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STATIC CONDENSATION PROCEDURES FOR HYBRIDIZED MIXED FINITE ELEMENT METHODS

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ABSTRACT. Stemming from the characterization of the static condensation procedure for mixed hybridized methods introduced in [9, 10], in this paper we use Helmholtz decompositions to obtain a substructuring of the local mapping problems, in order to end up with simpler systems of reduced size. This procedure is effective especially when dealing with high degree or variable degree approximations. Moreover, we extend the variational characterization of static condensation to more general saddle–point formulations. Two relevant examples of mixed–hybridized methods are considered, namely, the classical Galerkin Dual-Mixed Hybridized scheme and the Discontinuous Petrov–Galerkin (DPG) scheme of [7].

1. INTRODUCTION

We consider the following elliptic model problem written in mixed form on the polygonal domain $\Omega \subset \mathbb{R}^2$ with boundary Γ:
Given $f \in L^2(\Omega)$, find $(\boldsymbol{q}, u) \in (H(\mathrm{div}\,;\Omega) \times L^2(\Omega))$, such that

$$\left\{ \begin{array}{ll} \boldsymbol{q} = -\kappa \nabla u & \text{in } \Omega, \\ \mathrm{div}\, \boldsymbol{q} + du = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{array} \right. \tag{1.1}$$

where $\kappa(\boldsymbol{x})$ is a bounded diffusion coefficient, such that $\kappa(\boldsymbol{x}) \geq \kappa_0 > 0$ almost everywhere in Ω, and $d(\boldsymbol{x})$ is a nonnegative bounded reaction coefficient. Homogeneous Dirichlet boundary conditions are considered here for sake of simplicity, as all subsequent results can be extended straightforwardly to more general boundary conditions. In the numerical discretization of problem (1.1), we seek a simultaneous approximation of both primal and dual variables. This approach gives rise to a non-definite linear algebraic system of large size, the solution of which is computationally expensive. The so-called "hybridization" procedure introduces an additional field to relax the interelement continuity requirement for the normal component of the vector field [2], in such a way that static elimination of primal and dual variables can be performed, yielding a much smaller matrix equation in the sole hybrid field. The resulting method is called Dual–Mixed Hybridized (DMH). Hybridization is traditionally carried out via an algebraic manipulation of the full linear system acting on the primal, dual and hybrid variables; however, such a manipulation may become rather involved and dissuade from its actual use in computer implementation. In [9, 10], a novel characterization of the static condensation is proposed, showing that the reduced system may be derived from the solution of suitable local

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mapping problems. This interpretation provides a variational framework to the hybridized formulation and an explicit expression of the entries of the condensed local coefficient matrix and right-hand side. Moreover, the novel characterization may be profitably employed to establish connections between approximation methods obtained by using different discrete spaces, in the same spirit as in [17].

In the first part of this article, based on the above variational procedure, we use two different orthogonal decompositions of the local finite element space for the flux variable, in order to single out a substructuring of the elementwise mapping problems. As a matter of fact, ending up with smaller and easier local matrices is a desirable property, especially if one wants to deal with high order approximations, for example in the context of p–type adaptive refinement. The first decomposition "horizontally" splits the space into a solenoidal part and a weakly irrotational complement of the same polynomial degree. The second (alternative) decomposition "vertically" splits the space into a two-level hierarchy. One level has a lower degree, while the other level is a "surplus" space of functions having the same polynomial degree as the original space. Additionally, the surplus space is "horizontally" decomposed via a Helmholtz strategy [25]. The above two kinds of decomposition lead to simplified local mapping problems, that can be solved with a reduced computational effort.

In the second part of the article, we propose an extension of the variational characterization of the static condensation to more general saddle–point formulations, as discussed in [20]. As an example, we focus our attention on the Discontinuous Petrov–Galerkin (DPG) method introduced in [4] and subsequently analyzed in [7], providing a variational characterization of the local mappings and the expression of the entries of the local element stiffness matrix and load vector. As already done for the DMH formulation, we use a Helmholtz decomposition technique to identify a substructure in the local mapping problems, that leads to an easier implementation of the DPG method.

The paper is organized as follows. In Sect. 2 we introduce the geometrical quantities and the approximation spaces used in the discretization. In Sect. 3 we recall the DMH formulation of (1.1). In Sect. 4 the steps of the abstract variational characterization given in [9] are discussed, emphasizing the connection with nonconforming formulations. In Sect. 5 we introduce two variants of the Helmholtz decomposition of the flux discrete space and we discuss the structure of the resulting set of reduced mapping problems. In Sect. 6 we apply the variational characterization to the DPG formulation of lowest order. In Sect. 7 we briefly address the issue of dealing with more general nonhomogeneous Dirichlet–Neumann boundary conditions, and in Sect. 8 we draw some final conclusions.

2. Mathematical Preliminaries

In this section, we provide the basic notation that will be used throughout the article and we introduce the finite element approximation spaces.

2.1. **Notation.** In view of the finite element discretization of problem (1.1), we let $\overline{\Omega} = \bigcup \overline{K}$ be a regular [8] partition $\mathcal{T}_h$ of the domain Ω into triangular elements K of area $|K|$, diameter h_K and barycenter $\boldsymbol{x}_{CG} = (x_{CG}, y_{CG})^T$. For each $K \in \mathcal{T}_h$, we denote by ∂K and $\boldsymbol{n}_{\partial K}$ the boundary of the element and its outward unit normal vector (according to a counterclockwise orientation along ∂K), respectively. If v

is any function defined in Ω, we denote by v^K its restriction to K and by $v_{\partial K}$ its restriction on ∂K. We denote by $\mathcal{E}_h$ the set of all the $\mathtt{Ned}$ edges of $\mathcal{T}_h$, with $\mathtt{Ni}$ internal edges and $\mathtt{N}_b$ boundary edges, respectively. The set of the internal edges of $\mathcal{E}_h$ is denoted by $\mathcal{E}_{h,i}$. For each edge $e \in \partial K$, we indicate by $|e|$ the length of e and by $\boldsymbol{n}_{\partial K}|_e$ the restriction of $\boldsymbol{n}_{\partial K}$ on $e \in \mathcal{E}_{h,i}$ and by $\boldsymbol{n}_\Gamma|_e$ the restriction of $\boldsymbol{n}_{\partial K}$ on $e \in \Gamma$. Let w and $\boldsymbol{q}$ be piecewise smooth (p.s.) scalar and vector–valued functions on $\mathcal{T}_h$, respectively. For each internal edge $e = \partial K_1 \cap \partial K_2 \in \mathcal{E}_{h,i}$, we define the *jump* $[\![\cdot]\!]$ and the *average* $\{\cdot\}$ operators (the latter only for w) as

$$[\![w]\!] = w^{K_1}\boldsymbol{n}_{\partial K_1}|_e + w^{K_2}\boldsymbol{n}_{\partial K_2}|_e, \qquad [\![\boldsymbol{q}]\!] = \boldsymbol{q}^{K_1}\cdot\boldsymbol{n}_{\partial K_1}|_e + \boldsymbol{q}^{K_2}\cdot\boldsymbol{n}_{\partial K_2}|_e,$$

$$\{w\} = \frac{w^{K_1}|_e + w^{K_2}|_e}{2},$$

while on each boundary edge $e = \partial K_1 \cap \Gamma$, we set

$$[\![w]\!] = w^{K_1}\boldsymbol{n}_{\partial K_1}|_e, \qquad [\![\boldsymbol{q}]\!] = \boldsymbol{q}^{K_1}\cdot\boldsymbol{n}_{\partial K_1}|_e,$$

$$\{w\} = \frac{w^{K_1}|_e}{2}.$$

Notice that the jump of a vector-valued function is a scalar quantity, while the jump of a scalar-valued function is a vector quantity. Notice also that the above definitions are invariant if we exchange K_1 with K_2. Using the previous definitions, the following identity can be proved to hold [3]

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K} w\,\boldsymbol{q}\cdot\boldsymbol{n}_{\partial K}\,ds = \sum_{e\in\mathcal{E}_h}\int_e [\![\boldsymbol{q}]\!]\,\{w\}\,ds + \sum_{e\in\mathcal{E}_h}\int_e \{\boldsymbol{q}\}\cdot[\![w]\!]\,ds. \tag{2.1}$$

In view of the analysis of the DPG formulation, it is useful to rewrite (2.1) in an equivalent form, by introducing on each element boundary the new quantity

$$\mu_{\partial K} = \boldsymbol{q}^K\cdot\boldsymbol{n}_{\partial K} \qquad \forall K\in\mathcal{T}_h. \tag{2.2}$$

If w has the meaning of a displacement, and $\boldsymbol{q}$ is the associated stress, then μ has the meaning of normal stress. We define the jump and average operators for μ as

$$\begin{aligned}
\{\mu\} &= \frac{\mu_{\partial K_1}\boldsymbol{n}_{\partial K_1}|_e + \mu_{\partial K_2}\boldsymbol{n}_{\partial K_2}|_e}{2}, & [\![\mu]\!] &= \mu_{\partial K_1}|_e + \mu_{\partial K_2}|_e, & \forall e\in\mathcal{E}_{h,i},\\
\{\mu\} &= \frac{\mu_{\partial K_1}\boldsymbol{n}_{\partial K_1}|_e}{2}, & [\![\mu]\!] &= \mu_{\partial K_1}|_e, & \forall e\in\Gamma.
\end{aligned} \tag{2.3}$$

Notice that the jump of the scalar function μ is a scalar quantity, while the average of μ is a vector-valued quantity. Then, using (2.2) and (2.3) in (2.1), we easily get

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K} w\,\mu\,ds = \sum_{e\in\mathcal{E}_h}\int_e [\![\mu]\!]\,\{w\}\,ds + \sum_{e\in\mathcal{E}_h}\int_e \{\mu\}\cdot[\![w]\!]\,ds. \tag{2.4}$$

2.2. **Finite element spaces.** We now define the basic polynomial spaces and projection operators that will be used in the numerical approximation of (1.1). For $k \geq 0$, we let $\mathbb{P}_k(K)$ to be the space of polynomials in two variables of total degree at most k on K and by $R_k(\partial K)$ the space of polynomials in two variables of total degree at most k on each edge of ∂K. Notice that functions belonging to $R_k(\partial K)$

are not necessarily continuous at the vertices of K. Furthermore, we denote by $\mathbb{RT}_k(K)$, $k \geq 0$, the k-order Raviart–Thomas finite element space [22] defined as

$$\mathbb{RT}_k(K) = (\mathbb{P}_k(K))^2 \oplus \mathcal{P}_k(K)\,\boldsymbol{x}, \tag{2.5}$$

where $\boldsymbol{x} = (x,\, y)^T$ and where $\mathcal{P}_k(K) = \mathrm{span}\{x^\alpha y^\beta, \alpha + \beta = k\}$. The degrees of freedom of the space $\mathbb{RT}_k(K)$ are given by (see [5])

$$\begin{cases} \displaystyle\int_{\partial K} \mathbf{q} \cdot \mathbf{n}_{\partial K}\, \xi \, ds & \forall \xi \in \mathbb{R}_k(\partial K), \qquad k \geq 0, \\ \displaystyle\int_K \mathbf{q} \cdot \mathbf{p}\, dx & \forall \mathbf{p} \in (\mathbb{P}_{k-1}(K))^2, \quad k \geq 1. \end{cases} \tag{2.6}$$

Notice that if a function $\boldsymbol{\tau}^K$ belongs to $\mathbb{RT}_k(K)$, then its divergence belongs to $\mathbb{P}_k(K)$ and its normal trace on ∂K belongs to $R_k(\partial K)$. We also introduce the nonconforming Crouzeix–Raviart finite element space [11]

$$\mathbb{P}_1^{nc}(K) = \mathrm{span}\,\{\widetilde{\varphi}_i\}_{i=1}^3\,, \qquad \widetilde{\varphi}_i(\boldsymbol{x}_{m,j}) = \delta_{ij}, \quad i, j = 1, 2, 3 \quad \forall K \in \mathcal{T}_h \tag{2.7}$$

where each basis function $\widetilde{\varphi}_i \in \mathbb{P}_1(K)$ and $\boldsymbol{x}_{m,j}$ is the midpoint of each edge $e_j \in \partial K$. Notice that $\widetilde{\varphi}_i = 1 - 2\,\varphi_i$, φ_i being the i-th nodal basis function for $\mathbb{P}_1(K)$, and $\widetilde{\varphi}_i(x)|_{e_i} = 1$ for each fixed $i = 1, 2, 3$. This implies that

$$\nabla \widetilde{\varphi}_i = \frac{2\,\boldsymbol{n}_{e_i}}{H_i} = \frac{|e_i|\,\boldsymbol{n}_{e_i}}{|K|} \qquad i = 1, 2, 3, \tag{2.8}$$

H_i being the height associated with edge e_i. We denote by $\mathbb{P}_1^{nc}(\mathcal{T}_h)$ the space of functions whose restriction to each triangle K of $\mathcal{T}_h$ belongs to $\mathbb{P}_1^{nc}(K)$. Notice that $\mathbb{P}_1^{nc}(\mathcal{T}_h)$ is not a subspace of $H^1(\Omega)$. Finally, for any set $S \subseteq \Omega$, we denote by $\mathcal{P}_S^k\,\eta$ the L^2 projection of a function $\eta \in L^2(S)$ onto $\mathbb{P}_k(S)$.

