\rho}_x} (\frac{\bar{\rho}_x}{1+\bar{\rho}_x})^n$ (for instance if we are dealing with the discrete counterpart of a continuous-time BRW, see equation (2.6)), we have $G(z|x) = \frac{1}{1+\bar{\rho}_x(1-Pz(x))}$, that is,

$$G(z) = \frac{1}{1 + M(\mathbf{1} - z)}. \tag{3.11}$$

where $\mathbf{1}(x) := 1$ for all $x \in X$ and $Mv(x) := \sum_{y \in X} m_{xy} v(y)$ for all $v \in [0,1]^X$ (and m_{xy} is given by equation (2.4)).

3.3. **Coupling.** The family of BRWs can be extended to the more general class of truncated BRWs where a maximum of $m \in \mathbb{N} \cup \{\infty\}$ particles per site are allowed (we denote this process as a BRW$_m$). the general dynamics is given by the following recursive relation

$$\eta^m_{n+1}(x) = m \wedge \sum_{y \in X} \sum_{i=1}^{\eta^m_n(y)} f_{i,n,y}(x) = m \wedge \sum_{y \in X} \sum_{j=0}^{\infty} \mathbb{1}_{\{\eta^m_n(y)=j\}} \sum_{i=1}^{j} f_{i,n,y}(x). \tag{3.12}$$

Clearly the BRW$_\infty$ is the usual BRW.

In the following sections we want to compare two (or more) truncated BRWs. More precisely, suppose we have two families $\mu = \{\mu_x\}_{x \in X}$ and $\nu = \{\nu_x\}_{x \in X}$ such that $\mu_x(F_x^{-1}(\cdot)) = \nu_x(\cdot)$ for all $x \in X$ and for some family of functions $\{F_x\}_{x \in X}$ such that $F_x : \mathrm{supp}(\mu_x) \to \mathrm{supp}(\nu_x)$ and $F_x(f) \le f$ for all $f \in \mathrm{supp}(\mu_x)$. Then, given $k \le m \le \infty$, it is possible to construct a process $\{(\eta^m_n, \xi^k_n)\}_{n \in \mathbb{N}}$ such that

(1) $\{\eta^m_n\}_{n \in \mathbb{N}}$ is a BRW$_m$ behaving according to μ;
(2) $\{\xi^k_n\}_{n \in \mathbb{N}}$ is a BRW$_k$ behaving according to ν;
(3) $\eta_0 \ge \xi_0$ implies $\eta_n \ge \xi_n$ for all $n \in \mathbb{N}$ a.s.

Indeed for any $x \in X$, given the family of random variables $\{f_{i,n,x}\}_{i,n \in \mathbb{N}}$ (with law μ_x) then $\{F_x(f_{i,n,x})\}_{i,n \in \mathbb{N}}$ are iid with common law ν_x. Whence the evolution equation of $\{\eta_n\}_{n \in \mathbb{N}}$ is (3.12) and, similarly, $\{\xi_n\}_{n \in \mathbb{N}}$ satisfies

$$\xi^k_{n+1}(x) = k \wedge \sum_{y \in X} \sum_{i=1}^{\xi_n(y)} F_x \circ f_{i,n,y}(x). \tag{3.13}$$

It is easy to show by induction, using equations (3.12) and (3.13), that $\eta^m_0 \ge \xi^k_0$ implies $\eta^m_n \ge \xi^k_n$ for all $n \in \mathbb{N}$. A typical choice for the family of functions $\{F_x\}_{x \in X}$ is $F_x(f) := f|_Y$ (where $Y \subseteq X$)

which can be seen as a (truncated) BRW restricted to Y, that is, all the offsprings sent outside Y are killed.

We call this procedure of comparison a *coupling* between $\{\eta_n^m\}_{n\in\mathbb{N}}$ and $\{\xi_n^k\}_{n\in\mathbb{N}}$. We note that if $\{\eta_n^m\}_{n\in\mathbb{N}}$ dies out locally (resp. globally) a.s. then $\{\xi_n^k\}_{n\in\mathbb{N}}$ dies out locally (resp. globally) a.s.

More generally a *coupling* between $\{\eta_n^m\}_{n\in\mathbb{N}}$ and $\{\xi_n^k\}_{n\in\mathbb{N}}$ is a choice of a common law $\{\zeta_x\}_{x\in X}$ for the BRW $\{(\eta_n^m, \xi_n^k)\}_{n\in\mathbb{N}}$ such that $\zeta_x((f,g) : f \geq g, f,g \in S_X) = 1$ for all $x \in X$ and $\sum_{g\in S_X} \zeta_x((f,g)) = \mu_x(f)$, $\sum_{f\in S_X} \zeta_x((f,g)) = \nu_x(g)$. In many situations this construction of $\{\zeta_x\}_{x\in X}$ can be carried out effortlessly.

4. Local and global survival

Consider the discrete time BRW generated by the family $\mu = \{\mu_x\}_x$ of probabilities and suppose now that the process starts with one particle at x_0, hence $\eta_0 = \delta_{x_0}$. In this section we want to find conditions for global and local survival. Recall that if X is finite then local survival is equivalent to global survival: this is trivial for an irreducible matrix M; in the general case global survival, starting from x_0, is equivalent to local survival at some $y \in X$ (the same arguments of [2, Remark 4.4] apply).

Theorem 4.1. *Let (X,μ) be a BRW.*

1. *There is local survival starting from x_0 if and only if* $\limsup_{n\to\infty} \sqrt[n]{m_{x_0x_0}^{(n)}} > 1$.
2. *There is global survival starting from x_0 if and only if there exists $z \in [0,1]^X$, $z(x_0) < 1$, such that $G(z|x) \leq z(x)$, for all x.*
3. *If there is global survival starting from x_0, then there exists $v \in [0,1]^X$, $v(x_0) > 0$, such that*
   a) $Mv \geq v$
   b) *for all x, $Mv(x) = v(x)$ if and only if* $G(\mathbf{1} - (1-t)v; x) = 1 - (1-t)v(x), \forall t \in [0,1]$.
4. *If there is global survival starting from x_0, then* $\liminf_{n\in\mathbb{N}} \sqrt[n]{\sum_{x\in X} m_{x_0x}^{(n)}} \geq 1$.
5. *If X is finite there is global survival starting from x_0 if and only if* $\liminf_{n\in\mathbb{N}} \sqrt[n]{\sum_{x\in X} m_{x_0x}^{(n)}} > 1$.

Proof.

(1) Fix $x_0 \in X$, consider a path $\Pi := \{x_0, x_1, \ldots, x_n = x_0\}$ and define its number of cycles $\mathbb{L}(\Pi) := |\{i = 1, \ldots, n : x_i = x_0\}|$; the expected number of trails based on such a path is $\prod_{i=0}^{n-1} m_{x_ix_{i+1}}$. This is the expected number of particles living at x_0, descending from the original particle at x_0 and whose genealogy is described by the path Π, that is, their mothers were at x_{n-1}, their grandmothers at x_{n-2} and so on). We associate to the BRW a Galton-Watson branching process (with a different time scale): given any particle p in x_0 (corresponding to a trail with n cycles), define its children as all the particles whose trail

is a prolongation of the trail of p and is associated with a spatial path with $n+1$ cycles. Hence a particle belongs to the k-th generation if and only if the corresponding trail is based on a path with k cycles; moreover it has one (and only one) parent in the $(k-1)$-th generation. Since each particle behaves independently of the others then this branching process is markovian. It is clear that the BRW survives locally if and only if this branching process does.

The expected number of children of each particle of this new branching process is the sum over n of the expected number of trails based on paths of length n and having only one cycle, that is, $\sum_{n=1}^{\infty} \varphi^{(n)}_{x,x} = \Phi(x,x|1)$. Thus we have a.s. local extinction if and only if $\Phi(x,x|1) \leq 1$, that is, $\limsup_{n \in \mathbb{N}} \sqrt[n]{m^{(n)}_{xy}} \leq 1$.

(2) Let $q_n(x)$ and $q(x)$ be the probability of global extinction before or at the n-th generation and the probability of global extinction respectively, starting from a single initial particle at x. Clearly $q_{n+1} = G(q_n)$ and $q_n(x) \to q(x)$ as $n \to +\infty$. Since G is nondecreasing on the partially ordered set $[0,1]^X$, it is easy to show that $q(x_0) < 1$ if and only if there exists $v \in [0,1]^X$ such that $v(x_0) < 1$ and $G(v) \leq v$ (see also [2, Section 3] for more details).

(3) Let z such that $G(z) \leq z$, $z(x_0) < 1$. Define $v = \mathbf{1} - z_0$, take the derivative of the convex function $\phi(t) := G(\mathbf{1} - (1-t)v; x) - 1 + (1-t)v(x)$ at $t = 1$ and remember that $\phi(0) \leq \phi(1) = 0$.

(4) If there is global survival starting from $x_0 \in X$ then there exists $v \in [0,1]^X$ such that $v(x_0) > 0$ and $Mv \geq v$. Hence $M^n v \geq v$ for all $n \in \mathbb{N}$, that is, $\sum_{y \in X} m^{(n)}_{xy} v(y) \geq v(x)$; in particular, $\sum_{y \in X} m^{(n)}_{x_0 y} \geq v(x_0) > 0$ and this implies $\liminf_{n \to \infty} \sqrt[n]{\sum_y m^{(n)}_{x_0 y}} \geq 1$.

(5) Since X is finite there is global survival starting from x_0 if and only if there is local survival starting from some $x \in X$ such that $x_0 \to x$, thus there is global survival starting from x_0 if and only if there exists $x \in X$ such that $\limsup_{n \in \mathbb{N}} \sqrt[n]{m^{(n)}_{ww}} > 1$. Since M is finite, it is easy to show that

$$\liminf_{n \in \mathbb{N}} \sqrt[n]{\sum_{x \in X} m^{(n)}_{x_0 x}} = \max_{w \in X : x \to w} \limsup_{n \in \mathbb{N}} \sqrt[n]{m^{(n)}_{ww}}$$

whence there is global survival if and only if $\liminf_{n \in \mathbb{N}} \sqrt[n]{\sum_{x \in X} m^{(n)}_{x_0 x}} > 1$.

□

Theorem 4.1 extends [2, Theorems 4.1, 4.3 and 4.7]. Indeed in term of survival, studying a continuous-time BRW with rates $\{\lambda k_{xy}\}_{x,y \in X}$ is equivalent to studying its discrete counterpart (that is, a BRW where $\{\mu_x\}_{x \in X}$ is given by equations (2.3) and (2.6)). Moreover, according to [2, Theorem 4.2](c), for the discrete counterpart of a continuous-time BRW, global survival starting from x_0 is equivalent to the existence of $v \in [0,1]^X$, $v(x_0) > 0$, such that $Mv \geq v$. This is a necessary condition for global survival for all discrete-time BRWs. Finally, the proof of part (1) of the previous theorem (which holds as well even if $\{m_{xy}\}$ is unbounded or $m_{xy} = +\infty$ for some $x, y \in X$) is a natural adaptation of the proofs of [1, Theorem 3.1] and [2, Theorems 4.1 and 4.7]; an independent proof was given in [17, Theorem 2.4] using a different (analytic) technique.