3. The Dual–Mixed Hybridized Method

For each $K \in \mathcal{T}_h$, we define the local finite element spaces

$$V_h(K) = \mathbb{RT}_k(K), \quad W_h(K) = \mathbb{P}_k(K), \quad L_h(\partial K) = R_k(\partial K). \tag{3.1}$$

The corresponding global finite element spaces are

$$V_h = \prod_{K \in \mathcal{T}_h} V_h(K), \quad W_h = \prod_{K \in \mathcal{T}_h} W_h(K), \quad L_h = \prod_{K \in \mathcal{T}_h} L_h(\partial K), \tag{3.2}$$

with the constraint

$$\lambda_{\partial K_1}|_e = \lambda_{\partial K_2}|_e \quad \forall e \in \mathcal{E}_{h,i} \quad \text{and} \quad \lambda_{\partial K_1}|_e = 0 \quad \forall e \in \Gamma, \qquad \lambda \in L_h. \tag{3.3}$$

Conditions (3.3) express the fact that functions belonging to L_h are single–valued on $\mathcal{E}_{h,i}$ and satisfy the Dirichlet boundary condition of problem (1.1) in an essential manner on Γ.

Then, the Dual–Mixed Hybridized formulation of (1.1), (see [2]), reads: Find $(\boldsymbol{q}_h,\, u_h,\, \lambda_h) \in (V_h \times W_h \times L_h)$ such that for all $(\boldsymbol{v}_h,\, w_h,\, \eta_h) \in (V_h \times W_h \times L_h)$ we have

$$
\begin{aligned}
&\int_\Omega \mathcal{K}\, \boldsymbol{q}_h \cdot \boldsymbol{v}_h \, dx - \sum_{K \in \mathcal{T}_h} \int_K u_h \operatorname{div} \boldsymbol{v}_h \, dx + \sum_{e \in \mathcal{E}_{h,i}} \int_e \lambda_h \ [\![\boldsymbol{v}_h]\!] \, ds = 0, \\
&- \sum_{K \in \mathcal{T}_h} \int_K w_h \operatorname{div} \boldsymbol{q}_h \, dx - \int_\Omega d\, w_h\, u_h \, dx = - \int_\Omega f\, w_h \, dx, \\
&\sum_{e \in \mathcal{E}_{h,i}} \int_e \eta_h \ [\![\boldsymbol{q}_h]\!] \, ds = 0,
\end{aligned} \tag{3.4}
$$

where we have set $\mathcal{K} := \kappa^{-1}$. Equation $(3.4)_3$ expresses the fact that functions in V_h, that are *a-priori* fully discontinuous over $\mathcal{T}_h$, satisfy in weak form an interelement compatibility condition, physically corresponding to the action-reaction principle. Problem (3.4) admits a unique solution (see [5] for a proof).

4. Variational Characterization of the DMH Method

In this section, we briefly recall the steps of the abstract variational characterization of the mixed–hybridized method (3.4) given in [9], providing some new insights.

4.1. **Generalized displacement DMH problem from superposition of effects.** The triple $(\boldsymbol{q}_h,\, u_h,\, \lambda_h)$, solution of (3.4), can be characterized as follows.

The pair $(\boldsymbol{q}_h,\, u_h) \in (V_h \times W_h)$ is given by

$$
(\boldsymbol{q}_h,\, u_h) = (\boldsymbol{q}_{\lambda_h},\, u_{\lambda_h}) + (\boldsymbol{q}_f,\, u_f), \tag{4.1}
$$

where the two pairs at the right–hand side are suitable lifting operators associated with λ_h and f, respectively.

The Lagrange multiplier $\lambda_h \in L_h$ is the unique solution of

$$
a_h(\lambda_h,\, z_h) = b_h(z_h) \qquad \forall z_h \in L_h, \tag{4.2}
$$

where

$$
a_h(\lambda_h,\, z_h) = \int_\Omega \mathcal{K}\, \boldsymbol{q}_{\lambda_h} \cdot \boldsymbol{q}_{z_h} \, dx + \int_\Omega d\, u_{\lambda_h}\, u_{z_h} \, dx, \quad b_h(z_h) = \int_\Omega f\, u_{z_h} \, dx. \tag{4.3}
$$

The variational problem (4.2) is a generalized displacement approximation of the model problem (1.1) and can be written in matrix form as

$$
\mathbb{E}\, \Lambda = \mathbb{H}, \tag{4.4}
$$

where $\mathbb{E}$ is the global, symmetric and positive definite stiffness matrix, Λ is the vector of the degrees of freedom of λ_h over $\mathcal{E}_{h,i}$ and $\mathbb{H}$ is the load vector.

4.2. **Local mappings.** The pair $(\boldsymbol{q}_{\mathtt{m}}, u_{\mathtt{m}})$ is the local lifting of a given hybrid variable function $\mathtt{m} \in L_h(\partial K)$, such that $\mathtt{m} = 0$ for each edge $e = \partial K \cap \Gamma$, while $(\boldsymbol{q}_f, u_f)$ is the local mapping of a given source term $f \in L^2(K)$.

We introduce the following local bilinear forms

$$
\begin{aligned}
a^K(\boldsymbol{q},\, \boldsymbol{v}) &= \int_K \mathcal{K}\, \boldsymbol{q}\cdot \boldsymbol{v}\, dx &&: \quad (V_h(K)\times V_h(K)) \to \mathbb{R},\\
b^K(u,\, \boldsymbol{v}) &= -\int_K u \operatorname{div} \boldsymbol{v}\, dx &&: \quad (U_h(K)\times V_h(K)) \to \mathbb{R},\\
c^K(u,\, w) &= \int_K d\, u\, w\, dx &&: \quad (U_h(K)\times U_h(K)) \to \mathbb{R}.
\end{aligned}
$$

The first local mapping reads: Given $\mathtt{m} \in L_h(\partial K)$, find $(\boldsymbol{q}_{\mathtt{m}},\, u_{\mathtt{m}}) \in (V_h(K)\times W_h(K))$ such that for all $K \in \mathcal{T}_h$

$$
\begin{aligned}
a^K(\boldsymbol{q}_{\mathtt{m}},\, \boldsymbol{v}_h) + b^K(u_{\mathtt{m}},\, \boldsymbol{v}_h) &= G^K_{\mathtt{m}}(\boldsymbol{v}_h) && \forall \boldsymbol{v}_h \in V_h(K),\\
b^K(w_h,\, \boldsymbol{q}_{\mathtt{m}}) - c^K(w_h,\, u_{\mathtt{m}}) &= 0 && \forall w_h \in W_h(K),
\end{aligned} \tag{4.5}
$$

where we have introduced the local linear form

$$
G^K_\xi(\boldsymbol{v}_h) = -\int_{\partial K} \xi\, \boldsymbol{v}_h \cdot \boldsymbol{n}_{\partial K}\, ds : V_h(K) \to \mathbb{R},
$$

which is parametrically depending on the given function $\xi \in L_h(\partial K)$.

The second local mapping reads: Given $f \in L^2(K)$, find $(\boldsymbol{q}_f,\, u_f) \in (V_h(K) \times W_h(K))$ such that for all $K \in \mathcal{T}_h$

$$
\begin{aligned}
a^K(\boldsymbol{q}_f,\, \boldsymbol{v}_h) + b^K(u_f,\, \boldsymbol{v}_h) &= 0 && \forall \boldsymbol{v}_h \in V_h(K),\\
b^K(w_h,\, \boldsymbol{q}_f) - c^K(w_h,\, u_f) &= F^K_f(w_h) && \forall w_h \in W_h(K),
\end{aligned} \tag{4.6}
$$

where we have introduced the local linear form

$$
F^K_\phi(w_h) = -\int_K \phi\, w_h\, dx : W_h(K) \to \mathbb{R},
$$

which is parametrically depending on the given function $\phi \in L^2(K)$.

Using superposition of effects in $(3.4)_3$, yields

$$
\sum_{e\in\mathcal{E}_{h,i}} \int_e \eta_h\ [\![\boldsymbol{q}_{\lambda_h}]\!]\, ds = -\sum_{e\in\mathcal{E}_{h,i}} \int_e \eta_h\ [\![\boldsymbol{q}_f]\!]\, ds \qquad \forall \eta_h \in L_h. \tag{4.7}
$$

Substituting into (4.7) the characterizations of $\boldsymbol{q}_{\lambda_h}$ and $\boldsymbol{q}_f$, and using Lemma 2.2 of [9], leads to the generalized displacement formulation (4.2), or, equivalently, to the algebraic form (4.4).

4.3. **Local matrices in the lowest–order case.** As an example, in this section we provide the explicit expressions of the entries of the element matrix and load vector $\mathbb{E}^K$ and $\mathbb{H}^K$ in the lowest-order case $k = 0$. These expressions can be obtained from the finite element discretization of the local mappings.

We start computing the local mapping associated with a given function $\mathtt{m} \in R_0(\partial K)$. With this aim, we express the solution $(\boldsymbol{q}_{\mathtt{m}},\, u_{\mathtt{m}})$ of (4.5) in the basis of $R_0(\partial K)$ as [9]

$$
(\boldsymbol{q}_{\mathtt{m}},\, u_{\mathtt{m}}) = \sum_{i=1}^{3} (\boldsymbol{q}_{\mathtt{m},e_i},\, u_{\mathtt{m},e_i})\, \lambda_i, \tag{4.8}
$$

where $\lambda_i \in \mathbb{R}$ is the constant value of $\mathtt{m}$ on the edge e_i, and $(\boldsymbol{q}_{\mathtt{m},e_i}, u_{\mathtt{m},e_i})$ is the solution of problem (4.5) with $\mathtt{m}_{h\,\partial K} = \mathbf{1}_i$, $\mathbf{1}_i \in \mathbb{R}^3$ being the unit vector of the axis x_i, $i = 1, 2, 3$. The discretization of the local mapping problem with right–hand sides $\mathtt{m}_{h,\,\partial K} = \mathbf{1}_i$, $i = 1, 2, 3$, gives rise to three 2-by-2 block diagonal systems, the solution of which is

(4.9)

$$u_{\mathtt{m},e_i} = \frac{1}{3(1+\overline{d}^K\,\mathtt{h}^2)},$$

$$\boldsymbol{q}_{\mathtt{m},e_i} = -\overline{\kappa}^K\,\frac{\boldsymbol{n}_{\partial K,e_i}|e_i|}{|K|} - \frac{\overline{d}^K\,u_{\mathtt{m},e_i}}{2}(\boldsymbol{x}-\boldsymbol{x}_{CG}) = -\overline{\kappa}^K\,\nabla\widetilde{\varphi}_i - \frac{\overline{d}^K\,u_{\mathtt{m},e_i}}{2}(\boldsymbol{x}-\boldsymbol{x}_{CG}),$$

where $\overline{\kappa}^K := \left(\overline{\mathcal{K}}^K\right)^{-1} = \left(\mathcal{P}^0_K\mathcal{K}\right)^{-1}$ is the *harmonic average* of the diffusion coefficient κ over the element K and

$$\overline{d}^K = \mathcal{P}^0_K\,d, \qquad \mathtt{h}^2 := \frac{\displaystyle\int_K \mathcal{K}^K(\boldsymbol{x}-\boldsymbol{x}_{CG})\cdot(\boldsymbol{x}-\boldsymbol{x}_{CG})\,dx}{4\,|K|} \qquad \forall K \in \mathcal{T}_h.$$

Notice that in the case $k = 0$, the *average values* of $\mathcal{K}$ and d are an outcome of the computation.

Remark 4.1. Eq. (4.9) reveals that for the quantity ρ_i introduced in [9] Sect. 3.2, the relation $\rho_i = 1/3$ holds irrespectively of the shape of the mesh element K.

Remark 4.2. Substituting (2.8) into (4.9) allows to write the local mapping associated with a given $\mathtt{m} \in R_0(\partial K)$ as

$$u_{\mathtt{m}} = \frac{\mathcal{P}^0_K u^*_{\mathtt{m}}}{1+\overline{d}^K\,\mathtt{h}^2}, \qquad \boldsymbol{q}_{\mathtt{m}} = -\overline{\kappa}^K\,\nabla u^*_{\mathtt{m}} - \frac{\overline{d}^K\,u_{\mathtt{m}}}{2}(\boldsymbol{x}-\boldsymbol{x}_{CG}), \tag{4.10}$$

where $u^*_{\mathtt{m}}$ is the $\mathbb{P}_1$–nonconforming interpolant of $\mathtt{m}$ within K.

Let us now compute the local mapping associated with $f \in L^2(K)$. The corresponding discrete system is 2-by-2 block diagonal, the solution of which is

$$u_f = \frac{\mathtt{h}^2\,\mathcal{P}^0_K f}{1+\overline{d}^K\,\mathtt{h}^2}, \qquad \boldsymbol{q}_f = \frac{u_f}{2\,\mathtt{h}^2}(\boldsymbol{x}-\boldsymbol{x}_{CG}) \qquad \forall K \in \mathcal{T}_h. \tag{4.11}$$

Eventually, the entries of $\mathbb{E}^K$ and $\mathbb{H}^K$ for each element $K \in \mathcal{T}_h$ are computed as

(4.12)

$$(\mathbb{E}^K)_{ij} = \int_K \left(\overline{\kappa}^K\right)^{-1}\boldsymbol{q}_{\mathtt{m},e_i}\cdot\boldsymbol{q}_{\mathtt{m},e_j}\,dx + \int_K \overline{d}^K\,u_{\mathtt{m},e_i}\,u_{\mathtt{m},e_j}\,dx \qquad i,j = 1,2,3,$$

$$(\mathbb{H}^K)_i = \int_K f\,u_{\mathtt{m},e_i}\,dx = \int_K \mathcal{P}^0_K f\,u_{\mathtt{m},e_i}\,dx \qquad i = 1,2,3.$$

Remark 4.3. Assume that $d = 0$, and define the finite element space

$$\mathcal{V}_{h,0} = \{v_h \in \mathbb{P}^{nc}_1(\mathcal{T}_h),\ v_h = 0 \text{ at the midpoint of each edge } e \in \Gamma\}.$$

Then, using (4.9) into (4.12), it can be shown that problem (4.2) is equivalent to: Find $u^*_{\lambda_h} \in \mathcal{V}_{h,0}$ such that

$$\mathcal{A}_h(u^*_{\lambda_h}, \phi_h) = \mathcal{F}_h(\phi_h) \qquad \forall \phi_h \in \mathcal{V}_{h,0}, \tag{4.13}$$

where

$$\mathcal{A}_h(u^*_h, \phi_h) = \sum_{K \in \mathcal{T}_h} \int_K \overline{\kappa}^K \, \nabla u^*_{\lambda_h} \cdot \nabla \phi_h \, dx : \mathcal{V}_{h,0} \times \mathcal{V}_{h,0} \to \mathbb{R},$$

and $\mathcal{F}_h(\cdot)$ is the approximate computation of $\int_\Omega f \, \phi_h \, dx$ using the two–dimensional midpoint quadrature rule. The variational problem (4.2) in the case $d = 0$ is thus a generalized displacement finite element scheme of nonconforming type (cf. [22] and [5] Chapt. V) with two differences. The first difference is that the harmonic average of the diffusion coefficient is automatically performed over each mesh element K, instead of the usual average. The second difference is that f is replaced by its projection over constant functions on each element $K \in \mathcal{T}_h$. For a further connection between DMH methods and (extended) nonconforming formulations, we refer to Lemma 2.3 and 2.4 of [2] and Sect. 4 of [14].

5. Substructuring of Local Mappings

As the finite element degree k increases, the solution of the local problems (4.5) and (4.6) becomes involved. The characterization of the local mappings provided in this section can be used to single out a further substructuring of the local mapping problems. This turns out to be of interest in view of p-type refinement or variable degree formulations (see [13], [16],[12]).

5.1. One–level Helmholtz decomposition.

Throughout this section and the next one, we assume that the diffusion coefficient κ and the reaction term d are piecewise constant (cf.[25]). For all $K \in \mathcal{T}_h$, we introduce the affine manifold (see [21], Chpt. 7)

$$V^0_h(K) := \{\boldsymbol{v} \in V_h(K) \,|\, b^K(w, \, \boldsymbol{v}) = 0, \, \forall w \in W_h(K)\} = \{\boldsymbol{v} \in V_h(K) \,|\, \mathrm{div}\, \boldsymbol{v} = 0\}.$$

The space $V^0_h(K)$ induces the following "horizontal" Helmholtz decomposition of $V_h(K)$

$$V_h(K) = V^0_h(K) \oplus (V^0_h(K))^\perp \qquad \forall K \in \mathcal{T}_h, \tag{5.1}$$

where

$$(V^0_h(K))^\perp = \left\{\boldsymbol{v} \in V_h(K) \,|\, a^K(\boldsymbol{v}, \boldsymbol{v}^0) = 0, \, \forall \boldsymbol{v}^0 \in V^0_h(K)\right\}. \tag{5.2}$$

The representation (5.1) of the space $V_h(K)$ into a solenoidal and a weakly irrotational part is unique (see also [25]).

We have the following result.

Proposition 5.1. Let $d \geq 0$. Then

$$\boldsymbol{q}_f = \boldsymbol{q}_f^\perp \in (V^0_h(K))^\perp. \tag{5.3}$$

Moreover, if $d = 0$, then

$$\boldsymbol{q}_{\mathtt{m}} = \boldsymbol{q}^0_{\mathtt{m}} \in V^0_h(K). \tag{5.4}$$

Proof. Set $d = 0$ and consider the local mapping problem (4.5). Surjectivity of the divergence operator in $(4.5)_2$ immediately implies that $\boldsymbol{q}_{\mathtt{m}} \equiv \boldsymbol{q}_{\mathtt{m}}^0 \in V_h^0(K)$. Consider now the second local mapping. Taking $\boldsymbol{v}_h = \boldsymbol{v}_h^0 \in V_h^0(K)$ in $(4.6)_1$, immediately yields $\boldsymbol{q}_f^0 = \mathbf{0}$, which implies that $\boldsymbol{q}_f \equiv \boldsymbol{q}_f^{\perp} \in (V_h^0(K))^{\perp}$. If $d > 0$, it is immediate to check that both non-zero $\boldsymbol{q}_{\mathtt{m}}^0$ and $\boldsymbol{q}_{\mathtt{m}}^{\perp}$ contribute to the mapping $\boldsymbol{q}_{\mathtt{m}}$, whilst the same property $\boldsymbol{q}_f \equiv \boldsymbol{q}_f^{\perp}$ holds as in the case $d = 0$. □

The result of Prop. 5.1 suggests that the local problems (4.5) and (4.6) may be further split into smaller subproblems, the computation of which turns out to be easier than the solution of the full size problem. We introduce the following notation

$$\underbrace{\dim V_h(K)}_{M_k} = \underbrace{\dim V_h^0(K)}_{M_k^0} + \underbrace{\dim (V_h^0(K))^{\perp}}_{M_k^{\perp}}.$$

We start considering the local problem (4.5). In the case $d = 0$, using (5.1), relation $(4.5)_1$ can be written as

$$\begin{aligned} a^K(\boldsymbol{q}_{\mathtt{m}}, \boldsymbol{v}_h^0) &= G_{\mathtt{m}}^K(\boldsymbol{v}_h^0) \quad \forall \boldsymbol{v}_h^0 \in V_h^0(K), \\ b^K(u_{\mathtt{m}}, \boldsymbol{v}_h^{\perp}) &= G_{\mathtt{m}}^K(\boldsymbol{v}_h^{\perp}) \quad \forall \boldsymbol{v}_h^{\perp} \in (V_h^0(K))^{\perp}, \end{aligned} \tag{5.5}$$

which yields the block diagonal system

$$\begin{bmatrix} A^0 & 0_{(M_k^0, M_k^{\perp})} \\ 0^T_{(M_k^0, M_k^{\perp})} & A^{\perp} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{\mathtt{m}} \\ u_{\mathtt{m}} \end{bmatrix} = \begin{bmatrix} g_{\mathtt{m}}^0 \\ g_{\mathtt{m}}^{\perp} \end{bmatrix}, \tag{5.6}$$

where A^0 and $A^{\perp}$ are square matrices of size M_k^0 and $M_k^{\perp}$, respectively. Each sub–block of (5.6) is invertible because a^K is coercive on $V_h^0(K)$ and b^K satisfies the inf–sup condition. Solving system (5.6) completely determines the M_k^0 non-zero degrees of freedom of $\boldsymbol{q}_{\mathtt{m}}$ and the $M_k^{\perp} = \dim(\mathbb{P}_k)$ degrees of freedom of $u_{\mathtt{m}}$.

In the case $d > 0$, the unknown $\boldsymbol{q}_{\mathtt{m}}^0$ can be obtained from the square invertible system of size M_k^0

$$A^0 \boldsymbol{q}_{\mathtt{m}}^0 = g_{\mathtt{m}}^0,$$

whilst the (coupled) variables $\boldsymbol{q}_{\mathtt{m}}^{\perp}$ and $u_{\mathtt{m}}$ are the unique solution of the system

$$\begin{bmatrix} A^{\perp} & B^T \\ B & C \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{\mathtt{m}}^{\perp} \\ u_{\mathtt{m}} \end{bmatrix} = \begin{bmatrix} g_{\mathtt{m}}^{\perp} \\ 0_{(M_k^{\perp}, 1)} \end{bmatrix}.$$

Each matrix in the above system is square of size equal to M_k^0.

We consider now the local problem (4.6). Due to the fact that $\boldsymbol{q}_f = \boldsymbol{q}_f^{\perp}$, such a problem can be rewritten as

$$\begin{aligned} b^K(w_h, \boldsymbol{q}_f) \quad -c^K(w_h, u_f) &= F_f^K(w_h) \qquad \forall w_h \in W_h(K), \\ a^K(\boldsymbol{q}_f, \boldsymbol{v}_h^{\perp}) \quad +b^K(u_f, \boldsymbol{v}_h^{\perp}) &= 0 \qquad \forall \boldsymbol{v}_h^{\perp} \in (V_h^0(K))^{\perp}, \end{aligned}$$

or, equivalently, as the system

$$\begin{bmatrix} B^{\perp} & C \\ \\ A^{\perp} & (B^{\perp})^T \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_f \\ \\ u_f \end{bmatrix} = \begin{bmatrix} F \\ \\ 0_{(M_k^{\perp},1)} \end{bmatrix}. \tag{5.7}$$

Each matrix in (5.7) is square of size equal to $M_k^{\perp}$. Solving system (5.7) completely determines the $M_k^{\perp}$ non-zero degrees of freedom of $\boldsymbol{q}_f$ and the $M_k^{\perp}$ degrees of freedom of u_f. Notice that in the case $d = 0$, we have that $C = 0_{(M_k^{\perp}, M_k^{\perp})}$ and system (5.7) is block lower triangular.

5.2. **Construction of a basis for V_h^0 and $(V_h^0)^{\perp}$.** In order to compute the solution of the various systems introduced in Sect. 5.1, we need a basis for functions belonging to $V_h^0(K)$ and $(V_h^0(K))^{\perp}$ defined in (5.1) and (5.2), respectively.