Speaking of global survival, it is easy to show that, given any solution of $G(z) \leq z$, then $z(x)$ is an upper bound for the probability of extinction, say $q(x)$. Moreover the existence of a solution as in Theorem 4.1(2) is equivalent to the existence of a solution of $G(z) = z$ such that $z(x_0) < 1$. In particular one can prove that if q is the (infinite-dimensional) vector of extinction probabilities then q is the smallest solution of $G(z) = z$; this is clearly a particular solution of $G(z) \leq z$ where $z(x) < 1$ simultaneously for all x such that there is global solution starting from x. Thus if a BRW is irreducible and there is global survival starting from one vertex then the solution q satisfies $q(x) < 1$ for all $x \in X$. for a more detailed discussion on the generating function G and its properties we refer to [2, Sections 2 and 3].

We call a BRW on X *locally critical* at $x_0 \in X$ if and only if $\limsup_{n\to\infty} \sqrt[n]{m^{(n)}_{x_0x_0}} = 1$. According to Theorem 4.1(1) any locally critical (at x_0) BRW which starts with one particle at x_0 dies out locally. One is tempted to give an analogous definition for the global behavior using $\liminf_{n\in\mathbb{N}} \sqrt[n]{\sum_{x\in X} m^{(n)}_{x_0x}}$ but, as we show in Example 4.4, this is not the case (see also Examples 2 and 3 of [2]).

We observe that, according to Theorem 4.1(1) the local survival depends only on M, hence if we have two BRWs, say (X,μ) and (X,ν) with first-moment matrices M and $\bar{M}$ respectively, satisfying $m_{xy} \geq \bar{m}_{xy}$ (for all $x, y \in X$) then the local survival at x_0 for (X,ν) implies the the local survival at x_0 for (X,μ). Later on we show that, for a general BRW, the global survival does not depend only on M (see Example 4.4) nevertheless a characterization of global survival in terms of M holds for special classes of BRWs. The first example is given by the class of discrete counterparts of continuous-time BRWs and this is due to Theorem 4.1(2) and equation (3.11). Another class is described by the following result.

Definition 4.2. *We say that a BRW (X,μ) is locally isomorphic to a BRW (Y,ν) if there exists a surjective map $g : X \to Y$ such that*

$$\nu_{g(x)}(f) = \mu_x\left(h : \forall y \in Y, f(y) = \sum_{z\in g^{-1}(y)} h(z)\right), \quad \forall f \in S_Y. \tag{4.14}$$

We say that (X,μ) is a $\mathcal{F}$-BRW if it is locally isomorphic to some BRW (Y,ν) on a finite set Y.

The idea behind the previous definition is that g acts like a projection from X onto Y and, from the point of view of the BRW, all the vertices in $g^{-1}(y)$ looks similar. We note that *quasi transitive* BRWs (see Section 6.2 for the formal definition) are $\mathcal{F}$-BRWs. Another example of an $\mathcal{F}$-BRW is given by a BRW satisfying equation (2.3) where ρ_x is independent of $x \in X$, say $\rho_x = \rho$ for all

$x \in X$; in this case one simply chooses $Y = \{0\}$, that is a branching process, with reproduction law ρ.

Clearly the map g induces a map $\pi_g : S_X \to S_Y$ defined as $\pi_g(f)(y) = \sum_{x \in g^{-1}(y)} f(x)$ hence equation (4.14) becomes $\nu_{g(x)}(\cdot) = \mu_x(\pi_g^{-1}(\cdot))$. Clearly if $\{\eta_n\}_{n\in\mathbb{N}}$ is a realization of (X,μ) then $\{\pi_g(\eta_n)\}_{n\in\mathbb{N}}$ is a realization of (Y,ν). Moreover it is easy to show that, for all $x \in X$, $j \in Y$, $\widetilde{m}_{g(x)j} := \sum_{w\in S_Y} w(g(x))\nu_j(w) = \sum_{f\in S_X} \sum_{y\in g^{-1}(j)} f(y)\mu_x(f) = \sum_{y\in g^{-1}(j)} m_{xy}$ (that is, $\widetilde{m}_{g(x)\,j} = \pi_g(m_{x\cdot})(j)$). This means that the expected number of offsprings at j of a particle living at $g(x)$ (on the projected BRW (Y,ν)) is the sum of all the expected numbers of offsprings at y of a particle living at x (on the BRW (X,μ)) over all $y \in X$ whose projection is j. Thus $\sum_{j\in Y} \widetilde{m}_{g(x)\,j} = \sum_{y\in X} m_{xy}$. By induction on $n \in \mathbb{N}$ one can prove that, for all $x \in X$, $j \in Y$, $n \in \mathbb{N}$, we have $\widetilde{m}^{(n)}_{g(x)j} = \sum_{y\in g^{-1}(j)} m^{(n)}_{xy}$ whence $\sum_{j\in Y} \widetilde{m}^{(n)}_{g(x)j} = \sum_{y\in X} m^{(n)}_{xy}$.

The following result characterizes the global survival for $\mathcal{F}$-BRWs in terms of M.

Theorem 4.3. *Let (X,μ) is locally isomorphic to (Y,ν) and consider the following:*

(1) there is global survival for (X,μ) starting from $x_0 \in X$,

(2) there is global survival for (Y,ν) starting from $g(x_0) \in Y$,

(3) $\liminf_{n\to\infty} \sqrt[n]{\sum_y m^{(n)}_{x_0 y}} > 1$*;*

then (1) $\iff$ (2)*. Moreover if Y is finite (hence X is an $\mathcal{F}$-BRW) then* (3) $\iff$ (2)*.*

Proof. (1) $\iff$ (2). Let $\{\eta_n\}_{n\in\mathbb{N}}$ the BRW on X and $\xi_n(y) := \sum_{x\in g^{-1}(y)} \eta_n(x)$ for all $y \in Y$. It is easy to show that $\{\xi_n\}_{n\in\mathbb{N}}$ is a BRW on Y behaving according to ν; moreover if $\eta_0 = \delta_{x_0}$ then $\xi_0 = \delta_{g(x_0)}$. Clearly $\{\eta_n\}_{n\in\mathbb{N}}$ survives globally if and only if $\{\xi_n\}_{n\in\mathbb{N}}$ does.

(1) $\iff$ (3). Since, for all $n \in \mathbb{N}$, we have $\sum_{z\in X} m^{(n)}_{xy} = \sum_{y\in Y} \widetilde{m}^{(n)}_{g(x)y}$ then

$$\liminf_{n\in\mathbb{N}} \sqrt[n]{\sum_{x\in X} m^{(n)}_{x_0 x}} = \liminf_{n\in\mathbb{N}} \sqrt[n]{\sum_{y\in Y} \widetilde{m}^{(n)}_{g(x_0)y}}.$$

The claim follows from Theorem 4.1(5) being Y finite. □

Since, within some classes, the global behavior can be characterized completely by M, one can wonder if the same holds for a general BRW or, alternatively, if two generic BRWs with the same first-moment matrix must have the same global behavior. In particular one could conjecture that at least one of the two necessary conditions given in Theorem 4.1(2) and (3) is sufficient. All these conjectures are false as the following example shows (the main tool is Theorem 4.1(2)).

Example 4.4. *Let $X = \mathbb{N}$ and consider the family of BRWs $(\mathbb{N},\mu)$ with $\mu_i = p_i \delta_{n_i \mathbb{1}_{\{i+1\}}} + (1-p_i)\delta_{\mathbf{0}}$ (where $\mathbb{1}_{\{i+1\}} \in S_\mathbb{N}$ is defined by $\mathbb{1}_{\{i+1\}}(x) = 1$ if $x = i+1$ and 0 otherwise). Roughly speaking, each particle at i has n_i children at $i+1$ with probability p_i and no children at all with probability $1-p_i$. According to Theorem 4.1(2) global survival starting from 0 is equivalent to the existence of*

$z \in [0,1]^{\mathbb{N}}$, $z(0) < 1$, such that $G(z|i) \le z(i)$, for all i where $G(z|i) = p_i z(i+1)^{n_i} + 1 - p_i$. Note that $pz^n + 1 - p \to 1$ if $p \to 0$ or $z \to 1$.

Clearly if $n_i = n$, $p_i = p$ and $np > 1$ the BRW survives globally (take for instance $n = 4$ and $p = 1/2$). Let us suppose that $p_i = 2/n_i$. We construct iteratively a sequence $\{n_i\}_{i \in \mathbb{N}}$ such that the unique solution of $G(z) \le z$ is $z(i) = 1$ for all $i \in \mathbb{N}$.

Clearly $G(z) \le z$ implies

$$\begin{cases} z(0) \ge \frac{2}{n_0} z(1)^{n_1} + 1 - \frac{2}{n_0} \\ z(1) \ge \frac{2}{n_1} z(2)^{n_2} + 1 - \frac{2}{n_1} \\ \dots \\ z(k) \ge \frac{2}{n_k} z(k+1)^{n_k} + 1 - \frac{2}{n_k} \\ z(k+1) \ge 1 - \frac{2}{n_{k+1}}. \end{cases} \tag{4.15}$$

for all $k \in \mathbb{N}$. Let $n_0 = 4$ and suppose we already fixed $\{n_i\}_{i=0}^k$. If $n_{k+1} \to \infty$ then a solution of equation (4.15) satisfies $z(i) \to 1$ for all $i \le k+1$. Choose n_{k+1} such that $z(i) \ge k/(k+1)$ for all $i \le k$. This implies that the unique solution of the family of systems (dependent on k) given by equation (4.15) is $z(i) = 1$ for all $i \in \mathbb{N}$. Thus this is the only solution of $G(z) \le z$ and the BRW does not survive globally a.s. This example shows in particular that $\liminf_{n\to\infty} \sqrt[n]{\sum_y m^{(n)}_{x_0 y}} > 1$ does not imply, in general, global survival.

The first-moment matrix of the BRW above is not irreducible and the BRW can be identified with a time-inhomogeneous branching process; a slight modification allows us to construct an irreducible BRW. We just sketch the main steps.