In the following, we indicate by $\widehat{K}$ the reference triangle, by $\widehat{\boldsymbol{x}} = (\widehat{x}, \widehat{y})^T$ the coordinate vector on $\widehat{K}$ and by $F_K : \widehat{K} \to K$ the affine transformation

$$\boldsymbol{x} = F_K(\widehat{\boldsymbol{x}}) = \mathbf{b}_K + B_K \widehat{\boldsymbol{x}}, \quad \mathbf{b}_K = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad B_K = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix},$$

which transforms $\widehat{K}$ into the element $K \in \mathcal{T}_h$ of vertices (x_i, y_i), $i = 1, 2, 3$, labelled with a counterclockwise orientation. Accordingly, for a scalar–valued function v and a vector–valued function $\boldsymbol{\tau}$ defined on K, we set

$$v(\boldsymbol{x}) = \widehat{v}(\widehat{\boldsymbol{x}}), \qquad \boldsymbol{\tau}(\boldsymbol{x}) = \mathfrak{P}_K \widehat{\boldsymbol{\tau}}(\widehat{\boldsymbol{x}}) = \frac{1}{m_K} B_K \, \widehat{\boldsymbol{\tau}}(\widehat{\boldsymbol{x}}), \tag{5.8}$$

where $\mathfrak{P}_K$ is the Piola transformation (cf. [5], Sect. III.1.3) and $m_K := |\det B_K|$.

Let $\widehat{\phi} \in H^1(\widehat{K})$ and let $\mathbf{curl} : \widehat{\phi} \to \mathbf{curl}\, \widehat{\phi} = \left(\partial\widehat{\phi}/\partial\widehat{y}, \, -\partial\widehat{\phi}/\partial\widehat{x}\right)^T$. Then, we have (see [5], Corollary 3.2)

$$V_h^0(\widehat{K}) = \mathbf{curl}\, \mathbb{P}_{k+1}(\widehat{K}). \tag{5.9}$$

We have that $\dim(\mathbb{RT}_k(\widehat{K})) = (k+1)(k+3) = M_k$, $M_k^0 = \frac{1}{2}(k+1)(k+4)$, $M_k^{\perp} = \frac{1}{2}(k+1)(k+2) \equiv \dim(\mathbb{P}_k(\widehat{K}))$. The values of M_k, M_k^0 and $M_k^{\perp}$ are summarized in Tab. 1 for $k \in [0, 3]$.

k	M_k	M_k^0	$M_k^{\perp}$
0	3	2	1
1	8	5	3
2	15	9	6
3	24	14	10

TABLE 1. Values of M_k, M_k^0 and $M_k^{\perp}$ for the space $\mathbb{RT}_k(\widehat{K})$ as a function of the degree k.

The computation of the basis for $V_h^0(\widehat{K})$ follows directly from (5.9). The computation of the basis for the orthogonal complement $(V_h^0(\widehat{K}))^{\perp}$ requires solving the following linear algebraic system associated with (5.2)

$$\mathcal{V}^{\perp}\,\mathbf{x}^{\perp} = \mathbf{0}_{(M_k^0,1)}, \tag{5.10}$$

where $\mathcal{V}^{\perp} \in \mathbb{R}^{M_k^0 \times M_k^{\perp}}$ and $\mathbf{x}^{\perp} \in \mathbb{R}^{M_k^{\perp}}$. For a given $k \geq 0$, the solution of (5.10) can be found once for all, for example, by computing an orthonormal basis for the null space of $\mathcal{V}^{\perp}$ via a singular value decomposition.

In the case $k = 0$, it is easy to check that the above procedure yields

$$V_h^0(\widehat{K}) = \mathbf{curl}\,\mathbb{P}_1(\widehat{K}) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}, \quad (V_h^0(\widehat{K}))^{\perp} = \begin{pmatrix} \widehat{x} - \widehat{x}_{CG} \\ \widehat{y} - \widehat{y}_{CG} \end{pmatrix}.$$

In the case $k = 1$, we have

$$V_h^0(\widehat{K}) = \mathbf{curl}\,\mathbb{P}_2(\widehat{K}) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \widehat{x} \\ -\widehat{y} \end{pmatrix}, \begin{pmatrix} 0 \\ \widehat{x} \end{pmatrix}, \begin{pmatrix} \widehat{y} \\ 0 \end{pmatrix}\right\},$$

and using the `Matlab` command `null(A,'r')` to solve (5.10), we obtain

$$(V_h^0(\widehat{K}))^{\perp} = \operatorname{span}\left\{\widehat{\boldsymbol{\tau}}_1,\, \widehat{\boldsymbol{\tau}}_2,\, \widehat{\boldsymbol{\tau}}_3\right\},$$

where

$$\widehat{\boldsymbol{\tau}}_1 = \begin{pmatrix} -\dfrac{1}{2} + \widehat{x} + \widehat{y} \\[2ex] -\dfrac{1}{2} + \dfrac{1}{2}\widehat{x} + \widehat{y} \end{pmatrix}, \quad \widehat{\boldsymbol{\tau}}_2 = \begin{pmatrix} -\dfrac{3}{5}\widehat{x} + \dfrac{1}{10}\widehat{y} + \widehat{x}^2 \\[2ex] -\dfrac{1}{20} - \dfrac{1}{10}\widehat{x} + \widehat{x}\,\widehat{y} \end{pmatrix}, \quad \widehat{\boldsymbol{\tau}}_3 = \begin{pmatrix} \dfrac{7}{20} + \dfrac{3}{5}\widehat{x} + \dfrac{1}{5}\widehat{y} + \widehat{x}\,\widehat{y} \\[2ex] -\dfrac{3}{10} + \dfrac{2}{5}\widehat{x} + \widehat{y}^2 \end{pmatrix}.$$

5.3. **Hierarchical–Helmholtz splitting.** The approach discussed in the previous section is the most straightforward way to build a basis for $V_h^0(\widehat{K})$ and $(V_h^0(\widehat{K}))^{\perp}$. However, it yields a computation technique that may become involved as the degree k grows. In this section, we discuss an alternative approach, based on a hierarchical splitting of the Raviart-Thomas finite element space (see [19],[1],[25]). The advantage of this approach is that computations already carried out for lower degrees can be profitably reused, yielding more effective algorithms.

Throughout this section, for a given polynomial vector space $Q_p(\widehat{K})$ of degree p, $p \geq 0$, the spaces $Q_p^0(\widehat{K})$ and $Q_p^{\perp}(\widehat{K})$ will denote the divergence–free part and its complement, respectively, of the *Helmholtz decomposition*

$$Q_p(\widehat{K}) = Q_p^0(\widehat{K}) \oplus Q_p^{\perp}(\widehat{K}). \tag{5.11}$$

Moreover, for a given polynomial scalar space $W_{r+1}(\widehat{K})$ of degree $r+1$, $r \geq 0$, the spaces $W_r(\widehat{K})$ and $\widetilde{W}_{r+1}(\widehat{K})$ will denote the scalar space of degree r and the scalar surplus polynomial space of degree $r+1$, respectively, of the *scalar hierarchical decomposition*

$$W_{r+1}(\widehat{K}) = W_r(\widehat{K}) \oplus \widetilde{W}_{r+1}(\widehat{K}), \tag{5.12}$$

whilst for a given polynomial vector space $Q_{s+1}(\widehat{K})$ of degree $s+1$, $s \geq 0$, the spaces $Q_s(\widehat{K})$ and $\widetilde{Q}_{s+1}(\widehat{K})$ will denote the vector space of degree s and the surplus, respectively, of the *vector hierarchical decomposition*

$$Q_{s+1}(\widehat{K}) = Q_s(\widehat{K}) \oplus \widetilde{Q}_{s+1}(\widehat{K}). \tag{5.13}$$

We have the following result, that generalizes to a generic degree k the analysis of [25], Sect. 2.

Proposition 5.2. In the Helmholtz decomposition

$$\mathbb{RT}_{k+1}(\widehat{K}) = \mathbb{RT}^0_{k+1}(\widehat{K}) \oplus \mathbb{RT}^{\perp}_{k+1}(\widehat{K}), \tag{5.14}$$

the following hierarchical structure can be identified

$$\begin{cases} \mathbb{RT}^0_{k+1}(\widehat{K}) = \mathbb{RT}^0_0(\widehat{K}) \oplus \displaystyle\sum_{l=0}^{k} \widetilde{\mathbb{RT}}^0_{l+1}(\widehat{K}), \\ \mathbb{RT}^{\perp}_{k+1}(\widehat{K}) = \mathbb{RT}^{\perp}_0(\widehat{K}) \oplus \displaystyle\sum_{l=0}^{k} \widetilde{\mathbb{RT}}^{\perp}_{l+1}(\widehat{K}). \end{cases} \tag{5.15}$$

Proof. Set $k = 0$. Applying the decomposition of type (5.13) to the space $\mathbb{RT}^0_{k+1}(\widehat{K}) = \mathbb{RT}^0_1(\widehat{K})$ yields

$$\mathbb{RT}^0_1(\widehat{K}) = \mathbb{RT}^0_0(\widehat{K}) \oplus \widetilde{\mathbb{RT}}^0_1(\widehat{K}).$$

Set now $k = 1$. Applying the decomposition of type (5.13) to the space $\mathbb{RT}^0_{k+1}(\widehat{K}) = \mathbb{RT}^0_2(\widehat{K})$ yields

$$\mathbb{RT}^0_2(\widehat{K}) = \mathbb{RT}^0_1(\widehat{K}) \oplus \widetilde{\mathbb{RT}}^0_2(\widehat{K}) = \mathbb{RT}^0_0(\widehat{K}) \oplus \sum_{l=0}^{1} \widetilde{\mathbb{RT}}^0_{l+1}(\widehat{K}).$$

Then, induction on k gives $(5.15)_1$. Set again $k = 0$. Applying the decomposition of type (5.13) to the space $\mathbb{RT}_{k+1}(\widehat{K}) = \mathbb{RT}_1(\widehat{K})$ and then a decomposition of type (5.11) to each of the spaces $\mathbb{RT}_0(\widehat{K})$ and $\widetilde{\mathbb{RT}}_1(\widehat{K})$ yields

$$\begin{aligned} \mathbb{RT}_1(\widehat{K}) &= \mathbb{RT}_0(\widehat{K}) \oplus \widetilde{\mathbb{RT}}_1(\widehat{K}) = (\mathbb{RT}^0_0(\widehat{K}) \oplus \mathbb{RT}^{\perp}_0(\widehat{K})) \oplus (\widetilde{\mathbb{RT}}^0_1(\widehat{K}) \oplus \widetilde{\mathbb{RT}}^{\perp}_1(\widehat{K})) \\ &= (\mathbb{RT}^0_0(\widehat{K}) \oplus \widetilde{\mathbb{RT}}^0_1(\widehat{K})) \oplus (\mathbb{RT}^{\perp}_0(\widehat{K}) \oplus \widetilde{\mathbb{RT}}^{\perp}_1(\widehat{K})). \end{aligned}$$

Using $(5.15)_1$ (with $k = 0$), the first term of the right–hand side of the previous expression is equal to $\mathbb{RT}^0_1(\widehat{K})$; this leads to recognize

$$\mathbb{RT}^{\perp}_1(\widehat{K}) = \widetilde{\mathbb{RT}}^{\perp}_0(\widehat{K}) \oplus \widetilde{\mathbb{RT}}^{\perp}_1(\widehat{K})$$

Induction on k yields relation $(5.15)_2$.

□

Remark 5.3. Notice that the result of Prop. 5.2 represents at the same time a "horizontal" (relation (5.14)) and a nested "vertical" (relation (5.15)) decomposition of the space $\mathbb{RT}_{k+1}(\widehat{K})$.

In order to characterize the space $\widetilde{\mathbb{RT}}^0_{l+1}(\widehat{K})$, $l \geq 0$, of Prop. 5.2, we observe that, from [5], Corollary 3.2, we have

$$\mathbb{RT}^0_{l+1}(\widehat{K}) = \mathbf{curl}\, \mathbb{P}_{l+2}(\widehat{K}),$$

and from

$$\mathbf{curl}\, \mathbb{P}_{l+2}(\widehat{K}) = \mathbf{curl}\, (\mathbb{P}_{l+1}(\widehat{K})) \oplus \mathbf{curl}\, (\widetilde{\mathbb{P}}_{l+2}(\widehat{K})), \tag{5.16}$$

we have

$$\widetilde{\mathbb{RT}}^0_{l+1}(\widehat{K}) = \mathbf{curl}\, \widetilde{\mathbb{P}}_{l+2}(\widehat{K}). \tag{5.17}$$

We are left with the issue of characterizing the space $\widetilde{\mathbb{RT}}^{\perp}_{l+1}(\widehat{K})$. The basis for this space can be built by properly picking independent functions from the span of $\widetilde{\mathbb{RT}}_{l+1}(\widehat{K})$. Before proceeding, following [19], it is convenient to replace the standard internal degrees of freedom of the Raviart–Thomas finite element space of degree k, given by $(2.6)_2$, with the alternative (and equivalent) unisolvent set of internal degrees of freedom

$$
\left\{
\begin{array}{lll}
\displaystyle\int_{\widehat{K}} \operatorname{div} \mathbf{q}\, r\, dx & \forall r \in \mathbb{P}_k(\widehat{K}) \setminus \mathbb{R}, & k \geq 1, \\
\displaystyle\int_{\widehat{K}} \operatorname{curl} \mathbf{q}\, s\, dx & \forall s \in \mathbb{P}_{k-2}(\widehat{K}), & k \geq 2,
\end{array}
\right.
\tag{5.18}
$$

where $\operatorname{curl} : \mathbf{q} \to \operatorname{curl} \mathbf{q} = (\frac{\partial q_x}{\partial y} - \frac{\partial q_2}{\partial y})$, for any $\mathbf{q} = (q_x, q_y)^T \in (H^1(\widehat{K}))^2$.
We need first to give a constructive characterization of the space $\widetilde{\mathbb{RT}}_{l+1}(\widehat{K})$. With this aim, we introduce the interpolation operator $\rho_l : \mathbb{RT}_{l+1}(\widehat{K}) \to \mathbb{RT}_l(\widehat{K})$, such that for all $\mathbf{q} \in \mathbb{RT}_{l+1}(\widehat{K})$

$$
\begin{array}{lclll}
\displaystyle\int_{\partial\widehat{K}} (\rho_l \mathbf{q}) \cdot \mathbf{n}_{\partial\widehat{K}}\, \xi\, ds & = & \displaystyle\int_{\partial\widehat{K}} \mathbf{q} \cdot \mathbf{n}_{\partial\widehat{K}}\, \xi\, ds & \forall \xi \in R_l(\partial\widehat{K}), & l \geq 0, \\
\displaystyle\int_{\widehat{K}} \operatorname{div}(\rho_l \mathbf{q})\, r\, dx & = & \displaystyle\int_{\widehat{K}} \operatorname{div} \mathbf{q}\, r\, dx & \forall r \in \mathbb{P}_l(\widehat{K}) \setminus \mathbb{R}, & l \geq 1, \\
\displaystyle\int_{\widehat{K}} \operatorname{curl}(\rho_l \mathbf{q})\, s\, dx & = & \displaystyle\int_{\widehat{K}} \operatorname{curl} \mathbf{q}\, s\, dx & \forall s \in \mathbb{P}_{l-2}(\widehat{K}), & l \geq 2,
\end{array}
\tag{5.19}
$$

and then, we set $\widetilde{\mathbb{RT}}_{l+1}(\widehat{K}) = (\mathrm{Id} - \rho_l)\mathbb{RT}_{l+1}(\widehat{K})$. Due again to [5], Corollary 3.2, we have

$$
\dim(\mathbf{curl}\, \mathbb{P}_{l+2}(\widehat{K})) = \dim(\mathbb{P}_{l+2}(\widehat{K})) - 1,
$$

from which, using (5.16) and (5.17), we get

$$
\dim(\widetilde{\mathbb{RT}}^{0}_{l+1}(\widehat{K})) = \dim(\mathbf{curl}\, \widetilde{\mathbb{P}}_{l+2}(\widehat{K})) = \dim(\mathbb{P}_{l+2}(\widehat{K})) - \dim(\mathbb{P}_{l+1}(\widehat{K})) = l + 3,
$$

which eventually gives the number of independent functions in the span of $\widetilde{\mathbb{RT}}_{l+1}(\widehat{K})$

$$
\begin{aligned}
\dim(\widetilde{\mathbb{RT}}^{\perp}_{l+1}(\widehat{K})) &= \dim(\widetilde{\mathbb{RT}}_{l+1}(\widehat{K})) - \dim(\widetilde{\mathbb{RT}}^{0}_{l+1}(\widehat{K})) \\
&= (\dim(\mathbb{RT}_{l+1}(\widehat{K})) - \dim(\mathbb{RT}_l(\widehat{K}))) - \dim(\widetilde{\mathbb{RT}}^{0}_{l+1}(\widehat{K})) \\
&= (2l + 5) - (l + 3) = l + 2.
\end{aligned}
$$

Functions $\widetilde{\mathbf{q}} \in \widetilde{\mathbb{RT}}^{\perp}_{l+1}(\widehat{K})$ can be characterized by enforcing the following additional constraints

$$
\begin{array}{lll}
\displaystyle\int_{\partial\widehat{K}} \widetilde{\mathbf{q}} \cdot \mathbf{n}_{\partial\widehat{K}}\, \widetilde{\xi}\, ds = 0 & \forall \widetilde{\xi} \in \widetilde{R}_{l+1}(\partial\widehat{K}), & l \geq 0, \\
\displaystyle\int_{\widehat{K}} \operatorname{curl} \widetilde{\mathbf{q}}\, \widetilde{s}\, dx = 0 & \forall \widetilde{s} \in \widetilde{\mathbb{P}}_{l-1}(\widehat{K}), & l \geq 1,
\end{array}
\tag{5.20}
$$

where we set $\widetilde{\mathbb{P}}_0(\widehat{K}) \equiv \mathbb{P}_0(\widehat{K})$ and $\widetilde{R}_{l+1}(\partial\widehat{K})$ is the surplus space such that

$$R_{l+1}(\partial\widehat{K}) = R_l(\partial\widehat{K}) \oplus \widetilde{R}_{l+1}(\partial K). \tag{5.21}$$

We observe that relation $(5.20)_1$ provides 3 constraints, irrespectively of the degree, while relation $(5.20)_2$ provides l constraints, their sum being the necessary amount required to filter out from the space $\widetilde{\mathbb{RT}}_{l+1}(\widehat{K})$ the $l+3$ divergence–free functions belonging to $\widetilde{\mathbb{RT}}^0_{l+1}(\widehat{K})$. Tab. 2 summarizes the degrees of freedom and the constraints enforced by relations $(5.20)_1$ and $(5.20)_2$ for $l \in [0,3]$. We refer to [15] for an alternative characterization of the space $\widetilde{R}^{\perp}_{l+1}(\partial K)$.

space	l	$\dim(\widetilde{\mathbb{RT}}_{l+1}) = 2l+5$	$\dim(\widetilde{\mathbb{RT}}^0_{l+1}) = l+3$	constraints from $(5.20)_1$	constraints from $(5.20)_2$
$\mathbb{RT}_1$	0	5	3	3	-
$\mathbb{RT}_2$	1	7	4	3	1
$\mathbb{RT}_3$	2	9	5	3	2
$\mathbb{RT}_4$	3	11	6	3	3

TABLE 2. Summary of the degrees of freedom and of the number of constraints necessary to build the basis of $\widetilde{\mathbb{RT}}^{\perp}_{l+1}(\widehat{K})$ from the basis of $\widetilde{\mathbb{RT}}_{l+1}(\widehat{K})$.

The degrees of freedom of each subspace in the hierarchical–Helmholtz decomposition $\mathbb{RT}_{k+1}(K) = \mathbb{RT}_k(K) \oplus \widetilde{\mathbb{RT}}^0_{k+1}(K) \oplus \widetilde{\mathbb{RT}}^{\perp}_{k+1}(K)$, $k \geq 0$, on the generic element K, are depicted in Fig. 1 in the cases $k = 0, 1, 2$.

5.4. **Approximate hierarchical substructuring.** The spaces $\mathbb{RT}_k(\widehat{K})$, $\widetilde{\mathbb{RT}}^0_{k+1}(\widehat{K})$ and $\widetilde{\mathbb{RT}}^{\perp}_{k+1}(\widehat{K})$ in the decomposition

$$\mathbb{RT}_{k+1}(\widehat{K}) = \mathbb{RT}_k(\widehat{K}) \oplus \widetilde{\mathbb{RT}}^0_{k+1}(\widehat{K}) \oplus \widetilde{\mathbb{RT}}^{\perp}_{k+1}(\widehat{K}) \tag{5.22}$$

are not orthogonal with respect to the bilinear form $a^K(\cdot,\cdot)$. In order to end up with a computable hierarchical counterpart of problems (5.6) and (5.7), we replace the original local bilinear form $a^K(\cdot,\cdot)$ with a modified local bilinear form $\widetilde{a}^K(\cdot,\cdot)$ spectrally equivalent, and defined as (see [25],[15]; see also [24] for an alternative approach)

$$\widetilde{a}^K(\boldsymbol{q},\boldsymbol{v}) = a^K(\boldsymbol{q}_k,\boldsymbol{v}_k) + a^K(\widetilde{\boldsymbol{q}}^0_{k+1},\widetilde{\boldsymbol{v}}^0_{k+1}) + a^K(\widetilde{\boldsymbol{q}}^{\perp}_{k+1},\widetilde{\boldsymbol{v}}^{\perp}_{k+1}), \tag{5.23}$$

where

$$\boldsymbol{q} = \boldsymbol{q}_k + \widetilde{\boldsymbol{q}}^0_{k+1} + \widetilde{\boldsymbol{q}}^{\perp}_{k+1}, \quad \boldsymbol{v} = \boldsymbol{v}_k + \widetilde{\boldsymbol{v}}^0_{k+1} + \widetilde{\boldsymbol{v}}^{\perp}_{k+1}, \tag{5.24}$$

and $\boldsymbol{q}_k,\boldsymbol{v}_k \in \mathbb{RT}_k(\widehat{K})$, $\widetilde{\boldsymbol{q}}^0_{k+1},\widetilde{\boldsymbol{v}}^0_{k+1} \in \widetilde{\mathbb{RT}}^0_{k+1}(\widehat{K})$ and $\widetilde{\boldsymbol{q}}^{\perp}_{k+1},\widetilde{\boldsymbol{v}}^{\perp}_{k+1} \in \widetilde{\mathbb{RT}}^{\perp}_{k+1}(\widehat{K})$. The decomposition (5.22) is orthogonal with respect to the modified bilinear form $\widetilde{a}^K(\cdot,\cdot)$. Lemma 2.1 and Theorem 2.2 of [15] ensure the equivalence of the solutions of the modified mapping problems, where $a^K(\cdot,\cdot)$ is replaced by $\widetilde{a}^K(\cdot,\cdot)$, with problems (4.5) and (4.6).

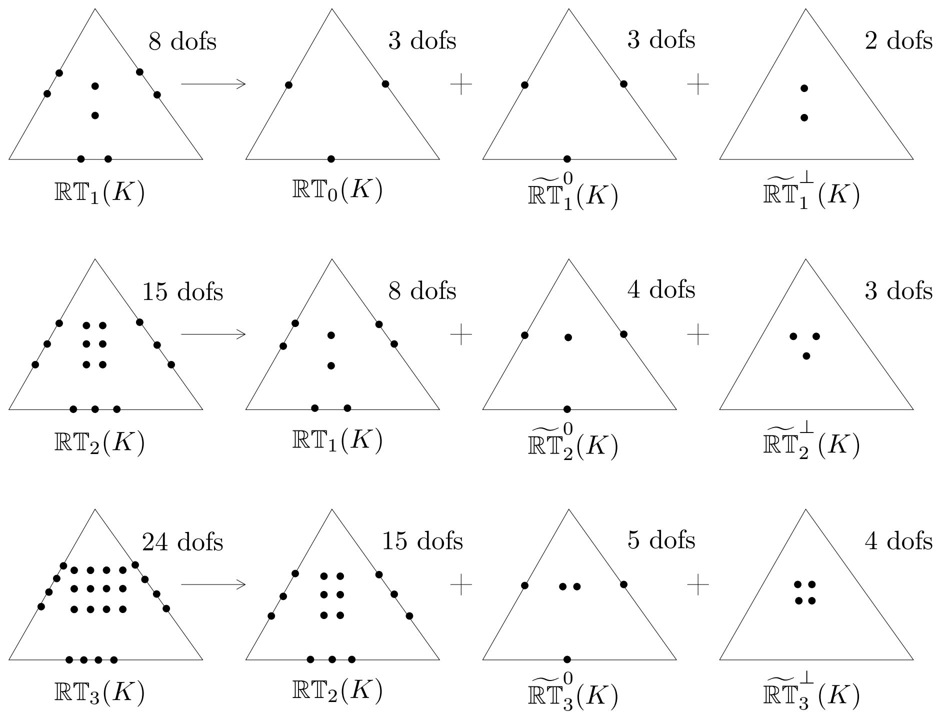

FIGURE 1. Degrees of freedom of the subspaces of the decompositions $\mathbb{RT}_1(K) = \mathbb{RT}_0(K) \oplus \widetilde{\mathbb{RT}}_1^0(K) \oplus \widetilde{\mathbb{RT}}_1^\perp(K)$ (top), $\mathbb{RT}_2(K) = \mathbb{RT}_1(K) \oplus \widetilde{\mathbb{RT}}_2^0(K) \oplus \widetilde{\mathbb{RT}}_2^\perp(K)$ (middle) and $\mathbb{RT}_3(K) = \mathbb{RT}_2(K) \oplus \widetilde{\mathbb{RT}}_3^0(K) \oplus \widetilde{\mathbb{RT}}_3^\perp(K)$ (bottom) for element K (upon mapping from $\widehat{K}$).