Again let $X = \mathbb{N}$ and consider the family of BRWs $\mu_i = p_i \delta_{n_i \mathbb{1}_{\{i+1\}} + \mathbb{1}_{\{i-1\}}} + (1-p_i)\delta_{\mathbf{0}}$ (for all $i \ge 1$) and $\mu_0 = p_0 \delta_{n_0 \mathbb{1}_{\{1\}}} + (1-p_0)\delta_{\mathbf{0}}$. In this case each particle at $i \ge 1$ has n_i children at i and 1 at $i-1$ with probability p_i and no children at all with probability $1-p_i$; each particle at 0 has the same behavior as in the previous example. The generating function G is

$$G(z|i) = \begin{cases} p_i z(i+1)^{n_i} z(i-1) + 1 - p_i & i \ge 1 \\ p_0 z(1)^{n_0} + 1 - p_0 & i = 0. \end{cases}$$

$G(z) \le z$ implies, for all k,

$$\begin{cases} z(0) \ge p_0 z(1)^{n_0} + 1 - p_0 \\ z(1) \ge p_1 z(2)^{n_1} z(0) + 1 - p_1 \\ \dots \\ z(k) \ge p_k z(k+1)^{n_k} z(k-1) + 1 - p_k \\ z(k+1) \ge 1 - p_{k+1}. \end{cases} \tag{4.16}$$

It is not difficult to prove that, if $p_{k+1} \to 0$ (and hence $z(k+1) \to 1$) then the set of solutions of equation (4.16) is eventually contained in any ε-enlargements of the set of vectors

$(z_0(1), z_0(2), \ldots, z_0(k), 1)$, *where* $(z_0(1), z_0(2), \ldots, z_0(k))$ *is ranging in the set of solutions of*

$$\begin{cases} z(0) \geq p_0 z(1)^{n_0} + 1 - p_0 \\ z(1) \geq p_1 z(2)^{n_1} z(0) + 1 - p_1 \\ \ldots \\ z(k-1) \geq p_{k-1} z(k)^{n_{k-1}} z(k-2) + 1 - p_{k-1} \\ z(k) \geq p_k z(k-1) + 1 - p_k. \end{cases} \tag{4.17}$$

Let us study this last equation. We note that if $n_i p_i p_{i+1} \leq (1-\varepsilon)/2$ *for all* $i \in \mathbb{N}$ *(for some* $\varepsilon > 0$*) then there is a unique solution of equation* (4.17)*, that is* $z(i) = 1$ *for all* $i = 0, \ldots, k$*. Indeed equation* (4.17) *represents the system* $\widetilde{G}(z) \leq z$ *for an irreducible BRW on* $\{0, 1, \ldots, k\}$ *where*

$$\widetilde{\mu}_i = \begin{cases} p_0 \delta_{n_0 \mathbb{1}_{\{1\}}} + (1-p_0)\delta_{\mathbf{0}} & \text{if } i = 0 \\ p_i \delta_{n_i \mathbb{1}_{\{i+1\}} + \mathbb{1}_{\{i-1\}}} + (1-p_i)\delta_{\mathbf{0}} & \text{if } i = 1, \ldots, k-1 \\ p_k \delta_{\mathbb{1}_{\{k-1\}}} + (1-p_k)\delta_{\mathbf{0}} & \text{if } i = k. \end{cases}$$

Indeed, since the graph is finite and connected, according to Theorem 4.1(2) and (5) there exists a solution $z \neq \mathbf{1}$ *of* $\widetilde{G}(z) \leq z$ *if and only if* $\liminf_{n\to\infty} \sqrt[n]{\sum_j \widetilde{m}_{ij}^{(n)}} > 1$ *for some (*$\Longleftrightarrow$ *for all)* $i \in \{0, 1, \ldots, k\}$*; but, again since the graph is finite, the previous conditions are equivalent to* $\limsup_{n\to\infty} \sqrt[n]{\widetilde{m}_{ii}^{(n)}} > 1$ *for some (*$\Longleftrightarrow$ *for all)* $i \in \{0, 1, \ldots, k\}$*. Elementary computations show that*

$$\widetilde{m}_{ii}^{(n)} \leq \begin{cases} \frac{1}{n+1}\binom{n+1}{n/2}\left(\frac{1-\varepsilon}{2}\right)^n \leq \binom{n}{n/2}\left(\frac{1-\varepsilon}{2}\right)^n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

(remember that $\widetilde{m}_{i\,i+1}\widetilde{m}_{i+1\,i} = p_i n_i p_{i+1} < (1-\varepsilon)/2$*) which implies* $\limsup_{n\to\infty} \sqrt[n]{\widetilde{m}_{ii}^{(n)}} \leq 1 - \varepsilon$*. This proves that the unique solution of equation* (4.17) *is* $z(i) = 1$ *for all* $i = 0, \ldots, k$*.*

As before, the trick to prove our goal is to choose the sequences $\{p_i\}_{i\in\mathbb{N}}$ *and* $\{n_i\}_{i\in\mathbb{N}}$ *such that* $p_i \to 0$ *fast enough and* $p_i n_i = 2$ *for all* $i \in \mathbb{N}$*. Note that if* $p_i = 2/n_i < (1-\varepsilon)/4$ *for all* $i \in \mathbb{N}$ *then* $p_{i+1} p_i n_i < (1-\varepsilon)/2$*.*

If $k = 1$ *then we can choose* n_1 *such that* $z(i) > 1/2$ *for all* $i \leq 1$*. Indeed if* $n_1 \to \infty$ *then* $p_1 \to 0$ *and both* $z(1), z(0) \to 1$*.*

Suppose we fixed $n_0, \ldots, n_k$ *such that any solution of equation* (4.16) *satisfies* $z(i) \geq k/(k+1)$ *for all* $i \leq k$ *and such that* $p_i < (1-\varepsilon)/4$ *for all* $i = 0, \ldots, k$*. If* $n_{k+1} \to \infty$ *then* $z(k+1) \to 1$ *hence any solution of equation* (4.16) *must converge as before to a solution of equation* (4.17)*. Hence* $n_{k+1} \to \infty$ *implies* $z(i) \to 1$ *for all* $i \leq k+1$ *and we can choose* n_{k+1} *such that* $z(i) \geq (k+1)/(k+2)$ *for all* $i \leq k+1$*. This yields the conclusion.*

Finally we note that if the BRW is given by $\mu_i = 1/2\,\delta_{4\mathbb{1}_{\{i+1\}}} + p_i \delta_{\mathbb{1}_{\{i-1\}}} + (1-p_i)\delta_{\mathbf{0}}$ *(for all* $i \geq 1$*) and* $\mu_0 = 1/2\delta_{4\mathbb{1}_{\{1\}}} + (1-p_0)\delta_{\mathbf{0}}$ *(where* p_i *is the same as before) then it survives globally, hence, even for irreducible BRWs, global survival does not depend only on the first-moment matrix* M *and* $\liminf_{n\to\infty} \sqrt[n]{\sum_y m_{x_0 y}^{(n)}} > 1$ *does not imply, in general, global survival.*

Another possible question arises from Theorem 4.1: is it true that $\sum_{y \in X} m_{xy} < 1$ for all $x \in X$ implies global extinction? According to the following example (see also [2, Example 1]), the answer is negative.

Example 4.5. *As before, we start by giving an example which is not irreducible, later on we modify the process in order to obtain an irreducible BRW.*

Let $X = \mathbb{N}$, $\{p_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1]$ and suppose that a particle at n has one child at $n+1$ with probability $1-p_n$ and no children with probability p_n (this is the reducible process of the previous example with $n_i = 1$ for all $i \in \mathbb{N}$). The generating function of this process is $\widetilde{G}(z|n) = 1-p_n+p_n z(n+1)$. Again this BRW can be identified with a time-inhomogeneous branching process which has a probability of survival (starting with one particle at n) $z(n) = \prod_{i=n}^{\infty} p_i$; hence it survives with positive probability, if and only if $\sum_{i=1}^{\infty} p_i < +\infty$. It is straightforward to check that z is a solution of $G(z) = z$.

This process is stochastically dominated by the (irreducible) BRW where each particle at $n \geq 1$ has one child at $n+1$ with probability p_n, one child at $n-1$ with probability $(1-p_n)/2$ (if $n=0$ then it has one child of type 0 with probability $(1-p_0)/2$) and no children at all with probability $(1-p_n)/2$. The generating function G can be explicitly computed

$$G(z|n) = \begin{cases} \frac{1-p_n}{2} + \frac{1-p_n}{2} z(n-1) + p_n z(n+1) & n \geq 1 \\ \frac{1-p_0}{2} + \frac{1-p_0}{2} z(0) + p_0 z(1) & n = 0. \end{cases}$$

By coupling this process with the previous one or, simply, by applying Theorem 4.1(2) ($z(n) = \prod_{i=n}^{\infty} p_i$ is a solution of $G(z) \leq z$) one can prove that $\sum_{i=1}^{\infty} p_i < +\infty$ implies global survival. Note that here $\sum_{j\in\mathbb{N}} m_{ij} = (1+p_i)/2 < 1$; clearly, $\liminf_{n\to\infty} \sum_{j\in\mathbb{N}} m_{ij}^{(n)} = 1$.

Analogous examples could be constructed for continuous time BRWs as well. for instance, an example of a continuous-time BRW which survives globally at the global critical point $\lambda = \lambda_w$ can be found in [2, Example 3].

5. Spatial approximation

5.1. **Generalization of a Theorem of Sarymshakov-Seneta.** Given a matrix $M = (m_{ij})_{i,j\in I}$ (where the set I is at most countable), recall the usual classification of indices of a matrix as described in [19, Chapter 1]. For any index i we denote by $[i]$ its *class*, that is, the set of indices which communicate with i. We define the convergence parameters $R(i,j) := \left(\limsup_{n\in\mathbb{N}} \sqrt[n]{m_{ij}^{(n)}}\right)^{-1}$ and $R := \inf_{i,j\in I} R(i,j)$; it is well known that $\limsup_{n\in\mathbb{N}} \sqrt[n]{m_{ij}^{(n)}} = \limsup_{n\in\mathbb{N}} \sqrt[n]{m_{i_1 j_1}^{(n)}}$ if $[i] = [i_1]$ and $[j] = [j_1]$; in particular it is independent of i, j if the matrix is irreducible.

Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} I_n = \mathbb{N}$ and denote by ${}_nR$ the convergence parameter of $M_n = (m_{i,j})_{i,j\in I_n}$; clearly, if the sequence $\{I_n\}_{n\in\mathbb{N}}$ is nondecreasing, we have that ${}_nR \geq {}_{n+1}R$. The following theorem generalizes [19, Theorem 6.8] (note that the submatrices $\{M_n\}_{n\in\mathbb{N}}$ are not necessarily irreducible); it is the key to prove our main result about spatial approximation (Theorem 5.3).

Theorem 5.1. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a nondecreasing sequence of subsets of $\mathbb{N}$. If $M = (m_{i,j})_{i,j\in I}$ is irreducible and $M_n = (m_{i,j})_{i,j\in I_n}$ then ${}_nR \downarrow R$ as $n \to \infty$. In particular, for all $i_0 \in I$ we have ${}_nR(i_0, i_0) \to R$.*

Proof. If M_n are all irreducible then the claim follows easily from [19, Theorem 6.8]. In the general case, fix an index $i_0 \in I$ and consider the sequence of sets $\{J_n\}_{n\in\mathbb{N}}$ where J_n is the class of i_0 in I_n. Given any index i, we have that $i \in J_n$ eventually (as $n \to \infty$); indeed if A is the set of vertices in a path connecting i_0 to i and back (which exists since M is irreducible) then eventually $A \subseteq I_n$ which implies $A \subseteq J_n$ thus $\bigcup_n J_n = X$. Let us call ${}_n\widetilde{R}$ the convergence parameter of $\widetilde{M}_n = (m_{i,j})_{i,j\in J_n}$. Since $\widetilde{M}_n$ is irreducible then, according to [19, Theorem 6.8], ${}_n\widetilde{R} \downarrow R$. On the other hand $R \leq {}_nR \leq {}_nR(i_0, i_0) \leq {}_n\widetilde{R}(i_0, i_0) = {}_n\widetilde{R}$ which yields the conclusions. □

Note that in the previous theorem the elements $\{I_n\}_{n\in\mathbb{N}}$ can be chosen arbitrarily; in particular they may be finite subsets of indices.

Corollary 5.2. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a general sequence of subsets of $\mathbb{N}$ such that $\liminf_{n\to\infty} I_n = \mathbb{N}$. If M is irreducible and $M_n = (m_{i,j})_{i,j\in I_n}$ then ${}_nR \to R$ as $n \to \infty$. In particular, for all $i_0 \in I$ we have ${}_nR(i_0, i_0) \to R$.*

Proof. Let $\{I'_n\}$ a nondecreasing sequence of finite subsets of $\mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} I'_n = \mathbb{N}$. For any n there exists r_n such that for all $r \geq r_n$ we have $I_r \supseteq I'_n$. Clearly, according to Theorem 5.1, for all $r \geq r_n$,

$$R \leq {}_nR \leq {}_rR(i_0, i_0) \leq {}_nR'(i_0, i_0) \downarrow R$$

as $n \to \infty$ (where ${}_nR'$ is the convergence parameter of $M'_n = (m_{i,j})_{i,j\in I'_n}$). □

5.2. **Application to BRWs.** We stated the results of the previous section considering matrices with natural indices in order to keep the same notation as in [19]. Here we consider a generic (at most countable) set X instead of $\mathbb{N}$.

Given a sequence of BRWs $\{(X_n, \mu_n)\}_{n\in\mathbb{N}}$, we define $m(n)_{xy} := \sum_{f\in S_{X_n}} f(y)\mu_{n,x}(f)$ and the corresponding sequence of submatrices $\{M_n\}_{n\in\mathbb{N}}$ where $M_n = (m(n)_{xy})_{x,y\in X_n}$. The main goal of this section is to find sufficient conditions on the sequence $\{(X_n, \mu_n)\}_{n\in\mathbb{N}}$ and on the BRW (X, μ) such that, eventually as $n \to \infty$, the behaviors of (X_n, μ_n) and (X, μ) are similar. In particular we investigate if the survival of (X, μ) can guarantee the survival of (X_n, μ_n) for all sufficiently large n. The following theorem is the main result of this section.

Theorem 5.3. *Let (X, E_μ) be connected and fix a vertex $x_0 \in X$. If $m(n)_{xy} \leq m_{xy}$ for all $x, y \in X$, $n \in \mathbb{N}$ and $m(n)_{xy} \to m_{xy}$ as $n \to \infty$ then*

(1) (X_μ) dies out locally (resp. globally) a.s. starting from $x_0 \Longrightarrow (X_n, \mu_n)$ dies out locally (resp. globally) a.s starting from x_0 for all $n \in \mathbb{N}$;

(2) (X, μ) survives locally wpp starting from $x_0 \Longrightarrow (X_n, \mu_n)$ survives locally wpp starting from x_0 eventually as $n \to \infty$.

Proof.

(1) It follows by coupling the BRW(X_n, μ_n) with the (subcritical) BRW(X, μ) as described in Section 3.3.

(2) Let us fix a sequence $\{Y_n\}_{n\in\mathbb{N}}$ of finite subsets of X such that $\liminf_{n\to\infty} Y_n = X$ (take for instance an increasing sequence as in Theorem 5.1).By Theorem 4.1 there exists $\varepsilon > 0$ such that $\limsup_{i\to\infty} \sqrt[i]{m^{(i)}_{x_0x_0}} > 1+\varepsilon$. Consider the sequence of submatrices $A_n = (a(n)_{xy})_{x,y\in Y_n}$ where $a(n)_{xy} := m_{xy}/(1+\varepsilon)$. Using Theorem 5.1 and Corollary 5.2 we have that

$$\lim_{n\to\infty} \limsup_{i\to\infty} \sqrt[i]{a(n)^{(i)}_{x_0x_0}} = \limsup_{i\to\infty} \sqrt[i]{m^{(i)}_{x_0x_0}}/(1+\varepsilon) > 1,$$

as $n \to \infty$. Let $\bar{n}$ such that $\limsup_{i\to\infty} \sqrt[i]{a(\bar{n})^{(i)}_{x_0x_0}} > 1$. Using the Bounded Convergence Theorem $m(n)_{xy} \to m_{xy}$ for all $x, y \in X$. Moreover since $Y_{\bar{n}}$ is finite there exists n_0 such that for all $n \geq n_0$ we have $m(n)_{xy} \geq m_{xy}/(1+\varepsilon) = a(\bar{n})_{xy}$ for all $x, y \in Y_{\bar{n}}$, thus

$$\limsup_{i\to\infty} \sqrt[i]{m(n)^{(i)}_{x_0x_0}} \geq \limsup_{i\to\infty} \sqrt[i]{a(\bar{n})^{(i)}_{x_0x_0}} > 1$$

for all $n \geq n_0$. Theorem 4.1(1) yields the conclusion.

□

Note that in the language of continuous-time BRWs (see [1] and [2] for details), the claim of the previous theorem is $\lambda_s(X_n) \to \lambda_s(X)$; hence it is a generalization of [3, Theorem 3.2]. Indeed, using Theorem 4.1(1), Theorem 5.1(2) is equivalent to

$$\forall \lambda : \lambda \limsup_{i\to\infty} \sqrt[i]{m^{(i)}_{x_0x_0}} > 1 \Longleftrightarrow \exists n_0 : \forall n \geq n_0,\ \lambda \limsup_{i\to\infty} \sqrt[i]{m(n)^{(i)}_{x_0x_0}} > 1,$$

that is,

$$\lim_{n\to\infty} \limsup_{i\to\infty} \sqrt[i]{m^{(i)}_{x_0x_0}} = \limsup_{i\to\infty} \sqrt[i]{m(n)^{(i)}_{x_0x_0}}$$

(since, clearly, $m(n)^{(i)}_{x_0x_0} \leq m^{(i)}_{x_0x_0}$ for all $i \in \mathbb{N}$).

Among all the possible choices of the sequence $\{(X_n, \mu_n)\}_{n\in\mathbb{N}}$ there is one which is *induced* by (X, μ) on the subsets $\{X_n\}_{n\in\mathbb{N}}$; more precisely, one can take $\mu_n(g) := \sum_{f\in S_X : f|_{X_n}=g} \mu_x(f)$ for all $x \in X_n$ and $g \in S_{X_n}$. Roughly speaking, this choice means that all the reproductions outside X_n are suppressed. Note that, in this case, $m(n)_{xy} = m_{xy}$ for all $x, y \in X_n$ (the result in this particular case is used, for instance, in the proof of [6, Theorem 2.4]).

Remark 5.4. *Theorem 5.3 deals mainly with local survival. One can wonder what can be said about global survival. Clearly if the (X, μ) process survives both globally and locally then eventually (X_n, μ_n) survives locally and thus globally.*

The question is nontrivial when (X, μ) survives globally but not locally (which we assume henceforth in this remark).

In this last case, if X_n is finite for every $n \in \mathbb{N}$ and the graph (X_n, E_{μ_n}) is connected then there is no distinction between global and local survival for the process (X_n, μ_n); in particular (X_n, μ_n) dies out (locally and globally) a.s. for all values of $n \in \mathbb{N}$.

On the other hand, the case where X_n is finite for every $n \in \mathbb{N}$ and the graph (X_n, E_{μ_n}) is not connected is more complicated and can be treated as in [2, Remark 4.4].

When X_n is infinite for infinitely many values of n, one cannot expect always to have global survival for sufficiently large values of n. The counterexample that we are going to construct is a continuous-time BRW (the discrete counterexample is given by taking the discrete-time counterparts of the process as described in Section 2). Let us consider a homogeneous tree $\mathbb{T}_k$ (with $k \geq 3$) and the continuous-time BRW with rates λ on each edge. Let us fix a root $o \in \mathbb{T}_k$, an infinite ray γ starting from o and denote by X_n the union of the ball of center o and radius n and the set of vertices of the (infinite) ray γ. Let us take on X_n the reproduction rates λ on each edge. Clearly $\bigcup_{n \in \mathbb{N}} X_n = \mathbb{T}_k$. Using the same arguments as in [1, Remark 3.10] *one can prove that $\lambda_w(X_n) = \lambda_s(X_n)$ (remember that the ray is just a copy of $\mathbb{N}$ and there is no global survival on $\mathbb{N}$ with the choice of rates we just made), hence $\lambda_w(X_n) \to \lambda_s(\mathbb{T}_k) > \lambda_w(\mathbb{T}_k)$ If we take $\lambda \in (\lambda_w(\mathbb{T}_k), \lambda_s(\mathbb{T}_k))$ we have that there is no global survival for the BRWs on X_n despite the BRW on $\mathbb{T}_k$ survives globally.*

A possible application of Theorem 5.3 is based on the following definition.

Definition 5.5. *Let $\gamma : X \to X$ be an injective map. We say that $\mu = \{\mu_x\}_{x \in X}$ is γ-invariant if for all $x, y \in X$ and $f \in S_X$ we have $\mu_x(f) = \mu_{\gamma(x)}(f \circ \gamma^{-1})$.*

We note that if a continuous-time BRW is γ-invariant according to [3, Section 3] then the discrete-time counterpart is γ-invariant.

Consider now an injective map K and suppose that μ is K-invariant. If there exists $Y \subseteq X$ such that all finite subsets $A \subset X$ we have $X_n := K^{(n)}(Y) \supseteq A$ (for some $n \in \mathbb{N}$) then the BRW(X, μ) survives (locally) wpp if and only if (Y, ν) survives (locally) wpp (where ν is the law induced by μ on Y). Note that, if μ_n is the law induced by μ on X_n, since μ is K-invariant, the behavior of (X_n, μ_n) is the same as the behavior of (Y, ν). Clearly if (Y, ν) survives then (X, μ) survives. On the other hand if (X, μ) survives, according to Theorem 5.3, there exists a finite subset $A \subset X$ such that the induced BRW on A survives thus, for any n such that $X_n \supseteq A$ we have that (X_n, μ_n) survives and this implies the survival of (Y, ν).