In the following, we provide the details of the approximate solution of the local mapping problems in the case $d = 0$. We consider the hierarchical splittings

$$u_{k+1} = u_k + \widetilde{u}_{k+1}, \quad w_{k+1} = w_k + \widetilde{w}_{k+1}, \quad \mathtt{m}_{k+1} = \mathtt{m}_k + \widetilde{\mathtt{m}}_{k+1}, \tag{5.25}$$

where $u_{k+1}, w_{k+1} \in \mathbb{P}_{k+1}(\widehat{K})$, $u_k, w_k \in \mathbb{P}_k(\widehat{K})$, $\widetilde{u}_{k+1}, \widetilde{w}_{k+1} \in \widetilde{\mathbb{P}}_{k+1}(\widehat{K})$, and $\mathtt{m}_{k+1} \in R_{k+1}(\partial\widehat{K})$, $\mathtt{m}_k \in R_k(\partial\widehat{K})$, $\widetilde{\mathtt{m}}_{k+1} \in \widetilde{R}_{k+1}(\partial\widehat{K})$, respectively.

Using (5.23), (5.24) and (5.25) into (4.5), we obtain two independent systems. The first system is associated with the lower order space in the hierarchical decomposition and reads:

Given $\mathtt{m} \in R_k(\partial \widehat{K})$, find $(\mathbf{q}_{k,\mathtt{m}}, u_{k,\mathtt{m}}) \in (\mathbb{RT}_k(\widehat{K}) \times \mathbb{P}_k(\widehat{K}))$, such that
(5.26)

$$\begin{aligned} a^K(\mathbf{q}_{k,\mathtt{m}}, \mathbf{v}_h) + b^K(u_{k,\mathtt{m}}, \mathbf{v}_h) &= -\int_{\partial \widehat{K}} \mathtt{m}_k \, \mathbf{v}_h \cdot \mathbf{n}_{\partial \widehat{K}} \, ds & \forall \mathbf{v}_h \in \mathbb{RT}_k(\widehat{K}), \\ b^K(w_h, \mathbf{q}_{k,\mathtt{m}}) &= 0 & \forall w_h \in \mathbb{P}_k(\widehat{K}). \end{aligned}$$

System (5.26) is a uniquely solvable local saddle–point problem to which the hierarchical decomposition (5.22)–(5.25) can be recursively applied.
The second system reads:

Given $\widetilde{\mathtt{m}} \in \widetilde{R}_{k+1}(\partial \widehat{K})$, find $(\widetilde{\mathbf{q}}_{k+1,\mathtt{m}} \equiv \widetilde{\mathbf{q}}^0_{k+1,\mathtt{m}}, \widetilde{u}_{k+1,\mathtt{m}}) \in (\widetilde{\mathbb{RT}}_{k+1}(\widehat{K}) \times \widetilde{\mathbb{P}}_{k+1}(\widehat{K}))$, such that

$$\begin{aligned} a^K(\widetilde{\mathbf{q}}^0_{k+1,\mathtt{m}}, \mathbf{v}_h) &= -\int_{\partial \widehat{K}} \widetilde{\mathtt{m}}_{k+1} \, \mathbf{v}_h \cdot \mathbf{n} \, ds & \forall \mathbf{v}_h \in \widetilde{\mathbb{RT}}^0_{k+1}(\widehat{K}), \\ b^K(\widetilde{u}_{k+1,\mathtt{m}}, \mathbf{v}_h) &= -\int_{\partial \widehat{K}} \widetilde{\mathtt{m}}_{k+1} \, \mathbf{v}_h \cdot \mathbf{n}_{\partial \widehat{K}} \, ds & \forall \mathbf{v}_h \in \widetilde{\mathbb{RT}}^\perp_{k+1}(\widehat{K}). \end{aligned} \tag{5.27}$$

System (5.27) is uniquely solvable and it allows to completely determine $\widetilde{\mathbf{q}}_{k+1,\mathtt{m}}$, and $\widetilde{u}_{k+1,\mathtt{m}}$ (cf. (5.6)).

Using (5.23), (5.24) and (5.25) into (4.6), we obtain again two independent systems. The first system is associated with the lower order space in the hierarchical decomposition and reads:

Given $f \in L^2(\widehat{K})$, find $(\mathbf{q}_{k,f}, u_{k,f}) \in (\mathbb{RT}_k(\widehat{K}) \times \mathbb{P}_k(\widehat{K}))$, such that

$$\begin{aligned} a^K(\mathbf{q}_{k,f}, \mathbf{v}_h) + b^K(u_{k,f}, \mathbf{v}_h) &= 0 & \forall \mathbf{v}_h \in \mathbb{RT}_k(\widehat{K}), \\ b^K(w_h, \mathbf{q}_{k,f}) &= -\int_{\widehat{K}} \Pi_k f \, w_h \, dx & \forall w_h \in \mathbb{P}_k(\widehat{K}). \end{aligned} \tag{5.28}$$

System (5.28) is a uniquely solvable local saddle–point problem to which the hierarchical decomposition (5.22)–(5.25) can be recursively applied.
The second system reads:

Given $f \in L^2(\widehat{K})$, find $(\widetilde{\mathbf{q}}_{k+1,f} \equiv \widetilde{\mathbf{q}}^\perp_{k+1,f}, \widetilde{u}_{k+1,f}) \in (\widetilde{\mathbb{RT}}_{k+1}(\widehat{K}) \times \widetilde{\mathbb{P}}_{k+1}(\widehat{K}))$, such that
(5.29)

$$\begin{aligned} a^K(\widetilde{\mathbf{q}}^\perp_{k+1,f}, \mathbf{v}_h) + b^K(\widetilde{u}_{k+1,f}, \mathbf{v}_h) &= 0 & \forall \mathbf{v}_h \in \widetilde{\mathbb{RT}}^\perp_{k+1}(\widehat{K}), \\ b^K(w_h, \widetilde{\mathbf{q}}^\perp_{k+1,f}) &= -\int_{\widehat{K}} (f - \Pi_k f) \, w_h \, dx & \forall w_h \in \widetilde{\mathbb{P}}_{k+1}(\widehat{K}). \end{aligned}$$

System (5.29) is uniquely solvable and it allows to completely determine $\widetilde{\mathbf{q}}_{k+1,f}$ and $\widetilde{u}_{k+1,f}$ (cf. (5.7) with $C = \emptyset$).

6. The Discontinuous Petrov–Galerkin Method

In this section, we recall the Discontinuous Petrov–Galerkin formulation of lowest order (DPG_0) of (1.1), introduced in [4] and analyzed in [6, 7]. For each $K \in \mathcal{T}_h$, we introduce the local trial finite element spaces

$$Q_h(K) = (\mathbb{P}_0(K))^2, \qquad U_h(K) = \mathbb{P}_0(K), \qquad L_h(\partial K) = M_h(\partial K) = R_0(\partial K),$$

and the local test spaces

$$V_h(K) = \mathbb{RT}_0(K), \qquad W_h(K) = \mathbb{P}_1(K).$$

The global trial spaces are defined as

$$\begin{aligned} Q_h &= \prod_{K\in\mathcal{T}_h} Q_h(K), & U_h &= \prod_{K\in\mathcal{T}_h} U_h(K), \\ L_h &= \prod_{K\in\mathcal{T}_h} L_h(\partial K), & M_h &= \prod_{K\in\mathcal{T}_h} M_h(\partial K). \end{aligned}$$

The space L_h satisfies the constraint (3.3), while the space M_h satisfies the constraint

$$[\![\mu]\!] = 0 \qquad \forall e \in \mathcal{E}_{h,i}, \quad \mu \in M_h, \tag{6.1}$$

where the jump $[\![\cdot]\!]$ is defined in (2.3). Condition (6.1) states the fact that functions in M_h satisfy in an essential manner an interelement compatibility condition, that physically expresses the action–reaction principle.

The global test spaces are defined as

$$V_h = \prod_{K\in\mathcal{T}_h} V_h(K), \quad W_h = \prod_{K\in\mathcal{T}_h} W_h(K).$$

Remark 6.1. In the DPG_0 approximation of (1.1), the hybrid variable λ_h is an approximation of $u_{\partial K}$, while the hybrid variable μ_h is an approximation of $\mu_{\partial K} = \boldsymbol{q} \cdot \boldsymbol{n}_{\partial K}$ (see Sect. 2). As a consequence, two different sets of finite element spaces are needed, because the numerical approach is of Petrov–Galerkin type. Moreover, the global spaces Q_h, U_h, V_h and W_h are fully discontinuous on $\mathcal{T}_h$.

The DPG_0 finite element approximation of problem (1.1) reads:
Find $(\boldsymbol{q}_h, u_h, \lambda_h, \mu_h) \in (Q_h \times U_h \times L_h \times M_h)$ such that for all $(\boldsymbol{v}_h, w_h) \in (V_h \times W_h)$ we have

$$\begin{aligned} &\int_\Omega \mathcal{K}\, \boldsymbol{q}_h \cdot \boldsymbol{v}_h \, dx - \sum_{K\in\mathcal{T}_h} \int_K u_h \operatorname{div} \boldsymbol{v}_h \, dx + \sum_{e\in\mathcal{E}_{h,i}} \int_e \lambda_h \ [\![\boldsymbol{v}_h]\!] \, ds = 0, \\ &- \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{q}_h \cdot \nabla w_h \, dx + \int_\Omega d\, u_h \, w_h \, dx \\ &+ \sum_{e\in\Gamma} \int_e [\![\mu_h]\!] \ \{w_h\} \ ds + \sum_{e\in\mathcal{E}_h} \int_e \{\mu_h\} \cdot \ [\![w_h]\!] \, ds = \int_\Omega f \, w_h \, dx. \end{aligned} \tag{6.2}$$

The sum over boundary edges at the left–hand side in $(6.2)_2$ is obtained using (6.1) into (2.4).