This applies, in particular, when $X = \mathbb{Z}^d$, μ_x is translation invariant and Y is a cone, namely $Y = \{y \in \mathbb{Z}^d : \langle y, y_0 \rangle \geq \alpha \|y\| \cdot \|y_0\|\}$ for some fixed nontrivial $y_0 \in \mathbb{Z}^d$ and $\alpha < 1$ (where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ represent the usual scalar product and norm of $\mathbb{Z}^d$ respectively). In this case $K(x) := x - y_0$.

Example 5.6. *If M is reducible the convergence granted by Theorem 5.1 might fail. Take for instance $I = \mathbb{N}$, $I_n := \{i \in \mathbb{N}; i \leq n\}$ and*

$$m_{ij} := \begin{cases} 1 & j = i+1 \\ 0 & \textit{otherwise.} \end{cases}$$

Clearly $R = 1/2$ but ${}_nR = \infty$ for all n. This translates into an example of a reducible BRW on $\mathbb{N}$ where the conclusions of Theorem 5.3 do not hold. Let $X = \mathbb{N}$ and

$$\mu_x = \frac{2}{3}\delta_{2\mathbb{1}_{\{x+1\}}} + \frac{1}{3}\delta_{\mathbf{0}}.$$

Roughly speaking each particle at x has 2 offsprings in $x+1$ with probability $2/3$ and no offsprings at all with probability $1/3$. If we start with a finite number of particles, this BRW survives globally wpp, but any (nontrivial) spatially restricted BRW dies out a.s.

6. Approximation by truncated BRWs

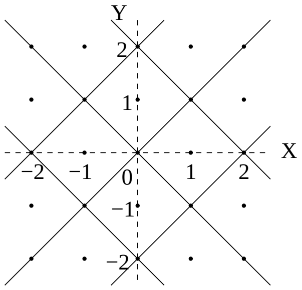
In this section we want to study the approximation of a BRW $\{\eta_n\}_{n\in\mathbb{N}}$ by means of the sequence of truncated BRWs $\{\{\eta_n^m\}_{n\in\mathbb{N}}\}_{m\in\mathbb{N}}$. We already know that if the BRW dies out locally (resp. globally) a.s. then any truncated BRW dies out locally (resp. globally) a.s. (this can be proved by coupling as explained in Section 3.3). On the other hand we would like to be able to prove a result similar to Theorem 5.3 as m tends to infinity. For continuous-time BRWs this has been done in [3]; the technique we use here is essentially the same. From now on the set X is assumed to be countable; indeed, if it is finite then there is no survival for the truncated BRW $\{\eta_n^m\}_{n\in\mathbb{N}}$ for any $m \in \mathbb{N}$. Moreover, for technical reasons we suppose that the graph (X, E_μ) has finite geometry (that is, $\sup_{x\in X} \deg(x) < +\infty$).

In the following (see Step 3 below) we need to find a measure ρ which dominates stochastically all the measures $\{\rho_x\}_{x\in X}$. It is straightforward to see that the existence of such a measure ρ is equivalent to $\sup_{x\in X} \rho_x([n, +\infty)) \to 0$ as $n \to +\infty$ (that we assume henceforth). In this case ρ can be chosen according to

$$\rho(n) = \sup_{x\in X} \rho_x([n, +\infty)) - \sup_{x\in X} \rho_x([n+1, +\infty)). \tag{6.18}$$

Moreover the measure ρ can be chosen with finite first (resp. k-th) moment if and only if $\sum_{n\geq 1} \sup_{x\in X} \rho_x([n, +\infty)) < +\infty$ (resp. $\int_0^\infty \sup_{x\in X} \rho_x([\sqrt[k]{t}, +\infty))\mathrm{d}t < +\infty$).

We assume that the matrix M is irreducible and we denote its convergence parameter by R_μ. We observe that, using this notation, according to Theorem 4.1(1), local survival is equivalent to $R_\mu < 1$. Remember that, in this case, $\liminf_{n\to\infty} \sqrt[n]{\sum_y m_{xy}^{(n)}}$ and $\limsup_{n\to\infty} \sqrt[n]{m_{xx}^{(n)}}$ do not depend on the choice of $x, y \in X$.



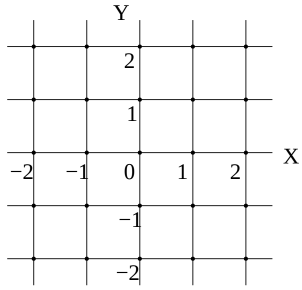


Figure 1: $X \times Y$ $(X = Y = \mathbb{Z})$.

Figure 2: $X \square Y$ $(X = Y = \mathbb{Z})$.

In the following, we need to define the product of two graphs (basically, these will be space/time products): given two graphs $(X, \mathcal{E})$, $(Y, \mathcal{E}')$ we denote by $(X, \mathcal{E}) \times (Y, \mathcal{E}')$ the weighted graph with set of vertices $X \times Y$ and set of edges $\mathcal{E} = \{((x, y), (x_1, y_1)) : (x, x_1) \in \mathcal{E}, (y, y_1) \in \mathcal{E}'\}$ (in Figure 1 we draw the connected component of $\mathbb{Z} \times \mathbb{Z}$ containing $(0, 0)$). Besides, by $(X, \mathcal{E}) \square (Y, \mathcal{E}')$ we mean the graph with the same vertex set as before and vertices $\mathcal{E} = \{((x, y), (x_1, y_1)) : (x, x_1) \in \mathcal{E}, y = y_1\} \cup \{((x, y), (x_1, y_1)) : x = x_1, (y, y_1) \in \mathcal{E}'\}$ (see Figure 2).

6.1. **The comparison with an oriented percolation.** First of all, remember the coupling between $\{\eta_n\}_{n\in\mathbb{N}}$ and $\{\eta_n^m\}_{n\in\mathbb{N}}$: the truncated process $\{\eta_n^m\}_{n\in\mathbb{N}}$ (satisfying equation (3.12)) can be seen as the BRW $\{\eta_n\}_{n\in\mathbb{N}}$ (satisfying equation (2.1)) by removing, at each step, all the births which cause more than m particles to live on the same site. As in [3] we need two other coupled processes. Fix $\widetilde{n} \in \mathbb{N}$ and let $\{\bar{\eta}_n\}_{n\in\mathbb{N}}$ be the process obtained from the BRW $\{\eta_n\}_{n\in\mathbb{N}}$ by removing all n-th generation particles with $n > \widetilde{n}$, that is

$$\bar{\eta}_n = \begin{cases} \eta_n & n \le \widetilde{n} \\ 0 & n > \widetilde{n}. \end{cases} \tag{6.19}$$

Define $\{\bar{\eta}_n^m\}_{n\in\mathbb{N}}$ analogously from $\{\eta_n^m\}_{n\in\mathbb{N}}$. Clearly, the following stochastic inequalities hold $\eta_n \ge \eta_n^m$and $\bar{\eta}_n \ge \bar{\eta}_n^m$ for all $n \in \mathbb{N}$. By construction, the progenies of a given particle in $\{\bar{\eta}_n\}_{n\in\mathbb{N}}$ or $\{\bar{\eta}_n^m\}_{n\in\mathbb{N}}$ lives at a distance from the ancestor not larger than $\widetilde{n}$.

Our proofs are essentially divided into four main steps. We report here shortly the essence of these steps and we refer to [3, Section 4] for further details.

Step 1. *Fix a graph $(I, \mathcal{E}(I))$ such that the Bernoulli percolation on $(I, \mathcal{E}(I)) \times \vec{\mathbb{N}}$ has two phases (where we denote by $\vec{\mathbb{N}}$ the oriented graph on $\mathbb{N}$, that is, (i, j) is an edge if and only if $j = i + 1$).*

The usual trick is to find a copy of the graph $\mathbb{Z}$ or $\mathbb{N}$ as a subgraph of I, since the (oriented) Bernoulli bond percolation on $\mathbb{Z} \times \vec{\mathbb{N}}$ and $\mathbb{N} \times \vec{\mathbb{N}}$ has two phases. In this paper, the main choices for I are $\mathbb{Z}$, $\mathbb{N}$ or X.

Step 2. *Given a globally (or locally) surviving BRW and for every $\varepsilon > 0$ there exists a collection of disjoint sets $\{A_i\}_{i\in I}$ ($A_i \subset X$ for all $i \in I$), $\bar{n} > 0$, and $k \in \mathbb{N} \setminus \{0\}$, such that, for all $i \in I$,*

$$\mathbb{P}\Big(\forall j : (i,j) \in \mathcal{E}(I), \sum_{x\in A_j} \eta_{\bar{n}}(x) \geq k \Big| \eta_0 = \eta\Big) > 1 - \varepsilon, \tag{6.20}$$

for all η such that $\sum_{x\in A_i} \eta(x) = k$ and $\eta(x) = 0$ for all $x \notin A_i$. The same holds, for $\widetilde{n} \geq \bar{n}$, for $\{\bar{\eta}_n\}_{n\in\mathbb{N}}$ in place of $\{\eta_n\}_{n\in\mathbb{N}}$.

In the following sections Step 2 will be established under certain conditions (and for suitable choices of $(I, \mathcal{E}(I))$). Basically we have to prove that, for a suitable surviving BRW, with a probability arbitrarily close to 1, given enough particles in A_i, after a fixed time $\bar{n}$, we have at least the same number of particles on every neighboring set A_j.

Step 3. *Let ε, $\{A_i\}_{i\in I}$, $\bar{n}$ and k be chosen as in Step 2. Then for all sufficiently large m we have that, for all $i \in I$,*

$$\mathbb{P}\Big(\forall j : (i,j) \in \mathcal{E}(I), \sum_{x\in A_j} \eta^m_{\bar{n}}(x) \geq k \Big| \eta^m_0 = \eta\Big) > 1 - 2\varepsilon, \tag{6.21}$$

for all η such that $\sum_{x\in A_i} \eta(x) = k$, $\eta(x) = 0$ for all $x \notin A_i$. The same holds, for $\widetilde{n} \geq \bar{n}$, for $\{\bar{\eta}^m_n\}_{n\in\mathbb{N}}$ in place of $\{\eta^m_n\}_{n\in\mathbb{N}}$.