In [7] it has been shown that problem (6.2) admits a unique solution in the case $d = 0$. The case $d(\boldsymbol{x}) \geq 0$ can be dealt with as follows. Set $f \equiv 0$ and let $u^*_{\lambda_h} \in \mathcal{V}_{h,0}$ be the piecewise linear nonconforming function such that

$$\mathcal{P}^0_e u^*_{\lambda_h} = \lambda_h \qquad \forall e \in \mathcal{E}_h. \tag{6.3}$$

Let us consider Eq. $(6.2)_1$. It can be checked that [7]

$$\boldsymbol{q}_h^K = -\overline{\kappa}^K \nabla u^*_{\lambda_h}, \quad u_h^K = \mathcal{P}^0_K u^*_{\lambda_h} \qquad \forall K \in \mathcal{T}_h. \tag{6.4}$$

Let us consider equation $(6.2)_2$, and take $w_h = u^*_{\lambda_h} \in \mathcal{V}_{h,0}$. This choice implies that

$$\sum_{e\in\Gamma} \int_e [\![\mu_h]\!]\ \{w_h\}\ ds + \sum_{e\in\mathcal{E}_h} \int_e \{\mu_h\} \cdot\ [\![w_h]\!]\, ds = 0 \qquad \mu_h \in M_h. \tag{6.5}$$

Assuming to replace d^K with its mean value $\overline{d}^K$ for all $K \in \mathcal{T}_h$, and using (6.4) and (6.5), yields

$$\sum_{K\in\mathcal{T}_h} \int_K \overline{\kappa}^K\, |\nabla u^*_{\lambda_h}|^2\, dx + \sum_{K\in\mathcal{T}_h} \int_K \overline{d}^K\, \mathcal{P}^0_K u^*_{\lambda_h}\, u^*_{\lambda_h}\, dx = 0.$$

Observing that

$$\int_K \overline{d}^K\, \mathcal{P}^0_K u^*_{\lambda_h}\, u^*_{\lambda_h}\, dx = \overline{d}^K\ \left(\mathcal{P}^0_K u^*_{\lambda_h}\right)^2\, |K| = \int_K \overline{d}^K\ \left(\mathcal{P}^0_K u^*_{\lambda_h}\right)^2\, dx \qquad \forall K \in \mathcal{T}_h,$$

we immediately get

$$0 = \sum_{K\in\mathcal{T}_h} \int_K \left(\overline{\kappa}^K\, |\nabla u^*_{\lambda_h}|^2 + \overline{d}^K\, (\mathcal{P}^0_K u^*_{\lambda_h})^2\right)\, dx \geq \kappa_0 \sum_{K\in\mathcal{T}_h} \int_K |\nabla u^*_{\lambda_h}|^2\, dx,$$

which implies $u^*_{\lambda_h} = 0$ and consequently $\boldsymbol{q}_h = \boldsymbol{0}$, $u_h = 0$ and $\lambda_h = 0$.
Eventually, consider again Eq. $(6.2)_2$ and take this time $w_h \in W_h(K)$ for all $K \in \mathcal{T}_h$. Using (2.3), we obtain

$$0 = \sum_{e\in\Gamma} \int_e [\![\mu_h]\!]\ \{w_h\}\ ds + \sum_{e\in\mathcal{E}_h} \int_e \{\mu_h\} \cdot\ [\![w_h]\!]\, ds = \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu_h\, w_h\, ds,$$

which implies $\mu_h = 0$ (see [23] for the proof).

6.1. **Local DPG mappings.** We define the two pairs $(\boldsymbol{q}_{\mathtt{m}}, u_{\mathtt{m}}, \mu_{\mathtt{m}})$ and $(\boldsymbol{q}_f, u_f, \mu_f)$ as the solutions of local dual–primal mixed problems, which provide the lifting of the given data $\mathtt{m}$ on ∂K and f in K, respectively. With this aim, let us introduce the following local bilinear forms

$$\begin{aligned}
A^K(\boldsymbol{q},\,\boldsymbol{v}) &= \int_K \mathcal{K}^K\,\boldsymbol{q}\cdot\boldsymbol{v}\,dx && :(Q_h(K)\times V_h(K))\to\mathbb{R},\\
C^K(u,\,w) &= \int_K \overline{d}^K\,u\,w\,dx && :(U_h(K)\times W_h(K))\to\mathbb{R},\\
\widehat{B}_1^K((u,\lambda),\,\boldsymbol{v}) &= -\int_K u\,\mathrm{div}\,\boldsymbol{v}\,dx \\
&\quad + \int_{\partial K}\lambda\boldsymbol{v}\cdot\boldsymbol{n}_{\partial K}\,ds && :((U_h(K)\times L_h(\partial K))\times V_h(K))\to\mathbb{R},\\
\widehat{B}_2^K((\boldsymbol{q},\mu),\,w) &= -\int_K \boldsymbol{q}\cdot\nabla w\,dx \\
&\quad + \int_{\partial K}\mu\,w\,ds && :((Q_h(K)\times M_h(\partial K))\times W_h(K))\to\mathbb{R}.
\end{aligned}$$

Then, the first local mapping reads: Given $\mathtt{m}\in L_h(\partial K)$, find $(\boldsymbol{q}_{\mathtt{m}},\,u_{\mathtt{m}},\,\mu_{\mathtt{m}})\in(Q_h(K)\times U_h(K)\times M_h(\partial K))$ such that for all $K\in\mathcal{T}_h$

(6.6)

$$\begin{aligned}
A^K(\boldsymbol{q}_{\mathtt{m}},\boldsymbol{v}_h) &+ \widehat{B}_1^K((u_{\mathtt{m}},0),\,\boldsymbol{v}_h) = G_{\mathtt{m}}^K(\boldsymbol{v}_h) && \forall \boldsymbol{v}_h\in V_h(K),\\
\widehat{B}_2^K((\boldsymbol{q}_{\mathtt{m}},\mu_{\mathtt{m}}),\,w_h) &+ C^K(u_{\mathtt{m}},\,w_h) = 0 && \forall w_h\in W_h(K),
\end{aligned}$$

where we have introduced the local linear form

$$G_\xi^K(\boldsymbol{v}_h) = -\int_{\partial K}\xi\,\boldsymbol{v}_h\cdot\boldsymbol{n}_{\partial K}\,ds : V_h(K)\to\mathbb{R},$$

which is parametrically depending on the given function $\xi\in L_h(\partial K)$.
This lifting can be thought as the discretization of the local problem

$$\begin{aligned}
\boldsymbol{q}_{\mathtt{m}} &= -\kappa\nabla u_{\mathtt{m}} && \text{in } K,\\
\mathrm{div}\,\boldsymbol{q}_{\mathtt{m}} + d u_{\mathtt{m}} &= 0 && \text{in } K,\\
\mu_{\mathtt{m}} &= \boldsymbol{q}_{\mathtt{m}}\cdot\boldsymbol{n}_{\partial K} && \text{on } \partial K,\\
u_{\mathtt{m}} &= \mathtt{m} && \text{on } \partial K.
\end{aligned}\tag{6.7}$$

The second local mapping reads: Given $f\in L^2(K)$, find $(\boldsymbol{q}_f,\,u_f,\,\mu_f)\in(Q_h(K)\times U_h(K)\times M_h(\partial K))$ such that for all $K\in\mathcal{T}_h$

(6.8)

$$\begin{aligned}
A^K(\boldsymbol{q}_f,\boldsymbol{v}_h) &+ \widehat{B}_1^K((u_f,0),\,\boldsymbol{v}_h) = 0 && \forall \boldsymbol{v}_h\in V_h(K),\\
\widehat{B}_2^K((\boldsymbol{q}_f,\mu_f),\,w_h) &+ C^K(u_f,\,w_h) = -F_f^K(w_h) && \forall w_h\in W_h(K),
\end{aligned}$$

where we have introduced the local linear form

$$F_\phi^K(w_h) = -\int_K \phi\,w_h\,dx : W_h(K)\to\mathbb{R},$$

which is parametrically depending on the given function $\phi \in L^2(K)$. This lifting can be thought as the discretization of the local problem

$$
\begin{aligned}
\boldsymbol{q}_f &= -\kappa \nabla u_f && \text{in } K,\\
\operatorname{div} \boldsymbol{q}_f + d u_f &= f && \text{in } K,\\
\mu_f &= \boldsymbol{q}_f \cdot \boldsymbol{n}_{\partial K} && \text{on } \partial K,\\
u_f &= 0 && \text{on } \partial K.
\end{aligned}
\tag{6.9}
$$

6.2. **DPG generalized displacement formulation.** In the case of a Petrov-Galerkin formulation, the formal approach adopted for the Galerkin problem does not apply straightforwardly. This difficulty requires to resort to the generalized saddle–point theory of [20]. The following result holds.

Theorem 6.2. *Let* $(\boldsymbol{q}_h, u_h, \lambda_h, \mu_h)$ *be the unique solution of problem* (6.2)*. Then*

$$
\boldsymbol{q}_h = \boldsymbol{q}_{\lambda_h} + \boldsymbol{q}_f, \qquad u_h = u_{\lambda_h} + u_f, \qquad \mu_h = \mu_{\lambda_h} + \mu_f,
\tag{6.10}
$$

and the Lagrange multiplier $\lambda_h \in L_h$ *is the unique solution of*

$$
A_h(\lambda_h, \zeta_h) = F_h(\zeta_h) \qquad \forall \zeta_h \in L_h,
\tag{6.11}
$$

with

$$
\begin{aligned}
A_h(\lambda_h, \zeta_h) &= \sum_{K \in \mathcal{T}_h} \left(A^K(\boldsymbol{q}_{\lambda_h}, \boldsymbol{q}_{\zeta_h}) + C^K(u_{\lambda_h}, w_{\zeta_h}) \right),\\
F_h(\zeta_h) &= -\sum_{K \in \mathcal{T}_h} F_f^K(w_{\zeta_h}),
\end{aligned}
\tag{6.12}
$$

$w_{\zeta_h} \in \mathcal{V}_{h,0}$ *being the nonconforming piecewise linear function such that* $\mathcal{P}_e^0 w_{\zeta_h} = \zeta_h$ *for all* $e \in \mathcal{E}_h$, $\zeta_h \in L_h$.

The following result is needed for the proof of Theorem 6.2.

Lemma 6.3. *Let* $\xi_h \in M_h$ *and* $\zeta_h \in L_h$*. Then, it is immediate to see that*

$$
\sum_{e \in \mathcal{E}_h} \int_e [\![\xi_h]\!]\, \zeta_h \, ds = \sum_{e \in \mathcal{E}_h} \int_e [\![\xi_h]\!] \{w_{\zeta_h}\} \, ds \qquad \forall e \in \mathcal{E}_h.
\tag{6.13}
$$

Proof of Theorem 6.2
Relation (6.10) is an immediate consequence of the linearity of problem (6.2).

Let $\mathtt{m}$ be a given function in L_h. Relation (6.1) implies

$$
\sum_{e \in \mathcal{E}_h} \int_e [\![\mu_{\mathtt{m}}]\!]\, \eta_h \, ds = 0 \qquad \forall \eta_h \in L_h,
$$

and, further, relation (6.10) gives

$$
\sum_{e \in \mathcal{E}_h} \int_e [\![\mu_{\mathtt{m}}]\!]\, \eta_h \, ds = -\sum_{e \in \mathcal{E}_h} \int_e [\![\mu_f]\!]\, \eta_h \, ds \qquad \forall \eta_h \in L_h.
\tag{6.14}
$$

Using Lemma 6.3 with $\xi_h = \mu_{\mathtt{m}} + \mu_f$, we obtain

$$
\sum_{e \in \mathcal{E}_h} \int_e \{w_{\zeta_h}\}\ [\![\mu_{\mathtt{m}}]\!]\, ds = -\sum_{e \in \mathcal{E}_h} \int_e \{w_{\zeta_h}\}\ [\![\mu_f]\!]\, ds.
\tag{6.15}
$$

To characterize the left–hand side of (6.15), we consider the mapping problem (6.6). Choose $w_h = w_{\zeta_h}$ in $(6.6)_2$ and, correspondingly, $\boldsymbol{v}_h = \nabla w_{\zeta_h}$ in $(6.6)_1$. Then, summing over mesh elements, we get

$$\sum_{e\in\mathcal{E}_h}\int_e \{w_{\zeta_h}\}\ [\![\mu_{\mathtt{m}}]\!]\,ds = -\sum_{K\in\mathcal{T}_h}\left(A^K(\boldsymbol{q}_{\mathtt{m}},\,\boldsymbol{q}_{\zeta_h}) + C^K(u_{\mathtt{m}},\,w_{\zeta_h})\right). \tag{6.16}$$

To characterize the right–hand side of (6.15), we consider the mapping problem (6.8). It is immediate to check that $\boldsymbol{q}_f = \boldsymbol{0}$ and $u_f = 0$ (from (6.4) with $\lambda_h = 0$). Choosing $w_h = w_{\zeta_h}$ in $(6.8)_2$, and summing over $\mathcal{T}_h$, we immediately obtain

$$-\sum_{e\in\mathcal{E}_h}\int_e \{w_{\zeta_h}\}\ [\![\mu_f]\!]\,ds = \sum_{K\in\mathcal{T}_h} F_f^K(w_{\zeta_h}). \tag{6.17}$$

□

Remark 6.4. Notice that while $\mu_{\mathtt{m}}$ and μ_f are not necessarily reciprocal across interelement boundaries, their sum μ_h is reciprocal.