Step 3 follows from Step 2; the proof is a natural adaptation of the same arguments of [3, Step 3]. Indeed let N_n be the total number of particles ever born in the BRW (starting from the configuration η) before time n; it is clear that N_n is a process stochastically dominated (the arguments are similar to the ones we used in Section 3.3) by a branching process with offspring law

$$\rho'(n) := \begin{cases} 0 & n = 0 \\ \rho(n-1) & n \geq 1 \end{cases}$$

and initial state N_0 (where ρ is given by equation (6.18)). If $N_0 < +\infty$ almost surely then for all $n > 0$ we have $N_n < +\infty$ almost surely; hence for all $n > 0$ and $\varepsilon > 0$ there exists $N(n, \varepsilon, k)$ such that, for all $i \in I$,

$$\mathbb{P}\Big(N_n \leq N(n, \varepsilon, k) \Big| \eta_0 = \eta\Big) > 1 - \varepsilon,$$

for all η such that $\sum_{x\in A_i} \eta(x) = k$, $\eta(x) = 0$ for all $x \notin A_i$. Define $\widetilde{N} = N(\bar{n}, \varepsilon, k)$. The conclusion follows, using elementary probability arguments, as in [3, Step 3].

Step 4. *Given a globally (or locally) surviving BRW, for every $\varepsilon > 0$ and for all sufficiently large m, there exists a one-dependent oriented percolation on $I \times \vec{\mathbb{N}}$ (with probability $1 - 2\varepsilon$ of opening simultaneously all edges from a vertex and 2ε of opening no edges) such that the probability of survival of the BRW$_m$ (starting at time 0 from a configuration η such that $\sum_{x\in A_{i_0}} \eta(x) = k$ and $\eta(x) = 0$ for all $x \notin A_{i_0}$) is larger than the probability that there exists an infinite cluster containing $(i_0, 0)$.*

Consider an edge $((i,n),(j,n+1))$ in $(I,\mathcal{E}(I))\times\vec{\mathbb{N}}$: let it be open if η_t^m has at least k individuals in A_i at time $n\bar{n}$ and in A_j at time $(n+1)\bar{n}$. Thus the probability of weak survival of η_t^m is bounded from below by the probability that there exists an infinite cluster containing $(i_0,0)$ in this percolation on $I\times\vec{\mathbb{N}}$, and, if A_{i_0} is finite, the probability of strong survival is bounded from below by the probability that the cluster contains infinitely many points in $\{(i_0,l):l\in\mathbb{N}\}$ (we suppose to start with k particles in A_{i_0}). Let ν_1 be the associated percolation measure. Unfortunately this percolation is neither independent nor one-dependent. In fact the opening procedure of the edges $((i,n),(j,n+1))$ and $((i_1,n),(j_1,n+1))$ may depend respectively on two different progenies of particles overlapping on a vertex x_0. This may cause dependence since if in x_0 there are already m particles then newborns are not allowed.

To avoid this difficulty we could adapt [3, Step 4] to a discrete-time process: the construction is made by means of the process $\{\bar{\eta}_n^m\}_{n\in\mathbb{N}}$ by choosing $m\geq 2\widetilde{N}H$ where $\widetilde{N}$ is the same as in Step 3 and H is the maximum of the number of paths of length $\widetilde{n}$ crossing a vertex (the assumption of bounded geometry that we made on the graph plays a fundamental role here). Note that with this choice of m we have that, starting with an initial condition η_0 (such that $\sum_{x\in X}\eta_0(x)=k$), $\eta_n^m=\eta_n$ and $\bar{\eta}_n^m=\bar{\eta}_n$ for all $n\leq\bar{n}$ on an event with probability at least $1-\varepsilon$ (namely, $\{N_{\bar{n}}\geq N(\bar{n},\varepsilon,k)\}$ as defined in Step 3). Step 4 follows then from Step 3.

Our next goals are to fix suitable graphs $(I,\mathcal{E}(I))$ and prove Step 2 for a large class of globally surviving BRWs: then by Steps 4 and 1, for all sufficiently large m, the corresponding truncated BRW_m survives globally with positive probability if m is sufficiently large. On the other hand, in order to show that, given a *locally* surviving BRW, the corresponding truncated BRW_m survives with positive probability if m is sufficiently large, we need to prove Step 2 with a choice of at least one A_i finite, say A_{i_0}, and I containing a copy of $\mathbb{Z}$ or $\mathbb{N}$ as a subgraph. Remember that, in a supercritical Bernoulli bond percolation in $\mathbb{Z}\times\vec{\mathbb{N}}$ or $\mathbb{N}\times\vec{\mathbb{N}}$, with probability 1 the infinite open cluster has an infinite intersection with the set $\{(0,n):n\in\mathbb{N}\}$. Thus, in the supercritical case we have, with positive probability, in the infinite open cluster an infinite number of vertices of the set $\{(0,n):n\geq 0\}$ including the origin. This (again by Steps 3 and 4) implies that, with positive probability, the BRW_m starting with k particles in A_{i_0} has particles alive in A_{i_0} at arbitrarily large times. Being A_{i_0} finite yields the conclusion.

Remark 6.1. *As in* [3]*, the previous set of steps represents the skeleton of the proofs of Theorems 6.5 and 6.9. In order to be able to prove Theorem 6.7 we need to modify this approach. Here are the main differences. We choose an oriented graph $(W,\mathcal{E}(W))$ and a family of subsets of X, $\{A_{(i,n)}\}_{(i,n)\in W}$ such that*

- *W is a subset of the set $\mathbb{Z}\times\mathbb{N}$ (the inclusion is between sets not between graphs);*
- *for all $n\in\mathbb{N}$ we have that $\{A_{(i,n)}\}_{i:(i,n)\in W}$ is a collection of disjoint subsets of X;*
- *$(i,n)\to(j,m)$ implies $m=n+1$.*

Step 2 translates into the following: given a (globally or locally) surviving BRW and for every $\varepsilon > 0$, there exists $\bar{n} > 0$ and $k \in \mathbb{N}$, such that, for all $n \in \mathbb{N}$, $i \in \mathbb{Z}$, and for all η such that $\sum_{x \in A_{(i,n)}} \eta(x) = k$,

$$\mathbb{P}\Big(\forall j : (i,n) \to (j, n+1), \sum_{x \in A_{(j,n)}} \eta_{(n+1)\bar{n}}(x) \geq k \Big| \eta_{n\bar{n}} = \eta\Big) > 1 - \varepsilon.$$

Step 3 is the same and the percolation in Step 4 now concerns the graph $(W, \mathcal{E}(W))$.

6.2. **Local survival.** Let us choose a vertex $o \in X$, fix the initial configuration as $\eta_0 := \delta_o$ and assume that the measure ρ as defined by equation (6.18) has finite second moment. The key to prove Step 2 is based on some estimates on the expected value $\mathbb{E}^{\delta_o}(\eta_n(x))$ of the number of individuals in a site. This expected value can be computed using equation (2.2): hence $\mathbb{E}^{\delta_o}(\eta_n(x)) = m^{(n)}_{o,x}$. It is clear that

$$\lim_{n \to \infty} \mathbb{E}^{\delta_o}(\eta_n(x)) = \begin{cases} 0 & \text{if } R_\mu > 1, \\ +\infty & \text{if } R_\mu < 1. \end{cases} \tag{6.22}$$

In the following lemma we prove that, when $R_\mu < 1$, if at time 0 we have one individual at each of the sites $x_1, \dots, x_l$, then, given any choice of l sites $y_1, \dots, y_l$, after some time the expected number of descendants in y_i of the individual in x_i exceeds any fixed $D \geq 1$ for all $i = 1, \dots, l$. The proof follows immediately from equation (6.22) we omit it; just note that, due to equation (6.19), the estimate on $\mathbb{E}^{\delta_{x_j}}(\bar{\eta}_n(y_j))$ follows immediately from the one on $\mathbb{E}^{\delta_{x_j}}(\eta_n(y_j))$.

Lemma 6.2. *Let us consider the finite set of couples $\{(x_j, y_j)\}_{j=0}^l$ and fix $D \geq 1$; if $R_\mu < 1$ then there exists $n > 0$ such that $\mathbb{E}^{\delta_{x_j}}(\eta_n(y_j)) > D$, $\forall j = 0, 1, \dots, l$. Moreover, $\mathbb{E}^{\delta_{x_j}}(\bar{\eta}_n(y_j)) > 1$ when $\widetilde{n} > n$.*

We show that, when $R_\mu < 1$, for all sufficiently large $k \in \mathbb{N}$, given k particles in a site x at time 0, "typically" (i.e. with arbitrarily large probability) after some time we will have at least Dk individuals in each site of a fixed finite set Y. Analogously, starting with l colonies of size k (in sites $x_1, \dots, x_l$ respectively), each of them will spread, after a sufficiently long time, at least Dk descendants in every site of a corresponding (finite) set of sites Y_i.

Lemma 6.3. *Suppose that $R_\mu < 1$.*

(1) Let us fix $x \in X$, Y a finite subset of X, $D \geq 1$ and $\varepsilon > 0$. Then there exists $\bar{n} = \bar{n}(x, Y) > 0$ (independent of ε), $k(\varepsilon, x, Y)$ such that, for all $k \geq k(\varepsilon, x, Y)$,

$$\mathbb{P}\left(\bigcap_{y \in Y} (\eta_{\bar{n}}(y) \geq Dk) \Big| \eta_0(x) = k\right) > 1 - \varepsilon.$$

The claim holds also with $\{\bar{\eta}_n\}_{n \in \mathbb{N}}$ in place of $\{\eta_n\}_{n \in \mathbb{N}}$ when $\widetilde{n} \geq \bar{n}$.

(2) Let us fix a finite set of vertices $\{x_i\}_{i=1,\dots,m}$, a collection of finite sets $\{Y_i\}_{i=1,\dots,l}$ of vertices of X, $D \geq 1$ and $\varepsilon > 0$. Then there exists $\bar{n} = \bar{n}(\{x_i\}, \{Y_i\})$ (independent of ε), $k(\varepsilon, \{x_i\}, \{Y_i\})$ such that, for all $i = 1, \dots, l$ and $k \geq k(\varepsilon, \{x_i\}, \{Y_i\})$,

$$\mathbb{P}\left(\bigcap_{y \in Y_i} (\eta_{\bar{n}}(y) \geq Dk) \Big| \eta_0(x_i) = k \right) > 1 - \varepsilon.$$

The claim holds also with $\{\bar{\eta}_n\}_{n \in \mathbb{N}}$ in place of $\{\eta_n\}_{n \in \mathbb{N}}$ when $\widetilde{n} \geq \bar{n}$.

Proof.

(1) If we denote by $\{\{\xi_{n,i}\}_{n \in \mathbb{N}}\}_{i \in \mathbb{N}}$ a family of independent BRWs behaving according to μ and starting from $\xi_{0,i} = \delta_x$ (for all $i \in \mathbb{N}$) then, by Lemma 6.2, we can choose $\bar{n}$ such that $\mathbb{E}^{\delta_x}(\xi_{\bar{n},i}(y)) > 2D$ for all $y \in Y$. We can chose a realization of $\{\eta_n\}_{n \in \mathbb{N}}$ such that $\eta_n(y) = \sum_{j=1}^k \xi_{n,j}(y)$; denote the variance $\mathrm{var}(\xi_{n,j}(y))$ by $\sigma^2_{n,y}$. Since $\xi_{n,j}$ is stochastically dominated by a branching process with offspring law ρ (where ρ is chosen as in equation (6.18)), it is clear that, for all y, $\sigma^2_{n,y} < \mathbb{E}(\rho)^{n-1}\mathrm{var}(\rho) < +\infty$ since we assumed at the beginning of this section that ρ has finite second moment. By using the one-sided Chebyshev inequality
$$\mathbb{P}\left(\eta_{\bar{n}}(y) \geq Dk\right) \geq \mathbb{P}\big(\eta_{\bar{n}}(y) \geq \mathbb{E}(\eta_{\bar{n}}(y))/2\big) \geq \frac{\mathbb{E}(\eta_{\bar{n}}(y))^2/4}{\mathbb{E}(\eta_{\bar{n}}(y))^2/4 + \sigma^2_{\bar{n},y}} \geq 1 - \frac{\sigma^2_{\bar{n},y}}{D^2k^2 + \sigma^2_{\bar{n},y}}$$
Whence, fixed any $\delta > 0$, there exists $k(\delta, x, y)$ such that, for all $k \geq k(\delta, x, y)$, $\mathbb{P}\left(\eta_{\bar{n}}(y) \geq Dk\right) \geq 1 - \delta$. For all $k \geq \max_{y \in Y} k(\delta, x, y) < +\infty$
$$\mathbb{P}\left(\bigcap_{y \in Y} (\eta_{\bar{n}}(y) \geq Dk) \Big| \eta_0(x) = k \right) \geq 1 - 2|Y|\delta,$$
where $|Y|$ is the cardinality of Y. The assertion for $\bar{\eta}_n$ follows from Lemma 6.2.

(2) Let $\{\{\xi_{n,i}\}_{t \geq 0}\}_{i \in \mathbb{N}}$ be as before and choose $\bar{n}$ such that $\mathbb{E}^{\delta_{x_i}}(\xi_{\bar{n},i}(y)) > 2D$ for all $y \in Y_i$ a nd for all $i = 1, \dots l$. According to (1) above we may fix k_i such that, for all $k \geq k_i$,
$$\mathbb{P}\left(\bigcap_{y \in Y_i} (\eta_{\bar{n}}(y) \geq Dk) \Big| \eta_0(x_i) = k \right) \geq 1 - \varepsilon.$$
Take $k \geq \max_{i=1,\dots,l} k_i$ to conclude. Again the assertion for $\bar{\eta}_n$ follows from Lemma 6.2.

□

The dependence of k on the offspring distribution μ is hidden in the term $\sigma^2_{\bar{n},y}$, that is, in $\bar{n}$ and in the dominating offspring law ρ. The key is to find a fixed k such that the lower bound in the previous theorem holds simultaneously for a family $\{(x_i, Y_i)\}$. One possibility is to choose a finite family (as we did in the previous lemma) but it is not the only one: one has to find a fixed $\bar{n}$ such that Lemma 6.2 holds (for all the couples (x_i, y) where $y \in Y_i$) and this gives immediately an upper bound for $\sigma^2_{\bar{n},y}$ (uniform with respect to y).

Remark 6.4. *Note that Lemmas 6.2 and 6.3 can be restated for the process $\{\bar\eta_n^m\}_{n\in\mathbb{N}}$ if m is sufficiently large. Indeed, when $m \geq 2N(\bar n, \varepsilon, Dk)H$ (as in Step 4) we have that $\bar\eta_n^m = \bar\eta_n$ for all $n \leq \bar n$ on an event with probability at least $1-\varepsilon$. In the rest of the paper, when not explicitly stated otherwise, Lemma 6.3 will be used by setting $D = 1$.*

We already know that if $\{\eta_n\}_{n\in\mathbb{N}}$ dies out locally (resp. globally) a.s. then $\{\eta_n^m\}_{n\in\mathbb{N}}$ dies out locally (resp. globally) a.s. The following theorem states the converse.

We recall that (X,μ) is *quasi transitive* if and only if there exists a finite subset $X_0 \subseteq X$ such that for all $x \in X$ there exists a bijective map $\gamma : X \to X$ and $x_0 \in X$ satisfying $\gamma(x_0) = x$ and μ is γ-invariant.

Theorem 6.5.
If at least one of the following conditions holds

(1) (X,μ) is quasi transitive and connected;
(2) (X,μ) is connected and there exists γ bijection on X such that
  (a) μ is γ-invariant;
  (b) for some $x_0 \in X$ we have $x_0 = \gamma^n x_0$ if and only if $n = 0$;

then if $\{\eta_n\}_{n\in\mathbb{N}}$ survives locally (starting from x_0) then $\{\eta_n^m\}_{n\in\mathbb{N}}$ survives locally (starting from x_0) eventually as $m \to +\infty$.

Proof. (1) Let $R_\mu < 1$ and define, for any $x \in X_0$, $Y_x := \{y \in X : (x,y) \in E_\mu\}$. Fix $I = X$, $\mathcal{E}(I) = \{(x,y) : (x,y) \in E_\mu \text{ or } (y,x) \in E_\mu\}$ and $A_x = \{x\}$. Lemma 6.3 yields Step 2. To prove that the percolation on $(I,\mathcal{E}(I)) \times \vec{\mathbb{N}}$ has two phases (that is, $(I,\mathcal{E}(I))$ can be used in Step 1) we note that this follows from the fact that the graph $\mathbb{N}$ is a subgraph of X. Recall that in the supercritical Bernoulli percolation on $\mathbb{N}\times\vec{\mathbb{N}}$ with positive probability the infinite open cluster contains $(0,0)$ and intersects the y-axis infinitely often. Hence by Steps 3 and 4 we have that, for all sufficiently large m, $\{\eta_n^m\}_{n\in\mathbb{N}}$ survives locally.

(2) By Lemma 6.3, there exists $\bar n$ such that, for sufficiently large $\widetilde{n}$,

$$\begin{cases} \mathbb{P}\left(\bar\eta_{\bar n}(\gamma x_0) \geq k \middle| \bar\eta_0(x_0) = k\right) > 1-\varepsilon \\ \mathbb{P}\left(\bar\eta_{\bar n}(x_0) \geq k \middle| \bar\eta_0(\gamma x_0) = k\right) > 1-\varepsilon. \end{cases}$$

This implies easily

$$\begin{cases} \mathbb{P}\left(\bar\eta_{\bar n}(\gamma^n x_0) \geq k \middle| \bar\eta_0(\gamma^{n-1} x_0) = k\right) > 1-\varepsilon \\ \mathbb{P}\left(\bar\eta_{\bar n}(\gamma^{n-1} x_0) \geq k \middle| \bar\eta_0(\gamma^n x_0) = k\right) > 1-\varepsilon \end{cases}$$

for all $n \in \mathbb{Z}$ since μ is γ-invariant. Thus $\{\eta_n^m\}_{n\in\mathbb{N}}$ survives locally (for sufficiently large m) applying Step 3 and 4 (here $I = \mathbb{Z}$ and $A_i = \{\gamma^i x_0\}$).

□

6.3. **Global survival.** In this section we discuss how the global behaviors of $\{\eta_n^m\}_{n\in\mathbb{N}}$ and $\{\eta_n\}_{n\in\mathbb{N}}$ are related and when the global survival of $\{\eta_n\}_{n\in\mathbb{N}}$ implies eventually the global survival of $\{\eta_n^m\}_{n\in\mathbb{N}}$.

We start by noting that if (X,μ) is quasi transitive and $\liminf_{n\to\infty}\sqrt[n]{\sum_y m_{xy}^{(n)}} = \limsup_{n\to\infty}\sqrt[n]{m_{xx}^{(n)}}$ then the global survival of $\{\eta_n\}_{n\in\mathbb{N}}$ implies the global survival of $\{\eta_n^m\}_{n\in\mathbb{N}}$ for a sufficiently large $m\in\mathbb{N}$. Indeed it is easy to show, by supermultiplicative arguments, that $\liminf_{n\to\infty}\sqrt[n]{\sum_y m_{xy}^{(n)}} \geq \limsup_{n\to\infty}\sqrt[n]{m_{xx}^{(n)}}$; on the other hand, since a quasi-transitive BRW is an $\mathcal{F}-BRW$, according to Theorems 4.1 and 4.3 $\{\eta_n\}_{n\in\mathbb{N}}$ survives globally if and only if it survives locally. We proved in Theorem 6.5 that $\{\eta_n^m\}_{n\in\mathbb{N}}$ survives locally (for sufficiently large m) thus it survives globally.

Remark 6.6. *The basic idea of this section is to take a BRW (X,μ) which is locally isomorphic to a BRW (I,ν) (the projection map being g); we define $\{A_i\}_{i\in I}$ by $A_i := g^{-1}(i)$. We know that, if $\{\eta_n\}_{n\in\mathbb{N}}$ is a realization of (X,μ) then a realization of (I,ν) is given by the projection (on I) $\{\xi_n\}_{n\in\mathbb{N}}$ where $\xi_n = \pi_g(\eta_n)$ for all $n\in\mathbb{N}$. Clearly $\nu_{g(x)}(\cdot) = \mu_x(g^{-1}(\cdot))$ and we can easily compute the expected number of particles alive at time n in A_i starting from a single particle alive in x at time 0 as*

$$\sum_{z\in A_i}\mathbb{E}_\mu^{\delta_x}(\eta_n(z)) = \mathbb{E}_\nu^{\delta_{f(x)}}(\xi_n(i)). \tag{6.23}$$

Since $\{\eta_n^m\}_{n\in\mathbb{N}}$ and $\{\pi_g(\eta_n^m)\}_{n\in\mathbb{N}}$ have the same global behavior and $\{\pi_g(\eta_n^m)\}_{n\in\mathbb{N}}$ stochastically dominates $\{\xi_n^m\}_{n\in\mathbb{N}}$ then if the latter survives globally wpp then $\{\eta_n^m\}_{n\in\mathbb{N}}$ survives globally wpp.

Following the previous remark, we take $I=\mathbb{Z}$, $X=\mathbb{Z}\times Y$ (for some set Y) and we denote by $g: X\to\mathbb{Z}$ the usual projection from X onto $\mathbb{Z}$, namely $g(n,y) := n$.