6.3. **Local matrices for the DPG$_0$ method.** Proceeding as in Sect. 4.3, we express the solution of (6.6) in the basis of $L_h(\partial K)$ as

$$(\boldsymbol{q}_{\mathtt{m}},\,u_{\mathtt{m}},\,\mu_{\mathtt{m}}) = \sum_{i=1}^{3}(\boldsymbol{q}_{\mathtt{m},e_i},\,u_{\mathtt{m},e_i},\,\mu_{\mathtt{m},e_i})\,\lambda_i, \tag{6.18}$$

where $(\boldsymbol{q}_{\mathtt{m},e_i},\,u_{\mathtt{m},e_i},\,\mu_{\mathtt{m},e_i})$ is the solution of problem (6.6) with $\mathtt{m} = \mathbf{1}_i$, $i = 1,2,3$. Correspondingly, the discretization of (6.6) gives rise to three independent 2-by-2 block lower triangular systems.

For each fixed $i = 1,2,3$, the equations of the first block are

$$\int_K \mathcal{K}^K \mathbf{q}_{\mathtt{m},e_i}\cdot\mathbf{v}_h\,dx - \int_K u_{\mathtt{m},e_i}\,\mathrm{div}\,\mathbf{v}_h\,dx = -\int_{\partial K}\mathbf{1}_i\mathbf{v}_h\cdot\mathbf{n}_{\partial K}\,ds,\quad \mathbf{v}_h\in\mathbb{RT}_0(K), \tag{6.19}$$

and their solution is the pair

$$\boldsymbol{q}_{\mathtt{m},e_i} = -\overline{\kappa}^K\nabla u^*_{\mathtt{m},e_i} = -\overline{\kappa}^K\nabla\widetilde{\varphi}_i,\qquad u_{\mathtt{m},e_i} = \mathcal{P}^0_K u^*_{\mathtt{m},e_i} = \frac{1}{3}. \tag{6.20}$$

Remark 6.5. Comparing (4.9) and (6.20), it is immediate to see that the difference between the values of $u_{\mathtt{m},e_i}$ computed by the lowest–order Galerkin DMH formulation and the Petrov-Galerkin DPG$_0$ formulation is $\mathcal{O}(h_K^2)$. This result extends to the case $d \geq 0$ previous relations proved in [17] and [7] in the case $d = 0$.

For each fixed $i = 1,2,3$, the equations of the second block are

$$\int_{\partial K}\mu_{\mathtt{m},e_i}\,w_h\,ds = \int_K \mathbf{q}_{\mathtt{m},e_i}\cdot\nabla w_h\,dx - \int_K \overline{d}^K\,u_{\mathtt{m},e_i}\,w_h\,dx,\qquad w_h\in\mathbb{P}_1(K). \tag{6.21}$$

Choosing $w_h \in \mathbb{P}_1^{nc}(K)$, system (6.21) becomes diagonal and its solution is

$$\mu_{\mathtt{m},e_i} = -\kappa^K\nabla u^*_{\mathtt{m},e_i}\cdot\mathbf{n}_{e_i} - \frac{\overline{d}^K}{3}\frac{|K|}{|e_i|}\mathcal{P}^0_K u^*_{\mathtt{m},e_i} = -\kappa^K\nabla\widetilde{\varphi}_i\cdot\mathbf{n}_{e_i} - \frac{\overline{d}^K}{9}\frac{|K|}{|e_i|}. \tag{6.22}$$

The discretization of (6.8) gives rise to a single 2-by-2 block lower triangular system, the solution of which is

$$
\begin{aligned}
&(\boldsymbol{q}_f,\, u_f) = [(0,\,0)^T,\, 0],\\
&\mu_{f,e_i} = \frac{1}{3}\frac{|K|}{|e_i|}\mathcal{P}^0_K f, \qquad i = 1,2,3.
\end{aligned}
\tag{6.23}
$$

Remark 6.6. Relation $(6.23)_2$ immediately yields

$$
\sum_{i=1}^{3} \mu_{f,e_i}\,|e_i| = \int_{\partial K} \mu_f\, ds = \int_K \mathcal{P}^0_K f\, dx = \int_K f\, dx \qquad \forall K \in \mathcal{T}_h,
$$

which expresses the local conservation property enjoyed by the DPG_0 approximation.

The corresponding entries of the local stiffness matrix and the load vector associated with the displacement–based formulation of the DPG_0 method (6.2) are given by

$$
\begin{aligned}
&(\mathbb{E}^K)_{i,j} = \int_K \overline{\kappa}^K \nabla\widetilde{\varphi}_j \cdot \nabla\widetilde{\varphi}_i\, dx + \frac{1}{3}\int_K \overline{d}^K\, \widetilde{\varphi}_i\, dx \qquad i,j = 1,2,3,\\
&(\mathbb{H}^K)_i = \int_K \mathcal{P}^0_K f\, \widetilde{\varphi}_i\, dx \qquad i = 1,2,3.
\end{aligned}
\tag{6.24}
$$

Remark 6.7. We observe that in the case $d = 0$, formulation (6.11) is a nonconforming scheme with harmonic average of the diffusion coefficient on each mesh element. (cf. Remark 6.5).

Remark 6.8. The discretization of the reaction term in $(6.24)_1$ does not yield a diagonal matrix, but a full 3×3 matrix, and the resulting scheme does not enjoy a discrete maximum principle for any $d \geq 0$ (see [18] for a discussion of this issue in the case of Galerkin dual-mixed methods using $\mathbb{RT}_0$ finite elements). In the present formulation, a simple way to overcome this problem is to perform a diagonal of the local reaction matrix. This procedure leads to a nonconforming monotone scheme, the global stiffness matrix $\mathbb{E}$ of which is a symmetric positive definite M-matrix.

6.4. **Substructuring of local mappings in the Petrov–Galerkin setting.** In this section we use again the tool of orthogonal decomposition in order to single out a further substructuring in the local mappings (6.6) and (6.8).
Let us introduce the null space [20]

$$
\mathcal{Z}^K_{1,h} = \ker \widehat{B}^K_1((u,\,0),\,\boldsymbol{v}) = \{\boldsymbol{v} \in V_h(K) \,|\, \mathrm{div}\,\boldsymbol{v} = 0\}.
$$

Then, we have the decomposition $V_h(K) = \mathcal{Z}^K_{1,h} \oplus (\mathcal{Z}^K_{1,h})^\perp$, orthogonal with respect to the L^2 inner product induced by the bilinear form $A^K(\cdot,\cdot)$. Moreover, let us introduce the affine manifold

$$
(\mathcal{Z}^K_{2,h})^\sigma := \{\boldsymbol{q} \in Q_h(K),\, \mu \in M_h(\partial K) \,|\, \widehat{B}^K_2((\boldsymbol{q},\,\mu),\, w) = (\sigma,\, w)_K \;\forall w \in W_h(K)\}.
$$

Finally, we let $\sigma_d := -\overline{d}^K\, u_{\mathtt{m}}$. Then, we have the following result.

Proposition 6.9. The local liftings $(\boldsymbol{q}_{\mathtt{m}}, \mu_{\mathtt{m}})$ and $(\boldsymbol{q}_f, \mu_f)$ are such that

$$(\boldsymbol{q}_{\mathtt{m}}, \mu_{\mathtt{m}}) \in (\mathcal{Z}^K_{2,h})^{\sigma_d}, \qquad (\boldsymbol{q}_f, \mu_f) \in (\mathcal{Z}^K_{2,h})^f. \tag{6.25}$$

Proof. Consider the mapping problem (6.6). Taking in $(6.6)_1$ $\boldsymbol{v}_h \in \mathcal{Z}^K_{1,h}$ and then $\boldsymbol{v}_h \in (\mathcal{Z}^K_{1,h})^\perp$, respectively, allows to completely determine $\boldsymbol{q}_{\mathtt{m}}$ and $u_{\mathtt{m}}$. Replacing these quantities into $(6.6)_2$, immediately gives $\mu_{\mathtt{m}}$ and $(6.25)_1$. Consider now the mapping problem (6.8). Taking in $(6.8)_1$ $\boldsymbol{v}_h \in \mathcal{Z}^K_{1,h}$ and $\boldsymbol{v}_h \in (\mathcal{Z}^K_{1,h})^\perp$, respectively, yields $\boldsymbol{q}_f = \mathbf{0}, u_f = 0$ irrespectively of d. Replacing these quantities into $(6.6)_2$, immediately gives μ_f and $(6.25)_2$. □

Remark 6.10. The structure of the DPG formulation introduces a decoupling in the lifting of d and f when static condensation is carried out. Precisely, the reaction term d is accounted for by $\mu_{\mathtt{m}}$, while the source term f is accounted for by μ_f (see [17] and [7] for a discussion on the relation between the variable μ_h and the vector field $\mathbf{q}_h$ in the case of mixed and nonconforming methods).

Prop. 6.9 suggests that a block Gauss–Seidel approach can be adopted to solve the linear systems arising from the local mapping problems.

Eq. $(6.6)_1$ can be written as

$$\begin{aligned} A^K(\boldsymbol{q}_{\mathtt{m}}, \boldsymbol{v}_h) &= G^K_{\mathtt{m}}(\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \mathcal{Z}^K_{1,h}, \\ \widehat{B}^K_1(u_{\mathtt{m}}, \boldsymbol{v}_h) &= G^K_{\mathtt{m}}(\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in (\mathcal{Z}^K_{1,h})^\perp, \end{aligned} \tag{6.26}$$

which yields the block diagonal system

$$\begin{bmatrix} A^0 & 0_{(2,1)} \\ 0^T_{(2,1)} & A^\perp \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{\mathtt{m}} \\ u_{\mathtt{m}} \end{bmatrix} = \begin{bmatrix} \mathtt{m}^0 \\ \mathtt{m}^\perp \end{bmatrix}, \tag{6.27}$$

where A^0 and $A^\perp$ are square matrices of size 2×2 and 1×1, respectively. Each sub–block of (6.27) is invertible because A^K is coercive on $\mathcal{Z}^K_{1,h}$ and $\widehat{B}^K_1$ satisfies the inf–sup condition. Eventually, the variable $\mu_{\mathtt{m}}$ is eliminated in favour of $u^*_{\mathtt{m}}$ by solving a 3×3 system, $\boldsymbol{q}_{\mathtt{m}}$ and $u_{\mathtt{m}}$ being given data. As for μ_f, the same 3×3 system as above must be solved, the difference being in the right–hand side, which only depends on the source term f.

7. Dirichlet–Neumann Boundary Conditions

In order to deal with the case where nonhomogenous Dirichlet boundary conditions $u = g_D$ are assumed, we still use superposition of effects. We continue to denote by η the extension to zero of the function, where $\mathcal{F}_h \subset \mathcal{E}_h$ to $\mathcal{E}_h$ still by η. As a consequence, if $\mathtt{m} = \lambda_h$ on $\mathcal{E}_{h,i}$ and $\mathtt{m} = \mathcal{P}^k_\Gamma g_D$ on Γ, we write $\mathtt{m} = \lambda_h + \mathcal{P}^k_\Gamma g_D$ and we proceed in the same way as in the homogeneous case.

When mixed Dirichlet–Neumann boundary conditions are considered, that is $\boldsymbol{q} \cdot \boldsymbol{n}_\Gamma = j_N$ on Γ_N, with $j_N \in H^{-1/2}(\Gamma_N)$ and $\Gamma = \Gamma_D \cup \Gamma_N$, we have to distinguish the DMH method from the DPG_0 method. In the DMH method, the Neumann boundary condition is weakly enforced in the sum of $(3.4)_3$, extended to all the edges, while in the DPG_0 method, the Neumann condition is enforced in an essential manner by the variable μ_h.

In the right–hand side of the DMH nonconforming problem (4.2), we have the presence of an additional term, which derives from the Neumann datum entering the right–hand side of (4.7). In an analogous manner, in the right–hand side of the DPG problem (6.11), we have the presence of an additional term, which derives from the Neumann datum entering the right–hand side of (6.15).

8. Conclusions

In this paper, stemming from the characterization of the static condensation procedure for mixed hybridized methods introduced in [9, 10], we have used Helmholtz decompositions to obtain a substructuring of the local mapping problems. This characterization turns out to be of special interest in p-type refinement or a variable degree strategy. Moreover, we have extended the variational characterization of static condensation to the DPG_0 scheme, which represents a more general saddle point formulation. Also in this case, Helmholtz decomposition has been used to yield a substructuring of the local mappings.

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