We suppose that ν is translation invariant (that is, $\nu_i=\nu_0$ for all $i\in\mathbb{Z}$) and we denote by ρ and $\bar\rho = \sum_{y\in X} m_{xy} = \sum_{j\in\mathbb{Z}}\widetilde{m}_{g(x)j}$ the distribution and the expected number of offsprings of $\{\eta_n\}_{n\in\mathbb{N}}$ respectively (where, according to the notation of Section 3, $\widetilde{m}_{ij}$ is the expected number of offsprings in j of a particle in i of the BRW $\{\xi_n\}_{n\in\mathbb{N}}$). We note that, since ρ and $\bar\rho$ are the distribution and the expected number of offsprings of ν as well, they do not depend on $x\in X$ or $i\in\mathbb{Z}$ since ν is translation invariant.

Theorem 6.7. *Let $X=\mathbb{Z}\times Y$ and suppose that the BRW (X,μ) is locally isomorphic to $(\mathbb{Z},\nu)$ where ν is translation invariant. If $m_{xy}=0$ whenever $|g(x)-g(y)|>1$ then*

(1) the BRW survives globally starting from x if and only if $\bar\rho = \sum_{y\in\mathbb{Z}} m_{xy} > 1$;
(2) if the BRW survives globally (starting from x) then $\{\eta_n^m\}_{n\in\mathbb{N}}$ survives globally (starting from x) provided that m is sufficiently large.

Proof. (1) This follows from Theorem 4.3 since (X,μ) is an $\mathcal{F}$-BRW which can be mapped onto the branching processes with offspring distribution ρ and recalling that $\sum_{y\in\mathbb{Z}} m_{xy}^{(n)} = \bar\rho^n$.

(2) According to Remark 6.6 it would be enough enough to prove the claim for the BRW $\{\xi_n\}_{n\in\mathbb{N}}$ where $\xi_n = \pi_g(\eta_n)$ whose diffusion matrix satisfies

$$\widetilde{m}_{ij} = \begin{cases} p & j = i+1 \\ q & j = i-1 \\ 1-p-q & i = j \\ 0 & \text{otherwise.} \end{cases}$$

for some $p, q \in [0,1]$ $(p+q \le 1)$.

Nevertheless we prefer to prove the theorem in the general case. To this aim we prove the general version of Step 2 (see Remark 6.1). We start by defining $A_{(i,n)} = g^{-1}(i)$ and we fix $\alpha, \beta \in (0,1)$ such that $\alpha \le \beta \le (1+\alpha)/2$. Note that

$$\begin{aligned} \widetilde{p}^{(n)}(0, \alpha n) &= \sum_{i=\alpha n}^{(1+\alpha)n/2} \binom{n}{i, \quad i-\alpha n, \quad n-2i+\alpha n} p^i q^{i-\alpha n}(1-p-q)^{n-2i+\alpha n} \\ &\ge \binom{n}{\beta n, \quad (\beta-\alpha)n, \quad (1-2\beta+\alpha)n} p^{\beta n} q^{(\beta-\alpha)n}(1-p-q)^{(1-2\beta+\alpha)n} \\ &\overset{n\to\infty}{\sim} \frac{1}{2\pi n\sqrt{\beta(\beta-\alpha)(1-2\beta+\alpha)}} \left(\frac{p^\beta q^{\beta-\alpha}(1-p-q)^{1-2\beta+\alpha}}{\beta^\beta(\beta-\alpha)^{\beta-\alpha}(1-2\beta+\alpha)^{1-2\beta+\alpha}}\right)^n \end{aligned} \tag{6.24}$$

(to avoid a cumbersome notation we write $n\alpha$ instead of $\lfloor n\alpha \rfloor$).

Define

$$Q_{\bar\rho}(\alpha, \beta) = \frac{\bar\rho p^\beta q^{\beta-\alpha}(1-p-q)^{1-2\beta+\alpha}}{\beta^\beta(\beta-\alpha)^{\beta-\alpha}(1-2\beta+\alpha)^{1-2\beta+\alpha}};$$

if the BRW survives globally then $\bar\rho > 1$ and equation (6.24) implies

$$\begin{aligned} \mathbb{E}^{\delta_0}(\xi_n(\alpha n)) &= \bar\rho^n p^{(n)}(0, n\alpha) \\ &\ge \bar\rho^n \binom{n}{\beta n, \quad (\beta-\alpha)n, \quad (1-2\beta+\alpha)n} p^{\beta n} q^{(\beta-\alpha)n}(1-p-q)^{(1-2\beta+\alpha)n} \\ &\sim \frac{1}{2\pi n\sqrt{\beta(\beta-\alpha)(1-2\beta+\alpha)}} \left(Q_{\bar\rho}(\alpha,\beta)\right)^n \end{aligned}$$

as $n \to \infty$. This, along with equation (6.23), implies easily that $\sum_{x\in A_{\alpha n}} \mathbb{E}^{\delta_0}(\eta_n(x))$ has a lower bound which is asymptotic to $\frac{1}{(2\pi n)\sqrt{\beta(\beta-\alpha)(1-2\beta+\alpha)}} \left(Q_{\bar\rho}(\alpha,\beta)\right)^n$ as $n \to \infty$.

Note that $Q_{\bar\rho}(p-q, p) = \bar\rho > 1$, thus there exist $\alpha_1 < \alpha_2 \le \beta_1 < \beta_2$ (with $\beta_i \le (1+\alpha_i)/2$, $i = 1, 2$) such that $Q_{\bar\rho}(x, y) > 1$, for all $(x, y) \in [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]$. By taking $n = \widetilde{N}$ sufficiently large one can find three distinct integers d_1, d_2 and d_3 such that $\alpha_1 n \le d_1 < d_2 \le \alpha_2 n$, $\beta_1 n \le d_3 \le \beta_2 n$ and $Q_{\bar\rho}(d_l/n, d_3/n) > 1$, $l = 1, 2$.

By reasoning as in Lemma 6.3 we have that, for all $\varepsilon > 0$, there exists $\bar n$, $k = k(\varepsilon)$ such that, for all $i \in \mathbb{Z}$, for all $\widetilde{n}$ sufficiently large,

$$\mathbb{P}\left(\sum_{x\in A_{i+j}} \bar\eta_{\bar n}(x) \ge k, j = d_1, d_2 \,\middle|\, \bar\eta_0(i) = \eta\right) > 1 - \varepsilon$$

$\forall i \in \mathbb{Z}$ and for all η such that $\sum_{x\in A_i} \bar\eta(x) = k$. Since k and $\bar n$ are independent of i we have proven the general version of Step 2 using $W = \{a(d_1,1) + b(d_2,1) : a,b \in \mathbb{N}\}$ where $(i,n) \to (j,n+1)$ if and only if $j - i = d_1$ or $j - i = d_2$.

□

The previous theorem applies to translation invariant BRWs on two particular graphs: $\mathbb{Z}^d$ and the homogeneous tree $\mathbb{T}_r$ with degree r.

Corollary 6.8. *If the BRW $(\mathbb{Z}^d, \mu)$ is translation invariant then*

(1) the BRW survives globally (starting from x) if and only if $\bar\rho = \sum_{y\in\mathbb{Z}} m_{xy} > 1$;

(2) if the BRW survives globally (starting from x) $\{\eta_n^m\}_{n\in\mathbb{N}}$ survives globally (starting from x) provided that m is sufficiently large.

Proof. If $d = 1$ then the proof is trivial. If $d > 1$, the claim follows immediately from the fact that $\mathbb{Z}^d = \mathbb{Z} \times \mathbb{Z}^{d-1}$ and, since μ is translation invariant we have that (X, μ) is locally isomorphic to $(\mathbb{Z}, \nu)$ where the projection ν is translation invariant. □

Corollary 6.9. *Let $\mathbb{T}_r$ be a homogeneous tree and suppose that the BRW $(\mathbb{T}_r, \mu)$ is γ-invariant for every automorphism γ of $\mathbb{T}_r$. If $\mu_x(f) \neq 0$ implies* $\mathrm{supp}(f) \subseteq B(x,1)$ *(where $B(x,1)$ is the usual ball of radius 1 and center x of the graph $\mathbb{T}_r$) then*

(1) the BRW survives globally (starting from x) if and only if $\bar\rho = \sum_{y\in\mathbb{Z}} m_{xy} > 1$;

(2) if the BRW survives globally (starting from x) then $\{\eta_n^m\}_{n\in\mathbb{N}}$ survives globally provided that m is sufficiently large.

Proof. Fix an end τ in $\mathbb{T}_r$ and a root $o \in X$ and define the map $h : X \to \mathbb{Z}$ as the usual height (see [22] page 129). Let $A_k = h^{-1}(k)$ as $k \in \mathbb{Z}$ (these sets are usually referred to as horocycles). Since μ is invariant with respect to every automorphism then we have, as before, that $\mathbb{T}_r = \mathbb{Z} \times \mathbb{Z}$ is locally isomorphic to $(\mathbb{Z}, \nu)$ where the projection ν is translation invariant.

□

7. Final remarks

The paper is devoted to three main issues: finding conditions for the local (resp. global) survival of the process, discussing the spatial approximation and, finally, studying the approximation by means of truncated BRWs. This has been done for continuous BRWs in [1, 2, 3].

About the first issue, a question was left open in [2], namely if $\liminf_{n\to\infty} \sqrt[n]{\sum_y m_{xy}^{(n)}} > 1$ could imply global survival starting from x. A question, which is closely related to the previous one, arises if one looks at [2]: for a continuous-time BRW there is a characterization in terms of a functional inequality of the global survival and this inequality depends only on the matrix M, namely

$$\exists v \in [0,1] : v(x_0) > 0,\ Mv \geq \frac{v}{\mathbf{1} - v} \tag{7.25}$$

if and only if there is global survival starting from x_0. Hence, for a continuous-time BRW, local behavior and global behavior depend only on the first-moment matrix of the process. We saw in Theorem 4.1(1) that this dependence still holds for the local behavior of a discrete BRW (see also [17]). Nevertheless, according to Example 4.4, one cannot expect to find an equivalent condition to global survival for a discrete BRW involving only the first-moment matrix M (neither the previous functional equation nor the inequality $\liminf_{n\to\infty} \sqrt[n]{\sum_y m_{xy}^{(n)}} > 1$). Finding conditions similar to equation (7.25) for some classes of BRWs is possible but it goes beyond the scope of this paper.

As for the spatial approximation, the results of Section 5 are quite satisfactory. On the other hand, there is room for improvements in the approximations by truncated BRWs of Section 6. Indeed, one can hope to find more classes of BRWs which can be approximated by their truncations. In our results a key role was played by the similarity of the BRW under suitable automorphisms of the graph (such as translations, for instance), nevertheless the four steps described in Section 6.1 are quite general and can be applied to a variety of classes of BRWs, provided one can prove Step 2 (as we did in Sections 6.2 and 6.3, possibly using different techiques).

Finally, some of our results can be applied in a natural way to BRWs in random environment (as in [3, Section 7]) but, again, this goes beyond the purpose of the paper.

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